Edge Guarding Plane Graphs

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Abstract

Let $G = (V, E)$ be a plane graph. We say that a face f of G is *guarded* by an edge $vw \in E$ if at least one vertex from $\{v, w\}$ is on the boundary of f. For a planar graph class $\mathcal{G}$ we ask for the minimal number of edges needed to guard all faces of any n-vertex graph in $\mathcal{G}$. In this extended abstract we provide new bounds for two planar graph classes, namely the quadrangulations and the stacked triangulations.

1 Introduction

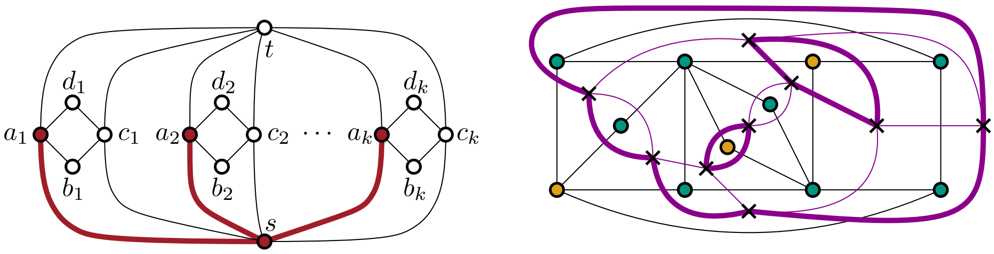
In 1975, Chvátal [4] laid the foundation for the widely studied field of *art gallery problems* by answering how many guards are needed to observe all interior points of any given n-sided polygon P. Here a guard is a point p in P and it can observe any other point q in P, if the line segment pq is fully contained in P. He shows that $\lfloor n/3 \rfloor$ guards are sometimes necessary and always sufficient. Fisk [7] revisited Chvátal's Theorem in 1978 and gave a very short and elegant new proof by introducing diagonals into the polygon P to obtain a triangulated, outerplanar graph. Such graphs are 3-colorable and in each 3-coloring all faces are incident to vertices of all three colors, so the vertices of the smallest color class can be used as guard positions. Bose et al. [3] studied the problem to guard the faces of a plane graph instead of a polygon. A *plane graph* is a graph $G = (V, E)$ with an embedding in $\mathbb{R}^2$ with not necessarily straight edges and no crossings in the interior of any two edges. Here a face f is guarded by a vertex v, if v is on the boundary of f. They show that $\lfloor n/2 \rfloor$ vertices (so called *vertex guards*) are sometimes necessary and always sufficient for n-vertex plane graphs.

We consider a variant of this problem introduced by O'Rourke [9]. He shows that only $\lfloor n/4 \rfloor$ guards are necessary in Chvátal's original setting if each guard is assigned to an edge of the polygon that he can patrol along instead of being fixed to a single point. Considering plane graphs again, an *edge guard* is an edge $vw \in E$ and guards all faces having v and/or w on their boundary. For a given planar graph class $\mathcal{G}$, we ask for the minimal number of edge guards needed to guard all faces of every plane n-vertex graph in $\mathcal{G}$.

General (not necessarily triangulated) n-vertex plane graphs might need at least $\lfloor n/3 \rfloor$ edge guards, even when requiring 2-connectedness [3]. The best known upper bounds have recently be presented by Biniaz et al. [1] and come in two different fashions: First, any n-vertex plane graph can be guarded by $\lfloor 3n/8 \rfloor$ edge guards found in an iterative process. Second, a coloring approach yields an upper bound of $\lfloor n/3 + \alpha/9 \rfloor$ edge guards where α counts the number of quadrangular faces in G. Looking at n-vertex triangulations, Bose et al. [3] provide a lower bound of $\lfloor (4n - 8)/13 \rfloor$ edge guards. A corresponding upper bound of $\lfloor n/3 \rfloor$ edge guards was published earlier in the same year by Everett and Rivera-Campo [6].

This note is based on the master's thesis of the first author [8] and we present our results on quadrangulations and stacked triangulations. For both planar graph classes we give a lower and an upper bound for the number of edge guards. All graphs considered below are assumed to be *plane*, i.e. given with a fixed plane embedding.

36th European Workshop on Computational Geometry, Würzburg, Germany, March 16–18, 2020.
This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.



(a) A quadrangulation with $4k + 2$ vertices needing k edge guards (drawn thick and red).

