Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

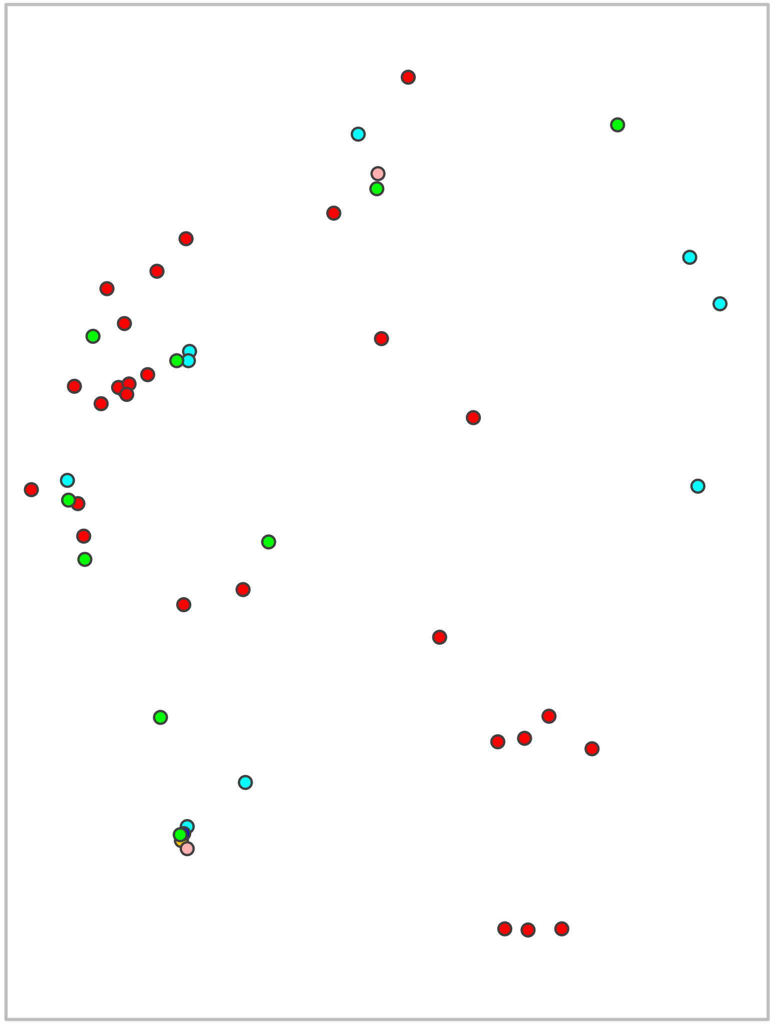
Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Cluster Minimization in Geometric Graphs

Jakob Geiger

Motivation



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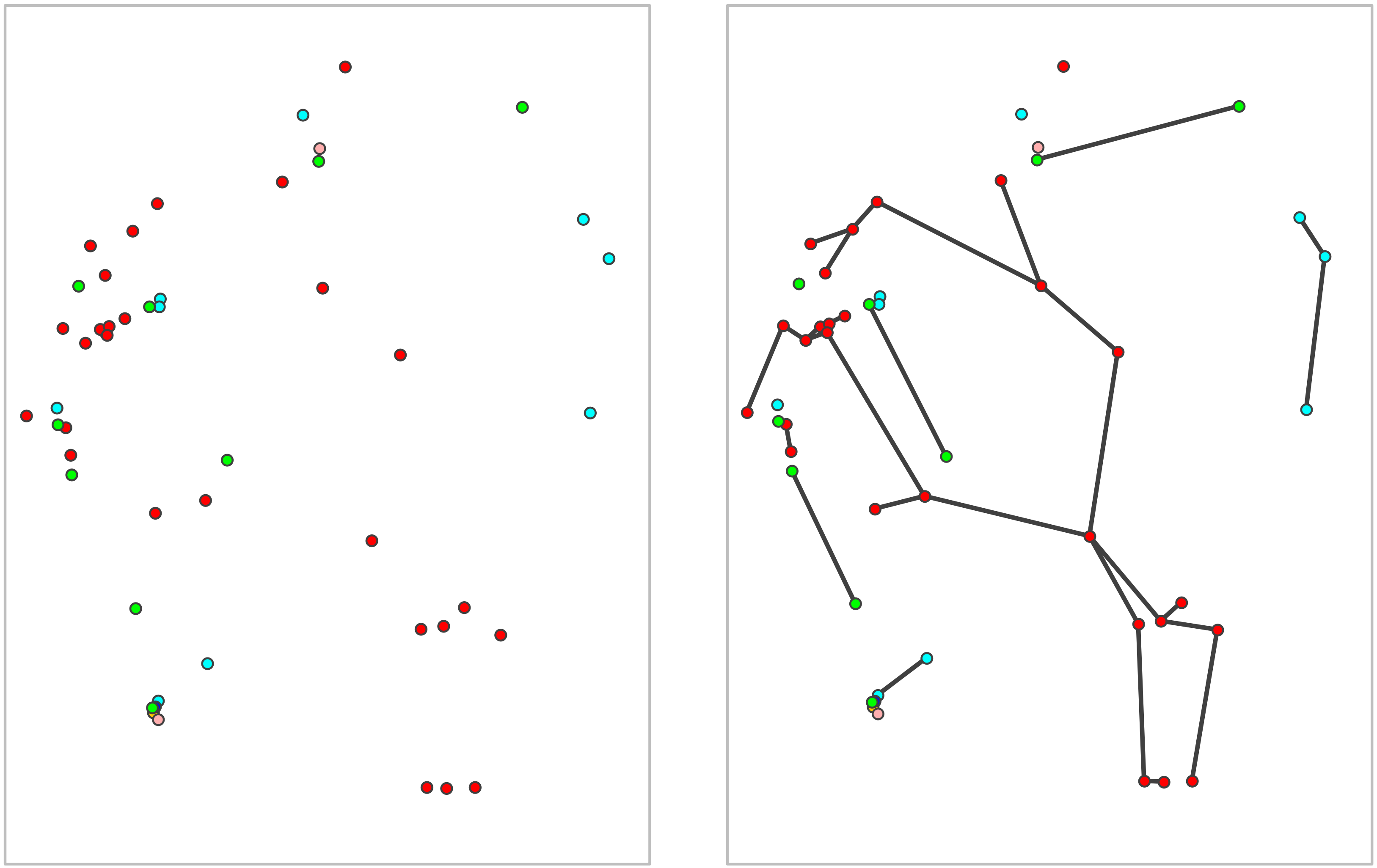
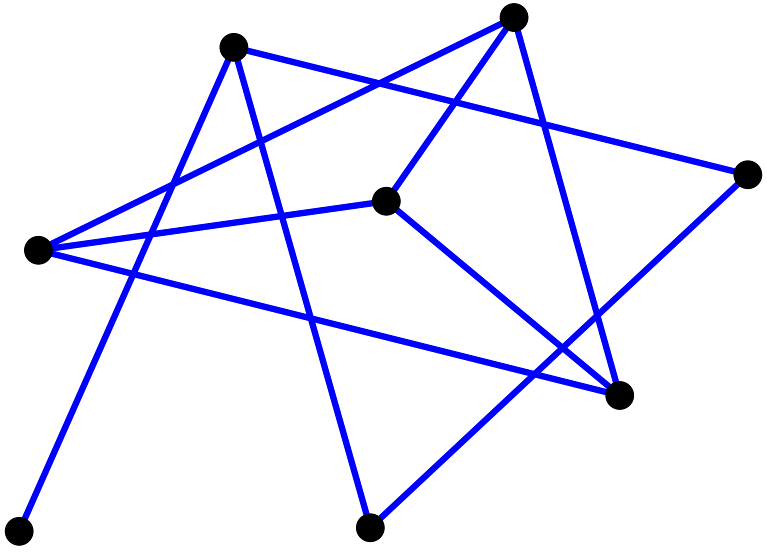


Cluster Minimization

Given: Geometric graph $G = (V, E)$



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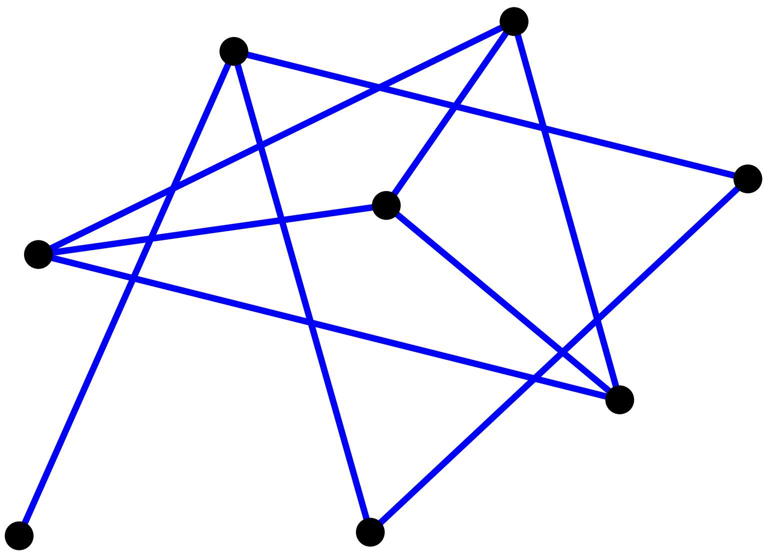
Goal: Find a subgraph $H = (V, E')$ of G such that no two edges in E' cross and the number of connected components in H is minimized.



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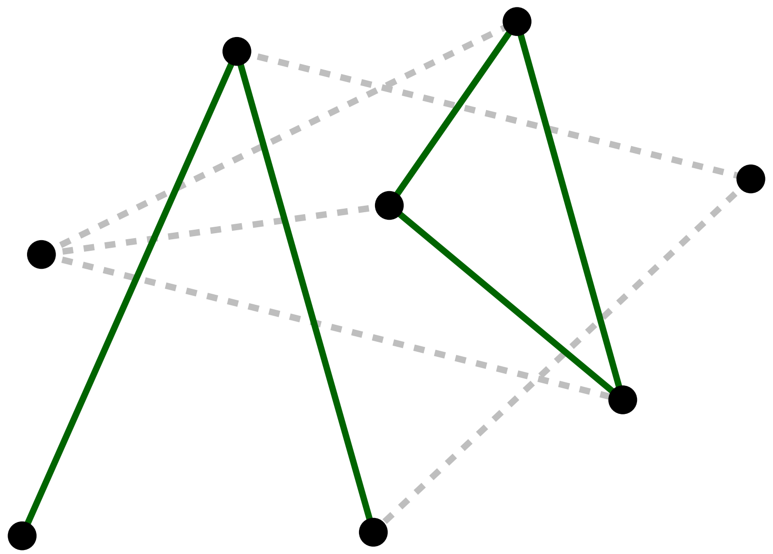

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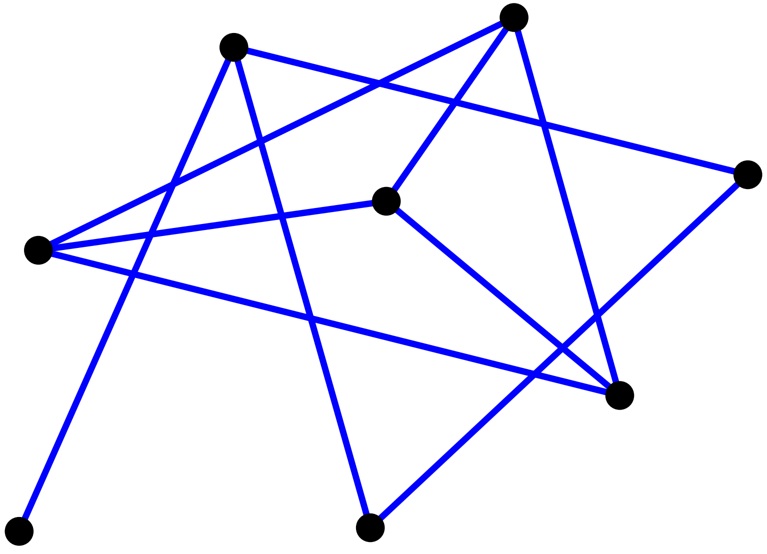

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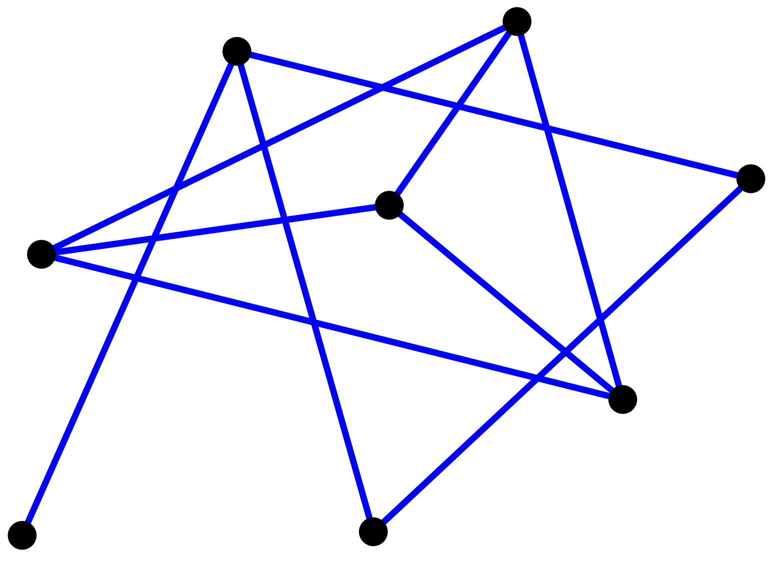

