Master Thesis

On the Segment Number of 4-Regular Planar Graphs

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Abstract

The segment number of a planar graph is the smallest number of line segments needed to draw the graph plane with straight-line edges. Using a technique of Hong and Nagamochi [HN10] about convex drawings, we prove that every 3-connected 4-regular planar graph can be realised such that every inner vertex is placed in the interior of some line segment. This yields that the segment number of such a graph G is at most $|V(G)|+3$. In contrast, there is an infinite family of 3-connected 4-regular planar graphs with segment number of at least $|V(G)|$. The class of 2-connected 4-regular planar graphs contains a family of graphs where each graph G has segment number of at least $7|V(G)|/6$.

Zusammenfassung

Die Streckenzahl eines Graphen ist die kleinste Anzahl von gerade gezeichneten Strecken, die benötigt wird um den Graphen geradlinig und planar darzustellen. In dieser Arbeit konzentrieren wir uns auf 3-zusammenhängende, 4-reguläre, planare Graphen und zeigen mit Hilfe einer Beweistechnik von Hong und Nagamochi [HN10] über konvexe Graphzeichnungen, dass für jeden derartigen Graphen eine Zeichnung existiert, in der jeder innere Knoten in der Zeichnung auf dem Inneren einer Strecke liegt. Mit diesem Resultat folgern wir, dass die Streckenzahl solcher Graphen durch $|V(G)| + 3$ nach oben beschränkt ist. Weiterhin zeigen wir, dass es unendlich viele 3-zusammenhängende, 4-reguläre, planare Graphen gibt, die jeweils mindestens die Streckenzahl $|V(G)|$ haben. Darüber hinaus behandeln wir die Menge der 2-zusammenhängenden, 4-regulären, planaren Graphen und geben eine Teilmenge dieser an, in der jeder Graph G eine Streckenzahl von mindestens $7|V(G)|/6$ hat.

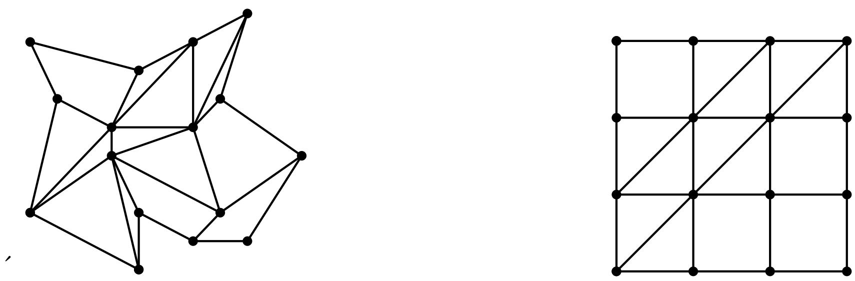
Contents

1 Introduction

Graphs are a widely used and beneficial method to model relations between different entities. As in every presentation of data, it is important to keep the comprehensibility high for the user. In terms of graphs this means to optimize the design and drawing according to some aesthetic criteria. A frequently used layout for graph drawings, which is also used in this thesis, is the representation of the vertices as dots and of the edges between them as lines. Well-studied aesthetic criteria for graph drawings are that crossings and bends in the drawing of edges should be minimized. As experimentally verified by Helen Purchase, Robert Cohen and Murray James [PCJ95], increasing the number of edge crossings or the number of edge bends decreases the understandability of the graph. For the sake of brevity, we refer in this thesis to a straight-line, crossing-free drawing just as a drawing.

Segments A *segment* in a straight-line drawing is a maximal set of edges that form a straight line segment [DMNW13]. The visual complexity of a drawing is defined as the total number of geometric objects (such as segments) that are used in the drawing. An example of two drawings of the same graph with different visual complexities is illustrated in Figure 1.1.

Kindermann, Meulemans and Schulz [KMS17] verified experimentally that users without mathematical background show a preference for graphs with a lower visual complexity. This motivates to study the minimum number of segments that is needed in any drawing of a planar graph G. This number is called the *segment number* of G. For example the segment number of the octahedron is 9 as shown by Kryven, Ravsky and Wolff [KRW19]. A drawing of the octahedron with 9 segments is illustrated in Figure 1.2.



(a) Drawing with a high number of segments and therefore a high visual complexity.

(b) Drawing with a lower visual complexity.

Fig. 1.1: Two drawings with different visual complexities of the same graph.

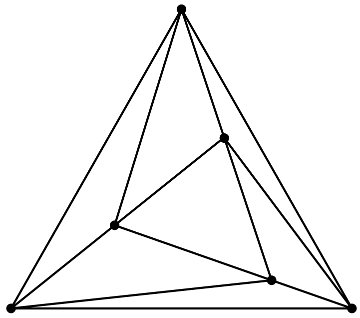


Fig. 1.2: Drawing with nine segments of the octahedron

Bounds In order to analyse the segment number of graph families, we use three different bounds:

- The *existential lower bound* $\mathfrak{e}$ for the segment number of a graph family states that there exists a family member G with a segment number of at least $\mathfrak{e}$.
- The *universal lower bound* $\mathfrak{s}$ for the segment number of a graph family states that each family member G has a segment number of at least $\mathfrak{s}$.
- The *upper bound* $\mathfrak{u}$ for the segment number of a graph family states that every family member has a segment number of at most $\mathfrak{u}$.

Related Work Durocher, Mondal, Nishat and Whitesides [DMNW13] showed that it is NP-hard to determine whether a plane graph G with maximum degree four can be drawn with $k \geq 3$ segments even when the drawing is addionally convex. This result indicates that it is not a trivial problem to determine the segment number of a graph efficiently. Therefore a couple of authors already studied the segment number of special types of graphs. An overview of the results is given in Table 1.1.

Dujmović, Eppstein, Suderman and Wood [DESW07] observed that every drawing of a planar graph G needs at least $\eta(G)/2$ segments, where $\eta(G)$ is the number of odd-degree vertices in G. Another lower bound for the segment number is given by the so-called slope number of G. The slope number of a drawing is the number of different slopes of the edges that are used in the drawing. The slope number of G is the minimum of the slope numbers of any drawing of G. Furthermore, they proved that every tree T has the segment number $\eta(T)/2$.

Dujmović et al. [DESW07] also studied maximal outerplanar graphs and showed that every outerplanar graph G has an outerplanar drawing with at most n segments and if $n \geq 3$, any drawing of G has at least n segments. For 2-trees Dujmović et al. [DESW07] proved that there exists a drawing with at most $3n/2$ segments and that the upper bound for the segment number of plane 3-trees is $2n - 2$, which is tight. Moreover Dujmović et al. [DESW07] studied 3-connected plane graphs and showed that every 3-connected plane graph has a plane drawing with at most $5n/2 - 3$ segments. The results were used by Heigl [Hei21] to prove that every 3-connected 4-regular planar graph G with

Graph class	upper bound	ex. lower bound	univ. lower bound
planar connected	$\frac{n-3m-28}{3}$ [DM19] $\frac{8}{3}n - \frac{14}{3}$ [KMSS19]	$2n-2$ [DESW07]	$\frac{\eta}{2}$ [DESW07]
planar 2-conn.	–	$\frac{5}{2}n-4$ [DESW07]	–
planar 3-conn.	$\frac{5}{2}n-3$ [DSW04]	$2n-6$ [DESW07]	$\sqrt{2n}$ [DESW07]
planar 3-conn. 4-reg.	$n+3$ 20	n 28	$\Theta(\sqrt{n})$ 23¸
planar 3-conn. 3-reg.	$\frac{n}{2}+3$ [BMNR10] [IMS17]	–	$\frac{n}{2}+3$ [DESW07]
triangulation	$\frac{7}{3}n - \frac{10}{3}$ [DM19]	$2n-2$ [DESW07]	$\Omega(\sqrt{n})$ [DESW07]
trian. max-deg 6	$\Omega(\sqrt{n})$ [DESW07]	$2n-6$ [DESW07]	–
trian. 4-conn.	$\frac{9}{4}n - \frac{9}{4}$ [DM19]	$2n-6$ [DESW07]	$\Omega(\sqrt{n})$ [DESW07]
trees	$\frac{\eta}{2}$ [DESW07]	–	$\frac{\eta}{2}$ [DESW07]
2-trees	$\frac{3}{2}n$ [DESW07]	–	$\frac{3}{2}n-2$ [DESW07]
planar 3-trees	$2n-2$ [DESW07]	$2n-2$ [DESW07]	–
maximal outerplanar	n [DESW07]	n [DESW07]	–

Tab. 1.1: Overview of the results regarding the segment number. n is the number of vertices, m the number of edges and η the number of vertices of odd degree in a graph.

n vertices has a drawing with at most $5n/3$ segments. Durocher and Mondal [DM19] improved the upper bound of 3-connected plane graphs for triangulations to $7n/3-10/3$ and in the case of 4-connected triangulations to $9n/4-9/4$.

Biswas, Mondal, Nishat and Rahman [BMNR10] gave an algorithm that constructs a drawing with $n/2+3$ segments for every cubic planar 3-connected graph (except K_4). Igamberdiev, Meulemans and Schulz [IMS17] presented two new algorithms that also generates drawings of cubic planar 3-connected graphs (except K_4) with $n/2+3$ segments and compared the performance of all three algorithms.

Durocher and Mondal [DM19] proved that every planar, connected graph can be drawn with at most $(n-3m-28)/3$ segments. Kindermann, Mchedlidze, Schneck and Symvonis [KMSS19] expanded the argumentation and proofed an universal upper bound for the segment number of planar, connected graphs of $8n/3-14/3$

Contribution First, we establish used notations in Chapter 2. The main result Theorem 20 can be found in Chapter 3. In this chapter, we start with some preliminary results, which we use in Theorem 20 to show that the every 3-connected 4-regular planar graph has a convex drawing with at most $|V(G)|+3$ segments. This result improves the upper bound of Dujmović et al. [DESW07] and Heigl [Hei21] of $5|V(G)|/3$ to $|V(G)|+3$. The given proof is based on a technique of Hong and Nagamochi [HN10], who introduced an algorithm for constructing a convex drawing of 3-connected planar graphs and an improved version of their algorithm from Klemz [Kle21]. Both papers describe a recursive combinatorial construction of the convex drawing. Their main idea of the construction is to split the given graph into three subgraphs that are handled recursively by using so-called archfree paths.

In Section 4.2, we introduce a set of 3-connected 4-regular planar graphs whose segment number is at least $|V(G)|$. This shows that the upper bound of $|V(G)| + 3$ for the segment number of 3-connected 4-regular planar graphs is tight up to an additive constant.

Furthermore, in Section 4.1 we give an example for a 3-connected 4-regular planar graph set such that every graph G in this set can be drawn with at most $\sqrt{4|V(G)|}$ segments. In combination with results from Dujmović et al. [DESW07] this graph set can be used to show that the universal lower bound for the segment number of 3-connected 4-regular planar graphs can not be asymptotically better than $\Theta(\sqrt{|V(G)|})$

In Section 4.3 we analyse the segment number of a set of 2-connected 4-regular planar graphs and we prove that every graph G in this set has at least $7|V(G)|/6$ segments in any drawing.

Remark on the Publication Parts of the results in this thesis were submitted for publication in advance. Beside results of the other authors Jonathan Klawitter, Boris Klemz, Felix Klesen, Stephen Kobourov, Myroslav Kryven, Alexander Wolff and Johannes Zink, rewritten versions of Chapter 3 and Section 4.2 were part of the submitted paper.

2 Terminology

Notations Let G be a planar graph. We call the set of boundaries of each face in G the *combinatorial embedding* of G. The combinatorial embedding of a 3-connected graph is unique. A planar graph is *plane* if it is equipped with a combinatorial embedding and a selected outer face.

Let G be a plane graph and let f_0 denote its outer face. For each face f we denote by ∂f the counterclockwise sequence of edges on the boundary of face f. Analogously ∂G denotes the counterclockwise sequence of edges on the boundary of G and is defined as ∂f_0. Note that as long as G is 2-connected, ∂f and ∂G are simple cycles. A vertex v is *part of* ∂f (resp. *part of* ∂G) if it is a endvertex of an edge in the sequence ∂f. We denote this by writing $v \in \partial f$ (resp. $v \in \partial G$). A vertex v in G is an *outer vertex* if it is part of ∂G, otherwise v is an *inner vertex*. A path P is an *inner path* if every vertex on P is an inner vertex. If P is a path, $|P|$ is defined as the number of edges on the path. Every path has a start- and an endvertex. With $|f|$ we refer to the number of edges in the sequence ∂f. With $V(G)$ (resp. $E(G)$) we denote the set of vertices (resp. edges) in G.

Definition of used graph properties For the purpose of this thesis we assume that all graphs in this thesis are simple that means that they do not have parallel edges or self-loops. Furthermore, we ignore in the argumentation whether the graph is directed.