(b) A quadrangulation G (black edges) and its dual G^* (purple edges) with a 2-factor (thick edges). The vertex coloring is a guard coloring.

Figure 1 Lower and upper bound for quadrangulations.

2 Main Results

2.1 Quadrangulations

Quadrangulations are the maximal plane bipartite graphs and every face is bounded by exactly four edges. All coloring approaches developed previously [1, 6] fail on graphs containing quadrangular faces. The previously best known upper bounds are the ones given by Biniaz et al. [1] for general plane graphs, $\lfloor 3n/8 \rfloor$ respectively $\lfloor n/3 + \alpha/9 \rfloor$, where α is the number of quadrilateral faces. For n-vertex quadrangulations we have $\alpha = n - 2$, so $\lfloor n/3 + (n-2)/9 \rfloor = \lfloor (4n-2)/9 \rfloor > \lfloor 3n/8 \rfloor$ for $n \geq 4$. In this section we provide a better upper and a not yet matching lower bound. Closing the gap remains an open problem.

▶ **Theorem 2.1.** *For $k \in \mathbb{N}$ there exists a quadrangulation Q_k with $n = 4k + 2$ vertices needing $k = (n-2)/4$ edge guards.*

Proof. Define $Q_k = (V, E)$ with $V := \{s, t\} \cup \bigcup_{i=1}^{k} \{a_i, b_i, c_i, d_i\}$ and $E := \bigcup_{i=1}^{k} \{sa_i,\ sc_i,\ ta_i,\ tc_i,\ a_ib_i,\ a_id_i,\ c_ib_i,\ c_id_i\}$ as the union of k vertex disjoint 4-cycles and two extra vertices connecting them. Figure 1a shows this and a planar embedding. Now for any two distinct $i, j \in \{1, \ldots, k\}$ the two quadrilateral faces (a_i, b_i, c_i, d_i) and (a_j, b_j, c_j, d_j) are only connected via paths through s or t. Therefore, no edge can guard two or more of them and we need at least k edge guards for Q_k. On the other hand it is easy to see that $\{sa_1, \ldots, sa_k\}$ is an edge guard set of size k, so Q_k needs exactly k edge guards. ◀

The following Lemma is from Bose et al. [2] and we cite it using the terminology of Biniaz et al. [1]. A *guard coloring* of a plane graph G is a non-proper 2-coloring of its vertex set, such that each face f of G has at least one boundary vertex of each color and at least one monochromatic edge (i.e. an edge where both endpoints receive the same color). They prove that a guard coloring exists for all graphs without any quadrangular faces.

▶ **Lemma 2.2** ([2, Lemma 3.1]). *If there is a guard coloring for an n-vertex plane graph G, then G can be guarded by $\lfloor n/3 \rfloor$ edge guards.*

▶ **Theorem 2.3.** *Every quadrangulation can be guarded by $\lfloor n/3 \rfloor$ edge guards.*

Proof. Let G be a quadrangulation. We show that there is a guard coloring for G, which is sufficient by Lemma 2.2. Consider the dual graph $G^* = (V^*, E^*)$ of G with its inherited plane embedding, so each vertex $f^* \in V^*$ is placed inside the face f of G corresponding to it. Since every face of G is of degree four, its dual graph G^* is 4-regular. Using Petersen's

2-Factor Theorem [10][1] we get that G^* contains a 2-factor H (a spanning 2-regular subgraph). Any vertex of H is of degree 2, so H is a set of vertex-disjoint cycles that can be nested inside each other. Now define a 2-coloring $\text{col} : V \to \{0, 1\}$ for the vertices of G: For each $v \in V$ let c_v be the number of cycles C of H such that v belongs to the region of the embedding surrounded by C. The color of v is determined by the parity of c_v as $\text{col}(v) := c_v \mod 2$.

We claim that this yields a guard coloring of G: Any edge $e = ab \in E$ has a corresponding dual edge e^*. If $e^* \in E(H)$, then e crosses exactly one cycle edge, so $|c_a - c_b| = 1$ and therefore $\text{col}(a) \neq \text{col}(b)$. Otherwise $e \notin E(H)$, so its two endpoints are in the same cycles, thus $\text{col}(a) = \text{col}(b)$ and e is monochromatic. Because H is a 2-factor, each face has exactly two monochromatic edges. ◀

Figure 1b shows an example quadrangulation with a 2-factor in its dual graph. From here it is easy to color the vertices in green and orange to obtain a guard coloring.