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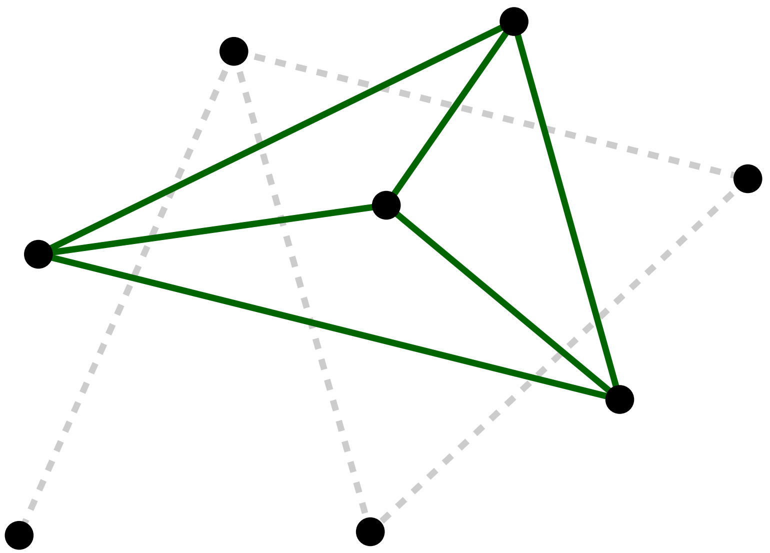

State of the art

Problem	Quality	Runtime
Cluster Minimization	exact	?
– Greedy	?	polynomial
– 1-plane graphs	exact	polynomial
Edge Maximization	exact	NP-hard

all results by [Akitaya et al. 2019]

My contribution

Problem	Quality	Complexity
Cluster Min.	exact	NP-hard
– Greedy	no const. factor	$n + k + m \log m$
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NP-Hardness

Independent Set $\leq_p$ Cluster Minimization

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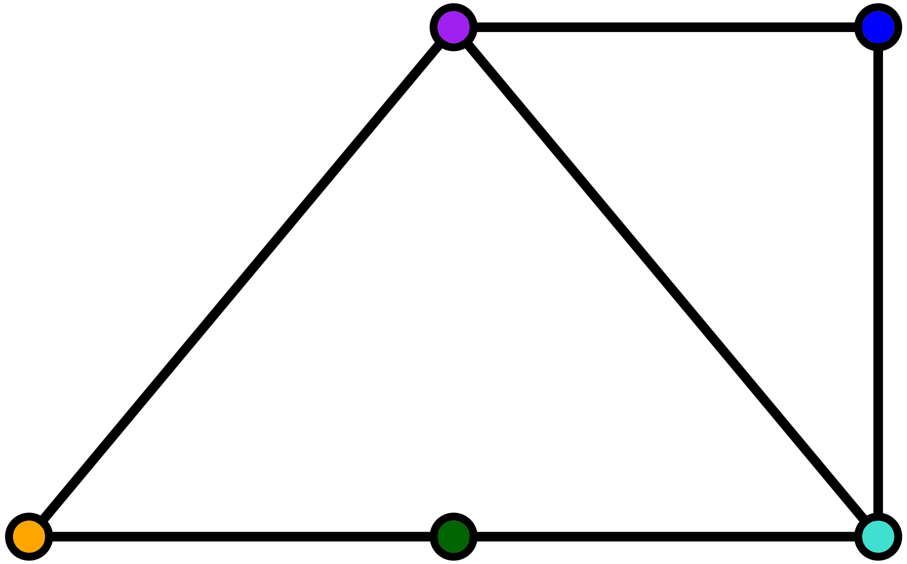
- Given an instance of Independent Set,

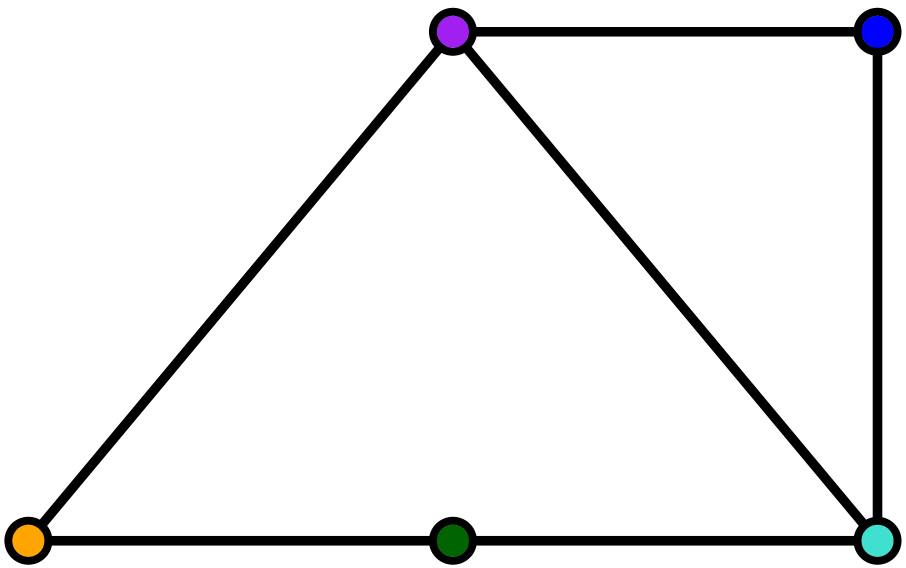


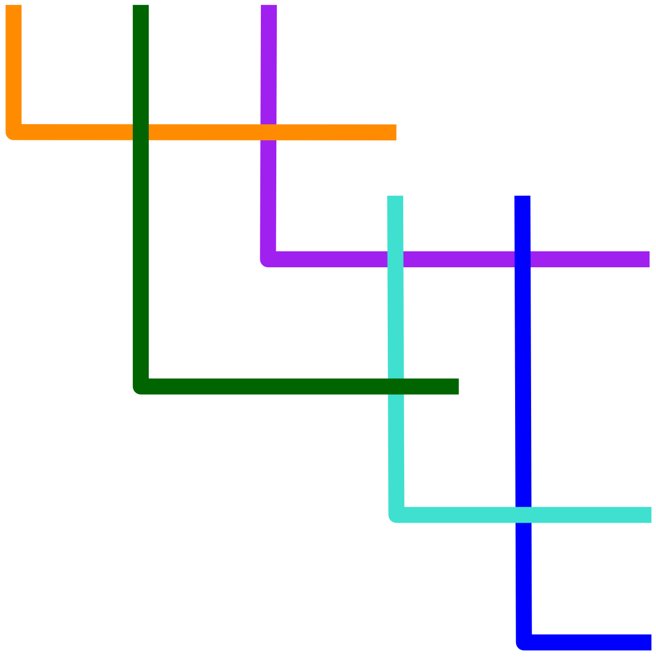
NP-Hardness

Independent Set $\leq_p$ Cluster Minimization

- Given an instance of Independent Set,
- Construct an equivalent L-shape intersection graph...

[Gonçalves et al. 2018]



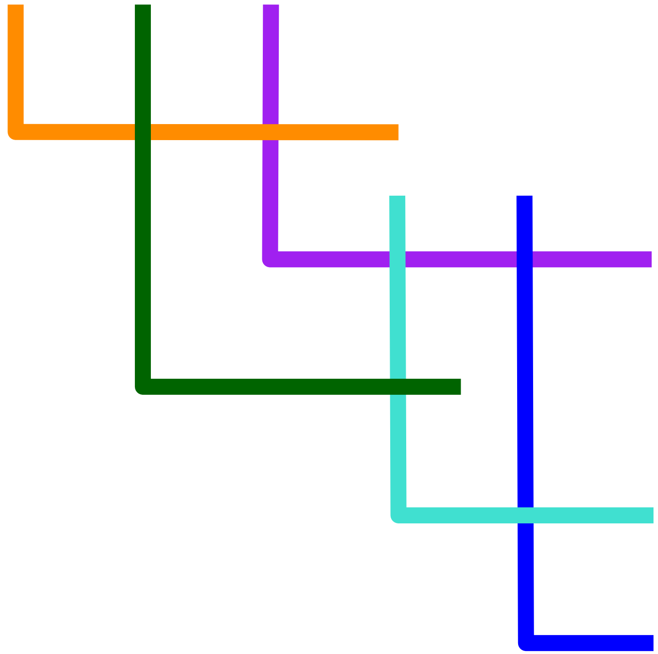


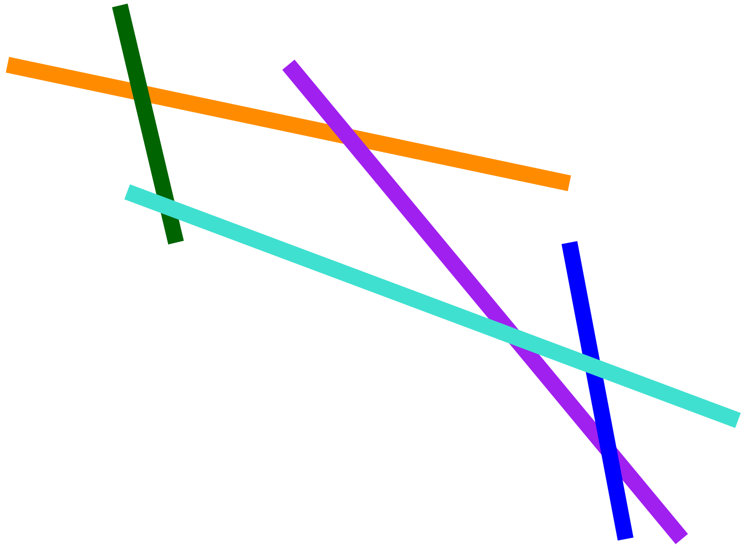
NP-Hardness

Independent Set $\leq_p$ Cluster Minimization

- Given an instance of Independent Set,
- Construct an equivalent L-shape intersection graph...
- ... then construct an equivalent segment intersection graph.

[Biedl 2020]





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Independent Set $\leq_p$ Cluster Minimization

- Given an instance of Independent Set,
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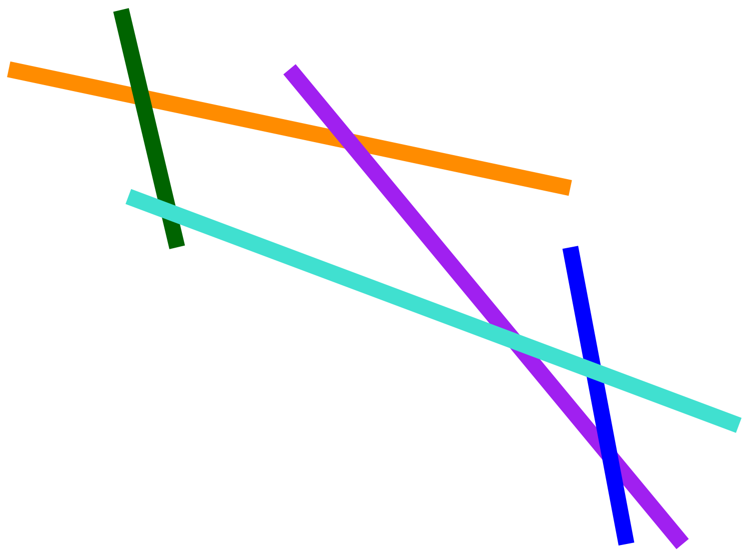
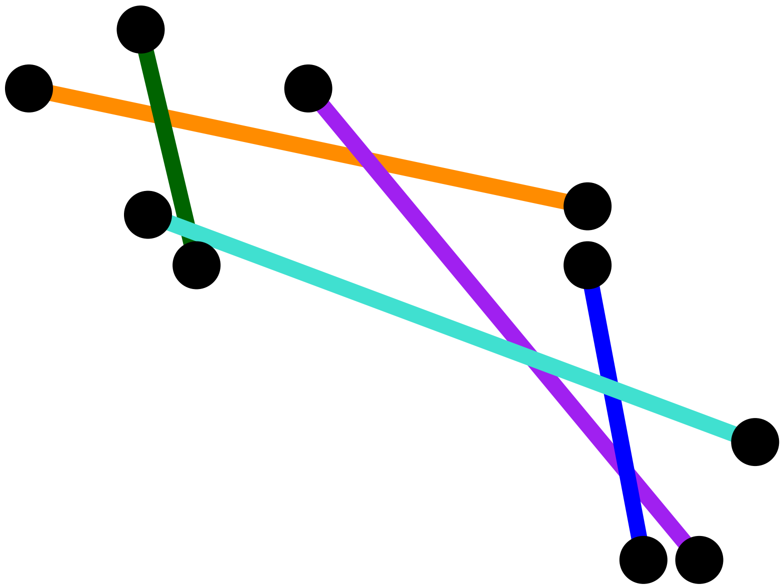
- Use the segments as edges in a geometric graph and place vertices at each endpoint.





