Homotopic Curve Shortening and the Affine Curve-Shortening Flow*

Sergey Avvakumov[1] and Gabriel Nivasch[2]

1 **Institute of Science and Technology Austria (IST Austria), Am Campus 1, 3400 Klosterneuburg, Austria**
sergey.avvakumov@ist.ac.at

2 **Ariel University, Ariel, Israel**
gabrieln@ariel.ac.il

Abstract

We define and study a discrete process that generalizes the convex-layer decomposition of a planar point set. Our process, which we call *homotopic curve shortening* (HCS), starts with a closed curve (which might self-intersect) in the presence of a set $P \subset \mathbb{R}^2$ of point obstacles, and evolves in discrete steps, where each step consists of (1) taking shortcuts around the obstacles, and (2) reducing the curve to its shortest homotopic equivalent.

We find experimentally that, if the initial curve is held fixed and P is chosen to be either a very fine regular grid or a uniformly random point set, then HCS behaves at the limit like the affine curve-shortening flow (ACSF). This connection between HCS and ACSF generalizes the link between "grid peeling" and the ACSF observed by Eppstein et al. (2017), which applied only to convex curves, and which was studied only for regular grids.

We prove that HCS satisfies some properties analogous to those of ACSF: HCS is invariant under affine transformations, preserves convexity, and does not increase the total absolute curvature. Furthermore, the number of self-intersections of a curve, or intersections between two curves (appropriately defined), does not increase. Finally, if the initial curve is simple, then the number of inflection points (appropriately defined) does not increase.

1 Introduction

Let $\mathbb{S}^1$ be the unit circle. In this paper we call a piecewise-smooth function $\gamma : [0,1] \to \mathbb{R}^2$ a *path*, and a piecewise-smooth function $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ a *closed curve*, or simply a *curve*. If γ is injective then the curve or path is said to be *simple*. We say that two paths or curves γ, δ are *ε-close* to each other if their Fréchet distance is at most ε, i.e. if they can be re-parametrized such that for every t, the Euclidean distance between the points $\gamma(t)$, $\delta(t)$ is at most ε.

1.1 Shortest Homotopic Curves

Let P be a finite set of points in the plane, which we regard as obstacles. Two curves γ, δ that avoid P are said to be *homotopic* if there exists a way to continuously transform γ into δ while avoiding P at all times. And two paths γ, δ that avoid P (except possibly at the endpoints) and satisfy $\gamma(0) = \delta(0)$, $\gamma(1) = \delta(1)$ are said to be *homotopic* if there exists a way to continuously transform γ into δ, without moving their endpoints, while avoiding P at all times (except possibly at the endpoints). We extend these definitions to the case where γ avoids obstacles but δ does not, by requiring the continuous transformation of γ into δ to avoid obstacles at all times except possibly at the last moment.

* Sergey Avvakumov was supported by the Austrian Science Fund (FWF), Project P31312-N35.

36th European Workshop on Computational Geometry, Würzburg, Germany, March 16–18, 2020.
This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

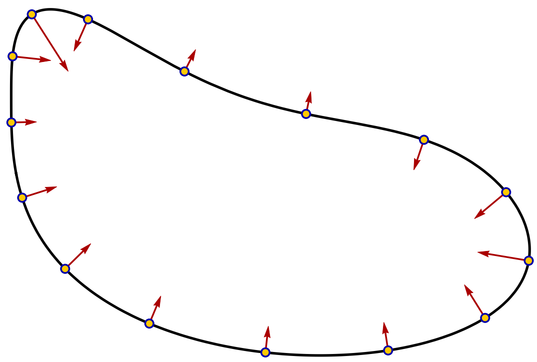

Figure 1 ACSF. Arrows indicate instantaneous velocity of points at the shown moment.

Then, for every curve (resp. path) γ in the presence of obstacles there exists a unique shortest curve (resp. path) δ that is homotopic to γ. The problem of computing the shortest path or curve homotopic to a given piecewise-linear path or curve, under the presence of polygonal or point obstacles, has been studied extensively [6, 8, 11, 17, 25, 26].

1.2 The Affine Curve-Shortening Flow

In the *affine curve-shortening flow*, a smooth curve $\gamma \subset \mathbb{R}^2$ varies with time in the following way. At each moment in time, each point of γ moves perpendicularly to the curve, towards its local center of curvature, with instantaneous velocity $r^{-1/3}$, where r is that point's radius of curvature at that time. See Figure 1.