In order to bridge the gap between the lower ($\lfloor (n-2)/4 \rfloor$) and the upper bound ($\lfloor n/3 \rfloor$), we also consider the subclass of 2-degenerate quadrangulations in the master's thesis [8, Theorem 5.9]:

▶ **Theorem 2.4.** *Every n-vertex 2-degenerate quadrangulation can be guarded by $\lfloor n/4 \rfloor$ edge guards.*

Note that this bound is best possible, as the quadrangulations constructed in Theorem 2.1 are 2-degenerate.

2.2 Stacked Triangulations

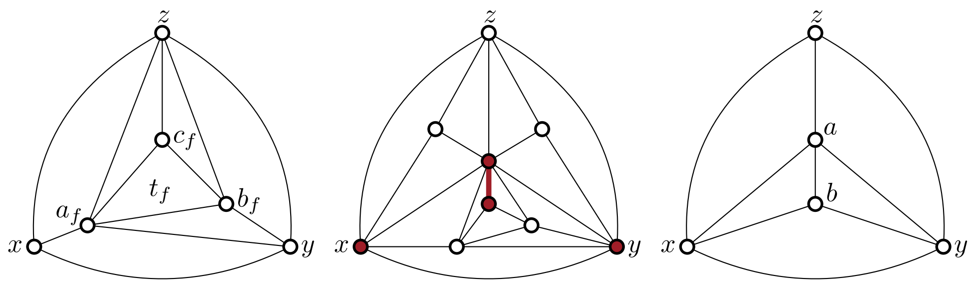
The stacked triangulations (also known as Apollonian networks or planar 3-trees) are a subclass of the triangulations that can recursively be formed by the following rules: (i) A triangle is a stacked triangulation and (ii) if G is a stacked triangulation and f an inner face, then the graph obtained by placing a new vertex into f and connecting it with all three boundary vertices is again a stacked triangulation. We shall prove that the stacked triangulations are a non-trivial subclass of the triangulations that need strictly less than $\lfloor n/3 \rfloor$ edge guards (which is the best known upper bound for general triangulations).

▶ **Theorem 2.5.** *For even $k \in \mathbb{N}$ there is a stacked triangulation G with $n = (7k + 4)/2$ vertices needing at least $k = (2n - 4)/7$ edge guards.*

Proof. Let S be a stacked triangulation with k faces and therefore $(k + 4)/2$ vertices (by Euler's formula). Subdivide each face f of S with three new vertices a_f, b_f, c_f such that the resulting graph is a stacked triangulation and these three vertices form a new triangular face t_f, i.e. f and t_f do not share any boundary vertices. This subdivision is shown in Figure 2a for a single face f. Then G has $n = (k + 4)/2 + 3k = (7k + 4)/2$ vertices. For any two distinct faces f, g of S the shortest path between any two boundary vertices of the new faces t_f and t_g has length at least 2, so no edge can guard both of them. Therefore G needs at least k edge guards. ◀

▶ **Theorem 2.6.** *Every n-vertex stacked triangulation can be guarded by $\lfloor 2n/7 \rfloor$ edge guards.*

[1] Diestel [5, Corollary 2.1.5] gives a very short and elegant proof of this theorem in his book. He only considers simple graphs there, but all steps in the proof (including the given proof of Hall's Theorem [5, 11, Theorem 2.1.2]) also work for multigraphs like G^* that have at most two edges between any pair of vertices.



(a) Face $f = (x, y, z)$ gets subdivided by three vertices a_f, b_f, c_f forming a new face t_f.

(b) If x, y are guarded, one additional edge suffices to guard all inner faces.

(c) Subdividing with two vertices allows to assume that $x, y \in V(\Gamma)$, where Γ is an edge guard set of minimum cardinality.

Figure 2 Lower and upper bound for stacked triangulations.

A proof of Theorem 2.6 is given in the master's thesis [8, Theorem 4.14] but it is too long for this extended abstract. We restrict ourselves to briefly describing the main idea: We do induction on n, the number of vertices. Given any n-vertex stacked triangulation, we find a triangle $\triangle := \{x, y, z\} \subseteq V(G)$ containing at least $k^- \geq 4$ vertices inside of it but among all possible candidates one where k^- is minimal. Let $V^- \subseteq V$ be the vertices in the interior of $\triangle$. We remove V^- from G, so $\triangle$ becomes a face and we subdivide it with $k^+ < k^-$ new vertices V^+. Call the resulting graph G'. Applying the induction hypothesis on G' provides us with an edge guard set Γ' of size at most $\lfloor 2|V(G')|/7 \rfloor$. We show that Γ' can be augmented to and edge guard set Γ for G with size $|\Gamma| = |\Gamma'| + \ell$, such that $\ell/(k^- - k^+) \leq 2/7$, so that Γ has size at most $\lfloor 2n/7 \rfloor$.