Definition 1. *Let $G = (V, E)$ be a plane graph and let f_0 denote its outer face. The Graph G is called* 3-connected *(resp. k-connected) if and only if the following equivalent statements are satisfied:*

- *If we remove two (resp. $k-1$) arbitrary vertices with the related edges from G, the resulting graph is always connected.*
- *For every vertex v in V, we can find three (resp. k) simple paths p_i ($i \in \{1, 2, 3\}$) which pairwise intersect only in v, start in v and end on the boundary of the outer face.*

Definition 2. *Let G be a plane 2-connected graph and let f_0 denote its outer face. Then G is called* internally 3-connected *if and only if the following equivalent statements are satisfied:*

- *Inserting a new vertex v in f_0 and adding edges between v and all vertices of f_0 results in a 3-connected graph.*
- *From each internal vertex w from G there exist three paths to f_0 that are pairwise disjoint except for the common vertex w*

- *Every separation pair u, v of G is external, meaning that u and v lie on ∂f_0 and every connected component of the subgraph of G induced by $V(G) \setminus \{u, v\}$ contains a vertex of ∂f_0.*

Definition 3. *Let G be a planar graph such that each vertex has degree* 4*, then G is called* 4-regular*. Let G' be a plane graph. If every inner vertex in G' has degree 4 and every outer vertex has maximum degree 4, then the graph is called* internally 4-regular*.*

Note that every 4-regular plane graph is also automatically internally 4-regular. Hence, results for the upper bound of internally 4-regular 3-connected planar graphs are transferable to 4-regular 3-connected planar graphs by choosing an outer face.

Definition of drawing properties A drawing of a plane graph G is called a *straight-line* drawing if each edge is realised as a straight line without bends. The drawing is *crossing-free* if the drawn edges intersect pairwise only in their endvertices. For simplicity, we refer to a straight-line, crossing-free drawing (in $\mathbb{R}^2$) just as a drawing.

Definition 4. *A drawing of a polygon is called* convex *if every internal angle of the polygon is at most π. A drawing Γ of a plane graph G is called* convex *if the boundary of each inner face is drawn as a convex polygon.*

3 Upper Bound for the segment number of 3-connected 4-regular planar graphs

In this chapter, we derive an upper bound for the segment number of 3-connected 4-regular planar graphs. In order to prove that an upper bound for the segment number of this graph class is $|V(G)| + 3$, we show that every member of this graph class has a drawing with the property that every vertex except three of them are drawn in the interior of a segment (see Theorem 18, Theorem 19). In this drawing, it holds that in each vertex (except three) maximal two segments end while in the other three vertices at most four segments end. Altogether, we obtain that the drawing of G contains maximal $|V(G)| + 3$ segments (see Theorem 20). Later in the thesis in Theorem 28, we will give an existential lower bound that shows that the obtained upper bound is tight up to an additive constant.

3.1 Preliminaries

In this section, we first introduce some additional definitions and results that will be helpful to prove Theorem 20.

Archfree paths First, we define the property "archfree" for paths, then we present some results how to construct such paths. Later in the proof of Theorem 18 we will use archfree paths to "cut" the graph into subgraphs, therefore it will be useful to have some strategies how to construct them.

Definition 5. *A path P is* arched *by a face f if P contains two distinct vertices a, b such that the subpath P_{ab} of P between a and b is not a subpath of the boundary of f (see Figure 3.1 for an example). A path P is called* archfree *if it is not* arched *by any internal face f.*

As it can easily be observed in Figure 3.1, an arched path P cannot be realised as a straight-line segment in a convex drawing because in this case f could not be drawn convex. Furthermore, we observe that an archfree path is automatically simple.

Now we focus on how to construct archfree paths in internally 3-connected plane graphs. The following Lemma presents a practical result. It states that the subpaths of the boundary of an inner face f that do not contain at least two of the edges in ∂f are archfree.

Lemma 6 ([HN10], Lemma 1)**.** *Let $G = (V, E)$ be an internally 3-connected plane graph and let f be an internal face of G. Any subpath Q of ∂f with $|Q| \leq |f| - 2$ is an archfree path.*

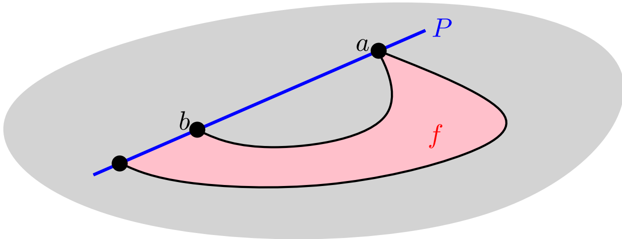


Fig. 3.1: Path P is arched by face f. The subpath P_{ab} of P between the two vertices a and b is not a subpath of the boundary of f.

Definition 7 ([HN08])**.** *A path Q between s and t is* extendible *in a plane graph $G = (V, E)$, if it is a subpath of a path between two outer vertices s' and t'. A face f* arches Q on the left side *if f is on the left side of Q. Analogously another face can* arch Q on the right side*.*

As illustrated in Figure 3.2, we define the left-aligned path $L(Q)$ *of Q as an inner path from s to t, obtained by replacing subpaths of Q with subpaths of the arching faces as follows: For each arching face f, let a_f and b_f be the first and last vertices in $V(f) \cap V(Q)$ when we walk along path Q from s to t, and f_Q be the subpath from a_f to b_f obtained by traversing f in the anticlockwise order. The path $L(Q)$ is the path obtained by replacing the subpath from a_f to b_f along Q with f_Q for all arching faces f.*

The right-aligned path $R(Q)$ *of Q is defined symmetrically to the left-aligned path.*

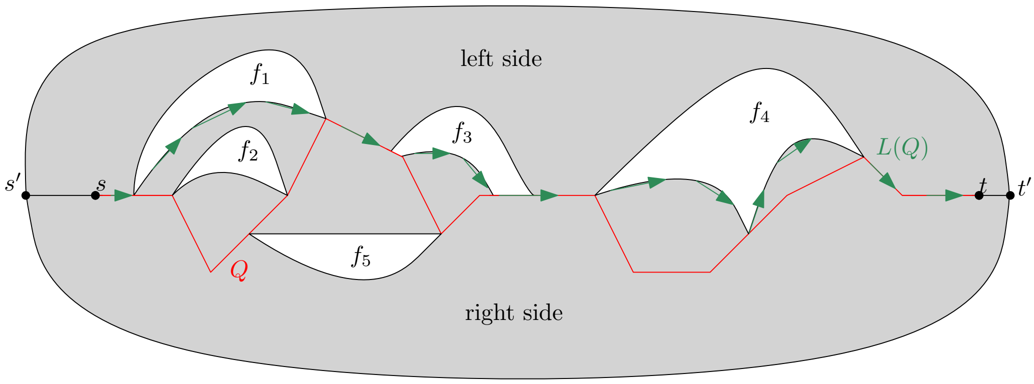


Fig. 3.2: The extendible path Q between s and t is marked red. The left-aligned path $L(Q)$ of Q is illustrated with green arrows.

Note that the definition replaces subpaths of Q with subpaths of each arching face in order to obtain $L(Q)$. In this process new arches that did not arch Q cannot occur, but it could be that the path is still arched on the left side after a replacement. For example if we start with the construction of $L(Q)$ in Figure 3.2 by dealing with archface f_2, the replaced subpath is still arched by face f_1. Therefore the definition replaces subpaths for "each" arching face.

If we want to imagine how to obtain $L(Q)$, we can deal with nested archfaces, like f_1 and f_2 in Figure 3.2 by just replacing the subpath with the "most outer" archface (here: f_1). With this strategy, we could save ourselves the need to replace the more inner arching faces (here: f_2) of the nest.

Furthermore, we observe that Q and $L(Q)$ have their start- and endvertex in common. We apply now our observations that the replacement process does not generate new arches to obtain the following Lemma:

Lemma 8 ([HN08], Lemma 5)**.** *Let $G = (V, E)$ be an internally 3-connected plane graph and Q be an extendible path from a vertex s to a vertex t such that every vertex on Q except s and t is an inner vertex. Then no inner face arches $L(Q)$ on the left side. Moreover, if no face arches Q on the right side, then $L(Q)$ is an archfree path.*

It can be observed that the left- and right-aligned path of an extendible path is still extendible. With that observation and Lemma 8, we can obtain an approach for the construction of archfree paths: If $L(Q)$ of an extendible path Q can just be arched from the right side $R(L(Q))$ can neither be arched from the right nor from the left side and is therefore archfree.

Lemma 9 ([HN08], Corollary 6)**.** *For any inner extendible path Q from s to t in an internally 3-connected plane graph G, the right-aligned path $R(L(Q))$ of the left-aligned path $L(Q)$ is an archfree path.*

Analogously: the left-aligned path $L(R(Q))$ of the right-aligned path $R(Q)$ is an archfree path.

Now we have a strategy how to construct an archfree path out of an arched path. For further argumentation we need to know some properties, the constructed archfree paths from Lemma 9 fulfill. By definition of the left-aligned and right-aligned paths we already observed that they have the same start- and endvertex as the base-path. The following Lemma describes another property regarding the connection between the intersection of two paths w_1 and w_2 and the intersection of w_1 with $L(w_2)$.

Lemma 10. *Let $G = (V, E)$ be a 3-connected plane graph, $a \in V(G)$ a vertex in G and b_1 and b_2 two different vertices on the boundary of the outer face (see Figure 3.3). Furthermore, let w_1 be a simple path between a and b_1 and w_2 is an extendible, simple path in G between a and b_2. Let the only common vertex of w_1 and w_2 be their startvertex a. Then the left-aligned path $L(w_2)$ of w_2 intersects with w_1 just in a as well.*

Analogously: $R(w_2)$ and w_1 have just vertex a as a common vertex as well as $L(R(w_2))$ (resp. $R(L(w_2))$) with w_1.

Proof. We prove the equality by showing $\subseteq$ and $\supseteq$.

$L(w_2) \cap w_1 \supseteq \{a\}$ The left-aligned path of w_2 has by construction the same start- and endvertex as w_2. Therefore a is still a vertex on both w_1 and $L(w_2)$.

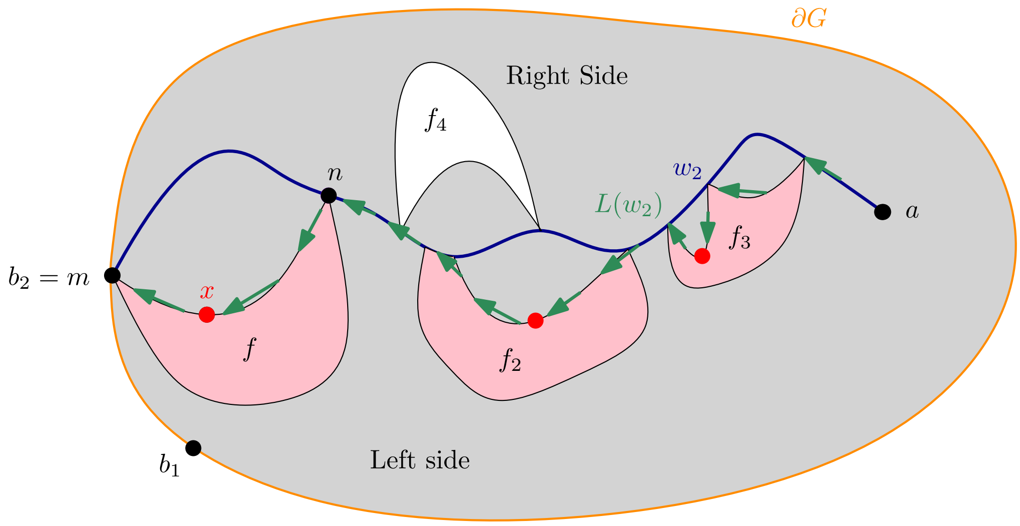


Fig. 3.3: The faces f_i are the archfaces of w_2 whereby the pink-coloured faces are on the left side. The left-aligned path $L(w_2)$ is marked with green arrows. Some possible positions of x are marked with red dots.

$L(w_2) \cap w_1 \subseteq \{a\}$ For a proof by contradiction, we assume that $L(w_2) \cap w_1$ contains a vertex x that is not in $\{a\}$. If $x \in L(w_2) \cap w_2$, it cannot be a vertex on w_1 because w_2 and w_1 intersect by definition just in vertex a and $x \neq a$. It follows that x is in $L(w_2) \setminus w_2$ as illustrated in Figure 3.3. We call the archface of w_2 on whose boundary x is located f. The subpath of w_2 that is replaced by the left-aligned path because of f is called l, the startvertex of l is called n and the endvertex m.

Consider the region that is bounded by l and the subpath of ∂f between m and n. Vertex x is by definition located inside the region and a can be on the boundary. Vertex b_1 could be part of the boundary of the region that is not part of w_2. By definition x is also a vertex on w_1 between a and b_1, therefore w_1 has to cross the boundary of the region in at least two vertices. Because of face f it cannot cross the subpath of ∂f between m and n and w_1 has to intersect with l in at least two vertices. This contradicts the part of the assumption that w_1 and w_2 intersect just in vertex a.