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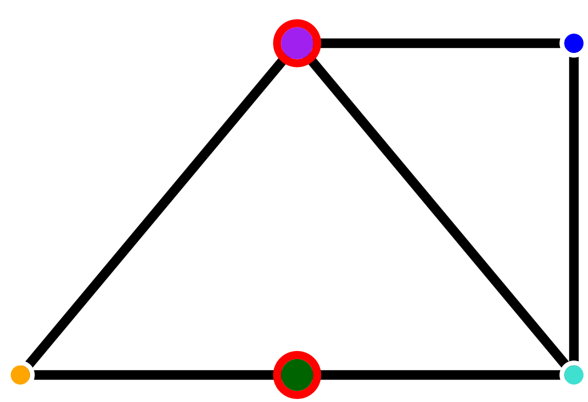
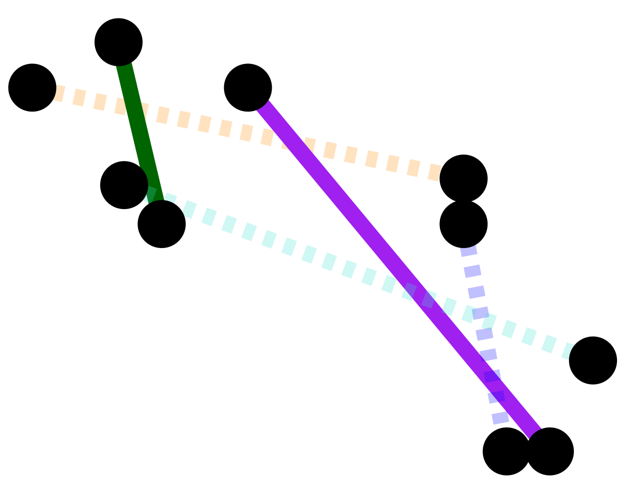
Independent Set $\leq_{\mathrm{p}}$ Cluster Minimization

- Given an instance of Independent Set,
- Construct an equivalent L-shape intersection graph...
- ... then construct an equivalent segment intersection graph.
- Use the segments as edges in a geometric graph and place vertices at each endpoint.
- In the resulting geometric graph, a solution with $2n - k$ clusters represents an independent set of size k.





Heuristics

Greedy: Iteratively select the least crossed edge

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Reverse Greedy: Iteratively delete the most crossed edge

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Preprocessing: compute all edge crossings

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$\Rightarrow O(k + m \log m)$ [Balaban 1995]

Greedy

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Use Union-Find to manage clusters $\Rightarrow O(n + m\alpha(m))$

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Remove	$O(\log n)^*$
ExtractMin	$O(\log n)^*$
DecreaseKey	$O(1)^*$

*amortized

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Overall Runtime: $O(n + k + m \log m)$

1-plane graphs: $m, k \in O(n)$

$\Rightarrow$ Overall runtime reduces to $O(n \log n)$!

Reverse Greedy

Iteratively delete the most crossed edge

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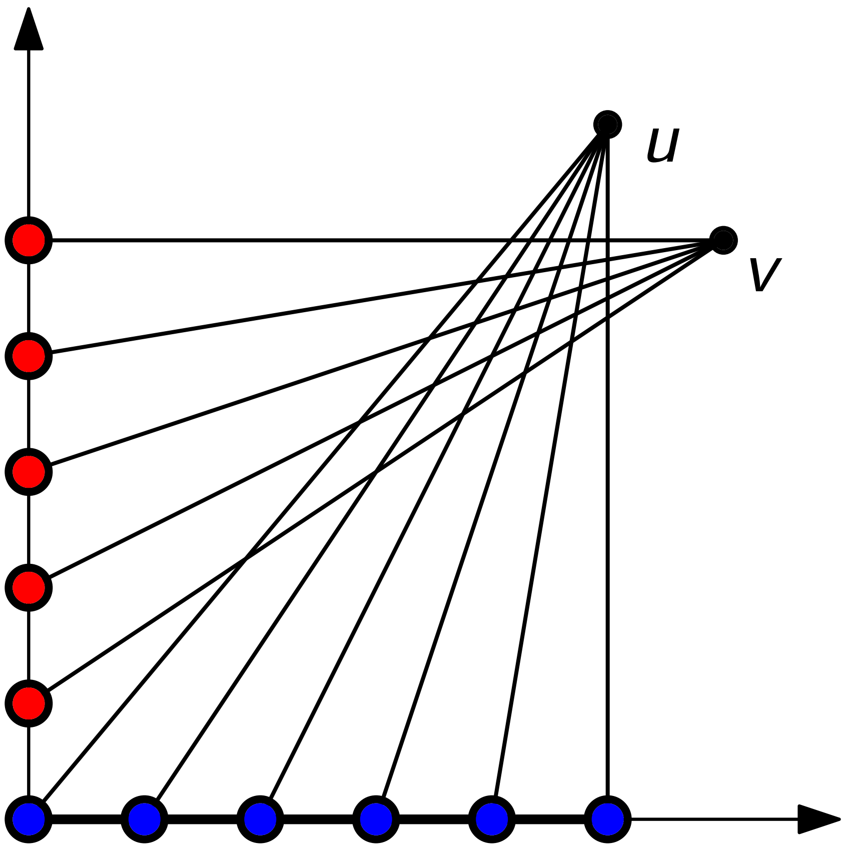
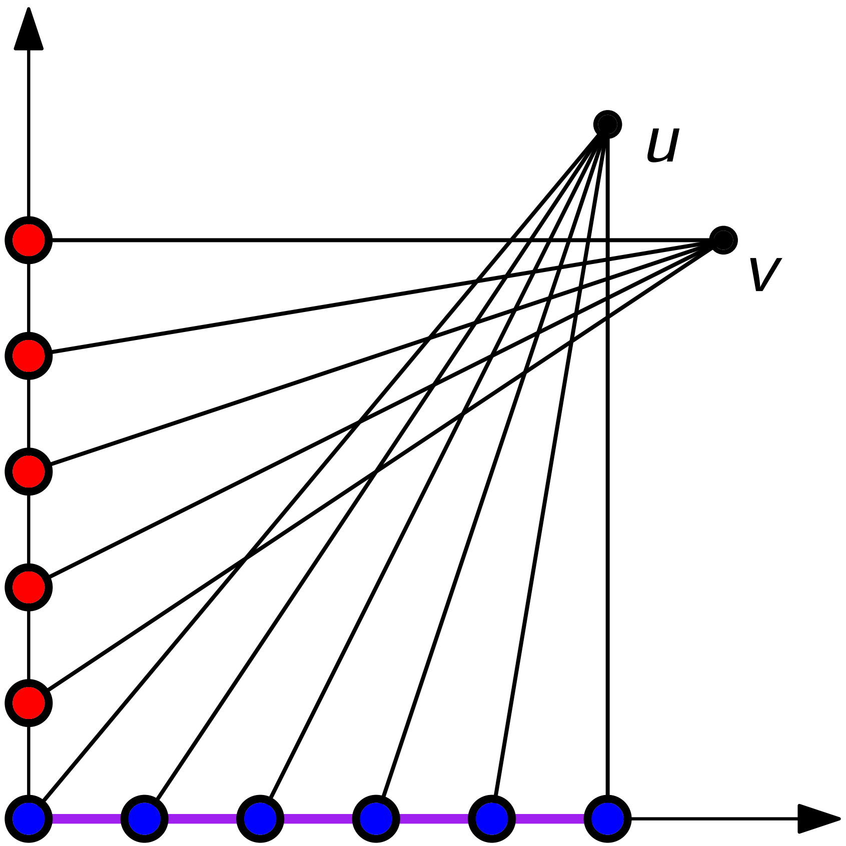
Overall Runtime: $O(n + k \log k + m \log m)$

Performance Analysis – Theoretical



n red, $n+1$ blue vertices

Performance Analysis – Theoretical



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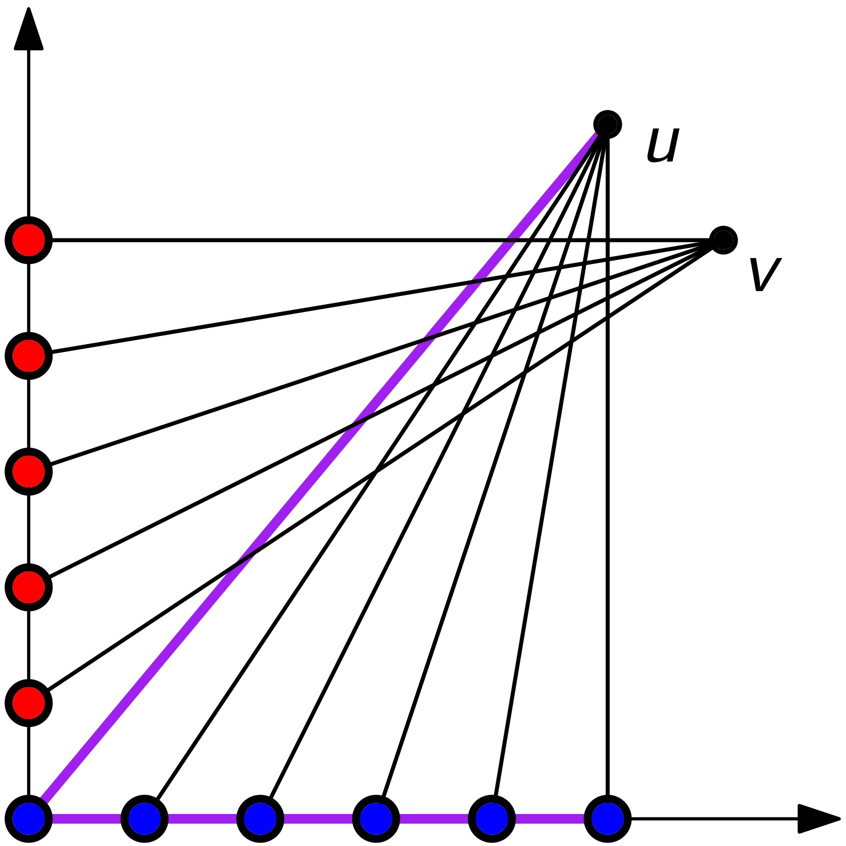

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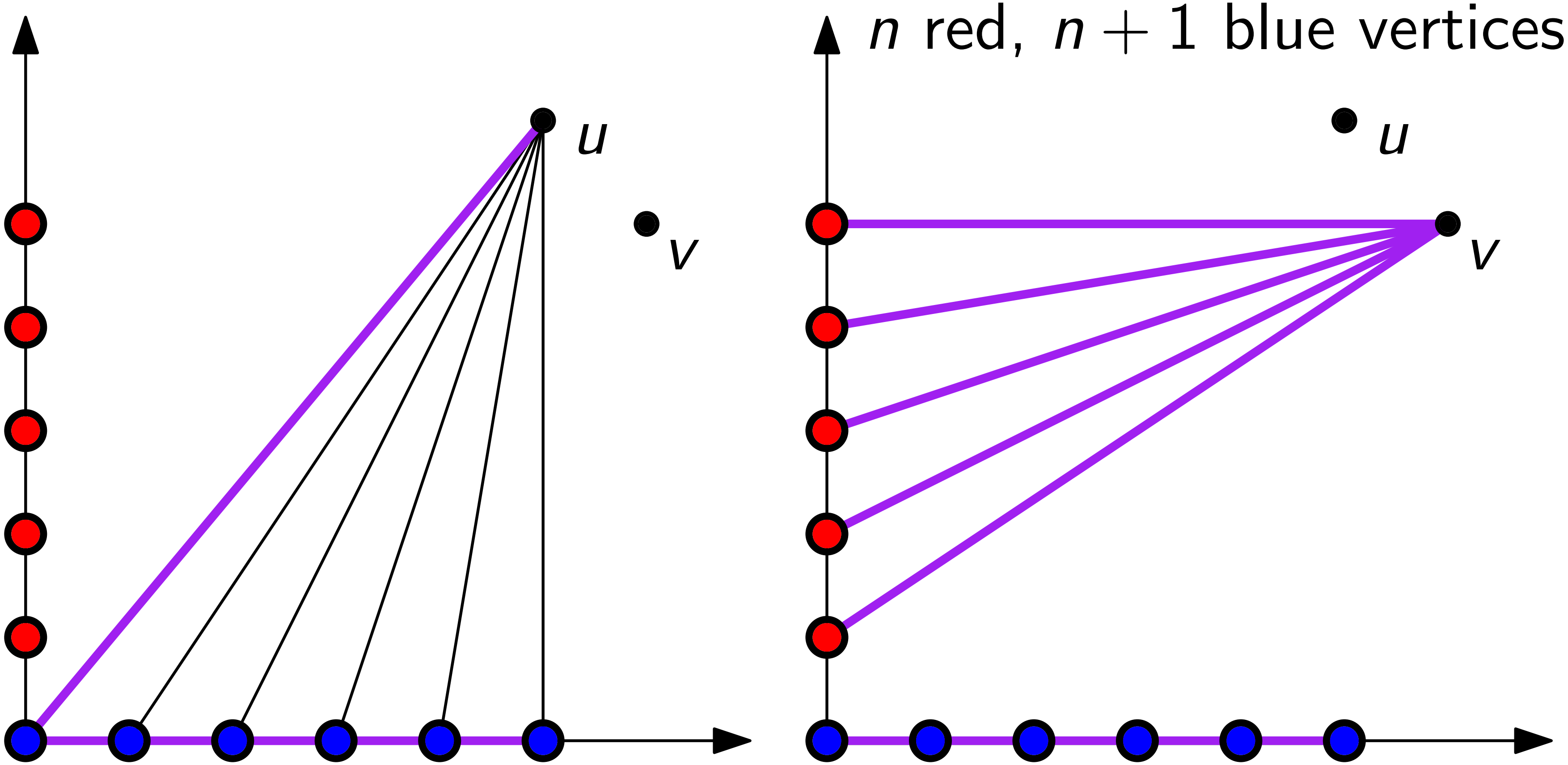