For example consider a stacked triangulation G with a separating triangle $\triangle = \{x, y, z\}$ as shown in Figure 2b with $k^- = 6$ vertices V^- inside (the figure only shows the separating triangle and its interior vertices). Assume for now that $V^+ = \emptyset$, so $\triangle$ is a face in G'. An edge guard set Γ' of G' guards $\triangle$, for example we could have $x \in V(\Gamma')$ and $y, z \notin V(\Gamma')$. But then – after reinserting the vertices of V^- – no single edge can guard all the remaining faces. So in this situation it is impossible to extend Γ' by a single edge to and edge guard set Γ for G. The following lemma tells us how to choose V^+ instead, such that such a situation cannot arise.

▶ **Lemma 2.7.** *Let $\{x, y, z\}$ be a face of a stacked triangulation G. By stacking two new vertices into $\{x, y, z\}$ we can obtain a stacked triangulation H such that for each edge guard set Γ of H there is an edge guard set Γ' with $x, y \in V(\Gamma')$ and $|\Gamma'| \leq |\Gamma|$.*

Proof. Add vertex a with edges xa, ya, za and then vertex b with edges ab, xb, yb to obtain H (see Figure 2c). Now let Γ be any edge guard set for H not yet fulfilling the requirements, so $|\{x, y\} \cap V(\Gamma)| \leq 1$. If $b \in V(\Gamma)$ as part of an edge vb, we can set $\Gamma' := (\Gamma \setminus \{vb\}) \cup \{xy\}$. This is possible, because for any possible neighbor v of b, edge xy guards a superset of the faces that vb guards. If otherwise $b \notin V(\Gamma)$, we assume without loss of generality that $x \in V(\Gamma)$ and $y \notin V(\Gamma)$. Note that $|\{x, y\} \cap V(\Gamma)| \geq 1$, because face $\{x, y, b\}$ must be guarded. Face $\{a, b, y\}$ can then only be guarded by edge va where $v \in \{x, z\}$. Since $N(a) \subseteq N(y)$ we can set $\Gamma' := (\Gamma \setminus \{va\}) \cup \{vy\}$. In both cases $x, y \in V(\Gamma')$ and $|\Gamma'| \leq |\Gamma|$. ◀

Let us go back to the example in Figure 2b: Using Lemma 2.7, we can now remove the six vertices in V^-, add two new ones $V^+ := \{a, b\}$ as in Figure 2c and assume that the induction hypothesis gives us an edge guard set Γ' with $x, y \in V(\Gamma')$. Then one additional edge is enough to guard the remaining inner faces and $\ell/(|V^-| - |V^+|) = 1/(6-2) \leq 2/7$ as desired. This guard set is shown in Figure 2b in red.

In addition to Lemma 2.7, we prove two more of this kind in the master's thesis [8] and which we list here without a proof. Like the lemma above, they describe how to add new vertices V^+ into a stacked triangulation, such that the resulting graph is still a stacked triangulation and that we can assume certain properties of minimal edge guard sets. Combining them, allows to handle all possible ways how the vertices V^- inside $\triangle$ can be connected.

▶ **Lemma 2.8.** *Let G be a stacked triangulation, v be a vertex of degree 3 and x, y, z its neighbors in G. Then for any edge guard set Γ guarding G we have $|\{v, x, y, z\} \cap V(\Gamma)| \geq 2$.*

▶ **Lemma 2.9.** *Let (x, y, z) be a face of a stacked triangulation G. By stacking three new vertices into (x, y, z) we can obtain a stacked triangulation H such that for each edge guard set Γ of H there is an edge guard set Γ' with $x \in V(\Gamma')$ and an edge $vw \in \Gamma'$ with $v \in \{x, y, z\}$ and w inside (x, y, z). Further $|\Gamma'| \leq |\Gamma|$.*

We conclude this note with the following open problems:

▶ **Open Problems.** How many edge guards are sometimes necessary and always sufficient for quadrangulations, (4-connected) triangulations and general plane graphs?

Acknowledgments

We thank Kolja Knauer for interesting discussions on the topic.

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