Therefore it is not possible that x existed and the Lemma is proven. □

We apply now Lemma 10 to prove the following Corollary:

Corollary 11. *Let $G = (V, E)$ be a 3-connected plane graph and w an extendible simple path with the startvertex s and the outer vertex t as the endvertex. Furthermore, let w be disjoint from ∂G except for the start- and endvertex (see Figure 3.4).*

Then the left-aligned (resp. right-aligned) path of w is also disjoint from ∂G except of the start- and endvertex.

In particular, the left-aligned (resp. right-aligned) path of a path w with at least one outer start- or endvertex with just inner vertices inbetween, intersects with ∂G also just in the outer start- or endvertex.

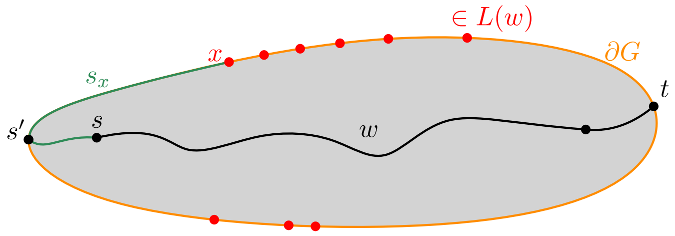


Fig. 3.4: The situation in Corollary 11: The extendible simple path w is illustrated in black colour. The green-marked path s_x is defined in the corresponding proof.

Proof. We proof the Corollary by contradiction. Since w is extendible, it is the subpath of a path Q between two outer vertices s' and t. First, we assume that there exists a vertex $x \notin \{s,t\}$, which is a common vertex of $L(w)$ and ∂G. We define the path s_x as the subpath of Q between s and s' linked with the subpath of ∂G between s' and x that does not contain t'. Without loss of generality, we assume that there is no other vertex x' between s' and x on s_x that is on $L(w)$. Vertex x is not on w because of the definition of w as a path without outer vertices except s and t.

The paths s_x and w intersect just in the startvertex s because the subpath of s_x between s and s' intersects with w just in s because of Lemma 10 and the second part of s_x is disjoint from w (except s in the case that $s' = s$) by definition. Furthermore, both end on ∂G. With Lemma 10, we deduce that $L(w)$ and s_x intersect just in the startvertex s. Therefore no such x can exist and the Corollary is proven. □

We have now a sufficient repertoire of construction strategies of disjoint archfree paths. Beside those, we will utilize some results about 3-connectivity and the existence of a special kind of faces.

Preservation of internally 3-connectivity First, we proof the following Lemma 12. If we have a plane internally 3-connected graph G and insert a vertex with three edges into one of the faces, the graph is still internally 3-connected.

Lemma 12. *Let $G = (V, E)$ be a plane internally 3-connected graph and f an inner face in G. We define G' as the graph that is emerged from G by adding a new vertex v in f with three new edges between v and three differerent vertices x_1, x_2 and x_3 on the boundary of f (See Figure 3.5). Then G' is an internally 3-connected graph.*

Proof. The property internally 3-connectivity is defined in Definition 2. For this proof we use the third property: every separation pair u, v of G is external.

We eliminate two arbitrary internal vertices in G'. If v is one of them, the resulting graph is connected because G was internally 3-connected. If v is not eliminated, the part of G' that corresponds to G is still connected. Furthermore, v is still connected with the rest of G' because at most two of the three neighbours of v are removed. Therefore, G' is an internally 3-connected graph as well. □

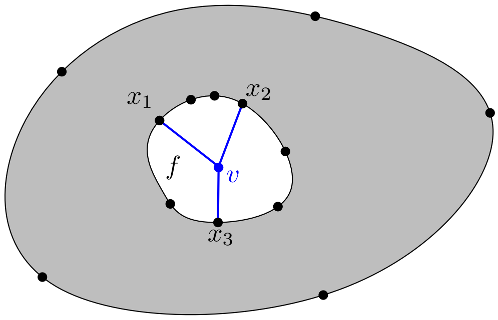


Fig. 3.5: Illustration of the construction of G' in Lemma 12: We insert a vertex v in face f in graph G. Furthermore, we add three edges between v and the boundary of f.

Lemma 13. *Let G be an internally 3-connected plane graph and let Γ be a simple cycle in G. The closed interior Γ^- of Γ is an internally 3-connected plane graph.*

Proof. The proof of the Lemma can be directly derived with the second statement in Definition 2 of internally 3-connectivity: let w be an inner vertex of Γ^-. Vertex w corresponds to an inner vertex in G and because of the internally 3-connectivity of G we find three disjoint paths p_1, p_2 and p_3 from w to the boundary of the outer face of G. Those paths intersect with Γ in at least one vertex. We define p'_i ($i \in \{1, 2, 3\}$) as the subpath of p_i between w and the first intersection vertex with Γ. Clearly, p'_1, p'_2 and p'_3 are still disjoint and paths from w to Γ, which is the boundary of the outer face of Γ^-. Therefore is Γ^- internally 3-connected and plane with Γ as the boundary of the outer face. □

Strictly inner faces and windmills We proceed now and define a special kind of face called "strictly inner face". Intuitively this is an inner face f without common vertices of the boundary ∂f with the boundary of the outer face. We will observe that this type of face has to exists in 3-connected internally 4-regular plane graphs with three vertices on the outer face.

Definition 14. *Let $G = (V, E)$ be a connected plane graph and let f_0 denote its outer face. An inner face f of G is called* strictly inner face *if its boundary ∂f is disjoint to the boundary of the outer face ∂f_0. An example for a strictly inner face can be found later in the chapter in Figure 3.7a.*

Not every graph contains strictly inner faces. The following Lemma assures the existence of a strictly inner face in a 3-connected internally 4-regular plane graph with three outer vertices.

Lemma 15. *Let $G = (V, E)$ be a 3-connected internally 4-regular plane graph with at least one inner vertex. Let f_0 denote the outer face of G and let the number of vertices on the boundary of f_0 be three. Then graph G has a strictly inner face.*

Proof. We prove this Lemma by contradiction. Therefore, we assume that G has no strictly inner face.

Let η be the number of vertices with odd degree, $|f|$ the amount of faces and k the number of inner vertices in G. Clearly, it holds that $k \geq 0$. The Handshaking-Lemma implies that η is even. Because of the 3-connectivity and internally 4-regularity of G, the vertices with odd degree are outer vertices with degree 3 and η is either 0 or 2.

Because of the definition of k, the number of vertices is $|V(G)| = 3 + k$. The amount of edges is dependent on η: every vertex in G is adjacent to four edges except the η outer vertices with just 3 edges. As every edge is between two vertices, we derive:

$$|E(G)| = \frac{1}{2}(4(k+3) - \eta) = 2k + 6 - \frac{1}{2}\eta$$

Since we assumed that G has no strictly inner face, we observe that every inner face has a common vertex with ∂f_0. Therefore, we know that the amount of faces $|f|$ is limited by the upper bound $7 - \eta$.

Now we apply the Euler Characteristic for connected plane graphs:

$$|V(G)| - |E(G)| + |f| = 2$$

With the observations from above, we derive:

$$\begin{aligned} & 3 + k - (2k + 6 - \frac{1}{2}\eta) + 7 - \eta \geq 2 \\ \Leftrightarrow \quad & -k \geq -2 + \frac{1}{2}\eta \\ \Rightarrow \quad & k \leq 2 - \frac{1}{2}\eta \end{aligned}$$

As mentioned above, η can be either 0 or 2. We discuss these two cases separately.

Case 1: $\eta = 0$. We derive that $k \leq 2$. If G has two inner vertices, it is isomorphic to K_5 and therefore not planar. If G has one inner vertex that vertex cannot have degree 4 because $|V(G)| = 4$. Altogether, this case cannot occur. ◁

Case 2: $\eta = 2$. We obtain that $k \leq 1$. If G has just one inner vertex that inner vertex cannot have 4 neighbours because $|V(G)| = 4$, therefore this case cannot appear either. ◁

Altogether, we obtain that G must have had a strictly inner face. □

Now we proceed with the definition of a special path set called "windmill". We will apply windmills in the last case of the proof of Theorem 18 where we construct archfree windmills. Those can be drawn as illustrated in Figure 3.18 and will come in handy to split up the graph into useful subgraphs.

Note that it is crucial for the existence of windmills in a graph that the graph has at least one strictly inner face.

Definition 16. *Let G be a planar graph and p_1, p_2 and p_3 three simple paths. The startvertex of p_i ($i \in \{1, 2, 3\}$) is called s_i, the corresponding endvertex e_i. A set of the three paths p_1, p_2, p_3 is called* windmill *of graph G, if the three paths fulfill the following properties:*

(W1) Each of the paths p_i has only the vertex s_i in common with ∂G

(W2) For each $i, j \in \{1, 2, 3\}$ with $i \neq j$, the two paths p_i and p_j have exactly one vertex in common, which is the endvertex e_i or e_j of exactly one of the two paths.

If p_1, p_2 and p_3 are additionally archfree the windmill is called an archfree windmill*.*

Note that the properties 1 and 2 in the definition already fix that two paths P_i and P_j cannot intersect in more than one vertex and the intersection vertex of the two paths is an inner vertex of one of the two paths (and the endvertex of the other one). Furthermore, the combination of both properties implies that none of the paths p_i and p_j can be empty. In the following Lemma we show the existence of archfree windmills in graphs with special properties.

Lemma 17. *Let G be an internally 3-connected plane graph of maximum degree 4 that contains a strictly inner face f. Then G contains an archfree windmill.*

Proof. For the sake of readability we consider all indices in this proof modulo 3. Let f_0 denote the outer face of G. We start and construct three disjoint archfree simple paths from ∂f to ∂f_0 by using the internally 3-connectivity of G: we add an auxiliary vertex v in f with edges to every vertex x_i on ∂f and call the resulting graph G'.

The graph G' is internally 3-connected because of Lemma 12 and the fact that inserting additional edges into an internally 3-connected graph does not harm the internally 3-connectivity. Since G' is internally 3-connected, we can find three simple paths w'_1, w'_2 and w'_3 from v to the outer face that pairwise intersect just in the vertex v, which is the startvertex of all three paths. We define the endvertices of w'_i as v_i. Note that the v_i are part of the boundary of the outer face and without loss of generality no other vertex on w'_i is on ∂f_0.

Those simple paths w'_1, w'_2 and w'_3 can now be used to construct three disjoint archfree paths $w_i (i \in \{1, 2, 3\})$ between ∂f and the three vertices v_1, v_2 and v_3 on the boundary of the outer face ∂f_0. Clearly w'_1, w'_2 and w'_3 are extendible paths. With Lemma 10, we conclude that $L(w'_1)$, $L(w'_2)$ and $L(w'_3)$ intersect pairwise just in vertex v. Furthermore, $R(L(w'_1))$, $R(L(w'_2))$ and $R(L(w'_3))$ are archfree because of Lemma 9, pairwise disjoint except for the startvertex v with Lemma 10 and contain no other outer vertex except their startvertex because of Corollary 11. As illustrated in Figure 3.6, we define for each $i \in \{1, 2, 3\}$ x_i as the first vertex on ∂f if we begin with vertex v_i and iterate $R(L(w'_i))$. As it can be seen in Figure 3.6, $R(L(w'_i))$ can have more than one intersection vertex with ∂f, but we are just interested in "the first one". The subpath of $R(L(w'_i))$ between v_i and x_i is called w_i. The w_i are pairwise disjoint by construction and each of them is archfree.

We process now the archfree paths w_i and define three paths s_1, s_2 and s_3 that build a windmill together:

For each $i \in \{1, 2, 3\}$, let s_i be the simple path between v_i and x_{i+1} that is constructed by linking w_i and the subpath of ∂f between x_i and x_{i+1} that does not contain x_{i+2} (Figure 3.7a).

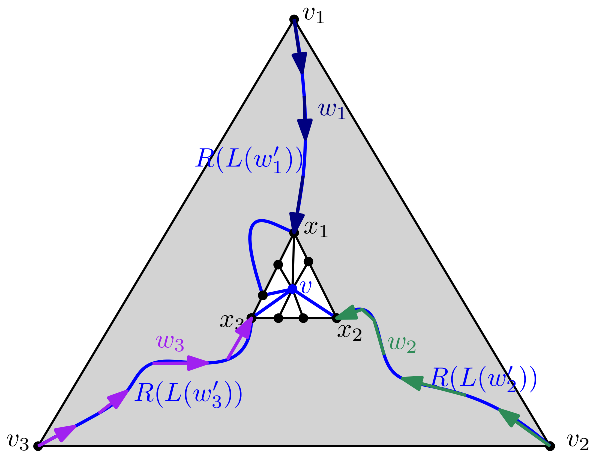


Fig. 3.6: Situation in the proof of Lemma 17: we inserted into G in face f an auxiliary vertex v with edges to ∂f. Then, we used the internally 3-connectivity to construct the three disjoint archfree paths w_1, w_2 and w_3 from ∂G to ∂f.

Note that the set of the three paths s_i is a windmill: by construction s_i intersects with ∂G just in the starvertex v_i. Two s_i and s_j intersect just in one vertex, which is exactly one of their endvertices x_i or x_j and the startvertex v_i is the only outer vertex on s_i.

