Hausarbeit für die Zulassung zur Ersten Staatsprüfung im Fach Informatik

Segment Number of maximal outerplanar Graphs

Streckenzahl maximaler außenplanarer Graphen

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Abstract

This thesis deals with the segment number of maximal outerplanar graphs. The segment number is the minimum number of segments necessary to draw a graph. We summarize results of related work on segment numbers and go into detail on the work of Dujmović et al.[DESW07] on maximal outerplanar graphs.

We examine the segment number of a subfamily of maximal outerplanar graphs, maximal outerpaths. We determine the segment number for the special case that except one vertex with a high degree all vertices have a maximum degree of three. We apply this knowledge to more general outerpaths by defining relations between vertices with high degree and prove a lower bound for the segment number of maximal outerpaths which does not allow one of these relations.

Finally, we discuss the question if there is a constant c, such that cn is a lower bound for the segment number of a n-vertex maximal outerplanar graph. We define a ratio for number of segments and vertices as well as a ratio for number of edges and segments. In the end we give a graph sequence which serves as an upper bound for c.

Zusammenfassung

Diese Arbeit beschäftigt sich mit der Streckenzahl von maximalen außerplanaren Graphen. Als Streckenzahl bezeichnet man die minimale Anzahl an Strecken, die benötigt wird, um einen Graphen zu zeichnen. Wir fassen Ergebnisse aus verwandten Arbeiten zur Streckenzahl zusammen und gehen im Detail auf die Arbeit zu maximalen außerplanaren Graphen von Dujmović et al.[DESW07] ein.

Wir untersuchen die Streckenzahl einer Teilfamilie der maximalen außenplanaren Graphen, der Außenpfade. Für den Spezialfall, dass bis auf einen Knoten mit hohem Grad, alle Knoten Maximalgrad 3 haben, bestimmen wir die Streckenzahl. Dieses Wissen wenden wir auf den allgemeineren Fall an, indem wir Beziehungen zwischen jenen Knoten mit hohem Grad definieren. Wir beweisen eine untere Schranke für die Streckenzahl maximaler Außenpfade, die einen dieser Beziehungstype ausschließen.

Zuletzt diskutieren wir die Frage, ob es eine Konstante c gibt, sodass cn eine untere Schranke für die Streckenzahl von einem maximalen außerplanaren Graphen mit n Knoten ist. Wir führen das Verhältnis von Strecken- und Knotenanzahl und das Verhältnis von Kanten- und Streckenanzahl ein und geben eine Graphenfolge, die uns eine obere Schranke für c liefert.

Contents

1 Introduction

Graphs serve as a useful tool for describing a set of related entities and analysing the structure of their relations. Be it a network of routers, a metro network or a neuronal network; in many scientific fields, as well as in daily life, graphs can be used to capture these relations.

Graphs need to be presented in a manner which allows an observer to easily extract the important information. The common method to do so is by graph drawings, in which the vertices are represented as discs or circles and the edges as some connection between them, e.g. lines, polylines or arcs. The scientific field of graph drawing analyses how to obtain understandable visualizations of graphs.

Drawings. There are several conventions for graph drawings. In this thesis we will focus on *crossing-free straight-line* drawings. For simplicity we will refer to them as drawings. A drawing is a straight-line drawing if all the edges are drawn as a straight line segment. It is crossing-free if edges do not intersect with each other or overlap each other.

Segments. One criterion for the readability of a drawing is its *visual complexity*. This is the number of geometric objects used to draw the edges. In the case of straight-line drawings, these objects can be lines or segments. The measurement with lines is captured in the *line cover number of G*, the minimum number of lines whose union contains a crossing-free straight-line drawing of G. In this thesis we will focus on segments as measurement. A *segment* is a maximal set of edges that form a straight-line segment. To obtain a low visual complexity, we therefore seek to find a *minimum-segment drawing* of a planar graph G, a drawing that among all possible drawings of G uses the least segments. The number of segments which are used in a minimum-segment drawing of G is called *segment number of G*.

Bounds. For analysing the segment number of graph families, there are three bounds:

- We call $\mathfrak{e}$ *existential lower bound* of the segment number of a graph family if there exists a G in the family which needs at least $\mathfrak{e}$ segments to be drawn.
- We call $\mathfrak{s}$ *universal lower bound* of the segment number of a graph family if for each G in the family the segment number is at least $\mathfrak{s}$.
- If we know that each graph of a graph family can be drawn with less than $\mathfrak{u}$ segments, then $\mathfrak{u}$ is an *upper bound* for the segment number of the graph family.

The first one is useful to analyse the worst cases in a graph family. The last two bounds help us determine whether a drawing D of a graph G is a minimum-segment drawing. If its amount of segments matches the lower universal bound for segments, it is; if the amount is greater than the upper bound, it is not. With no such bounds, in the first case, one would have to prove that there is no drawing of G which uses less segments than D; in the second case one would have to find a drawing of G which uses less segments than D. To illustrate this we will later consider an example.

Related Work. The segment number for planar graphs was introduced by Dujmović et al.[DESW07] along with the *planar slope number*. For this number, we consider a graph G and the number k of different slopes in each crossing-free straight-line drawing of G. The minimum of k over all these drawings is the planar slope number. In general they presented the obvious universal lower bound for the segment number. If η is the number of vertices with uneven degree in a graph $G = (V, E)$, the lower bound for the segment number is $\eta/2$. Further more they found bounds for segment and slope number among others of trees, maximal outerplanar and plane 3-connected cubic graphs. For trees in particular, they showed that for each tree there is a drawing which achieves both the segment and planar slope number simultaneously. Their existential lower bound for plane 3-connected cubic graphs was later improved by Mondal et al.[MNBR13] to $n/2+3$, which is known to be optimal.