The ACSF was first studied by Alvarez et al. [2] and Sapiro and Tannenbaum [27]. It differs from the more usual *curve-shortening flow* (CSF) [9, 13], in which each point is given instantaneous velocity r^{-1}. Unlike the CSF, the ACSF is invariant under affine transformations. It has applications in computer vision [9].

Under either the CSF or the ACSF, a simple curve remains simple, and its length decreases strictly with time ([13], [27], resp.). Furthermore, a pair of disjoint curves, run simultaneously, remain disjoint at all times ([28], [4], resp.). More generally, the number of intersections between two curves never increases ([3], [4], resp.). The total absolute curvature of a curve decreases strictly with time and tends to 2π ([20, 21], [4], resp.). The number of inflection points of a simple curve does not increase with time ([3], [4], resp.). Under the CSF, a simple curve eventually becomes convex and then converges to a circle as it collapses to a point [20, 21]. Correspondingly, under the ACSF, a simple curve becomes convex and then converges to an ellipse as it collapses to a point [4].

When the initial curve is not simple, a self-intersection might collapse and form a singularity that lasts for an instant. Unfortunately, unlike for the case of the CSF, for the ACSF no rigorous results have been obtained for self-intersecting curves [4]. Still, ACSF computer simulations can be run on curves that have self-intersections or singularities with little difficulty.

1.3 Relation to Grid Peeling

Let P be a finite set of points in the plane. The *convex-layer decomposition* (also called the *onion decomposition*) of P is the partition of P into sets $P_1, P_2, P_3, \dots$ obtained as follows: Let $Q_0 = P$. Then, for each $i \geq 1$ for which $Q_{i-1} \neq \emptyset$, let P_i be the set of vertices of the convex hull of Q_{i-1}, and let $Q_i = Q_{i-1} \setminus P_i$. In other words, we repeatedly remove from P the set of vertices of its convex hull. See [5, 12, 15, 16].

Eppstein et al. [18], following Har-Peled and Lidický [23], studied *grid peeling*, which is the convex-layer decomposition of subsets of the integer grid $\mathbb{Z}^2$. Eppstein et al. found an experimental connection between ACSF for convex curves and grid peeling. Specifically, let γ be a fixed convex curve. Let n be large, let $(\mathbb{Z}/n)^2$ be the uniform grid with spacing $1/n$, and let $P_n(\gamma)$ be the set of points of $(\mathbb{Z}/n)^2$ that are contained in the region bounded by γ. Then, as $n \to \infty$, the convex-layer decomposition of $P_n(\gamma)$ seems experimentally to converge to the ACSF evolution of γ, after the time scale is adjusted appropriately. They raised the question whether there is a way to generalize the grid peeling process so as to approximate ACSF for non-convex curves as well.

1.4 Our Contribution

In this paper we describe a generalization of the convex-layer decomposition to non-convex, and even non-simple, curves. We call our process *homotopic curve shortening*, or HCS. Under HCS, an initial curve evolves in discrete steps in the presence of point obstacles. We find that, if the obstacles form a uniform grid, then HCS shares the same experimental connection to ACSF that grid peeling does. Hence, HCS is the desired generalization sought by Eppstein et al. [18]. We also find that the same experimental connection between ACSF and HCS (and in particular, between ACSF and the convex-layer decomposition) holds when the obstacles are distributed uniformly at random, with the sole difference being in the constant of proportionality.

Although the experimental connection between HCS and ACSF seems hard to prove, we do prove that HCS satisfies some simple properties analogous to those of ACSF: HCS is invariant under affine transformations, preserves convexity, and does not increase the total absolute curvature. Furthermore, the number of self-intersections of a curve, or intersections between two curves (appropriately defined), does not increase. Finally, if the initial curve is simple, then the number of inflection points (appropriately defined) does not increase.

2 Homotopic Curve Shortening

Let P be a finite set of obstacle points. A *P-curve* (resp. *P-path*) is a curve (resp. path) that is composed of straight-line segments, where each segment starts and ends at obstacle points.

Homotopic curve shortening (HCS) is a discrete process that starts with an initial P-curve γ_0 (which might self-intersect), and at each step, the current P-curve γ_n is turned into a new P-curve $\gamma_{n+1} = \mathrm{HCS}_P(\gamma_n)$.