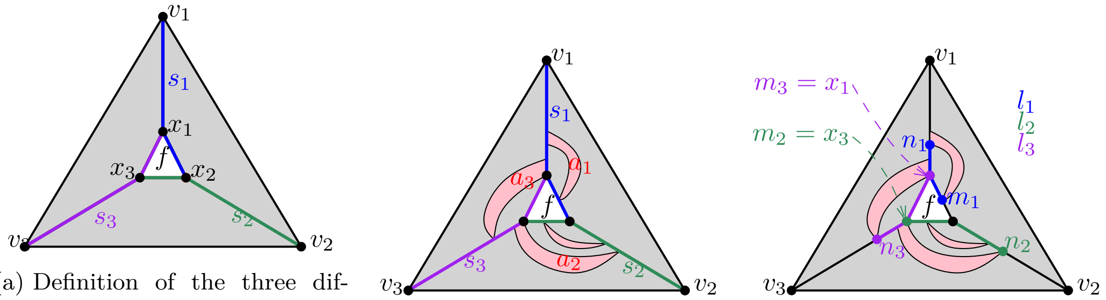
Clearly, the paths s_i may be arched. If they would be already archfree, we can define new paths s_i'' as s_i. The set of the s_i'' is then an archfree windmill in G and we are done. Otherwise we can assume that s_i are arched and we process them to receive an archfree windmill:

By construction the subpath of s_i that is arched must contain the linkvertex x_i as illustrated in Figure 3.7b. Furthermore, we can easily observe that s_i is an extendible simple path and can just be arched on the left side because both subpaths of s_i before and after x_i are by design archfree and there is a path from x_i to the outer face on the right side of s_i. If s_i is arched, we define l_i as the subpath of s_i, which is replaced in $L(s_i)$. If s_i is archfree, we define l_i as the empty path and define its start- and endvertex both as x_i. Since s_i is constructed by linking two archfree paths, l_i is unique (compare Figure 3.7c). Note that x_i has to be part of l_i. We define n_i as the startvertex (closer to v_i) of l_i and m_i as the endvertex of l_i. If s_i was archfree, n_i and m_i are both the same vertex as x_i.

Case 1: None of the m_i matches with x_{i+1}. An example for the following descriptions can be found in Figure 3.8.

In this case, we can define s_i'' as $L(s_i)$ linked with the subpath of s_{i+1} between x_{i+1} and m_{i+1}. The linkvertex x_{i+1} is part of both parts because the left-aligned path keeps the start- and endvertex of the original path fixed. Clearly, both parts intersect just in the linkvertex x_{i+1}.

The first part of s_i'' is archfree because of Lemma 8 and the fact that s_i was not arched from the right side. The second part of s_i'' is archfree with Lemma 6: the subpath of s_{i+1} between x_{i+1} and m_{i+1} is at the same time a subpath of the boundary of f and



(a) Definition of the three different s_i based on the three disjoint archfree w_i and the boundary of the strictly inner face f.

(b) Example of possible arches on the left side of s_i.

(c) Definition of l_i as the replaced subpath of s_i.

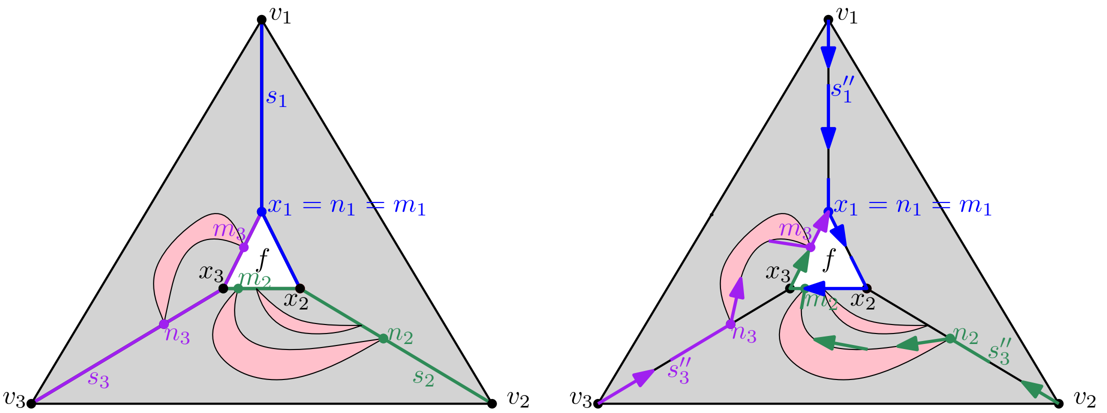
Fig. 3.7: Definition of s_i and preparations for the construction of the archfree windmill $\{s_1'', s_2'', s_3''\}$

at least two more edges are not part of that subpath. If s_i'' is arched by a face a, the arched subpath must contain x_{i+1} because both parts of s_i'' are already archfree. Face a cannot be on the right side of s_i'' because of f and s_{i+2}: face a would be crossed by s_{i+2}. Furthermore, face a cannot be on the left side of s_i'' because it would be crossed by s_{i+1}.

We show now that the set of the s_i'' is a windmill. Clearly, s_i'' are simple paths. The first property states that each of the s_i'' has exactly one common vertex with ∂G, which matches with the startvertex v_i. We know that the paths s_i'' have v_i as a startvertex on ∂G. The first subpath of s_i'', $L(s_i)$ does not intersect in more than the startvertex v_i with ∂G because of Corollary 11. The second subpath of s_i'' is also a subpath of the boundary of the strictly inner face f and therefore it does not contain any outer vertices.

The second property of a windmill states that two paths p_i and p_j have just one vertex in common, which is the endvertex from exactly one of them. By definition, s_i'' and s_{i+1}'' intersect in vertex m_{i+1}, which is the endvertex of s_i''. Without loss of generality s_1'' and s_2'' intersect in another vertex x_s than m_2. With Lemma 10, we can derive that the subpath of s_1'' between v_1 and x_2 is disjoint of the paths s_2 and w_3 except x_2. If s_2 was not arched, we are done with the argumentation because the subpath was the entire path s_1'' and $x_2 = m_2$. Otherwise, the subpath is linked with a subpath of ∂f, which is by its definition disjoint with s_2'' except m_2. Therefore, such a x_s cannot exist.

Altogether, we constructed a set of three simple paths $s_i''(i \in \{1, 2, 3\})$, which fulfill the windmill-properties. Furthermore, we observed that the s_i'' are archfree. ◁



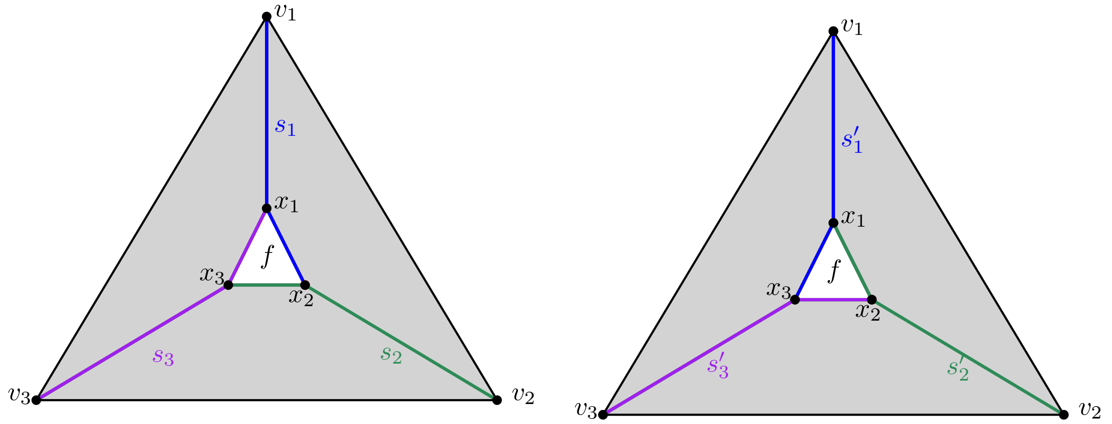
(a) Example for the situation with some arches

(b) Definition of the three archfree s''_i

Fig. 3.8: The situation in Case 1: On the left side an example for the initial situation in this case, on the right side the definition of the s''_i ($i \in \{1, 2, 3\}$), which build an archfree windmill together.

Case 2: One or more of the m_i match with x_{i+1} as in Figure 3.7c. In this case, we define s'_i almost like s_i (see Figure 3.9) just that we take the other direction of subpaths on ∂f. If it is not arched by a "big arch" that ends in x_i, we handle the case symmetrically to the first case. If it is arched as well, we introduce a new strategy to receive three archfree segments.

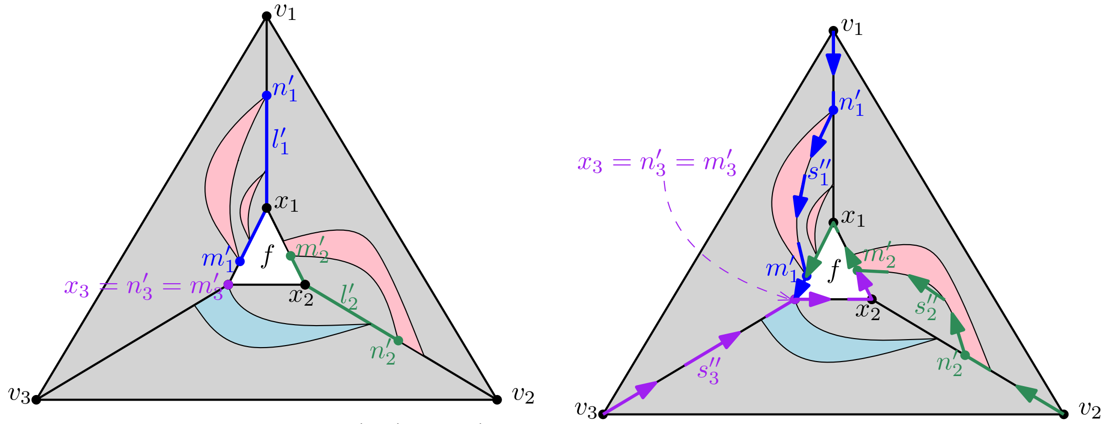
We begin with the definition of s'_i as visualized in Figure 3.9b: for each $i \in \{1, 2, 3\}$, let s'_i be the simple path between v_i and x_{i-1} that is constructed by linking w_i and the subpath of ∂f between x_i and x_{i-1} that does not contain x_{i+1}. We can easily see that for each $i \in \{1, 2, 3\}$, s'_i is an extendible simple path.



(a) Illustration of the three s_i as they were defined in the beginning of the proof.

(b) Definition of the three different s'_i

Fig. 3.9: Comparison of the definition of s_i and s'_i: for the definition of s'_i, we iterate ∂f "in the other direction" than before for the definition of s_i.



(a) Example for the definition of l'_i, n'_i and m'_i. The blue marked arch is responsible that we could not apply Case 1

(b) Definition of the three s''_i symmetrically to Case 1.

Fig. 3.10: Handling of the case, that the three s'_i were not arched with big arches.

Symmetrically to the first case we define l'_i (Here with the right-aligned path since every arch is on the right side of the path), n'_i and m'_i. The new situation is illustrated in Figure 3.10a.

If none of the m_i match with a vertex x_{i-1}, we can define the new s''_i symmetrically to the first case as illustrated in Figure 3.10b. In this case, we constructed a set of three simple paths s''_i, which fulfill the windmill property and which are additionally archfree.

If one or more of the m'_i match with a vertex x_{i-1}, we are in a similar situation as in Figure 3.11: We call the special kind of arches that end in x_i *big arches*. Since the graph is internally 4-regular, in each x_i can end at most one big arch. With this observation, we conclude that if there are $i, j \in \{1, 2, 3\}$ with s_i and s'_j are arched by big arches, there cannot be more big arches than those two in the graph. Furthermore, we can conclude that the only possible position of the two big arches is $i = j$.

Without loss of generality, we assume that $i = j = 1$. Additionally, we assume without loss of generality that the number of edges between n'_1 and v_1 are more (see Figure 3.12a) or equal (see Figure 3.12b) to the number of edges between n_1 and v_1. The other case — n'_1 was closer to v_i — is handled symmetrically.

The simple path s''_1 is defined as the right-aligned path $R(s'_1)$ linked with the subpath of s'_3 between x_3 and m'_3. The second simple path s''_2 is defined as the right-aligned path of the path s_{2a} which is defines as the path that is constructed by linking s'_2 with the subpath of s'_1 between x_1 and n'_1. The third simple path s''_3 is defined as the right-aligned path $R(s'_3)$.