Most recently, Okamoto et al.[ORW19] applied the concept of the segment numbers not only to planar graphs and their crossing-free straight-line drawings. They analysed the segment number of planar graphs and their crossing-free polyline drawings in 2D. For any, not necessarily planar graph they introduced the segment number for crossing-free straight-line drawings in 3D and straight-line drawings with crossings in 2D. To compare these segment numbers they constructed graphs with which they obtained existential lower-bounds for the original segment number of connected and biconnected cubic graphs. In this paper we will only consider the segment number of crossing-free straight-line drawings in 2D. Some known lower and upper bounds for that segment number are listed in Table 1.1.

Maximal outerplanar graphs. In this thesis we will focus on the segment numbers of maximal outerplanar graphs and their outerplanar drawings. A graph G is *outerplanar* if there is a 2D drawing D of G such it is crossing-free and all the vertices are on the boundary of the outer face. We call D a *outerplanar* drawing of G. An outerplanar graph $G = (V, E)$ is considered *maximal* if $(V, E \cup vw)$ is not outerplanar for any pair of non-adjacent vertices $v, w \in V$. For a n-vertex maximal outerplanar graph we know that its outerplanar embedding is unique [DESW07] and according to Euler's Theorem it has exactly $2n - 3$ edges. In Figure 1.1 (a) and (b) we illustrate two outerplanar graphs G_1 and G_2. They is outerplanar, because all vertices are on the boundary of the outer face; they is maximal because any new edge would either cause crossings or a vertex that is not on the boundary of the outer face.

Tab. 1.1: Overview of existing lower and upper bounds for segment numbers of several graph types and their 2D crossing-free straight-line drawings. The upper bounds are universal for all graphs. The lower bounds are existential, except for trees, for which the lower bounds are universal. Here n is the number of vertices and η is the number of vertices of odd degree.

graph type	lower bound	upper bound	source
trees	$\eta/2$	$\eta/2$	[DESW07]
maximal outerplanar	n	n	
plane 2-trees	$2n$	$2n$	
plane 3-trees	$2n$	$2n$	
plane 2-connected	$5n/2$	-	
plane 3-connected	$2n$	$5n/2$	
planar 2-connected	$2n$	-	
planar 3-connected	$2n$	$5n/2$	
1-connected cubic graph	$5n/6$	-	[ORW19]
2-connected cubic graph	$3n/4$	-	
3-connected cubic graph	$n/2+3$	$n/2+3$	[MNBR13]

Maximal outerpaths. In the main part of this thesis we will discuss a subfamily of maximal outerplanar graphs, the outerpaths. To define them we first need to define the weak dual graph. For a plane graph $G = (V, E)$ the *dual graph* $D = (U, F)$ is defined by $U = \{f \mid f \text{ face of } G\}$ and $F = \{ef \mid e, f \in U, e \text{ shares an edge with } f \text{ in } G\}$. If $f \in U$ is the outer face, we call $D \setminus f$ the *weak dual graph.* Note that the expression $H \setminus v$ for a graph $H = (W, J)$ is defined by $H \setminus v = (W \setminus v, J')$ with $J' = J \setminus \{e \in J \mid e \text{ incident to } v\}$.

Let G be a outerplanar graph with embedding and H its weak dual graph. We call G *outerpath*, if H is a path. We define maximal outerpath analogously to maximal outerplanar graph. Note that for a maximal outerplanar graph the embedding is unique and therefore the weak dual graph is well defined. In Figure 1.1 (c) we display a maximal outerpath G_2 and its weak dual graph H_2.

Examples for bounds. With the examples in Figure 1.1 we explain the use for bounds of the segment number. Given that the upper bound for the segment number of G_1 is 12, see Table 1.1, we know that D_1 with 13 segments is not a minimum-segment drawing of G_1. But for D_1', we don't know if it is a minimum-segment drawing. The drawing admits the upper bound with 9 segments; but given that we do not have an universal lower bound, we'd now have to prove that there is no possible drawing of G_1 which uses less segments. On the other hand, with Theorem 3.16 we will have shown a universal lower bound for some maximal outerpaths to which G_2 belongs. With this theorem, we know a lower bound of G_2 for the segment number is 8, proving that D_2 is a minimum-segment drawing.

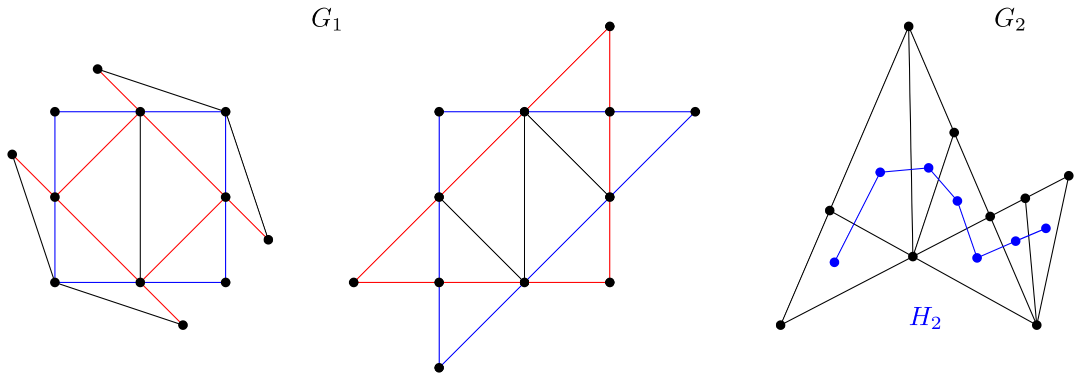


Fig. 1.1: A maximal outerplanar graph G_1 with 12 vertices drawn with (a) 13 segments in drawing D_1 and (b) 9 segments in drawing D'_1. (c) Furthermore a maximal outerpath G_2 drawn with 8 segments in D_2 in black; blue illustrates the weak dual graph H_2 of G_2.