The definition of $\gamma' = \mathrm{HCS}_P(\gamma)$ for a given P-curve γ is as follows. Let $(p_0, \ldots, p_{m-1})$ be the circular list of obstacle points visited by γ. Call p_i *nailed* if γ goes straight through p_i, i.e. if $\angle p_{i-1}p_ip_{i+1} = \pi$.[1] Let $(q_0, \ldots, q_{k-1})$ be the circular list of nailed vertices of γ. Suppose first that $k \geq 1$. Then γ' is obtained through the following three substeps:

1. *Splitting.* We split γ into k P-paths $\delta_0, \ldots, \delta_{k-1}$ at the nailed vertices, where each δ_i goes from q_i to q_{i+1}.
2. *Shortcutting.* For each non-endpoint vertex p_i of each δ_i, we make the curve avoid p_i by taking a small shortcut. Specifically, let $\varepsilon > 0$ be sufficiently small, and let C_{p_i} be a

[1] All indices in circular sequences are modulo the length of the sequence.

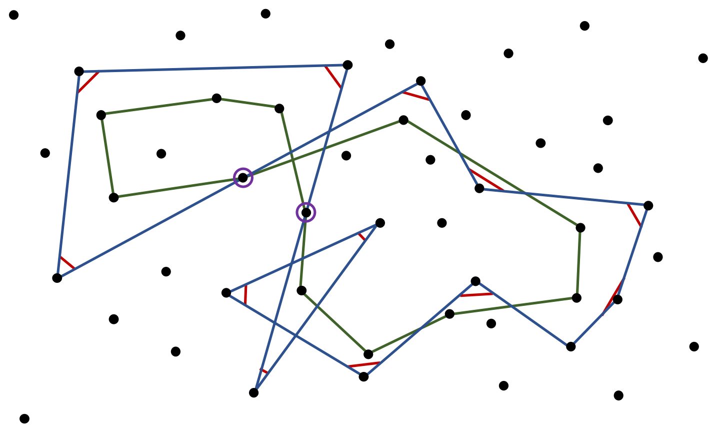

Figure 2 Computation of a single step of HCS: Given a P-curve γ (blue), we first identify its nailed vertices (purple). In this case, the two nailed vertices split γ into two paths δ_0, δ_1. In each δ_i we take a small shortcut around each intermediate vertex (red). Then we replace each δ_i by the shortest path homotopic to it, obtaining the new P-curve $\gamma' = \mathrm{HCS}_P(\gamma)$ (green).

circle of radius ε centered at p_i. Let e_i be the segment $p_{i-1}p_i$ of δ_i. Let $x_i = e_i \cap C_{p_i}$ and $y_i = e_{i+1} \cap C_{p_i}$. Then we make the path go straight from x_i to y_i instead of through p_i. Call the resulting path ρ_i, and let ρ be the curve obtained by concatenating all the paths ρ_i.

3. *Shortening.* Each ρ_i in ρ is replaced by the shortest P-path homotopic to it. The resulting curve is γ'.

If γ has no nailed vertices ($k = 0$) then γ' is obtained by performing the shortcutting and shortening steps on the single closed curve γ. Figure 2 illustrates one HCS step on a sample curve.

The process terminates when the curve collapses to a point. If the initial curve γ_0 is the boundary of the convex hull of P, then the HCS evolution of γ_0 is equivalent to the convex-layer decomposition of P.

3 Experimental Connection Between ACSF and HCS

Our experiments on a variety of curves show that HCS, using $P = (\mathbb{Z}/n)^2$ as the obstacle set, approximates ACSF at the limit as $n \to \infty$, just as grid peeling approximates ACSF for convex curves. Furthermore, we find that the connection between ACSF and HCS also holds if the uniform grid $(\mathbb{Z}/n)^2$ is replaced by a random point set, though with a different constant of time proportionality. See Figure 3 for an example.

4 Properties of Homotopic Curve Shortening

We prove that HCS satisfies some properties analogous to those of ACSF.

▶ **Theorem 4.1.** *HCS is invariant under affine transformations.*

▶ **Theorem 4.2.** *Under HCS, once a curve becomes the boundary of a convex polygon, it stays that way.*

The *total absolute curvature* of a piecewise-linear curve γ with vertices $(p_0, \ldots, p_{m-1})$ is the sum of the exterior angles $\sum_{i=0}^{m-1}(\pi - |\angle p_{i-1}p_ip_{i+1}|)$.