Path s''_1 is archfree because its first part is archfree with Lemma 9, the second part is archfree because it is a subpath of ∂f and there cannot be an arch at the link x_3 because of the big arch that ends in x_3. Path s'_2 is archfree because of the big arch that ends in x_2. The subpath of s'_1 between x_1 and n'_1 is archfree by construction. If the path s_{2a} that is constructed by linking those two archfree paths is arched, the archface has to be

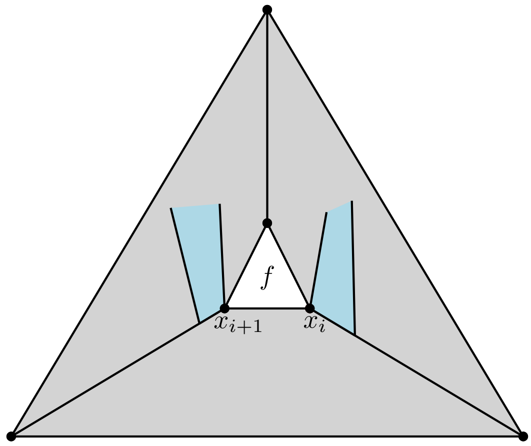


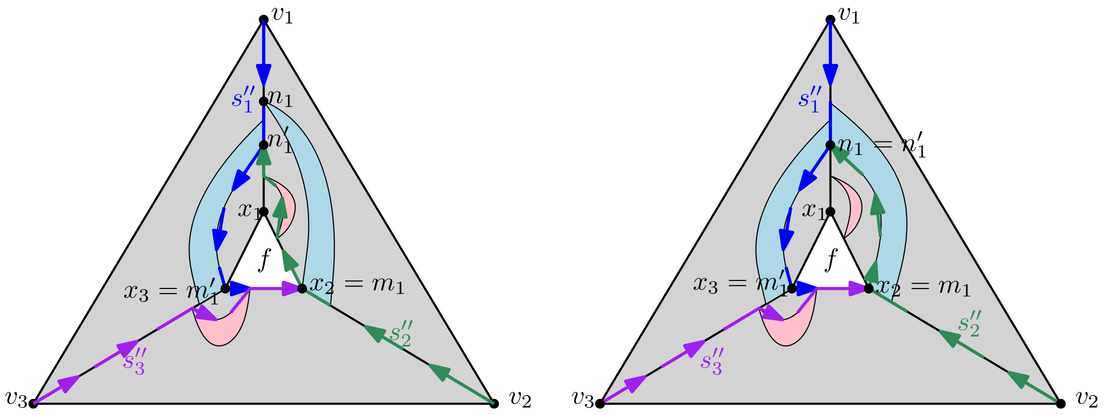
Fig. 3.11: Situation if there was a s_i and s'_j that were arched by a big face. The positions of the two big faces (sketched in blue) are restricted by the 4-regularity and the property, that the faces cannot overlap.

on the right side of the path s_{2a}. Therefore is s''_2 archfree as it is the right-aligned path of s_{2a}. Path s''_3 is archfree because s'_3 can just be arched from the right side and s''_3 is defined as the right-aligned path of s'_3.

We show now that the set of the three simple paths s''_i is a windmill. The first property states that each of the s''_i has exactly one common vertex with ∂G, which is the startvertex of s''_i. We know that the startvertex v_i of s''_i is a vertex on ∂G. The first part of s''_1 between v_1 and x_3 just intersects with ∂G in v_1 because of Corollary 11. The second part of s''_1 between x_3 and m'_3 is a subpath of the boundary of the strictly inner face f and therefore contains no outer vertex. Analogously, we can argument that s''_3 intersects with ∂G just in the startvertex v''_3. Path s''_2 was constructed by constructing the right-aligned path of the path s_{2a} that consisted out of s'_2 and the subpath of s'_1 between x_1 and n_1. Path s'_2 has v_2 as the startvertex, the subpath of s'_1 between x_1 and n_1 does not contain any outer vertices. With Corollary 11, we can deduce that s''_2 has no other common vertices with ∂G than the startvertex v_2.

The second property states that two paths p_i and p_j have just one vertex in common, which is the endvertex from exactly one of them. With a symmetric argumentation as in Case 1, we can show that s''_1 and s''_3 intersect just in vertex m'_3, which is the endvertex of s''_1 and an inner vertex of s''_3. Furthermore, s''_3 and s''_2 intersect in x_2, which is the endvertex of s''_3 and an inner vertex of s''_2. Another intersection vertex does not exist with an analogous argumentation as in Case 1.

The paths s''_1 and s''_2 intersect in n'_1, which is the endvertex of s''_2 and an inner vertex of s''_1. We assume that they intersect in another vertex v_s than n'_1. By definiton v_s is a vertex on s''_2. Consider the path s_{2a} that is formed by linking s'_2 with the subpath between x_1 and n'_1 of s'_1. Clearly, its only intersection vertex with s''_1 is n'_1. By applying Lemma 10 twice we obtain that s''_2, which is defined as $R(s_{2a})$, intersects with s''_1 just in vertex n'_1 as well.



(a) $n_1 \neq n_1'$ (b) $n_1 = n_1'$ (analog)

Fig. 3.12: Example for how to define s_i'' whereby both s_1 and s_1' are arched by big arches (here in blue).

Altogether, the set of the three simple paths s_1'', s_2'' and s_3'' forms an archfree windmill and the Lemma is proven. □

3.2 Drawings with segment constraints

In this section, we use the results of Section 3.1 to give a proof for Theorem 18: for every 3-connected internally 4-regular plane graph exists a drawing such that every inner vertex is placed in the interior of a segment.

In order to prove this idea, we apply a technique that was established by Hong and Nagamochi [HN10] and a runtime-improved version by Klemz [Kle21]. Both of them describe algorithms, which can be used to recursively construct a convex drawing of a 3-connected hierarchical plane st-graph G with a certain simple convex polygon as the realisation of the outer face. The algorithms can easily be adjusted for graphs, which are just plane. The main idea of their algorithms is to choose an inner vertex y and to construct three archfree paths from y to the boundary of the outer face. Those archfree paths are realised as straight line segments and used to split G into three subgraphs, which can be drawn recursively. We construct the subgraphs in Case 1 and Case 2 analogously to those algorithms however the last case of our proof Case 3 is quite different to their algorithm because in their strategy it cannot easily be ensured that the chosen inner vertex y is in the interior of a segment.

Theorem 18. *Let $G = (V, E)$ be an internally 3-connected internally 4-regular plane graph and let Γ^0 be a simple convex polygon that is compatible with G i.e. every segment of the polygon corresponds to an archfree path on ∂G. There exists a convex drawing of G with Γ^0 as the realisation of the outer face and the following property is fulfilled: For every inner vertex $v \in V$ there exists a segment s_v such that v is a vertex on s_v, but neither a start- nor an endvertex of s_v (see Figure 3.13).*

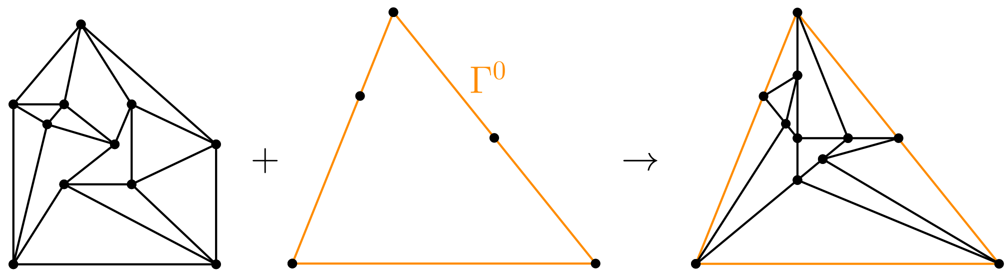
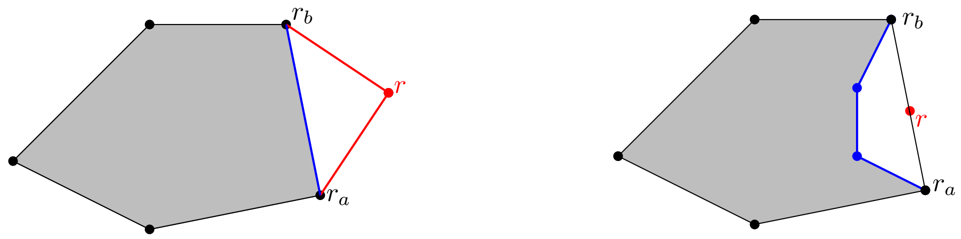


Fig. 3.13: To the left the 3-connected internally 4-regular plane graph, that will be drawn with the compatible polygon Γ^0 (middle) as the realisation of the outer face. On the right the drawing of the graph that fulfills the property in Theorem 18 and with the predefined realisation of the outer face Γ^0.



(a) Case 1.1

(b) Case 1.2

Fig. 3.14: The situation in Case 1: vertex r has degree 2 and is eliminated in this case. In order to do so, we remove the red marked parts in both illustrations in the proof and determine the convex drawing of the resulting graph. Later, we insert r on the position, which is defined by Γ^0.

Proof. As mentioned above, the idea of this proof is based on the "Hierarchical-Convex-Draw"-algorithm by Hong and Nagamochi [HN10] and a runtime-improved version from Klemz [Kle21].

Note that the coordinates of vertices on the outer face are fixed by the polygon Γ^0. Our goal is to determine the coordinates for each inner vertex such that the drawing fulfills the properties in the Theorem. We reach this goal by splitting the graph into several subgraphs. The coordinates of inner vertices of the subgraphs are computed recursively and combined to a drawing of the original graph. The base case of this recursion is a graph without any inner vertex. In this case, the property that every inner vertex is located on the interior of a segment is trivially fulfilled. Furthermore, the drawing is clearly convex. We can assume now that the graph has inner vertices and start with a case analysis:

Case 1: There exists a vertex r on ∂G with $\deg_G(r) = 2$. We define r_a and r_b as the two neighbors of r. Since r is part of ∂G, both r_a and r_b are on ∂G as well.

Case 1.1: The edge (r_a, r_b) exists in the edge set E. In this case, we set

$$G_1 = (V \setminus \{r\}, E \setminus \{(r_a, r), (r_b, r)\})$$

like in Figure 3.14a. We define a new realisation of the outer face Γ_1^0 by replacing the corresponding parts to (r_a, r) and (r_b, r) in Γ^0 by a new segment (r_a, r_b). The polygon Γ_1^0 is still a convex polygon because none of the inner angles can be greater than 180°. Clearly, Γ_1^0 is simple. Furthermore, Γ_1^0 is compatible with G_1 because (r_a, r_b) is archfree with Lemma 6. Additionally, the new graph is internally 3-connected and internally 4-regular. We determine the coordinates of the internal vertices in a convex drawing of G_1 with Γ_1^0 as the realisation of the outer face inductively. Afterwards we add r on the position, which is given by Γ^0. This does not add any inner vertex, therefore the property in Theorem 18 is not harmed. Since the face that is formed by the newly inserted vertex r is convex, the whole drawing is convex. ◁

Case 1.2: The edge (r_a, r_b) does not exist in the edge set E.. We define

$$G_1 = (V \setminus \{r\}, E \setminus \{(r_a, r), (r_b, r)\} \cup \{(r_a, r_b)\})$$

as visualized in Figure 3.14b. The new realisation of the outer face Γ_1^0 is defined analogously to Case 1.1 and is compatible because (r_a, r_b) is archfree in G_1 with Lemma 6. Furthermore, Γ_1^0 is a simple convex polygon and the new graph is internally 3-connected and internally 4-regular. We inductively determine the coordinates of the internal vertices of G_1 with Γ_1^0 as the realisation of the outer face. Afterwards we delete (r_a, r_b) from the drawing and add r with both the edges (r_a, r) and (r_b, r) on the position, which is given by the polygon Γ^0. The inner vertices that share a face with r have been inner vertices before and their position and neighbors did not change. Therefore, they still fulfill the property in Theorem 18 after this adjustment. Furthermore, the inner face that is adjacent to r is still convex because Γ^0 is convex and none of the angles in the face can be greater than 180° after the adjustment. ◁

Case 2: Every vertex on ∂G has more than two neighbors and G is not 3-connected. In this case, we know that graph G contains a separation pair of two vertices x_1 and x_2. Since G is internally 3-connected, both vertices have to be part of ∂G. Moreover, they are both on the boundary of inner face f as illustrated in Figure 3.15. We denote the two subpaths of ∂f between x_1 and x_2 by p_1 and p_2.

Without loss of generality, we assume that the vertices x_1 and x_2 are chosen such that p_1 contains no outer vertices except the start- and endvertex and p_2 contains more than one edge.

With Lemma 6 and the knowledge that p_2 contains at least two edges, we conclude that p_1 is archfree. We draw p_1 as a straight line and use it to split the graph into the two subgraphs G_1 and G_2 as illustrated in Figure 3.16.

In order to do this, we construct two new simple convex polygons Γ^1 and Γ^2 by linking the two parts of Γ^0 between x_1 and x_2 with the straight line p_1. Γ^1 and Γ^2 are illustrated in Figure 3.16. Clearly, both of them are simple convex polygons. The subgraph of G that is surrounded by Γ^1 is called G_1 as illustrated in Figure 3.16. It is internally 4-regular and with the definition of internally 3-connectivity it is easy to argue that G_1 is internally 3-connected. Furthermore, we know that Γ^1 is compatible with G_1 since p_1 is archfree.