Greedy/Reverse Greedy: $n+2$ clusters

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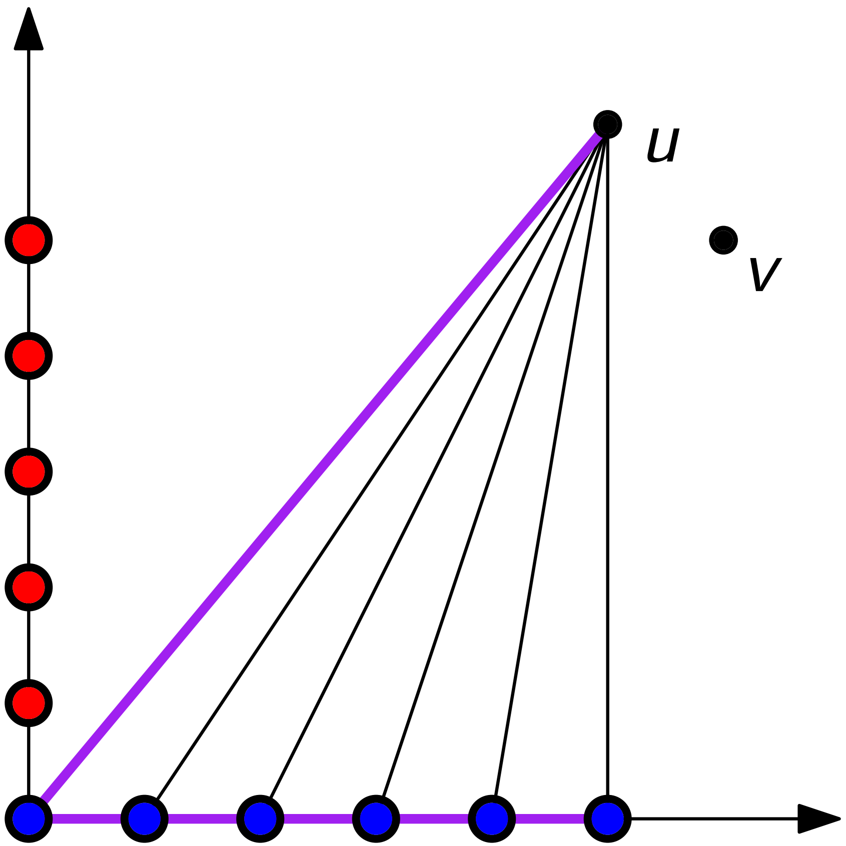
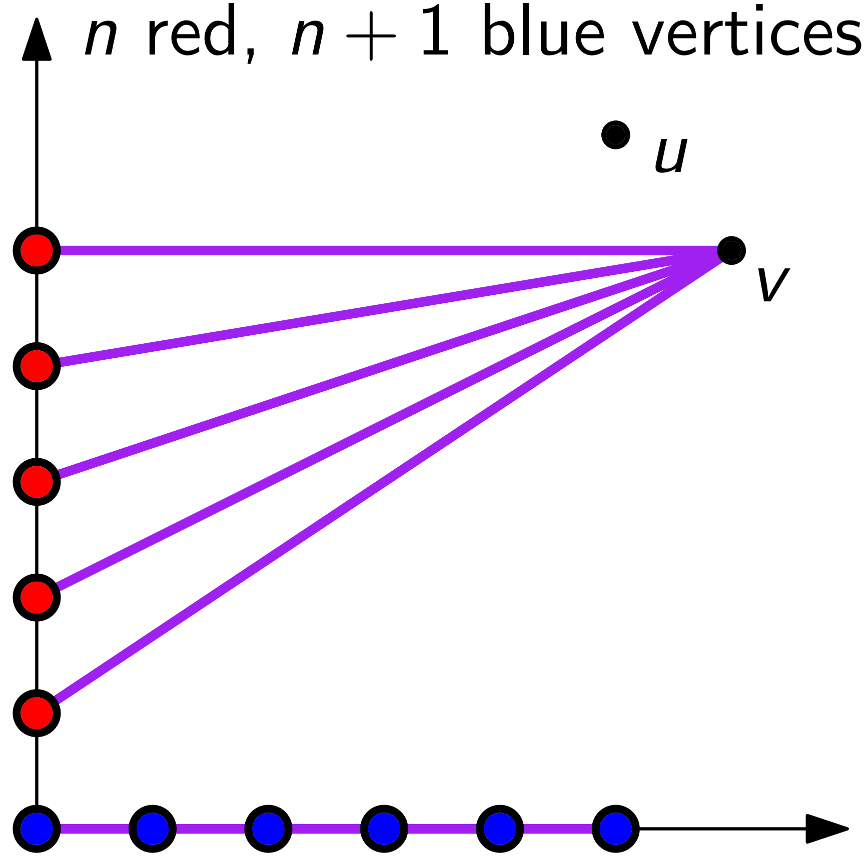
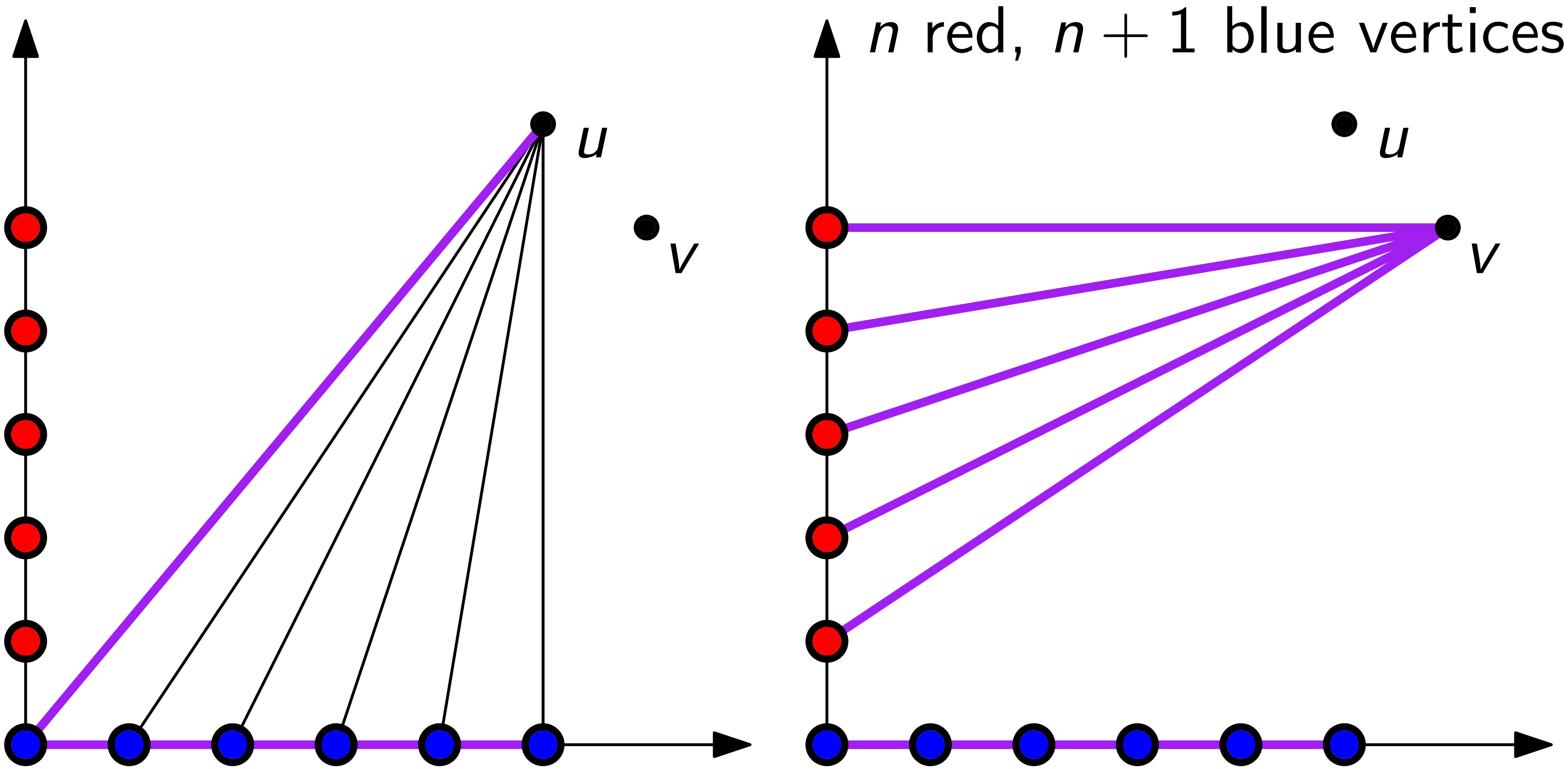




Greedy/Reverse Greedy: $n+2$ clusters

Optimal solution:

Performance Analysis – Theoretical

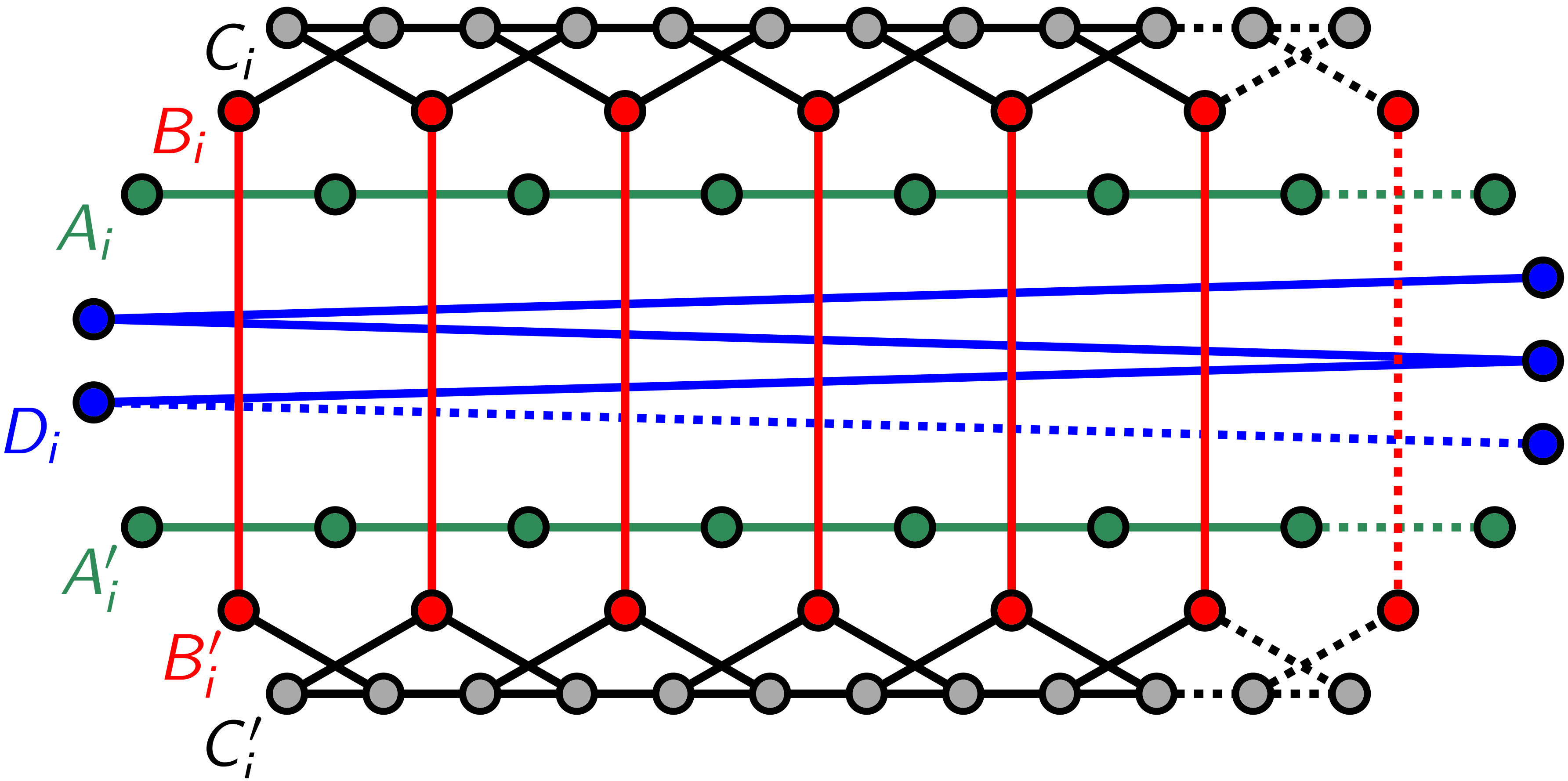


Greedy/Reverse Greedy: $n+2$ clusters

Optimal solution: 4 clusters

$\Rightarrow$ no constant approximation factor for both heuristics!