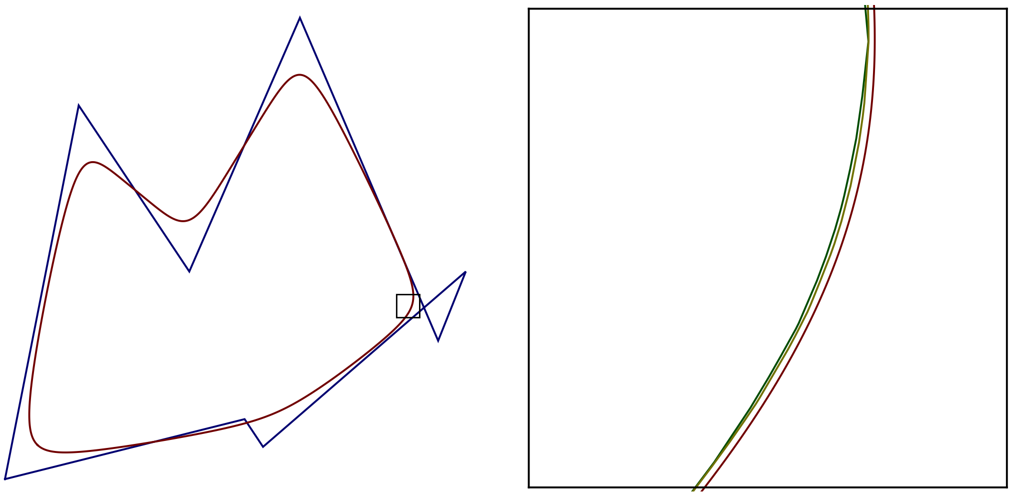

Figure 3 Left: Initial curve (blue) and simulated ACSF result after the curve's length reduced to 70% of its original length (red). Right: Comparison between ACSF approximation (red), HCS with $n = 10^7$ uniform-grid obstacles (green), and HCS with $n = 10^7$ random obstacles (yellow) on a small portion of the curve.

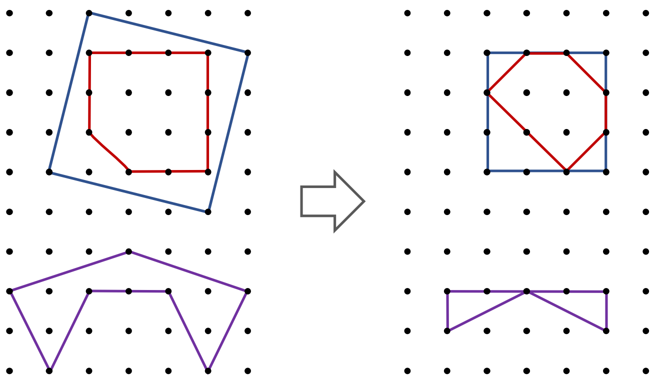

Figure 4 HCS might cause disjoint curves to intersect, or a simple curve to self-intersect.

▶ **Theorem 4.3.** *Under HCS, the total absolute curvature of a curve never increases.*

If γ, δ are disjoint P-curves, then $\mathrm{HCS}_P(\gamma), \mathrm{HCS}_P(\delta)$ are not necessarily disjoint. Similarly, if γ is a simple P-curve, then $\mathrm{HCS}_P(\gamma)$ is not necessarily simple. See Figure 4.

Curves γ, δ are called *disjoinable* if they can be made into disjoint curves by peforming on them an arbitrarily small perturbation. Similarly, a curve γ is called *self-disjoinable* if it can be turned into a simple curve by an arbitrarily small perturbation. (Akitaya et al. [1] recently found an $O(n \log n)$-time algorithm that can decide, in particular, whether a given curve is self-disjoinable.)

An intersection between two curves, or between two portions of one curve, is called *transversal*, if at the point of intersection both curves are differentiable and their normal vectors are not parallel at that point. If all intersections between curves γ_1 and γ_2 are transversal, then we say that γ_1, γ_2 are themselves *transversal*. Similarly, if all self-intersections of γ are transversal, then we say that γ is *self-transversal*. (Transversal and self-transversal curves are sometimes called *generic*, see e.g. [10].)