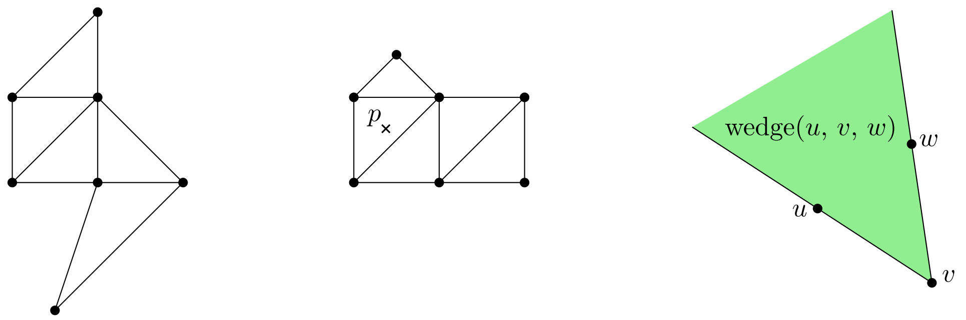
Contribution. First, we will revisit the results of Dujmović et al.[DESW07] for the segment number of maximal outerplanar graphs and present their proofs in Chapter 2. The main part of the thesis, Chapter 3, focuses on the segment number of maximal outerpaths. We first analyse in Section 3.1 the segment number of maximal outerpaths which have one vertex with a degree greater than 4, and all other vertices lower than 4. To turn to a more general case we then define strong and weak connections of Type A and Type B in Section 3.2 and provide some algorithms for maximal outerpaths in Section 3.4. We use these to prove the lower bound for maximal outerpaths with no strong Type A connections in Section 3.5. In the end, we consider the question if there is a constant c such that cn is a lower bound for the segment number of a n-vertex maximal outerplanar graph in Chapter 4. To illustrate the lower bound constant we define two ratios for a graph drawing Section 4.1. We then give a graph sequence, which provides an upper bound for the lower bound constant in Section 4.2.

2 Bounds for Maximal Outerplanar Graphs

In this chapter, we look at the work of Dujmović et al.[DESW07] on the number of segments in a drawing of a maximal outerplanar graph. They obtained an upper and an existential lower bound, see Theorem 2.4 and Theorem 2.5. In order to examine them in detail, we first need to define two terms, see Figure 2.1 for their illustration.

Definition 2.1 (Star-Shaped)**.** A drawing D of a graph is called *star-shaped*, if there exists a point p in some internal face of D, and every ray from p intersects the boundary of the outer face in exactly one point. We will call p a *star-point of* D.

Definition 2.2 (Wedge)**.** For three non-collinear points u, v and w in the plane, the *wedge(u, v, w)* is the infinite region that contains the interior of the triangle uvw, and is enclosed on two sides by the ray from v through u and the ray from v through w.



(a) (b) (c)

Fig. 2.1: Illustrations of (a) a non-star-shaped drawing of a graph G, (b) a star-shaped drawing of this graph G with star-point p and (c) the wedge(u, v, w)

2.1 Upper Bound

First we consider the upper bound. Obviously, the segment number of a graph cannot by greater than the number of vertices of this graph. As stated in Chapter 1, the number of edges is defined by the number of vertices:

Observation 2.3. Let G be a n-vertex maximal outerplanar graph. Then G has exactly $2n - 3$ edges.

Thus for a n-vertex maximal outerplanar graph a natural upper bound is $2n-3$. With the following theorem, we can show that there is an even sharper upper bound. For this we use the fact that for a given maximal outerplanar graph G with a vertex v with a degree of 2, the graph $G \setminus v$ is still a maximal outerplanar graph. Hence we can perform a proof by induction where the main idea for the inductive step is to position v such that one of its incident edges is sharing a segment with an edge in G' and does not cause any crossings.

Theorem 2.4. *Every n-vertex maximal outerplanar graph G has an outerplanar drawing with at most n segments.*

Proof. For $n \in \{0, 1, 2\}$ the theorem is trivial. For $n \geq 3$, we prove the theorem by induction over n and with the additional invariant that the drawing is star-shaped.

Initial case. The theorem holds for $n = 3$. G is a triangle, and any drawing of G is outerplanar with three segments. The invariant holds true, as any point in the inner face of the triangle is a star point.

Induction hypothesis. Any given maximal outerplanar graph with $n - 1 \geq 3$ vertices has an outerplanar drawing with at most $n - 1$ segments which is star-shaped.

Inductive step. Let G be a maximal outerplanar graph with n vertices. Since G is maximal, G has a vertex v with degree 2 whose neighbours x and y are adjacent. Thus $G' = G/v$ is maximal outerplanar graph with $n - 1$ vertices. Given the induction hypothesis, G' has a star-shaped outerplanar drawing D' with $n - 1$ segments and star-point p. In D' the edge xy separates the outer face and some internal face F. As G' is maximal, F is bound by a triangle xyr. Without loss of generality we can assume that yx is horizontal in D' and F is below xy. Due to induction the star-point p is either in wedge(y, x, r) or in wedge(x, y, r). We assume p is in wedge(y, x, r), the other case follows analogously. Let R be the area in the wedge(x, p, y) above xy, see Figure 2.2 (a).

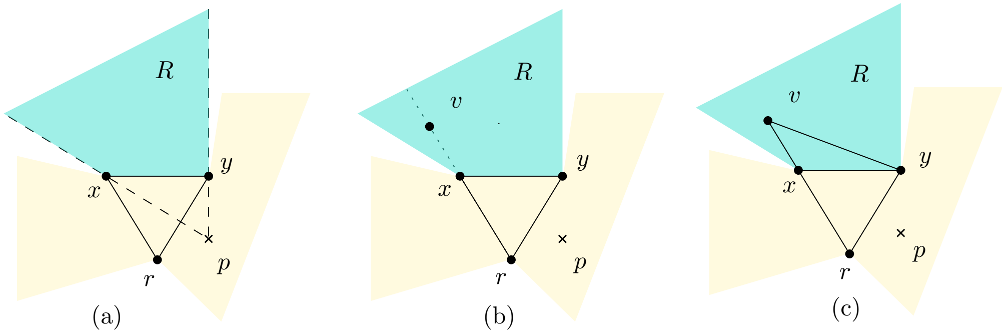


Fig. 2.2: Construction of a star-shaped drawing of an outerplanar graph.