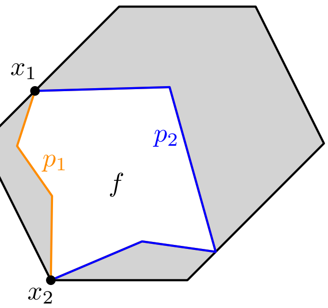


Fig. 3.15: The situation in Case 2: graph G is not 3-connected and has therefore a separation pair of the two vertices x_1 and x_2. Because of them, face f can be found and the two paths p_1 and p_2 on ∂f defined. Without loss of generality, x_1 and x_2 are chosen such that p_1 contains no outer vertices except x_1 and x_2 and p_2 contains more than one edge.

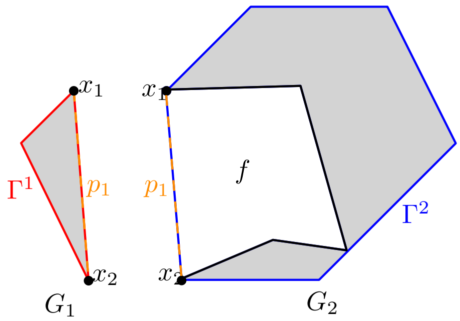


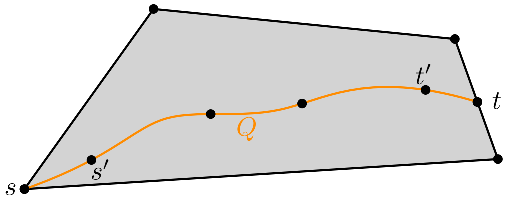
Fig. 3.16: We can now use the archfree path p_1 to split G into two subgraphs G_1 and G_2 with their realisations of the outer faces Γ^1 and Γ^2. The drawing of those subgraphs is determined inductively.

The subgraph G_2 is defined analogously with Γ^2 as the realisation of the outer face. Both subgraphs G_i ($i \in \{1, 2\}$) with their realisations of the outer face Γ^i are drawn inductively and the drawings are combined to a drawing of graph G. Every inner vertex is either an inner vertex of G_1 or G_2 or a vertex on the straight drawn path p_1. Therefore, all inner vertices fulfill the property in Theorem 18 for inner vertices. $\triangleleft$

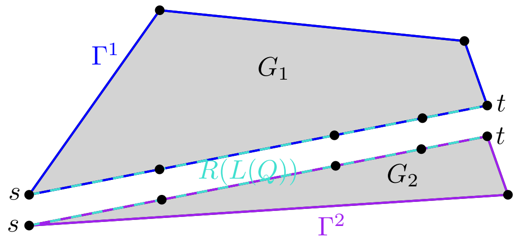
Case 3: G is 3-connected.

Case 3.1: The graph has more than 3 outer vertices as illustrated in Figure 3.17a. We can choose two vertices s and t on ∂G, which are not on the same segment of Γ^0. Both have a neighbour s' resp. t', which are inner vertices. Since G is 3-connected, there is an extendible simple path Q between s and t so that every vertex on Q except s and t is an inner vertex. The right-aligned path of the left-aligned path $R(L(Q))$ is archfree according to Lemma 9 and still an inner path because of Corollary 11.

The path $R(L(Q))$ splits the graph G into two subgraphs G_1 and G_2. We define the realisation of the outer face Γ^1 as the part of Γ^0 that corresponds to G_1 linked with the straight drawn segment $R(L(Q))$. Analogously, we define Γ^2 for G_2. Both Γ^1 and Γ^2



(a) Illustration of the situation in Case 3.1: G is 3-connected and has at least 4 vertices on the boundary of the outer face. We choose s and t on ∂G such that they are not positioned on one segment in the realisation of the outer face. Then, we construct a path between s and t, which contains just inner vertices except those two.



(b) Path $R(L(Q))$ is archfree and disjoint from the boundary of the outer face except s and t. We can use it as a cut through the graph and obtain the subgraphs g_1 and g_2 and their realisations of the outer faces Γ^1 and $\Gamma 2$.

Fig. 3.17

are simple convex polygons and compatible with G_1 resp. G_2 by construction. Clearly, both G_1 and G_2 are planar, internally 4-regular and internally 3-connected.

We determine the drawings of the two subgraphs G_1 and G_2 with the realisations of the outer faces Γ^1 and Γ^2 inductively and combine them afterwards. Every inner vertex in G is now either an inner vertex in G_1 or G_2 and is therefore drawn in the interior of a segment or it is a vertex on the straight drawn path $R(L(Q))$ and thus fulfills the property in Theorem 18. ◁

Case 3.2: The graph has three outer vertices v_1, v_2 and v_3. It holds that G has a strictly inner face f because of Lemma 15. As described in the proof of Lemma 17, we construct an archfree windmill $\{s_1'', s_2'', s_3''\}$.

We use the constructed paths in the archfree windmill to define the four subgraphs for the recursive calls. We draw the constructed s_i'' as straight lines into the given triangle-polygon Γ^0. The outcome of this is illustrated in Figure 3.18.

We define Γ^1 as the simple polygon that is surrounded by subpaths of s_1'', s_2'' and the straight line in Γ^0 between v_1 and v_2. Clearly, it is a simple convex polygon. The subgraph G_1 of G corresponding to Γ^1 contains all vertices and edges on Γ^1 and the subgraph in the inner of the polygon. With Definition 1, it follows that G_1 is internally 3-connected. Γ^1 is compatible with G_1 because of the property that Γ^1 was defined with archfree paths. We inductively determine the coordinates of the internal vertices of G_1 with Γ^1 as the realisation of the outerface.

The second and third subgraph G_2 and G_3 with their realisations of the outer faces Γ^2 and Γ^3 are defined analogously (see Figure 3.18). Analogous to G_1 we determine the coordinates of the internal vertices of both inductively. The fourth subgraph is in the middle of the graph and surrounded by subpaths of s_1'', s_2'' and s_3''. The related polygon Γ^4 is fixed by those three subpaths. With the same arguments as above, the polygon is compatible with G_4, simple and convex. We determine the coordinates of the inner vertices of G_4 inductively.

Every inner vertex on the three segments s_1'', s_2'' and s_3'' is drawn in the middle of a

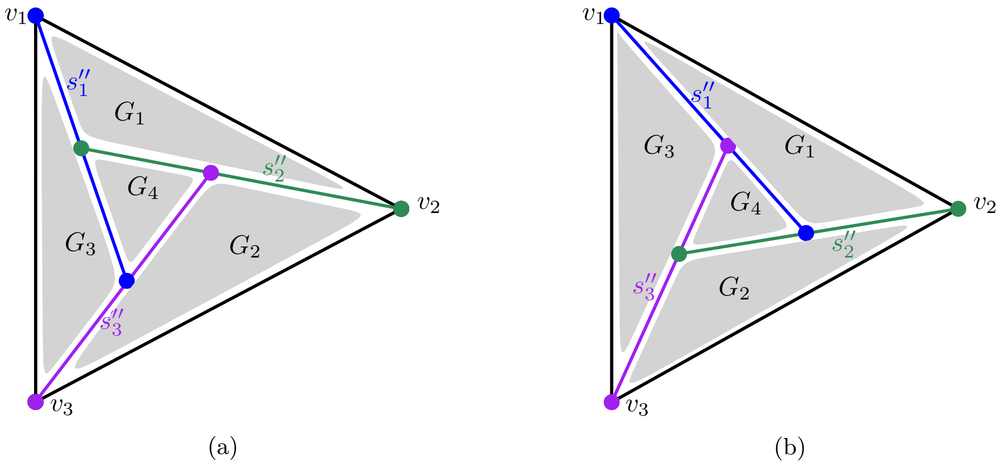


Fig. 3.18: In Case 3.2, we construct an archfree windmill $\{s_1'', s_2'', s_3''\}$. We use the windmill to split G into four subgraphs $G_i (i \in \{1, 2, 3, 4\})$, which are drawn recursively with a triangle-shaped polygon as the realisation of the outer face. The windmill can have two different orientations because of the case-analysis in the construction process in the proof of Lemma 17.

segment. After the recursive drawing of the four subgraphs this property is fulfilled for every inner vertex in the graph. □

3.3 Proof of the Main Theorem

In this section, we apply Theorem 18 to derive an upper bound of $|V(G)| + 3$ for the segment number of 3-connected 4-regular planar graphs (see Theorem 20).

Theorem 19. *Let $G = (V, E)$ be a 3-connected plane graph and let f_0 denote its outer face. Then the path ∂f_0 is not arched by any inner face.*

Proof. If ∂f_0 is arched by an inner face, G is not 3-connected. □

The combination of Theorem 19 and Theorem 18 are now used to prove our Main Theorem about the upper bound for the segment number of 3-connected 4-regular planar graphs.

Theorem 20. *The segment number of every 3-connected internally 4-regular plane graph G is at most $|V(G)| + 3$.*

Particulary: every 3-connected 4-regular planar graph G has a drawing with at most $|V(G)| + 3$ segments.

Note that the segment number is the lowest possible number of segments that is needed to draw the graph in any drawing. Since we do not have a denoted outer face, we cannot use "internally 4-regular" in combination with "planar". Nevertheless, we can generalize the result in Theorem 20 for 3-connected planar graphs with the the property that each vertex has degree 4 except the vertices on the boundary of one arbitrary face that have at most degree 4. This property can be seen as a degenerated version of internally 4-regularity for planar graphs without a denoted outer face.

Proof. Let G be a 3-connected internally 4-regular plane graph and let f_0 denote its outer face. Theorem 19 states that f_0 is not arched by any inner face. Therefore, we can realise the boundary of the outer face as a simple triangle-shaped polygon Γ^0.

The polygon Γ^0 is simple, convex and compatible with G. With Theorem 18, we see that G has a convex drawing D with Γ^0 as the realisation of the outer face such that every inner vertex is placed in the interior of a segment.

Let v_1, v_2 and v_3 be the three vertices in the angles of Γ^0. In the drawing D every vertex except v_1, v_2 and v_3 is in the interior of a segment and can therefore be the endvertex of at most two segments. In the vertices v_1, v_2 and v_3 four segments end.

Altogether, we observe that D contains at most

$$\frac{1}{2} \cdot (2 \cdot (|V(G)| - 3) + 4 \cdot 3) = |V(G)| + 3$$

segments.

If G was an 3-connected 4-regular planar graph, we can choose an outer face f_0. With the same argumentation as above and the observation that every 4-regular graph is also internally 4-regular, we can conclude that G has a drawing with at most $|V(G)| + 3$ segments. □

In fact, the given upper bound for the segment number of 3-connected 4-regular planar graphs in Theorem 20 is tight up to an additive constant. The corresponding existential lower bound is shown in Section 4.2.

4 Lower Bounds for the segment number of 4-regular planar graphs

In this chapter, we focus on the lower bounds for the segment number of 4-regular planar graphs. First, we will study the universal lower bound $\mathfrak{s}$ of the 4-regular planar graphs and show with an observation from Dujmović et al. [DESW07] that the universal lower bound of this graph set can not be asymptotically better than $\Theta(\sqrt{|V(G)|})$.

Afterwards, we will prove an existential lower bound of the 4-regular 3-connected planar graphs of $|V(G)|$ by analysing a suitable subset of this graph class. Finally, we will present a subset of the 4-regular 2-connected planar graphs that gives an existential lower bound of $7|V(G)|/6$ for the segment number of this graph set.

4.1 Universal Lower Bound

In this section, we study the universal lower bound for the segment number of 4-regular planar graphs. We will show with an observation from Dujmović et al. [DESW07] that this bound can not be asymptotically better than $\Theta(\sqrt{|V(G)|})$.

Theorem 21 ([DESW07], p. 207)**.** *Let $G = (V, E)$ be a graph without degree-1- and degree-2-vertices. Then any drawing of G contains at least $\sqrt{2|V(G)|}$ segments.*

Particulary, $\sqrt{2|V(G)|}$ is an universal lower bound for the segment number of 4-regular planar graphs.

Proof. Let s be the segment number of G and n the number of vertices in G. Since G has no degree-2-vertices, every vertex is located on at least two segments. Clearly, two segments can only cross once.

Therefore, a drawing with s segments can contain at most $\binom{s}{2}$ vertices and we get

$$n \leq \binom{s}{2} = \frac{s!}{2! \cdot (s-2)!} = \frac{s \cdot (s-1)}{2} = \frac{s^2}{2} - \frac{s}{2}.$$

We transform this with the quadratic formula and with the additional observation that s and n are positive integer values, we obtain

$$s \geq \frac{\frac{1}{2} + \sqrt{(\frac{1}{2})^2 - 4 \cdot \frac{1}{2} \cdot (-n)}}{2 \cdot \frac{1}{2}} = \frac{1}{2} + \sqrt{\frac{1}{4} + 2n} > \sqrt{2n}.$$

□

In order to show that this universal lower bound is tight up to a small constant factor, we now present a set of graphs such that every member can be drawn with at most $-1 + \sqrt{5 + 4|V(G)|}$ segments.