If γ is self-transversal, we denote by $\chi(\gamma)$ the number of self-intersections of γ. If γ is not self-transversal, then we define $\chi(\gamma)$ as the minimum of $\chi(\widehat{\gamma})$ among all self-transversal curves $\widehat{\gamma}$ that are ε-close to γ, for all small enough $\varepsilon > 0$. Hence, $\chi(\gamma) = 0$ if and only if γ is self-disjoinable. We define similarly the number of intersections $\chi(\gamma_1, \gamma_2)$ between two curves. Then, γ_1 and γ_2 are disjoinable if and only if $\chi(\gamma_1, \gamma_2) = 0$. (Fulek and Tóth recently proved that the problem of computing $\chi(\gamma)$ is NP-hard [19].)

▶ **Theorem 4.4.** *Under HCS, the intersection and self-intersection numbers never increase.*

We say that an obstacle set P is in *general position* if no three points of P lie on a line. Note that if P is in general position then there are no nailed vertices in HCS.

▶ **Theorem 4.5.** *Under HCS with obstacles in general position, a simple curve stays simple, and a pair of disjoint curves stay disjoint.*

Let γ be a simple piecewise-linear curve with vertices $(v_0, \ldots, v_{n-1})$. An *inflection edge* of γ is an edge $v_i v_{i+1}$ such that the previous and next vertices v_{i-1}, v_{i+2} lie on opposite sides of the line through v_i, v_{i+1}. Let $\varphi(\gamma)$ be the number of inflection edges of γ. Note that $\varphi(\gamma)$ is always even, since every inflection edge lies either after a sequence of clockwise vertices and before a sequence of counterclockwise vertices, or vice versa.

If γ is not simple but self-disjoinable, then we define $\varphi(\gamma)$ as the minimum of $\varphi(\gamma')$ over all simple piecewise-linear curves γ' that are ε-close to γ, for all sufficiently small $\varepsilon > 0$.

▶ **Theorem 4.6.** *Under HCS on a self-disjoinable curve, the curve's number of inflection edges never increases.*

5 Discussion

One of the reasons continuous curve-shortening flows were introduced and studied was to overcome the shortcomings of the *Birkhoff curve-shortening process* ([7], see also e.g. [14]), specifically the fact that it might cause the number of curve intersections to increase [22, 24]. As we have shown, HCS is a discrete process that overcomes this flaw without introducing analytical difficulties, at least in the plane. It would be interesting to check whether HCS can be applied on more general surfaces.

For more details see our full paper at [arXiv:1909.00263].

Acknowledgements. Thanks to Arseniy Akopyan, Imre Bárány, Jeff Erickson, Radoslav Fulek, Jeremy Schiff, Arkadiy Skopenkov, and Peter Synak for useful discussions. Thanks also to the referees for their useful comments.

References

1 Hugo A. Akitaya, Radoslav Fulek, and Csaba D. Tóth. Recognizing weak embeddings of graphs. In *Proc. 29th Symp. on Discrete Algorithms*, pages 274–292, 2018. doi:10.1137/1.9781611975031.20.

2 Luis Alvarez, Frédéric Guichard, Pierre-Luis Lions, and Jean-Michel Morel. Axioms and fundamental equations of image processing. *Arch. Rational Mech. Anal.*, 123(3):199–257, 1993. doi:10.1007/BF00375127.

3 Sigurd Angenent. Parabolic equations for curves on surfaces: Part II. Intersections, blow-up and generalized solutions. *Annals of Mathematics*, 133(1):171–215, 1991. doi:10.2307/2944327.

4 Sigurd Angenent, Guillermo Sapiro, and Allen Tannenbaum. On the affine heat equation for non-convex curves. *J. Amer. Math. Soc.*, 11(3):601–634, 1998. doi:10.1090/S0894-0347-98-00262-8.

5 Vic Barnett. The ordering of multivariate data. *J. Roy. Statist. Soc. Ser. A*, 139(3):318–355, 1976. doi:10.2307/2344839.

6 Sergei Bespamyatnikh. Computing homotopic shortest paths in the plane. *Journal of Algorithms*, 49(2):284–303, 2003. doi:https://doi.org/10.1016/S0196-6774(03)00090-7.