Performance – Greedy vs. Reverse Greedy



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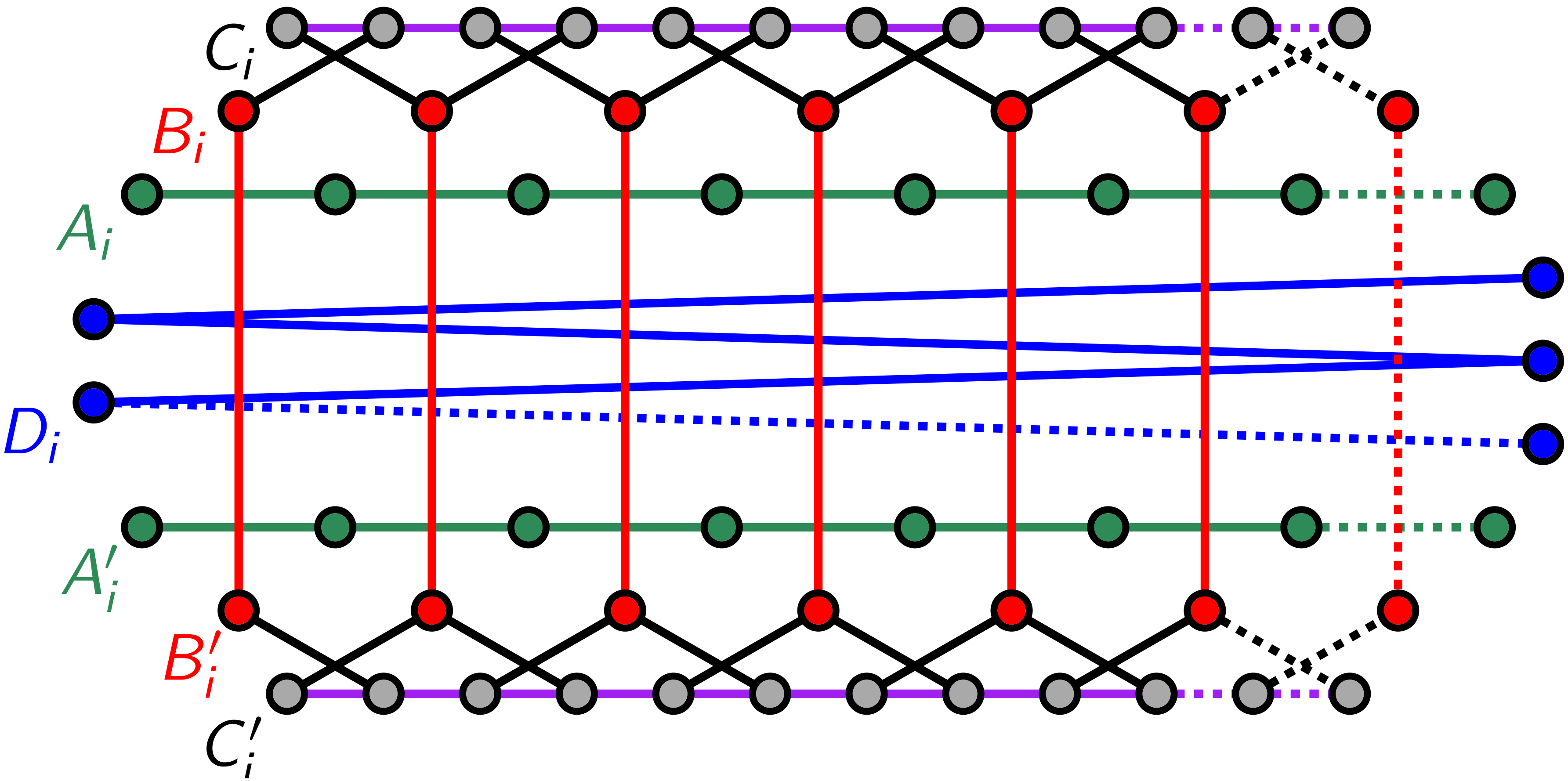

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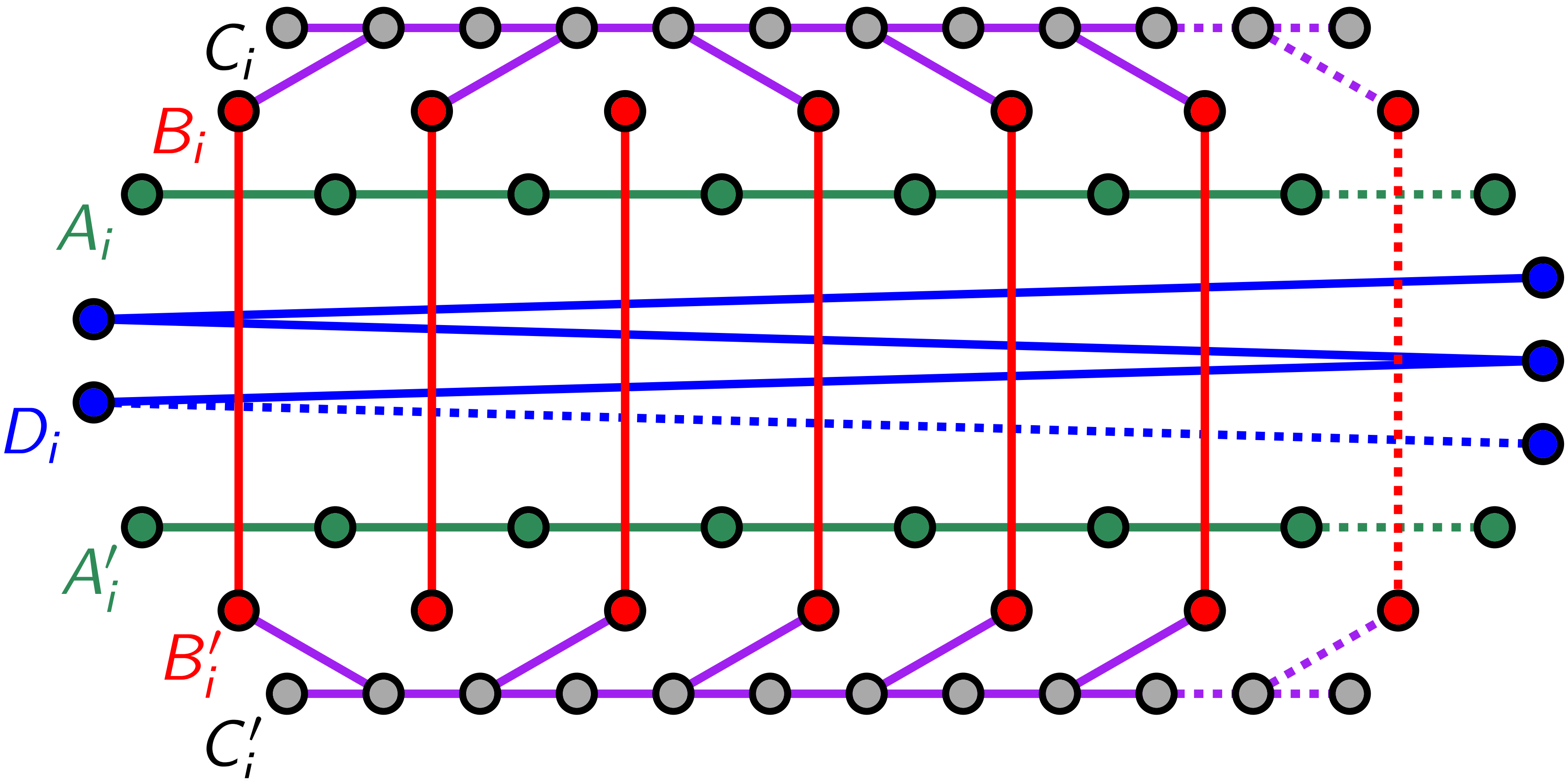

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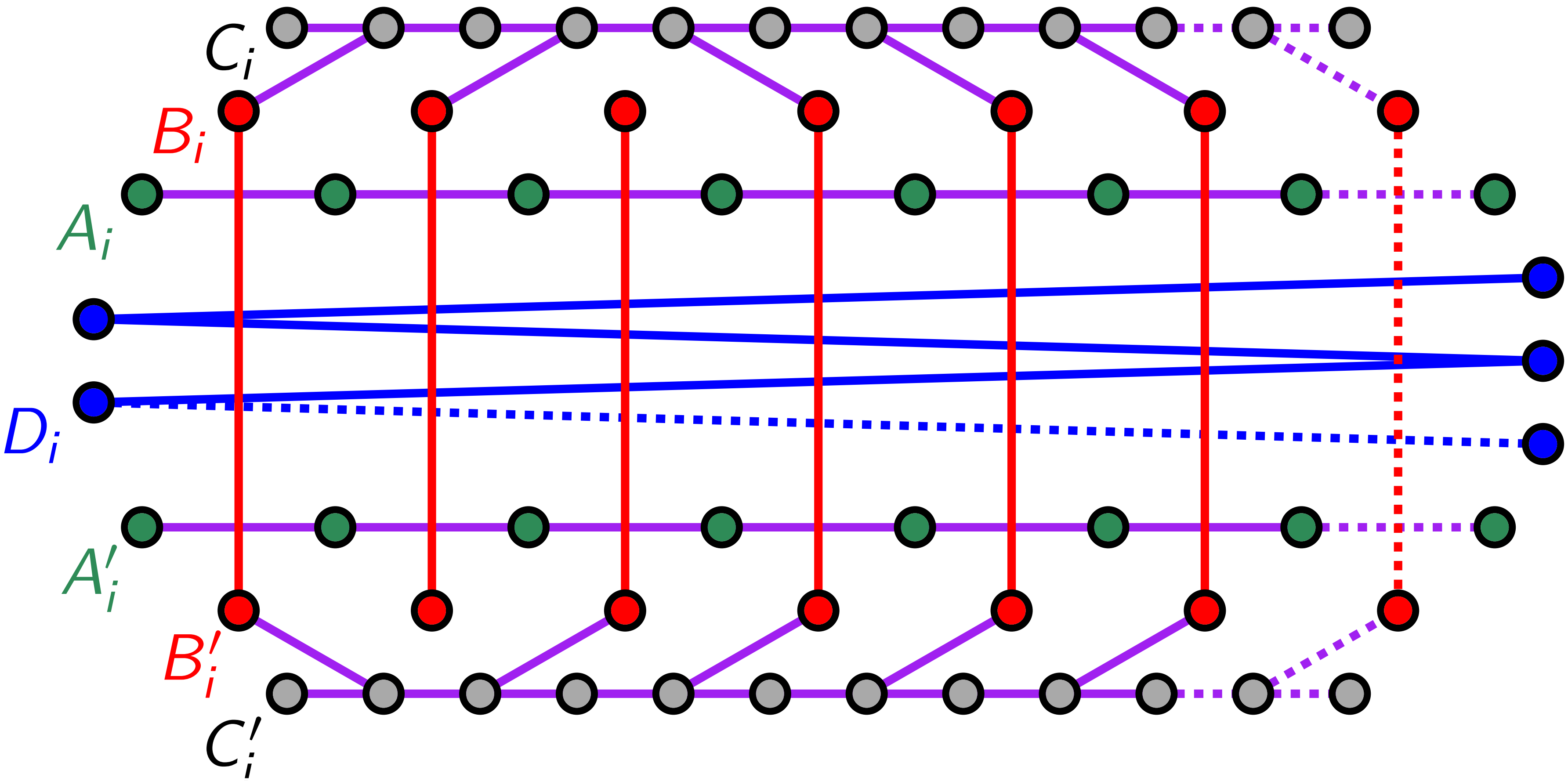

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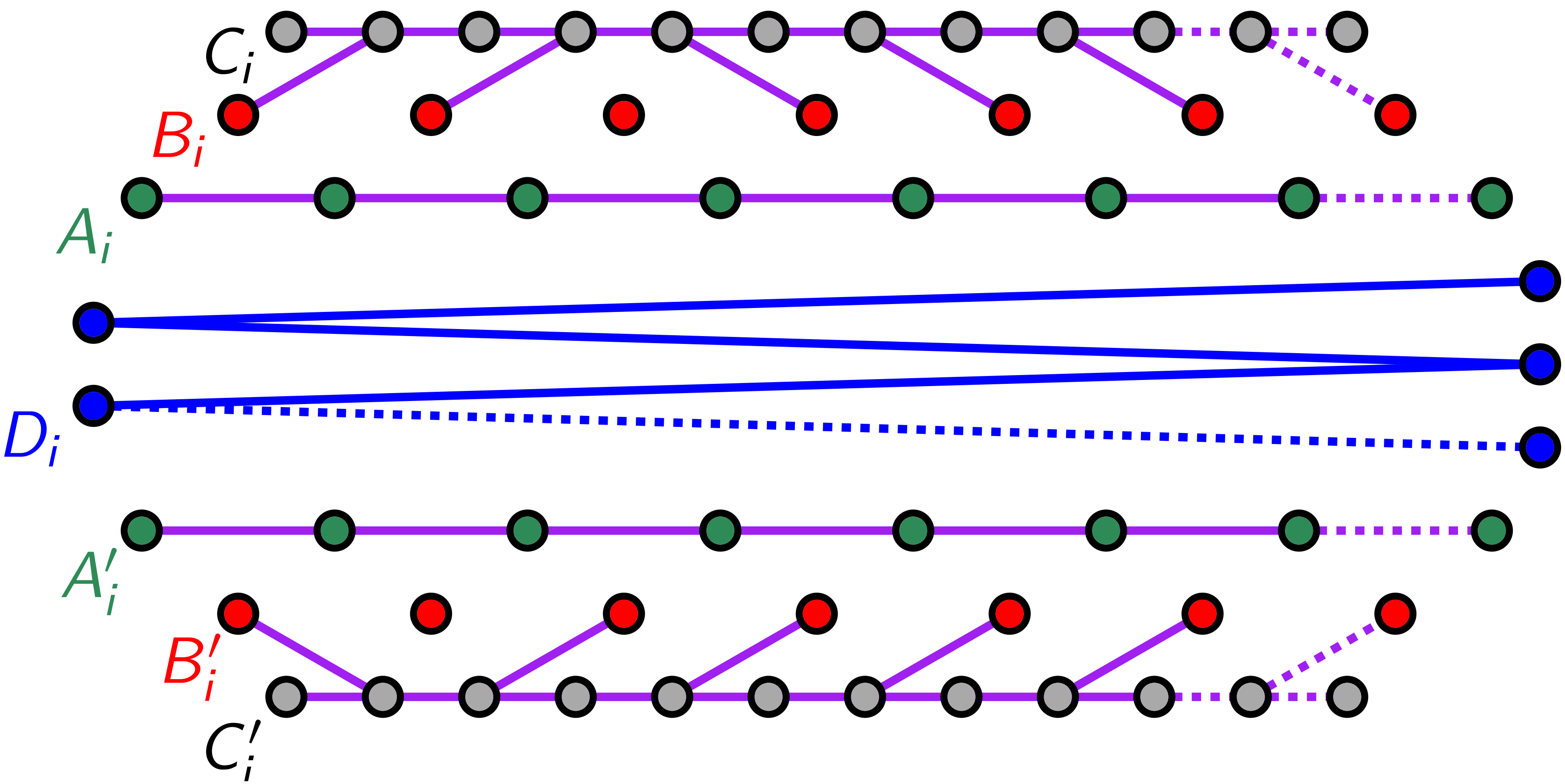