The ray rx is intersecting the area R, see Figure 2.2 (b). Let v be anywhere on rx and in R, and let D be the drawing resulting from D' with the additional straight lines vx

and yv, see Figure 2.2 (c). Note that D is a drawing of G with the required properties as the following conditions hold:

- Star-shaped: From the induction hypothesis we know that p is a star-point in D'. Any ray starting in p intersecting with R intersects with the boundary of the outer face only in xy. Therefore $R \cap D' = \emptyset$. Rays from p intersecting with vx or vy therefore have no further intersections with the outer face.

  As p is still in some internal face, p is still a star-point for D, and hence D is star-shaped.

- Number of segments: Given our induction hypothesis, D' has only $n-1$ segments. The segment covering xr is one of them. When adding v to the drawing xv shares the same segment as xr. So, we add at most the segment covering vy. We can conclude that D has at most n segments.

□

Note that the upper bound given in Theorem 2.4 is not sharp, see Figure 2.3 (a). The graph has seven vertices, but only needs six segments.

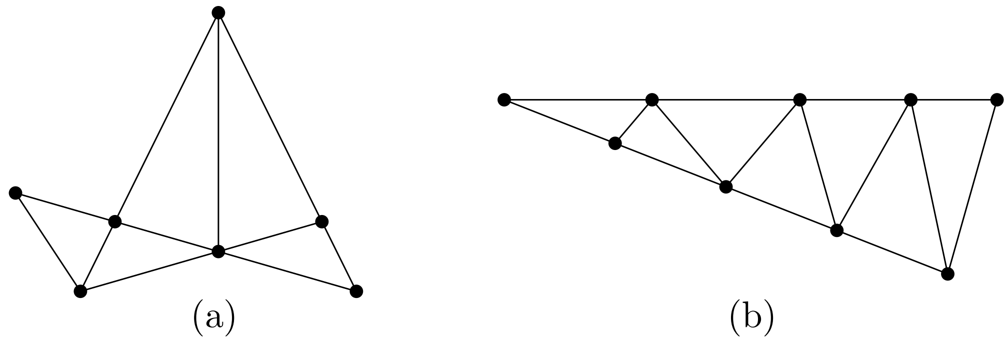


Fig. 2.3: (a) A drawing of a maximal outerplanar graph with 7 vertices that only needs 6 segments. (b) A drawing of the graph G_9 of Theorem 2.5.

2.2 Lower Bounds

As discussed in Chapter 1 there are two different lower bounds: Existential and Universal. For maximal outerplanar graphs so far no universal lower bound has been found except the natural universal lower bound for all graphs mentioned in Chapter 1. If η is the number of vertices with uneven degree in a graph $G = (V, E)$, the bound for the segment number is $\eta/2$. That is because there is one edge $e \in E$ incident to each $v \in V$ with uneven degree which cannot share a segment through v with another edge. This edge implies a start or ending for a segment. Obviously, this lower bound is not sharp. We will later give a sharper universal lower bound for a subfamily of maximal outerplanar graphs.

In terms of the existential lower bound, Dujmović et al.[DESW07] were able to find a graph sequence whose number of segments meets the upper bound. We therefore know that for a sharper upper bound than in Theorem 2.4, more than just the number of vertices has to be considered. Each graph G_n of this sequence is unique for each n. To do a proof by induction, we use the fact that $G_n \setminus v$ with $\deg(v) = 2$ is also part of the sequence and thus a drawing of G_n always contains a drawing of $G_n \setminus v$. In the inductive step we use that both neighbours of v have a maximum degree of 4 to conclude that a drawing of G_n needs one more segment than a drawing of $G_n \setminus v$.

Theorem 2.5. *For all $n \geq 3$, there is an n-vertex maximal outerplanar graph that has at least n segments in any drawing.*
More precisely, let G_n be a maximal outerplanar graph with $n \geq 3$ vertices with the following characteristics:

- *G_n is an outerpath.*
- *The maximum degree of G_n is at most four.*

Then every drawing of G_n needs at least n segments.

Proof. The proof is performed by induction over n.

Initial case. For $n = 3$ the graph G_n is a triangle, and therefore has $n = 3$ segments.

Induction hypothesis. For a fixed $n - 1 \geq 3$ every drawing of the graph G_{n-1} has at least $n - 1$ segments.

Inductive step. The inductive step is done via contradiction. Suppose for the sake of contradiction that there is a drawing D_n of G_n which has at most $n-1$ segments. Since G_n is maximal, it has a vertex v with degree 2 whose neighbours x and y are adjacent. Either $\deg(x) = 3$ or $\deg(y) = 3$ otherwise the weak dual graph of G would not be a path see Figure 2.4 (a). Without loss of generality we assume $\deg(x) = 3$.

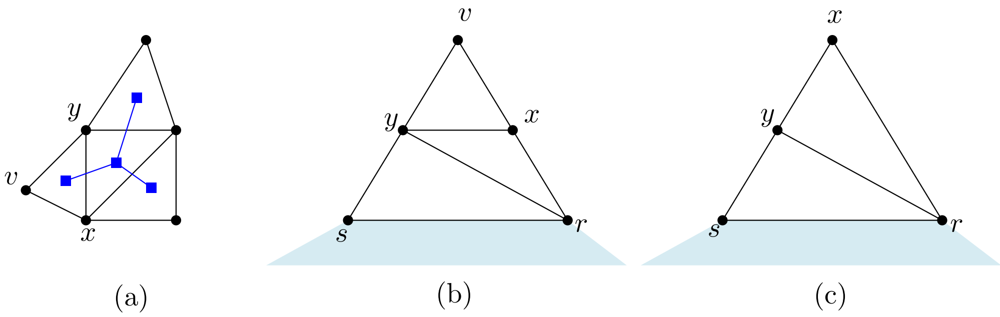


Fig. 2.4: (a) Weak dual graph of G is not a path if $\deg(x) = 4$ and $\deg(y) = 4$, (b) Situation in D_n, (c) Situation in D_{n-1}

Note that $G_n \setminus v$ is isomorph to G_{n-1}. Therefore D_n contains a drawing D_{n-1} of G_{n-1} which with the induction hypothesis has at least $n-1$ segments.