7 George D. Birkhoff. Dynamical systems with two degrees of freedom. *Trans. Amer. Math. Soc.*, 18:199–300, 1917.

8 Sergio Cabello, Yuanxin Liu, Andrea Mantler, and Jack Snoeyink. Testing homotopy for paths in the plane. *Discrete & Computational Geometry*, 31(1):61–81, 2004. doi:10.1007/s00454-003-2949-y.

9 Frédéric Cao. *Geometric Curve Evolution and Image Processing*, volume 1805 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003. doi:10.1007/b10404.

10 Hsien-Chih Chang and Jeff Erickson. Untangling planar curves. *Discrete & Computational Geometry*, 58:889–920, 2017. doi:10.1007/s00454-017-9907-6.

11 Bernard Chazelle. A theorem on polygon cutting with applications. In *Proc. 23rd Annual Symposium on Foundations of Computer Science (FOCS 1982)*, pages 339–349, 1982. doi:10.1109/SFCS.1982.58.

12 Bernard Chazelle. On the convex layers of a planar set. *IEEE Trans. Inform. Theory*, 31(4):509–517, 1985. doi:10.1109/TIT.1985.1057060.

13 Kai-Seng Chou and Xi-Ping Zhu. *The Curve Shortening Problem.* Chapman & Hall/CRC, Boca Raton, FL, 2001. doi:10.1201/9781420035704.

14 Cristopher B. Croke. Area and the length of the shortest closed geodesic. *J. Differential Geometry*, 27:1–21, 1988.

15 Ketan Dalal. Counting the onion. *Random Struct. Algor.*, 24(2):155–165, 2004. doi:10.1002/rsa.10114.

16 William F. Eddy. Convex Hull Peeling. In *COMPSTAT 1982 5th Symposium held at Toulouse 1982*, pages 42–47. Physica-Verlag, 1982. doi:10.1007/978-3-642-51461-6_4.

17 Alon Efrat, Stephen G. Kobourov, and Anna Lubiw. Computing homotopic shortest paths efficiently. *Computational Geometry*, 35(3):162–172, 2006. doi:https://doi.org/10.1016/j.comgeo.2006.03.003.

18 David Eppstein, Sariel Har-Peled, and Gabriel Nivasch. Grid peeling and the affine curve-shortening flow. *Experimental Mathematics*, page to appear, 2018. https://doi.org/10.1080/10586458.2018.1466379. doi:10.1080/10586458.2018.1466379.

19 Radoslav Fulek and Csaba D. Tóth. Crossing minimization in perturbed drawings. In T. Biedl and A. Kerren, editors, *Proc. 26th Symp. Graph Drawing and Network Visualization*, pages 229–241. Springer, 2018. doi:10.1007/978-3-030-04414-5_16.

20 Michael Gage and Richard S. Hamilton. The heat equation shrinking convex plane curves. *J. Differential Geom.*, 23(1):69–96, 1986. doi:10.4310/jdg/1214439902.

21 Matthew A. Grayson. The heat equation shrinks embedded plane curves to round points. *J. Differential Geom.*, 26(2):285–314, 1987. doi:10.4310/jdg/1214441371.

22 Matthew A. Grayson. Shortening embedded curves. *Annals of Mathematics*, 129(1):79–111, 1989.

23 Sariel Har-Peled and Bernard Lidický. Peeling the grid. *SIAM J. Discrete Math.*, 27(2):650–655, 2013. doi:10.1137/120892660.

24 Joel Hass and Peter Scott. Shortening curves on surfaces. *Topology*, 33:25–43, 1994. doi:10.1016/0040-9383(94)90033-7.

25 John Hershberger and Jack Snoeyink. Computing minimum length paths of a given homotopy class. *Computational Geometry*, 4(2):63–97, 1994. doi:https://doi.org/10.1016/0925-7721(94)90010-8.

26 Der-Tsai Lee and Franco P. Preparata. Euclidean shortest paths in the presence of rectilinear barriers. *Networks*, 14(3):393–410, 1984. doi:10.1002/net.3230140304.

27 Guillermo Sapiro and Allen Tannenbaum. Affine invariant scale-space. *Int. J. Comput. Vision*, 11(1):25–44, 1993. doi:10.1007/bf01420591.

28 Brian White. Evolution of curves and surfaces by mean curvature. In *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)*, pages 525–538, 2002. URL: https://www.mathunion.org/fileadmin/ICM/Proceedings/ICM2002.1/ICM2002.1.ocr.pdf.