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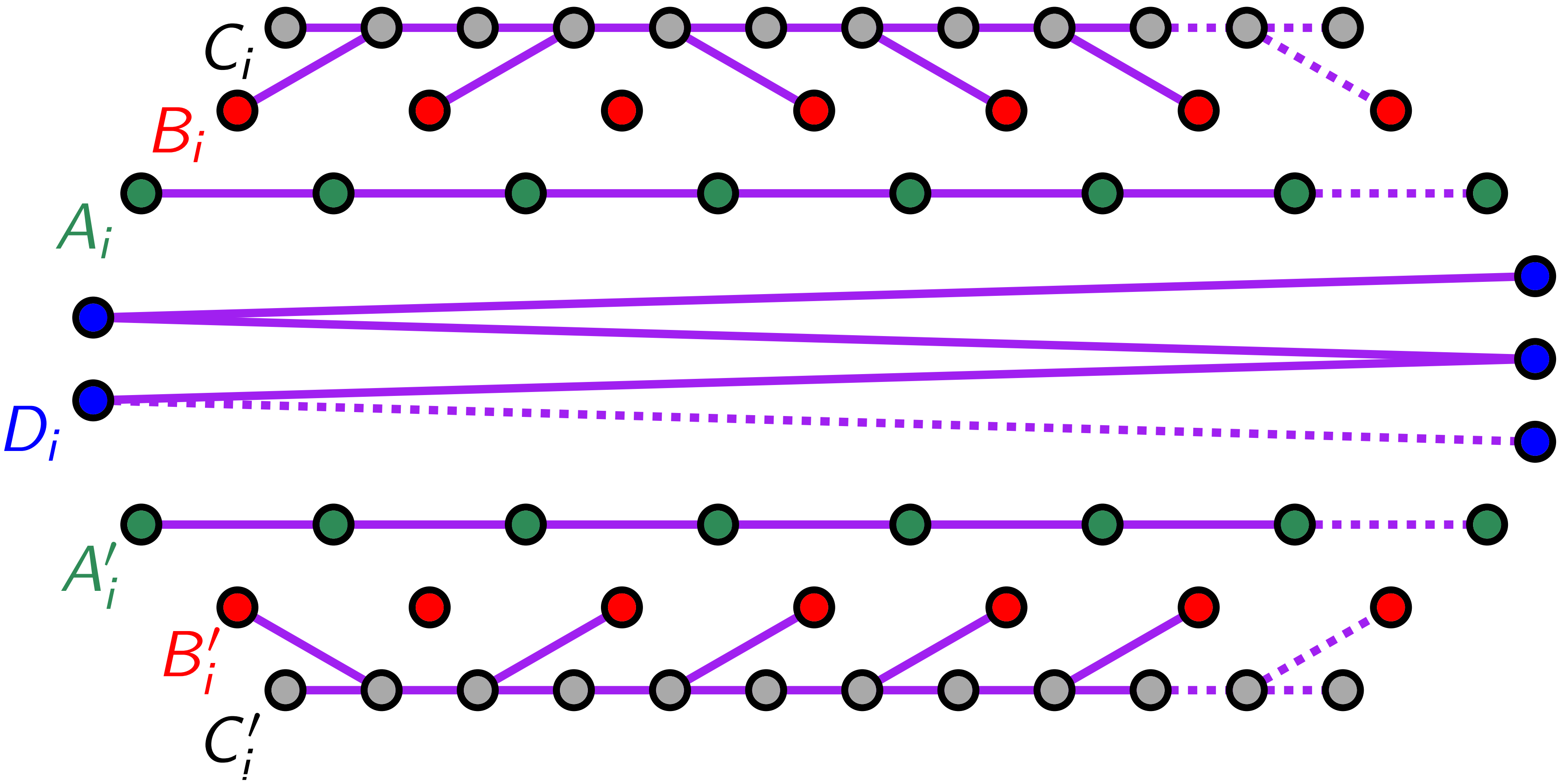

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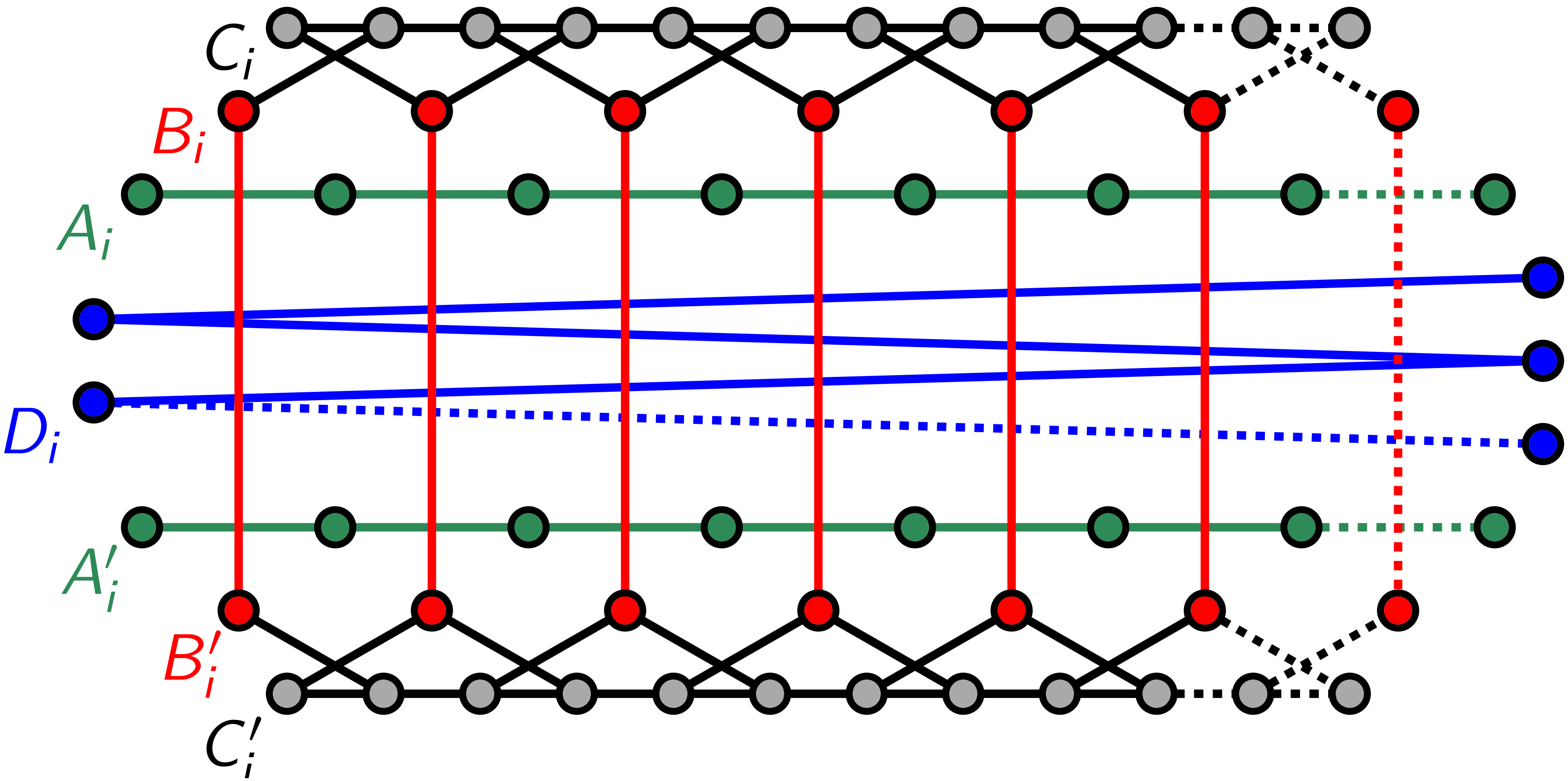

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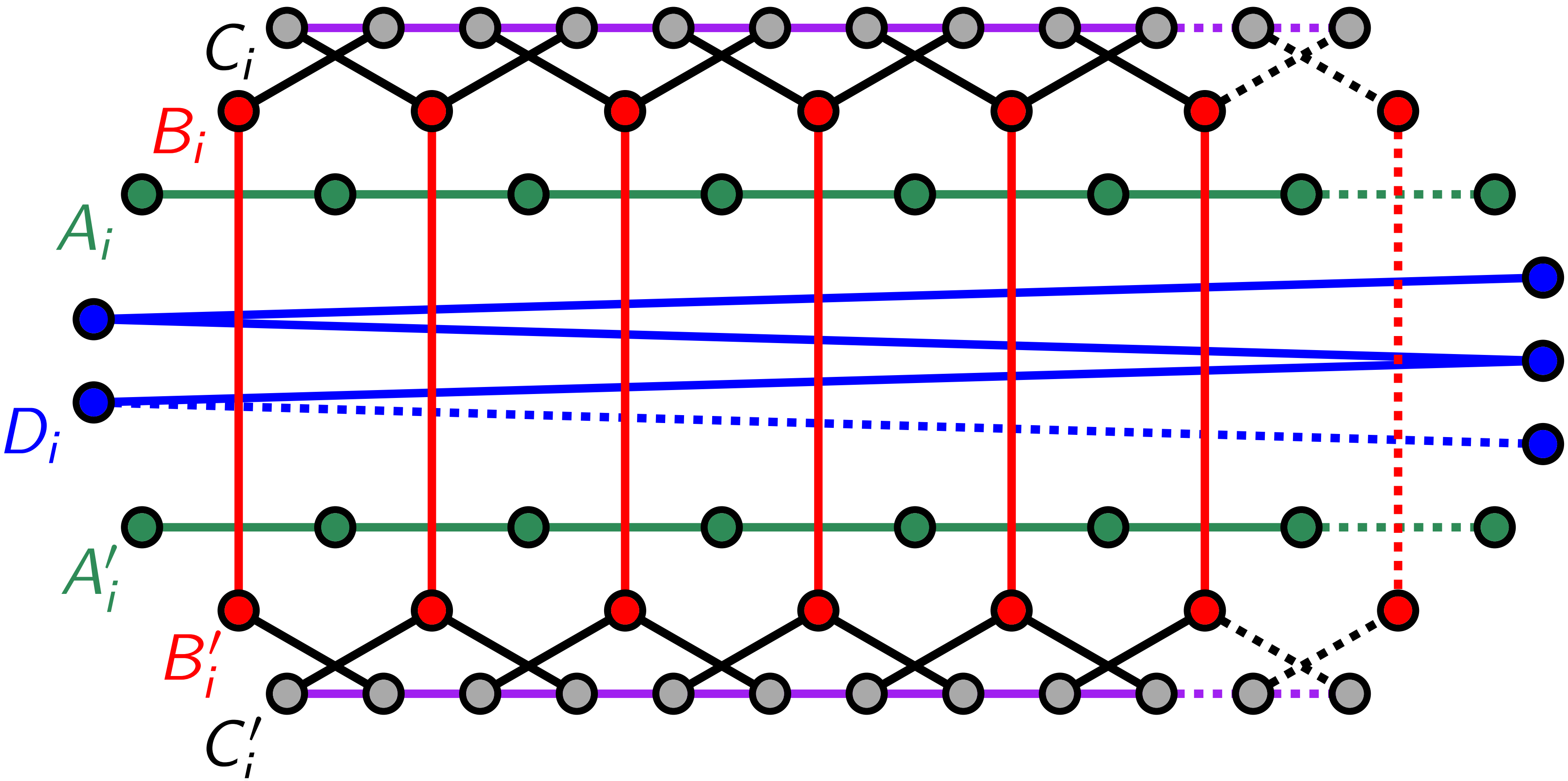