Therefore D_n has at least $n-1$ segments and by our assumption, in fact, exactly $n-1$ segments. Given that D_n contains D_{n-1}, the edges vx and vy share a segment with some edges in G_{n-1}. To be precise the segment covering vx has to contain another vertex r such that xr is an edge in G_{n-1}. Analogously, the segment covering vy has to contain another vertex s such that ys is an edge in G_{n-1}, see Figure 2.4 (b).

Given that v, x, y is a triangle, we know that x, s, r, y are pairwise different. The fact that $\deg(x) = 3$ allows the conclusion that y and r are neighbours. Thus xy cannot share a segment with xr. Since $\deg(y) \leq 4$, and y already has four incident edges, there is no edge with which xy can share a segment.

We can now produce a better drawing of G_{n-1} using D_n. Move x in D_{n-1} to the spot where v is in D_n. Then the edge xy is now where the edge vy was in D_n and like vy, the edge xy is on the same segment as ys, see Figure 2.4 (c). We showed before that xy is not sharing a segment with another edge in D_n.

Thus with this procedure we removed one segment from D_{n-1}. Therefore G_{n-1} can be drawn with only $n-2$ segments; a contradiction. □

For this sequence of graphs, we know due to Theorem 2.5 and Theorem 2.4 that the segment number is n for each G_n. The drawing of G_9 provided in Figure 2.3 (b) therefore is a minimum-segment drawing.

Note that for the inductive step we only use the fact that the dual graph is a path, and that one neighbour of v has the degree four. So the inductive step of the proof above also serves as proof of the following corollary.

Corollary 2.6. Let $G = (V, E)$ be maximal outerplanar graph. Let $\deg(v) = 2$ for a $v \in V$ and let v be adjacent to $w \in V$ with degree 4. Suppose there exists a lower bound $\mathfrak{s}'$ for segments of a drawing of $G' = G \setminus v$. Then a drawing D of G needs at least $\mathfrak{s}' + 1$ segments.

Additionally, the figure Figure 2.4 is a good illustration for the properties of the neighbours of v. It shows that only one of them can have a degree greater than three. For later use we put this observation down in a corollary.

Corollary 2.7. Let $G = (V, E)$ be maximal outerpath with $n \geq 4$. Let $v \in V$ have $\deg(v) = 2$. Then v is adjacent to a vertex x with degree 3.

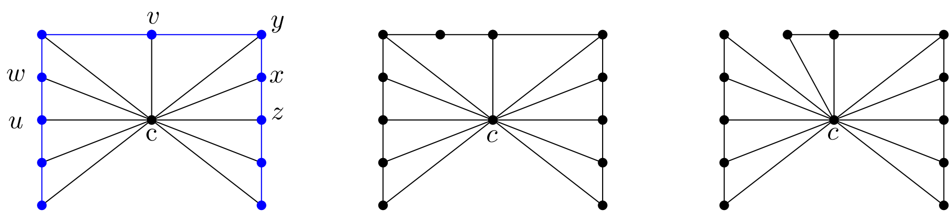
3 Segment Number of Maximal Outerpaths

As mentioned in Section 2.2, there is no sharp lower bound for maximal outerplanar graphs. In order to change that, we limit ourselves on one of its subfamilies, the maximal outerpaths. We first consider vertices with a high degree in Section 3.1 and then define relations between those vertices in Section 3.2. In Section 3.4 we present some algorithms we then use to show a lower bound for the segment number in Section 3.5.

3.1 Centered Outerpaths

In Theorem 2.5 we already analysed the segment number of a sequence of maximal outerpaths. They had the additional condition, that the maximum degree of G_n is at most four. Now we will consider graphs which allow higher degrees. To do so, we will start off with the special case that there is only one vertex with a degree greater than 4, and no vertex with degree of exactly 4:

Definition 3.1. Let $G = (V, E)$ be a maximal outerpath with n vertices. We call G *centered outerpath*, if $n \geq 6$ and there is a $c \in V$ with degree $n - 1$. We call c its *center*, $E_{\text{inner}} = \{e \in E \mid e \text{ incident to } c\}$ its *inner edges* and $E_{\text{outer}} = E \setminus E_{\text{inner}}$ its *outer edges.*



(a) (b) (c)

Fig. 3.1: (a) Centered outerpath with 12 vertices. $G \setminus c$ colored blue. Its inner edges are colored black, its outer edges colored blue.
(b) Graph where c is not connected to all other vertices.
(c) Graph where $G \setminus c$ is not a path.

Figure 3.1 (a) illustrates a centered outerpath with $n = 12$ and its inner and outer edges. We can use it to extract some simple observations:

Lemma 3.2. Let G be a centered outerpath with n vertices and center c.

(a) The graph $G \setminus c$ is a path.

(b) The center is adjacent to all other vertices.

(c) For a given n, the centered outerpath is unique.

For a better understanding, Figure 3.1 (b) and (c) display graphs where $G \setminus c$ is not a path or where c is not connected to all vertices. Both contradict the graph being maximal outerplanar.

Since $G \setminus c$ is a path, and c is connected to all vertices, we can easily determine the exact degrees of all vertices in a centred outerpath:

Corollary 3.3. Let $G = (V, E)$ be a centered outerpath with n vertices and center c. There are exactly two vertices w and w' both with degree 2. For all $u \in V \setminus \{c, w, w'\}$ applies $\deg(u) = 3$.

This corollary shows that the centered outerpath matches the requirements we had for our special case: A centered outerpath has one vertex with a degree greater than 4, and all other are lower than 4.

Now we draw conclusions about the edges in centered outerpaths and in what ways they can share segments. For a better understanding of the following proof, see Figure 3.1 (a).

Theorem 3.4. *Let $G = (V, E)$ be a centered outerpath with center c and D a drawing of G.*

(a) For any $e \in E_{inner}$ and any $e' \in E_{outer}$ there is no drawing where e and e' are sharing a segment.

(b) Any $e \in E_{inner}$ can share a segment with at most one other edge e'. In this case, e' is also in E_{inner}.