Theorem 22. *There is an infinite subset S of the 4-regular planar graphs that fulfills the following property: for each graph G in S, the segment number is at most $-1 + \sqrt{5 + 4|V(G)|}$*

Proof. First, we define a graph gadget G_g that is used later to construct the graphs in S. For illustrations refer to Figure 4.1. The gadget contains two vanishing vertices v_1 and v_2, which are the startvertices for four segments s_{i1}, s_{i2}, s_{i3}, s_{i4} $(i \in \{1, 2\})$ each. The segments are drawn as straight lines as illustrated in Figure 4.1 and the endvertices of s_{1k} and s_{2k} $(k \in \{1, 2, 3, 4\})$ match. Every intersection of two segments represents a vertex. G_g is drawn with eight segments and contains twelve vertices.

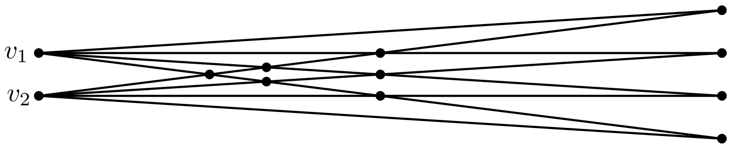


Fig. 4.1: Gadget G_g that is used to construct G_k. G_g consists out of eight segments and contains twelve vertice.

We define S as the set of graphs G_k $(k \in \{1, 2, 3, ...\})$ as illustrated in Figure 4.2. Graph G_k is constructed by arranging $2k$ gadgets as illustrated in the Figure. The vertices of G_k are exactly the intersection vertices of two or more segments. Clearly, graph G_k is planar and 4-regular. Let n_k denote the number of vertices in G_k and s_k the number of segments in the given drawing.

Graph G_k is drawn with $s_k := 2 \cdot k \cdot 8 = 16k$ segments. Every gadget in the graph consists of twelve vertices and none of those vertices except one belongs to two different gadgets. Additional vertices are generated by the intersections of segments of different gadgets. In total those are $(8k - 1)^2 - 1$ vertices. This leads to the number of vertices

$$\begin{aligned} n_k &= 12 \cdot 2k - 1 + (8k - 1)^2 - 1 \\ &= 64k^2 + 8k - 1 \end{aligned}$$

A transformation with the quadratic formula and the additional observation that k is positive, results in

$$\begin{aligned} k &= \frac{-8 + \sqrt{8^2 - 4 \cdot 64 \cdot (-n - 1)}}{2 \cdot 64} \\ &= \frac{-1 + \sqrt{5 + 4n}}{16} \end{aligned}$$

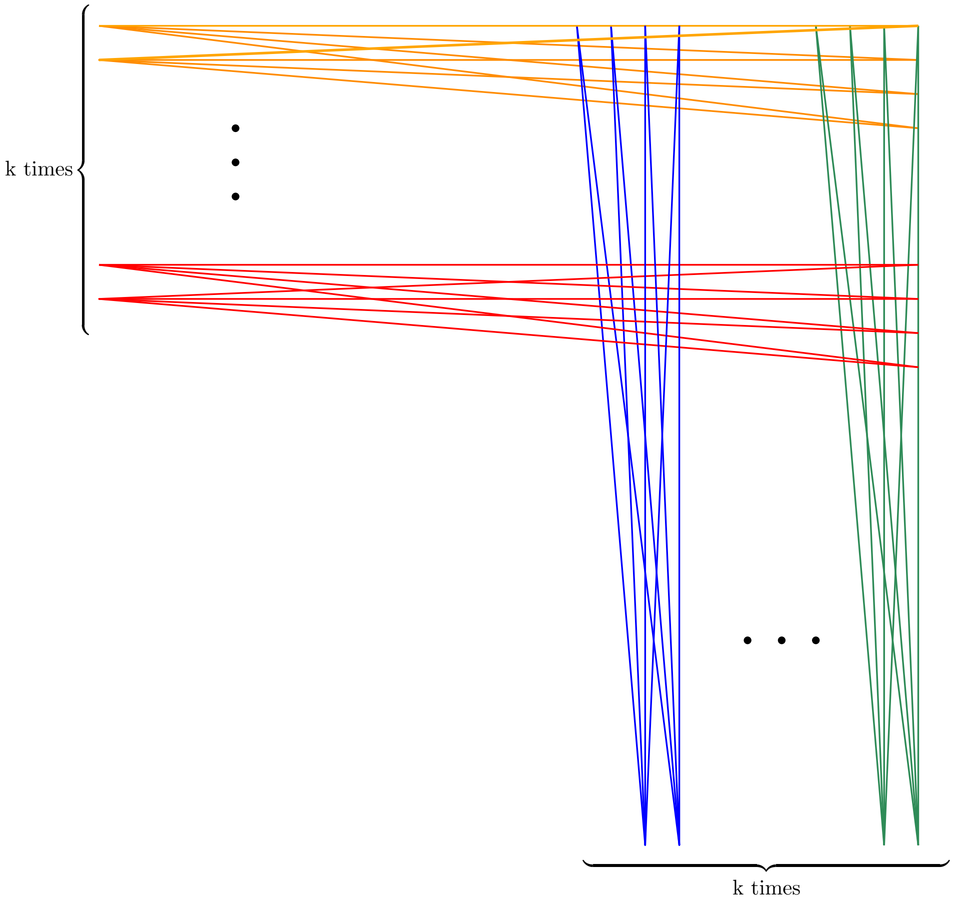


Fig. 4.2: Generic member G_k of the graph set S, which consists of $2k$ gadgets G_g. For reasons of clarity, the vertices in the graph are not specially marked. The vertices in the graph are exactly the intersections of two or more segments. The drawing contains $-1 + \sqrt{1 + 4|V(G_k)|}$ segments.

This leads to the equation

$$\begin{aligned} s &= 16k \\ &= 16 \cdot \frac{-1 + \sqrt{5 + 4n}}{16} \\ &= -1 + \sqrt{5 + 4n} \end{aligned}$$

□

Altogether, we showed that there is a drawing of G_k with $-1 + \sqrt{5 + 4|V(G_k)|}$ segments. We did not prove that the graph cannot be drawn with less segments, but this result suffices in combination with Theorem 21 for the following conclusion:

Corollary 23. *The asymptotically best universal lower bound for the segment number of 4-regular graphs is in* $\Theta(\sqrt{|V(G)|})$.

Proof. The Corollary follows directly from the results from the Theorems above: Theorem 21 states that the asymptotically best universal lower bound is in $\Omega(\sqrt{|V(G)|})$ and Theorem 22 implies that it is in $O(\sqrt{|V(G)|})$. □

4.2 Existential Lower Bound of 3-connected 4-regular planar graphs

In this section, we proof an existential lower bound of the graph set of 3-connected 4-regular planar graphs. In Theorem 28, we present a subset of the 3-connected 4-regular planar graphs with the property that every graph G in this subset cannot be drawn with less than $|V(G)|$ segments. This shows that the given upper bound for segments in 3-connected 4-regular planar graphs in Theorem 18 is tight up to an additive constant.

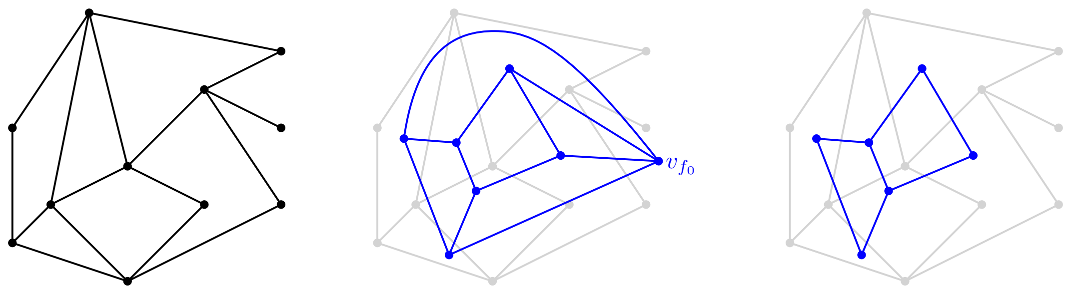
In order to prepare the proof of Theorem 28, we start with some definitions and preliminary results.

Definition 24. *A planar graph G is* outerplanar *if G has a drawing D such that all vertices are on the boundary of the outer face. An outerplanar graph $G = (V, E)$ is* maximal *if the graph $(V, E \cup \{(v, w)\})$ is not outerplanar for any pair of non-adjacent vertices $v, w \in V$.*

Drawing D is called outerplanar if all vertices are on the boundary of the outer face.

Definition 25. *Let G be a plane graph and let f_0 denote its outer face. The* dual *graph G_D of G is the graph that is constructed by inserting a vertex v_f for every face f in G into an empty graph and adding an edge between two vertices v_f and v_g if and only if the corresponding faces f and g have at least one common edge on their boundary (see Figure 4.3b).*

The weak dual *graph G_{WD} of G is the graph that arises if we remove vertex v_{f_0}, that corresponds to the outer face f_0 in G, from G_W.*



(a) The basic graph G (b) The dual graph G_D of G (c) The weak dual G_{WD} of G

Fig. 4.3

The definitions above can now be used to describe a graph set of graphs G_n that have a segment number of at least $|V(G)|$ as shown by Dujmović et al. in [DESW07].

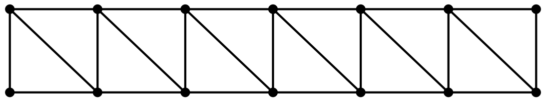


Fig. 4.4: The graph G_{14}. It is an example out of the set of graphs that is defined in Lemma 26 and has the segment number 14

Lemma 26 ([DESW07], Proof of Theorem 7)**.** *Let G_n be the maximal outerplanar graph on $n \geq 3$ vertices whose weak dual is a path and the maximum degree of G_n is at most four, as illustrated in Figure 4.4. Then G_n has at least n segments in any drawing.*

The described graph G_n has an encouraging high segment number of at least $|V(G_n)|$, but it is not yet 4-regular and 3-connected. We will solve this by extending the graph to a ring. In order to describe the new graph set, we will use the well-known definition of the k-th power of a graph G.

Definition 27. *Let G be a graph. The* k-th power of G*, G^k, is the graph with the same set of vertices $V(G)$ as G and an edge between two vertices v_1 and v_2 in $V(G)$ if and only if the distance of v_1 and v_2 in G is at most k.*

Finally, we can describe the graph set, that proofs the existential lower bound of $|V(G)|$ for the segment number of 3-connected 4-regular planar graphs.

Theorem 28. *For all $k \geq 3$, there is an $2k$-vertex 3-connected 4-regular planar graph that has at least $2k$ segments in every drawing, regardless of the choice of the outerface.*

Proof. For each $k \geq 3$, define Z_k as the second power of C_{2k}. An example drawing of Z_8 is illustrated in Figure 4.5. For the further argumentation we enumerate the vertices canonically with v_i for $i \in \{1, 2, ..., 2k\}$ as they occur in C_{2k}. All indices of vertices are considered modulo $2k$.

The smallest member of the graph set Z_3, which is an oktahedron, was already illustrated in Figure 1.2 and shortly mentioned in Chapter 1: Z_3 has six vertices and its segment number is nine as shown by Kryven, Ravsky and Wolff [KRW19].

Clearly, every Z_k is 3-connected 4-regular planar and contains $2k$ vertices. We assume that we found a drawing D_k of Z_k with less than $2k$ segments and prove the Theorem with a contradiction.

First, we categorise the vertices in D_k regarding the amount of segments, that end in the vertex. A vertex v in which i segments in D_k end is called *T_i-vertex.* As Z_k is a 4-regular graph, this leads to the three categories T_0, T_2 and T_4. A vertex v_x is *between* two vertices v_i and v_j if $v_x \in \{v_i, v_{i+1}, v_{i+2}, \ldots, v_j\}$.

Since there are less than $2k$ segments in D_k, the drawing contains more T_0-vertices than T_4-vertices. Clearly, the realisation of the boundary of the outer face in D_k has to contain at least three T_4 vertices. Therefore, we can find two T_0-vertices v_i and v_j such that there is no T_4-vertex between them. Without loss of generality, we assume that there is no other T_0-vertex between them except v_i and v_j.

We define the graph S as the subgraph of G that contains every vertex between v_{i-2} and v_{j+2} (see Figure 4.6a). Furthermore, S does not contain edges between the two

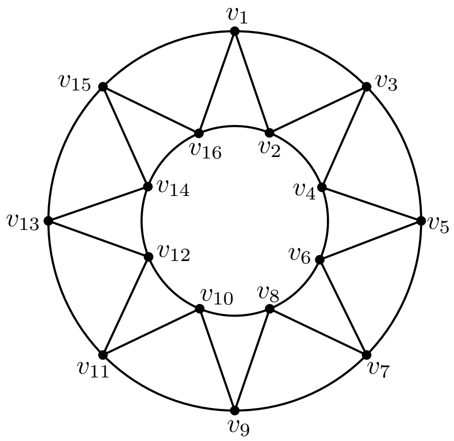


Fig. 4.5: The graph Z_8. This graph is part of the graph set in the proof of Theorem 28 and is a 3-connected 4-regular planar graph with a segment number of $|V(Z_8)| = 16$.

sets $\{v_{i-2}, v_{i-1}\}$ and $\{v_{j+2}, v_{j+1}\}$ as long as the two groups are disjoint as illustrated in Figure 4.6b. Let n be the number of vertices in S.