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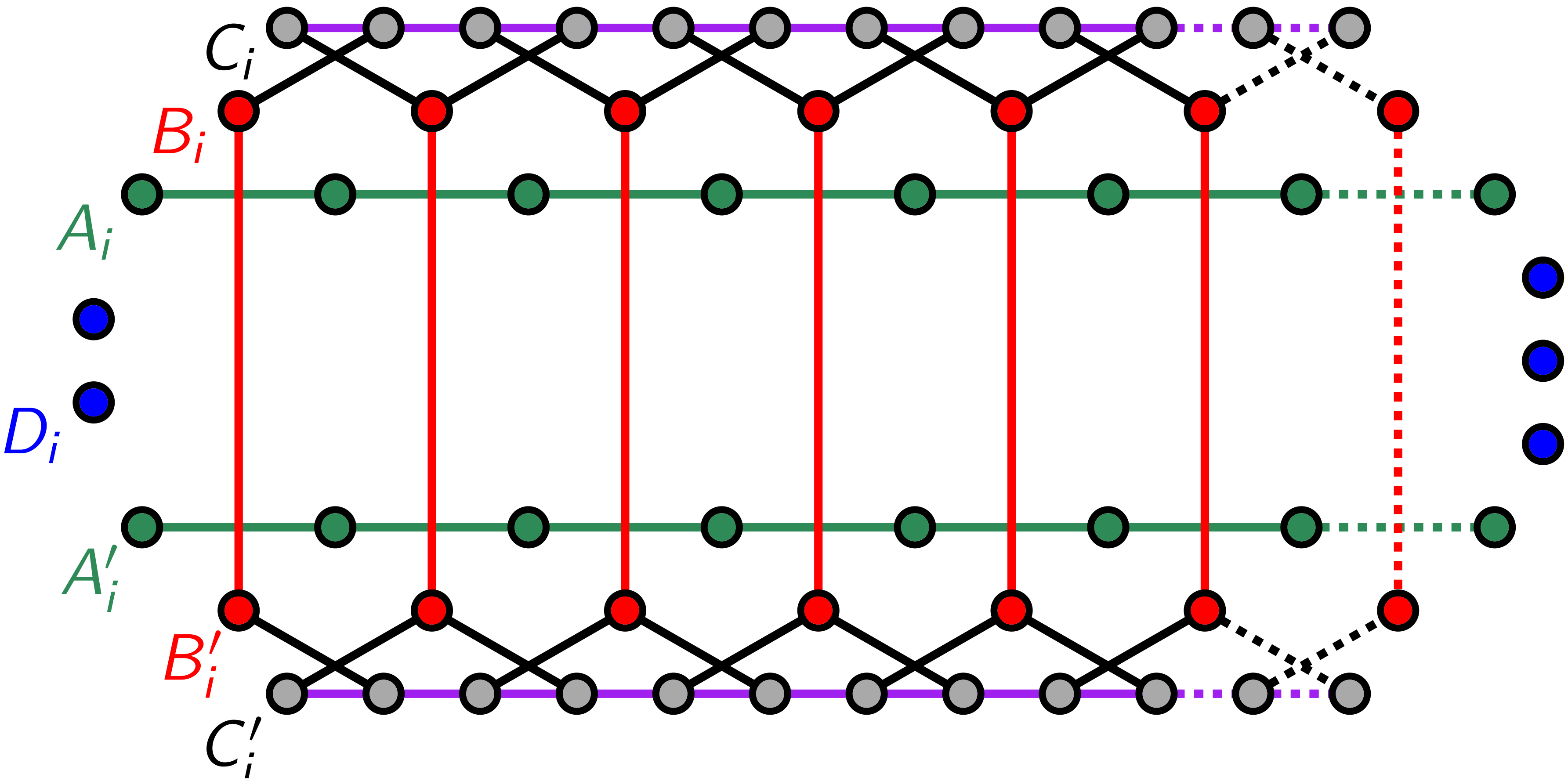

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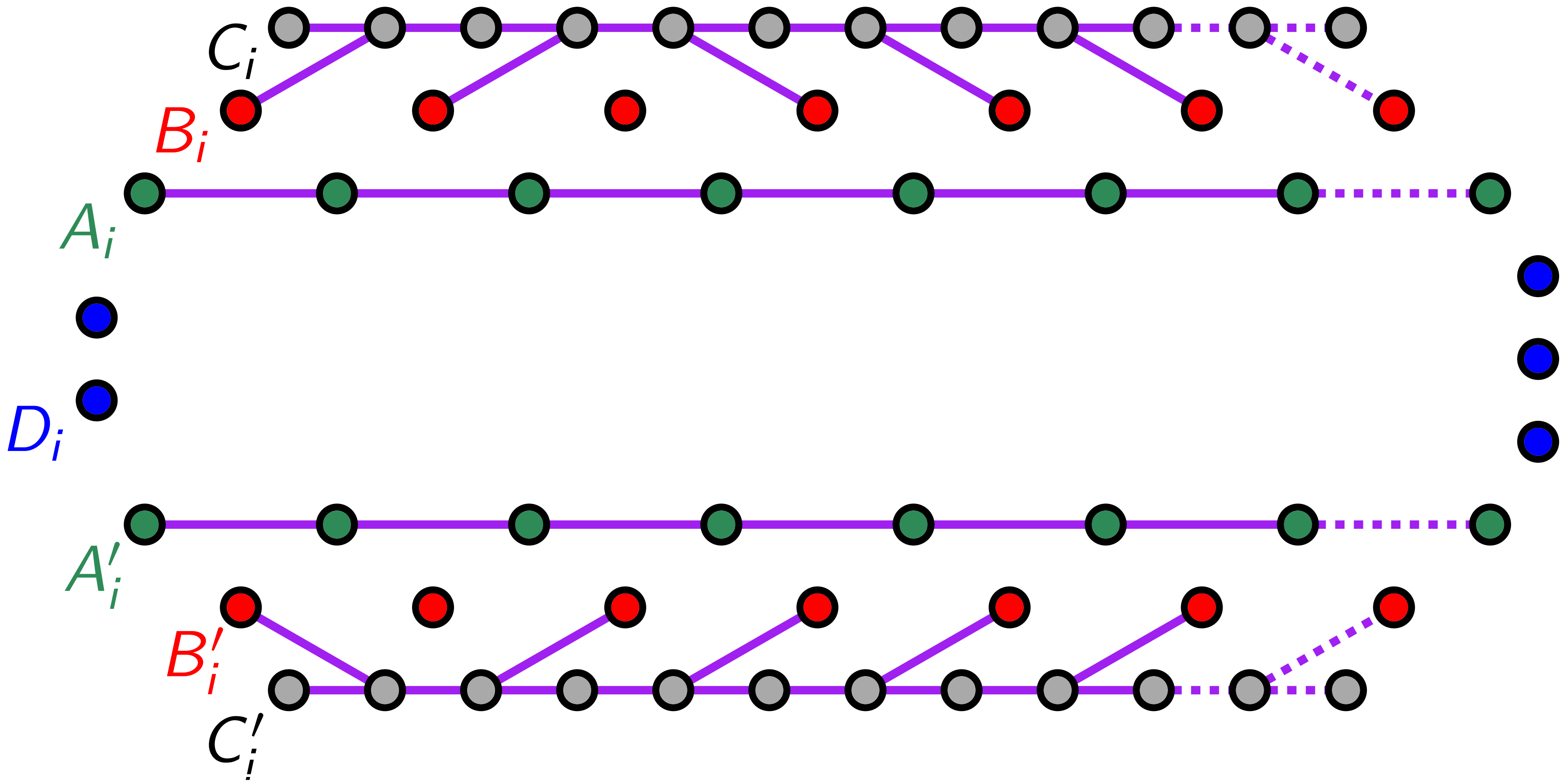

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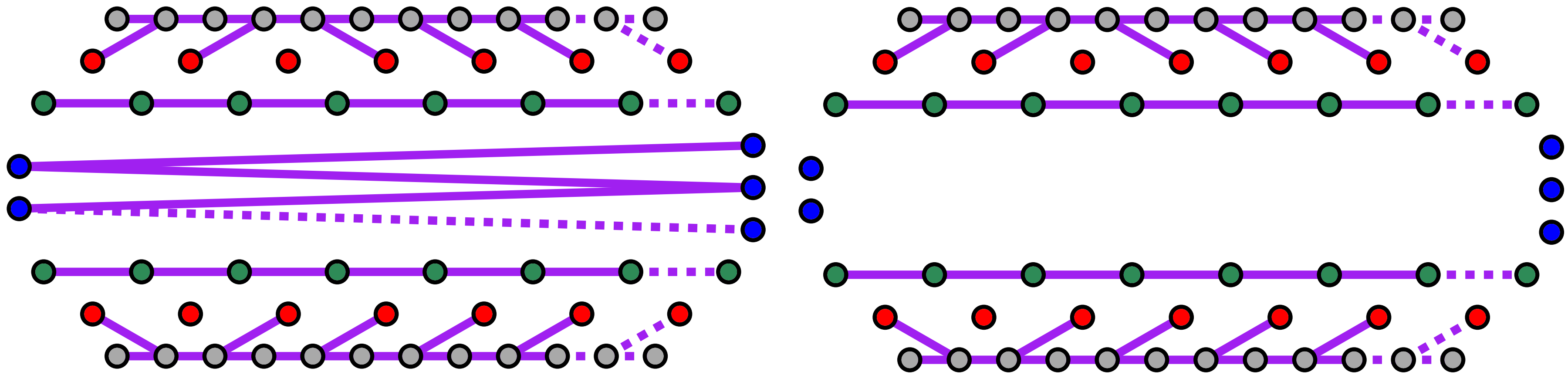

Greedy 7 clusters vs. Reverse Greedy k+7!

An ILP for Cluster Minimization

Sketch:

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Sketch:

- Model Cluster Minimization as a flow network.

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- Each sink represents the "center" of a cluster, connected nodes send the generated flow there.

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- ILP minimizes the number of sinks.

Experiment setup

- Use map of places of interest in a city.

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- Divide the map in quadrants of varying sizes.

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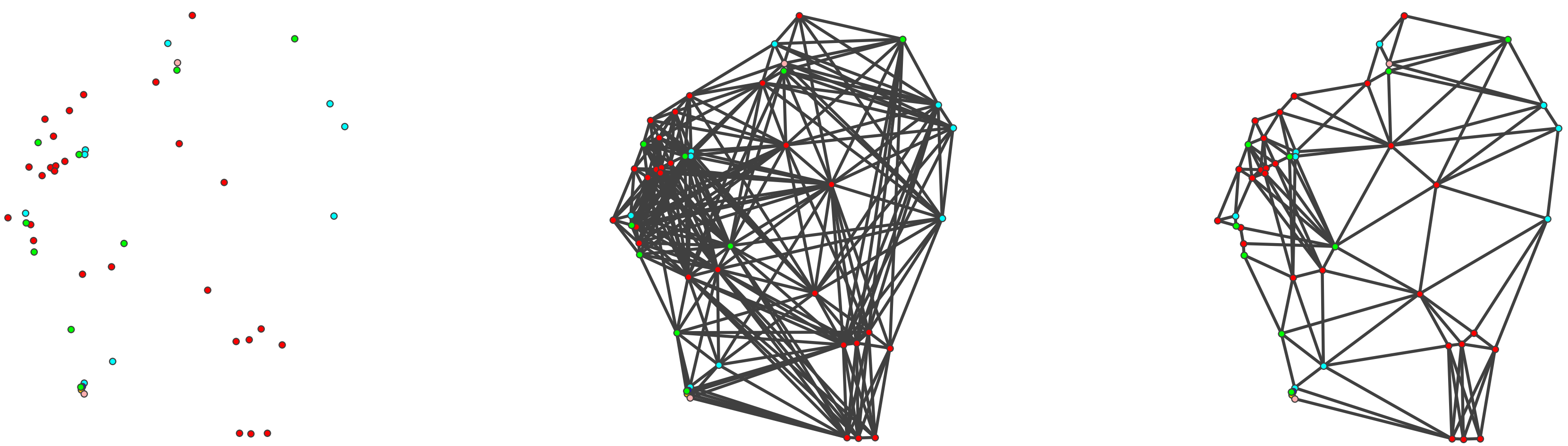

$\beta = 0.5$ $\beta = 0.9$

Experiment setup

- Use map of places of interest in a city.
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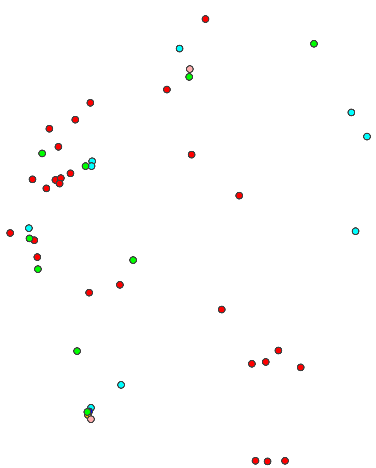
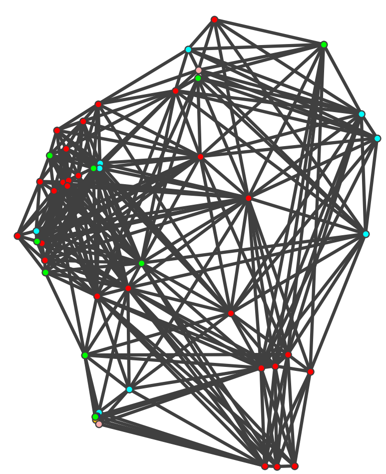
- Connect the vertices with β-skeletons.
- Run both heuristics, ILP where feasible.