Proof. Let us consider an edge $uc \in E_{\text{inner}}$. We know that if uc is sharing a segment with other edges, one of them needs to be incident either to c or to u.

(a) Let us first consider an edge $uw \in E_{\text{outer}}$ incident to u. Suppose uw and uc share the same segment s, then u, c and w are collinear. Since the center c is adjacent to w, we know uwc is a triangle but also collinear. A contradiction to a proper crossing-free drawing.

Now consider $xy \in E_{\text{outer}}$ which is not incident to the vertex u. For xy and uc to be on the same segment, the vertices x, y, c, u need to be collinear. Since the center c is adjacent to x and y, we know yxc is a triangle but also collinear; a contradiction to D being a proper crossing-free drawing.

(b) Consider an edge $cu \in E_{\text{inner}}$. By (a) we know that cu cannot share a segment with an edge in E_{outer}. Thus suppose there are $cv, cz \in E_{\text{inner}}$ with c, v, u, z pairwise different such that cu, cv, cz share a segment. Then one of these edges would be on top of another; a contradiction to D being a proper crossing-free drawing.

□

With Theorem 3.4 we can show a lower bound for the segment number of centered outerpaths.

Theorem 3.5. *Let $G = (V, E)$ be a centered outerpath with n vertices. A drawing D of G needs at least $\lceil n/2 \rceil + 2$ segments.*

Proof. Let c be the center of G. Due to the definition of centered outerpaths $|E_{\text{inner}}| = n - 1$. First consider E_{outer}. Because of Theorem 3.4 (a) no edge in E_{outer} can share a segment with edges of E_{inner}. Given that $E_{\text{outer}} \neq \emptyset$, we need at least one segment to cover E_{outer}.

Case 1: Let us assume we use only one segment a to cover E_{outer}. Then all vertices in $V \setminus c$ are on this segment. Obviously, no pair of edges in E_{inner} can then share a segment. This implies that each edge in E_{inner} needs its own segment. In this case we therefore need at least $1 + |E_{\text{inner}}| = 1 + n - 1 = n$ segments.

Case 2: Let us assume we use two segments to cover E_{outer}. For this case we need to consider two different cases:

(a) The number of vertices n is even. To cover E_{outer} we now use two segments. Thanks to Theorem 3.4 (b) we know that to cover E_{inner} we need at least $\lceil (n-1)/2 \rceil$ segments. Since $n - 1$ is odd we need at least $n/2$ segments to cover E_{inner}. In summary we need at least $n/2 + 2$ to draw G.

(b) The number of vertices n is odd. We suppose that we need only two segments s and t to cover E_{outer} and at the same time can use Theorem 3.4 (b) to cover E_{inner} with only $\lceil (n-1)/2 \rceil = (n-1)/2$ segments. Then we would only need $(n-1)/2+2$ to draw G.

Given that $G \setminus c$ is a path (see Lemma 3.2), there needs to be a vertex u, which has to be the intersection of s and t. Let us take a look at the edge uc. Because of our assumption, uc is sharing a segment r with another edge cw in E_{inner}. Due to Theorem 3.4 (a), the segment r is different to t and s. The vertex $w \in V \setminus c$ has to be either on s or t. Without loss of generality, we can say that w is on s. The vertices w and u therefore both would be on s and on r, two different segments; a contradiction. So one of our two assumptions was wrong.

i If the first assumption was wrong, we cannot use only two segments to cover E_{outer}. Hence we need three segments, see Figure 3.2 (b).

ii If the second assumption was wrong, uc and cw cannot share a segment. Hence we cannot use Theorem 3.4 (b) to cover E_{inner} with only $\lceil (n-1)/2 \rceil = (n-1)/2$ segments. We therefore need at least $(n-1)/2 + 1$ segments to cover E_{inner}, see Figure 3.2 (c).

In both cases we need one more segment than we supposed. Thus we need at least $(n + 1)/2 + 2$ segments.

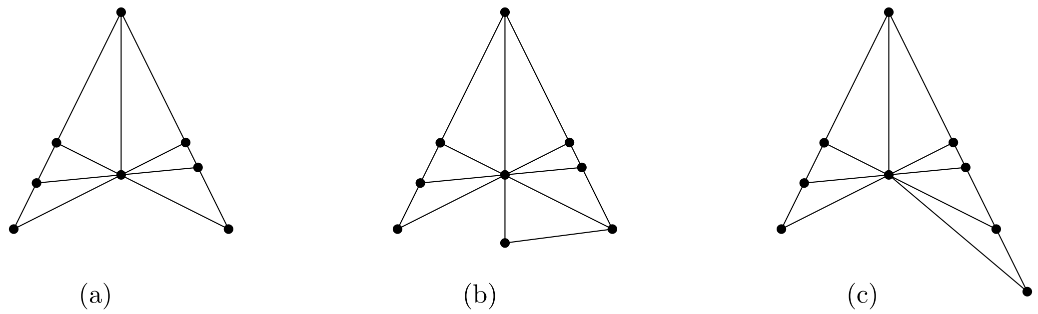


Fig. 3.2: Drawing of centered outerpaths with minimum segments: (a) n is even, (b) n is odd, using three segments to cover E_{outer}, (c) n is odd, using two segments to cover E_{outer}

Case 3: We use more segments to cover E_{outer} than discussed in case 2.

(a) For an even number n of vertices, we already assumed in case 2 (a) that we can use Theorem 3.4 (b) to its full extend. So, having more segments to cover E_{outer} does not reduce the number of segments for E_{inner}. We therefore end up with more than $n/2 + 2$ segments.

(b) For an odd number n of vertices, we ended up with another case distinction in case 2 (b):

  i In the first case, we have two edges $uc, cw \in E_{\text{inner}}$ that cannot share with another edge in E_{inner}. Adding another segment to cover E_{outer} is the same we did in case 2 (b) ii). We again end up with at least $(n+1)/2+2$ segments.

  ii In the other case, we use Theorem 3.4 (b) to its full extend, but need three segments to cover E_{outer}. As we already used the pairing of the inner edges to its full extend, adding another segment for the outer edges can not reduce the segments for E_{inner}. So, we would use more than $(n + 1)/2 + 2$.