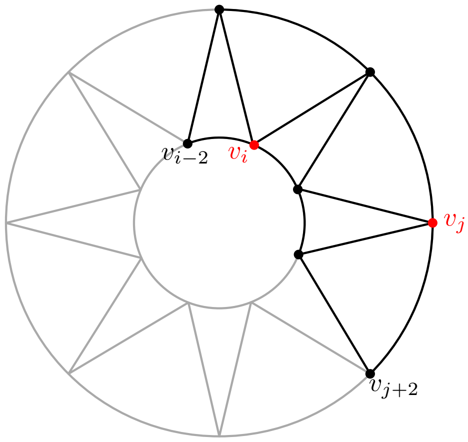
Case 1: S matches with G_n (defined in Lemma 26). We have a look at the part of the drawing of G that contains S and obtain the segments that are used to draw S: graph S contains two T_0-vertices. Maximal two segments can end in the vertices v_{i-2} and v_{j+2} because both of them have degree 2 in S. With the same argument, maximal three segments can end in v_{i-1} and v_{j+1}. The rest of the vertices in S are T_2-vertices because of the definiton of S. The sum of the number of segmentends over all vertices in S is $2n - 2$, therefore the drawing contains $n - 1$ segments. That contradicts Lemma 26: graph S matches with G_n and has therefore at least n segments in any drawing. ◁

Case 2: S does not match with G_n. In this case S is the same graph as G and the two sets $\{v_{i-2}, v_{i-1}\}$ and $\{v_{j+2}, v_{j+1}\}$ are not disjoint and the union of both sets contains at least 3 vertices.

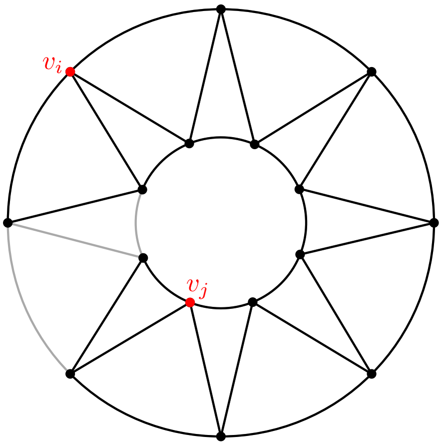
The drawing D_k contains at least three T_4-vertices on the boundary of the outer face. All of those are in $\{v_{i-2}, v_{i-1}\} \cup \{v_{j+2}, v_{j+1}\}$, which cannot contain more than those three T_4-vertices. Furthermore, we have the two T_0-vertices v_i and v_j. The remaining vertices are T_2 vertices. Altogether, we have $2k + 1$ segments in the drawing, which contradicts the assumption that the drawing was made with less than $2k$ segments. ◁

Therefore, such a drawing D_k of G_k with less than $2k$ segments could not exist. □

In fact, the segment number of Z_k with $k \geq 6$ is $2k$. An example of a drawing of Z_{10} with 20 segments can be found in Figure 4.7. It can be easily adjusted for drawings of Z_i with $i \geq 6$. With Theorem 28, we know that there does not exist a drawing of Z_k ($k \geq 6$) with less segments.



(a) Subgraph S of Z_k with the two T_0-vertices v_i and v_j.



(b) Subgraph of Z_k with the two T_0-vertices v_i and v_j and the feature that S does not contain every edge from G that is between two vertices of S

Fig. 4.6: Two examples how S is chosen depending on v_i and v_j

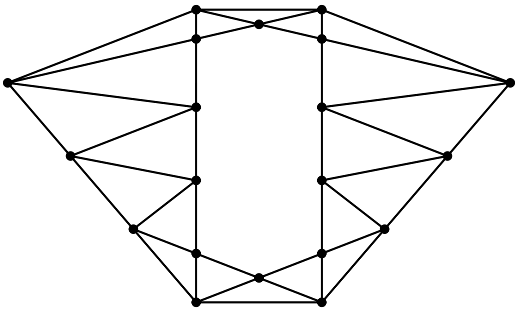


Fig. 4.7: A segment-optimal drawing of Z_{10} with 20 segments.

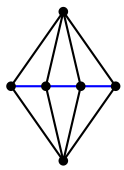
4.3 Existential Lower Bound of 2-connected 4-regular planar graphs

In this section, we discuss a 2-connected 4-regular planar graph set such that every graph G in this set has at least $7|V(G)|/6$ segments in any drawing. This graph set gives an example for the observation that Theorem 20 cannot directly be generalized for 2-connected 4-regular planar graphs.

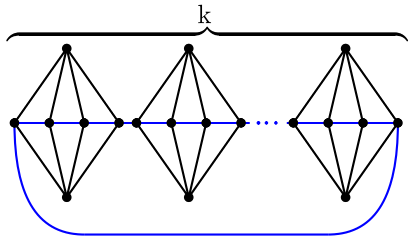
Theorem 29. *There is an infinite subset S of the 2-connected 4-regular planar graphs that fulfills the following property: every graph G in S has at least $7|V(G)|/6$ segements in any drawing.*

Proof. Consider the graph gadget G_g in Figure 4.8a. It consists of the six vertices $\{x_1, x_2, y_1, y_2, y_3, y_4\}$ and the set of edges

$$E(G_g) = \{(x_i, y_j) \mid i \in \{1,2\}, j \in \{1,2,3,4\}\} \cup \{(y_1,y_2),(y_2,y_3),(y_3,y_4)\}$$



(a) Gadget G_g for the construction of the graph set S.



(b) Member D_k of the graph set S with k gadgets G_g

Fig. 4.8: Definition of the graph set S with the property that each graph G in S has at least $7|V(G)|/6$ segments in any drawing.

We define D_k ($k \geq 2$) as the graph that contains k gadgets, which are arranged in one simple cycle with connection edges between the gadgets as illustrated in Figure 4.8b. Clearly, every member in this graph set is 4-regular, 2-connected and planar. Since every gadget contains 6 vertices and 11 edges, the whole graph contains $6k$ vertices and $12k$ edges. The set S is defined as $\{D_k \mid k \geq 2\}$.

First, we have a look at the gadget G_g in Figure 4.8a and analyse its segment number. We show, that G_g cannot be drawn with less than eight segments. The derived information can be used later to derive the segment number of D_k.

We define a *link* as a set $\{(u, v), (x, y)\}$ of two adjacent edges $(u, v), (x, y) \in E$. Two links are adjacent if and only if the sets intersect in one edge. In the following paragraph, we describe a segment as a set of links. A drawing contains a link if there exists a segment in the drawing, which contains the link.

There are four different types of links in the gadget that can be part of a segment. All of them are visualized in Figure 4.9. With the naming of the vertices as in Figure 4.9, they can be formally describe:

(Type 1) $\{(x_1, y_i), (y_i, x_2)\}$ with $i \in \{1, 2, 3, 4\}$

(Type 2) $\{(y_1, x_1), (x_1, y_3)\}$, $\{(y_2, x_1), (x_1, y_4)\}$, $\{(y_1, x_2), (x_2, y_3)\}$, $\{(y_2, x_2), (x_2, y_4)\}$

(Type 3) $\{(y_1, y_2), (y_2, y_3)\}$, $\{(y_2, y_3), (y_3, y_4)\}$

(Type 4) $\{(y_1, x_1), (x_1, y_4)\}$, $\{(y_1, x_2), (x_2, y_4)\}$.

We observe that any drawing of the gadget can contain maximal one type-1-link, two type-2-links, two type-3-links and one type-4-link.

Case 1: The drawing contains two type-2-links. Then, it cannot contain any type-4- or type-3-link, otherwise two different segments would intersect in two vertices. Additionally, it can contain one type-1-link. ◁

Case 2: The drawing contains one type-2-link. In this case, it can contain additionally one type-1-link. It can contain maximal one type-3-link, otherwise two segments would

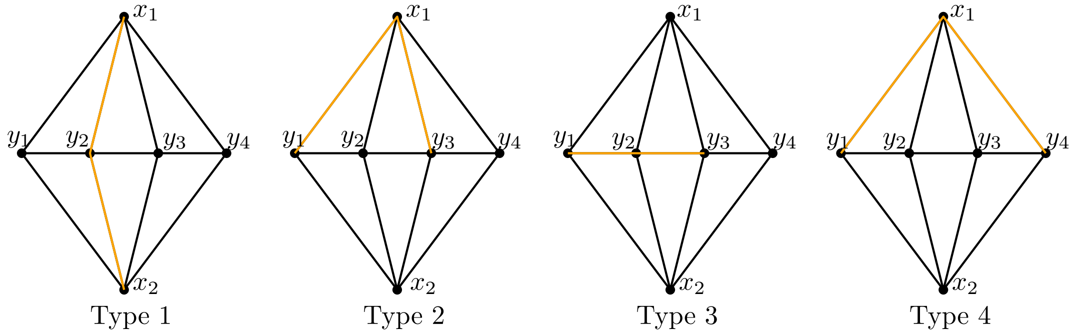


Fig. 4.9: The four types of links in the gadget.

intersect in two vertices. It cannot contain a type-4-link: without loss of generality, we assume that type-4-link $\{(y_1, x_1), (x_1, y_4)\}$ is contained, then x_2 has to be located on the type-2-link. The 180° from the type-4-link causes, that the angle $\measuredangle y_4 x_2 y_1$ is smaller than 180°. This contradicts the fact, that the type-2-link is located there and the embedding of G_g is unique because of its 3-connectivity. ◁

Case 3: The drawing contains no type-2-link. Then, it can contain one type-1-link and maximal two more links out of type-3- and type-4-links because if the drawing contains two type-3-links it cannot contain any type-4-link because of a similar argumentation as above that two segments cannot intersect in more than just one vertex. ◁

Altogether a drawing contains maximal three links. Since there are eleven edges in the gadget and every link connects exactly two edges in a segment, the gadget cannot be drawn with less than $8 = 11 - 3$ segments.

We use this information to analyse the segment number of the whole graph D_k. Two different drawings of adjacent gadgets in the graph can share maximal one segment and every gadget has exactly two adjacent gadgets, therefore the segment number of D_k has to be at least $k \cdot (8 - 1) = 7k$.

Graph D_k contains k gadgets and $|V(G)| = 6k$ vertices. Therefore, the segment number s in dependency of $|V(G)|$ is at least $s = 7k = 7|V(G)|/6$ and the Theorem is shown. □

5 Conclusion and Outlook

Summary In this thesis, we studied the upper bound, existential- and univeral lower bound for the segment number of 3-connected 4-regular planar graphs.

We showed that an upper bound for the segment number for those graphs is given by $|V(G)| + 3$. The proof was constructive and the constructed drawing was additionally a convex drawing. This improved the upper bound of Dujmović et al. [DESW07] and Heigl [Hei21] of $5|V(G)|/3$ to $|V(G)|+3$. In order to prove that our upper bound is tight up to an additive constant, we gave an example of a subset of the 3-connected 4-regular planar graphs that have at least $|V(G)|$ segments in any drawing.

Furthermore, we studied the universal lower bound for the segment number of the 3-connected, 4-regular planar graphs and showed that the universal lower bound, that was pointed out by Dujmović et al. [DESW07] was tight up to a small constant factor.

Finally, we gave a set of 2-connected 4-regular planar graphs such that every graph G in the set has a segment number of at least $7|V(G)|/6$.

Transferability of the upper bound The proven upper bound of $|V(G)| + 3$ for 3-connected 4-regular planar graphs (see Theorem 20) is not generalisable for 3-connected planar graphs because of the existential lower bound $2n - 6$ that was pointed out by Dujmović et al. [DESW07]. Furthermore, it is not transferable for 2-connected 4-regular planar graphs since we gave an example of a set of 2-connected 4-regular planar graphs with the property that each graph G in this set has the segment number $7|V(G)|/6$.

Even if the established technique to construct a drawing such that every (inner) vertex is drawn on the interior of a segment is not directly transferable it could be worth to apply the idea to other graph classes for example triangulated graphs with the property that every vertex has at least degree 4. While the first two cases of our proof will be easy to adjust to this graph class, the second part of case 3 will be challenging. In those graphs, the archfaces can be placed in unfavorable positions and make the construction of archfree windmills more difficult.

Future Work In this thesis, we made restrictive assumptions on the graphs, for which our results hold. Clearly, many problems concerning the segment number are still not solved.

Moreover, segments are just one example of a geometric object. As the visual complexity of a drawing is defined as the number of geometric objects in the drawing, it is interesting to study the number of other geometric objects, for example the number of circular arches as introduced by Schulz [Sch15].

As it is important to keep the visual complexity low for the user, further studies on the number of geometric objects will be beneficial.

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Erklärung

Hiermit versichere ich die vorliegende Abschlussarbeit selbstständig verfasst zu haben, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt zu haben, und die Arbeit bisher oder gleichzeitig keiner anderen Prüfungsbehörde unter Erlangung eines akademischen Grades vorgelegt zu haben.

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