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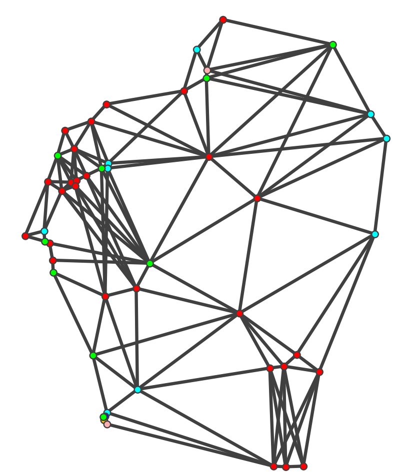

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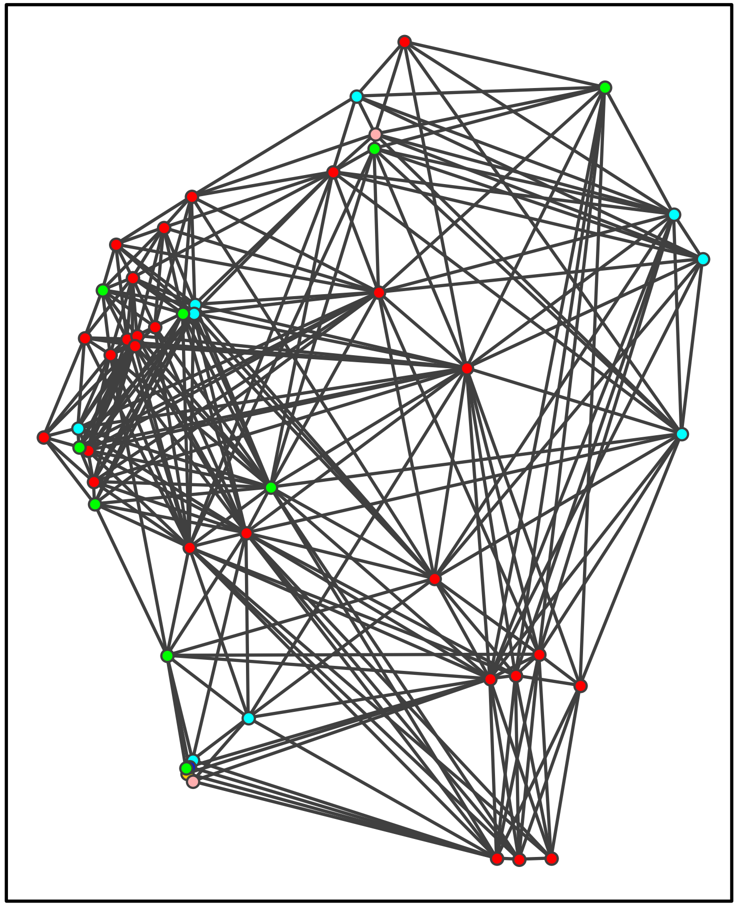

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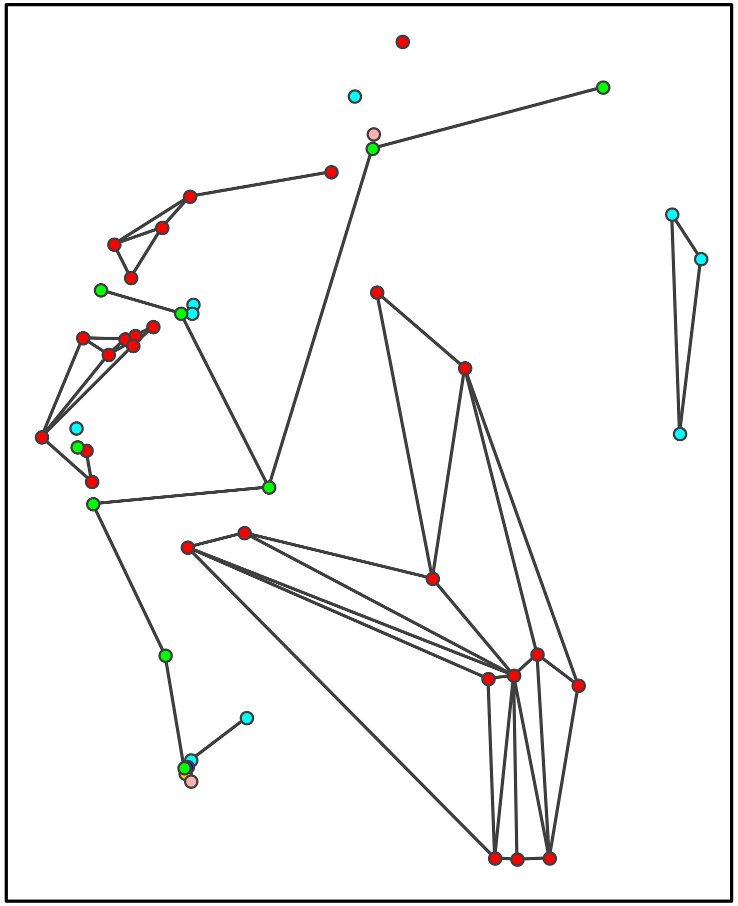
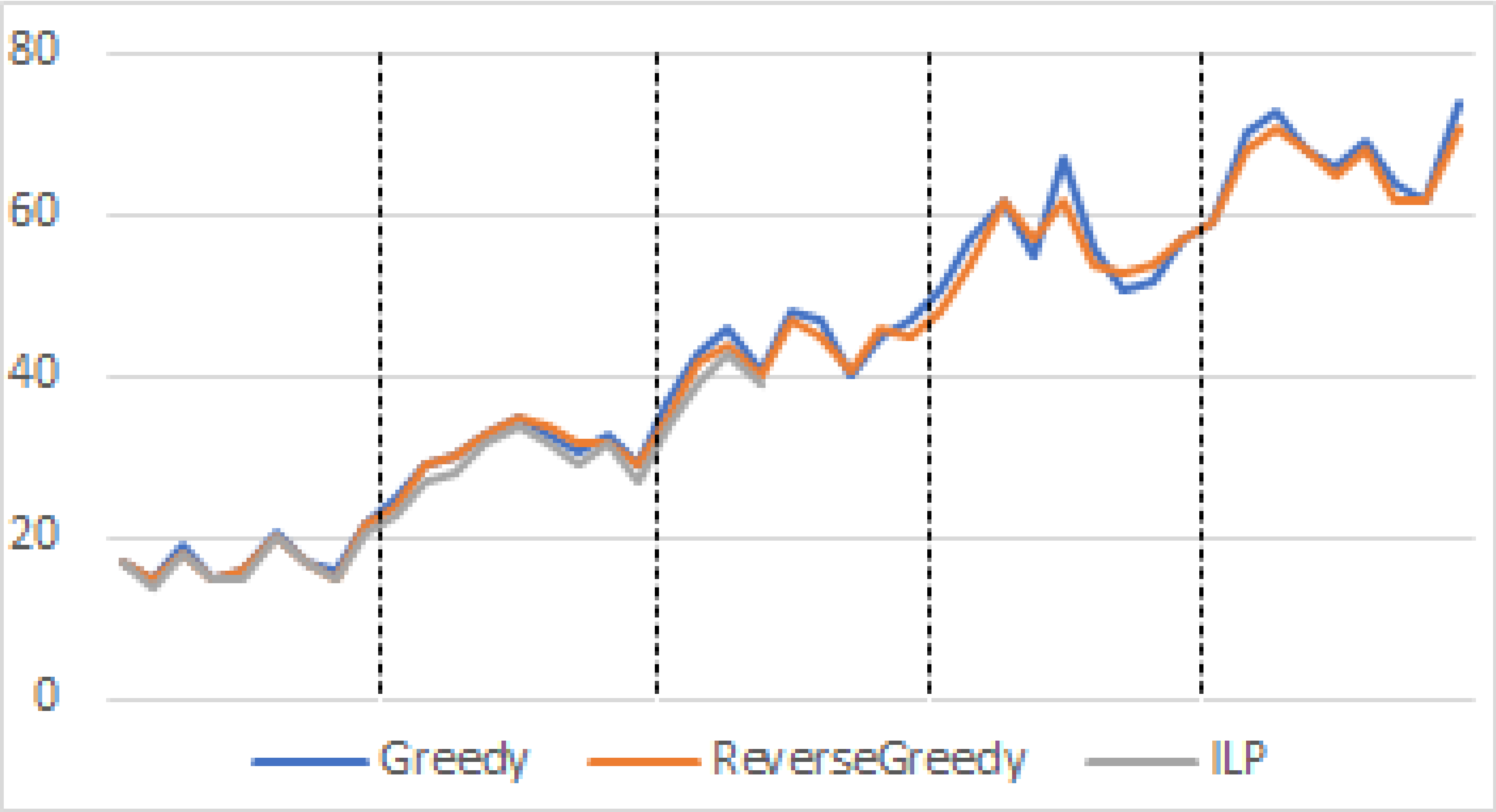


50 points, 15 clusters

Performance Analysis - Experiments



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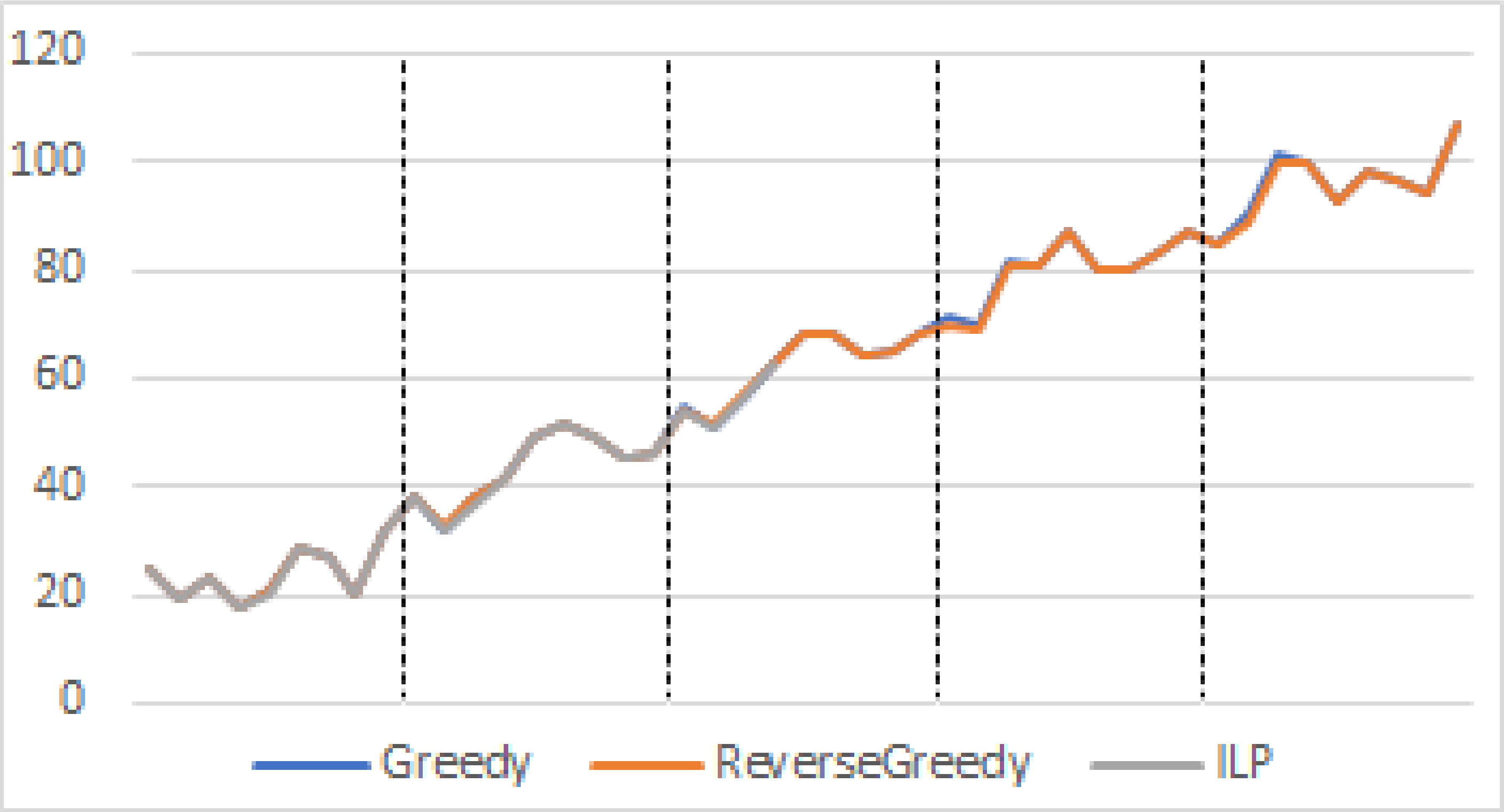

$\beta = 0.9$

Experiment Summary

Biggest difference: Greedy 37 clusters vs. ILP 34 clusters!

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Reverse Greedy tends to perform better than Greedy, but differences are marginal

Summary and Future Work

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Cluster Min.	exact	NP-hard
– Greedy	no const. factor	$n + k + m \log m$
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- Is there a constant factor approximation for Cluster Minimization?

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– 1-plane graphs	exact	$n \log n$
Edge Max.	exact	NP-hard

- There is a graph family on which the Greedy algorithm is arbitrarily better than the Reverse Greedy algorithm.
- Is there a graph family where the opposite is true?
- Is there a constant factor approximation for Cluster Minimization?
- How does the problem change if we allow *some* crossings?

Summary and Future Work

- Can we enhance the Greedy algorithm somehow?