So, using more segments to cover E_{outer} than discussed in case 2 does not lower the total number of segments.

Summary In conclusion we can say that in case 1 we need at least n segments. In case 2 we need at least $\lceil n/2 \rceil + 2$ segments, and case 3 is always worse than case 2. So in general we need at least $\min(\lceil n/2 \rceil + 2, n)$ segments. Since in Definition 3.1 we defined $n \geq 6$ we know $\min(\lceil n/2 \rceil + 2, n) = \lceil n/2 \rceil + 2$.

□

We now provide an algorithm to draw centered outerpaths while matching the lower bound. We split this process in two algorithms as one of them will be used in later proofs. For the first one we assume that three vertices are already positioned in $\mathbb{R}^2$.

Theorem 3.6. *Let $G = (V, E)$ be a n-vertex centered outerpath with center c and let $u_1, \ldots, u_{\deg(c)}$ be a path of $G \setminus c$. Let $m := \deg(c)$ be odd and the triangle $u_1 u_2 c$ already be* *positioned in $\mathbb{R}^2$. There is a drawing D of G such that E_{outer} is covered by two segments and there is only one edge in E_{inner} that does not share a segment with anther inner segment. It uses $n/2+3$ segments and hence is a minimum-segment drawing.*

Proof. Without loss of generality the segment x which covers u_1u_2 is vertical, the center c is on the right half-plane of x and u_2 is above u_1 as illustrated in Figure 3.3 (a).

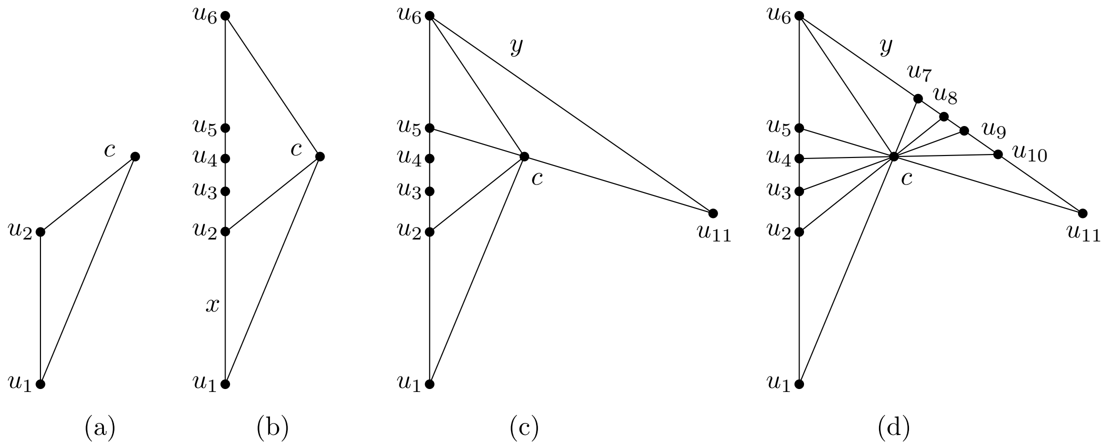


Fig. 3.3: Step by step algorithm to draw centered outerpath with 12 vertices.

- We extend x upwards, at its endpoint put the vertex $u_{(m+1)/2}$ and draw a segment from it to c. Between u_2 and $u_{(m+1)/2}$ we position the u_i with $2 < i < (m+1)/2$ in the proper order, see Figure 3.3 (b)
- We draw a segment from $u_{(m+1)/2-1}$ through c while exceeding c and position u_m on its endpoint. Now we draw a segment y from $u_{(m+1)/2}$ to u_m, see Figure 3.3(c)
- In the end we draw segments starting in u_i with $1 \leq i < (m+1)/2 - 1$ through c until they cross y, at this crossing point we put $u_{(m+1)/2+i}$.

For each u_i with $1 \leq i < (m+1)/2$ there is a segment through c that covers u_ic and $cu_{(m+1)/2+i}$. Furthermore we have drawn a segment to cover $cu_{(m+1)/2}$. These are $2 \cdot ((m+1)/2 - 1) + 1 = m$ inner edges, and thus all of them. Furthermore we have all outer edges, as can be seen in Figure 3.3 (d).

In summary we have used $(m+1)/2+2 = n/2+2$ segments to draw D which matches the lower bound of Theorem 3.5. It thus is a minimum-segment drawing. □

Theorem 3.7. *The segment number of a n-vertex centered outerpath is $\lceil n/2 \rceil + 3$.*

Proof. Let c be the center and let $u_1, \dots, u_{\deg(c)}$ be a path of $G \setminus c$. Let u_1, u_2c be positioned anywhere in $\mathbb{R}^2$. If $\deg(c) = n-1$ is odd, we know with Theorem 3.6 that there is minimum-segment drawing matching the lower bound of Theorem 3.5. If $\deg(c) = n-1$ even, we consider $G \setminus u_{\deg(c)}$ and use the drawing obtained by Theorem 3.6.

It uses $(n-1)/2+3$ segments. We add a segment starting in c and ending such that it crosses the outer edge $u_{\deg(c)-2}u_{\deg(c)-1}$. On that crossing point we position $u_{\deg(c)-1}$ and consider the old $u_{\deg(c)-1}$ as new $u_{\deg(c)}$, compare Figure 3.3 (d) with Figure 3.4 (a). That way we have $(n-1)/2+3+1 = \lceil n/2 \rceil + 3$ segments. In both cases we have achieved a drawing matching the lower bound of Theorem 3.5. □

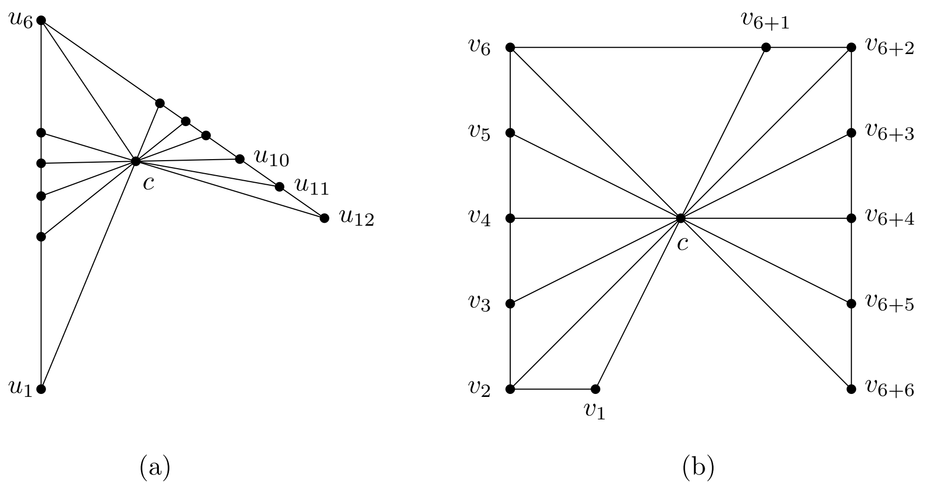


Fig. 3.4: (a) Minimum-segment drawing of a centered outerpath with 13 vertices. (b) Centered outerpath whose center c has $\deg(c) = 12$, and where all $e \in E_{\text{inner}}$ share a segment with another $f \in E_{\text{inner}}$.

Before turning to more general cases of maximal outerpaths, we make one more important observation for centered outerpaths with an odd number of vertices on how edges belonging to E_{inner} are paired optimally if we ignore E_{outer}.

Observation 3.8. Let G be a centered outerpath with center c. Let $\deg(c)$ be even and $v_1, \ldots, v_{\deg(c)}$ be a sequence of vertices for the path $G \setminus c$. If all $e \in E_{\text{inner}}$ share a segment with another $f \in E_{\text{inner}}$, then $v_i c$ is sharing a segment with $cv_{(\deg(c)/2)+i}$ where $1 \leq i \leq \deg(c)/2$.

Figure 3.4(b) illustrates Observation 3.8. Remember that using the pairing for all edges in E_{inner} requires at least three segments to cover E_{outer}. Apart from that the observation is independent of the number of segments being used to cover E_{outer}.

3.2 Connections Types between Centres

We can now turn away from this special case and apply our new knowledge to a more general case. To do so, we first need to apply the concept of centres to outerpaths in general.

Definition 3.9. Let $G = (V, E)$ be a maximal outerpath. We call $c \in V$ a *center*, if $\deg(c) \geq 5$. We call $S(c) = (U, F)$ with $U = \{w \in V \mid w \text{ adjacent to } c\} \cup c$ and $F = \{wu \mid w, u \in U\} \cap E$ the *c-centered subgraph of G*.

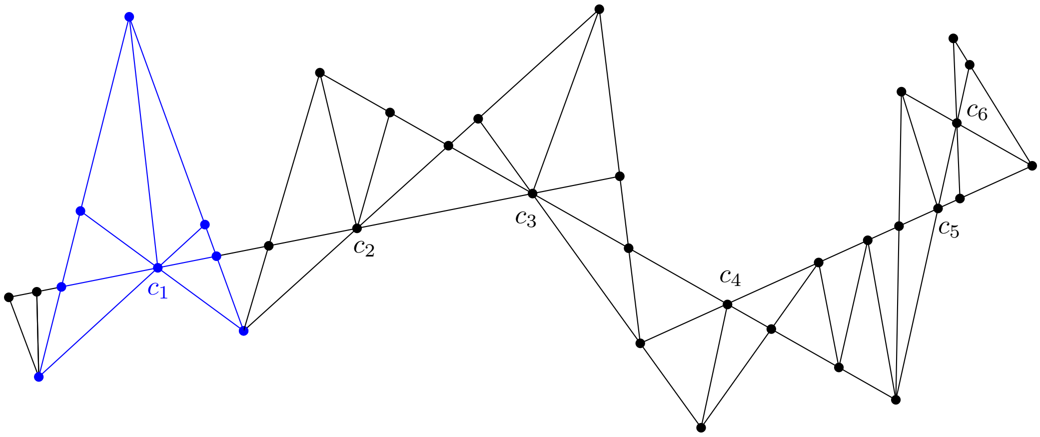


Fig. 3.5: A drawing of a maximal outerpath G with the centres $c_1, \ldots, c_5$ and with the c_1-centered subgraph of G coloured blue.

Obviously $S(c)$ is a centered outerpath, see Figure 3.5 and thus the results in Section 3.1 can be used.

One might hope that gaining a lower bound for maximal outerplanar graphs might now be fairly simple: For a given maximal outerplanar graph G with centres $c_1, \ldots, c_m$ one could combine the result of Theorem 3.5 and Corollary 2.6. For Corollary 2.6, one would count all vertices with degree four. To combine it with Theorem 3.5 one would then add the sum of lower bounds for the c_i-centered subgraphs for all centres. But this method is not sufficient, since the subgraphs of centres can share segments with each other, see Figure 3.5. The subgraphs of c_2 and c_3 for example share three segments with each other. We therefore need to analyse if and how subgraphs of centres share segments. To do so, we first define relations between centres. For illustrations, see Figure 3.6.

Definition 3.10. Let $G = (V, E)$ be a maximal outerpath and $v, w \in V$ be centres. We say v and w have a

(a) *Type A connection* if v and w are adjacent.

(b) *Type B connection* if v and w are not adjacent, but there is at least one $u \in V$ which is adjacent to v and w.

(c) *strong* connection if there are two vertices $u_1, u_2 \in V$ which are adjacent to both v and w

(d) *weak* connection if there is only one vertex $u_3 \in V$ which is adjacent to both v and w.

If one of the these cases applies, we say v and w are *connected*.

Let us revisit Figure 3.5 and analyse which connection types these centres have:

- The centres c_1 and c_2 are not adjacent, but have one vertex they both are adjacent to. Therefore they have a weak Type B connection.

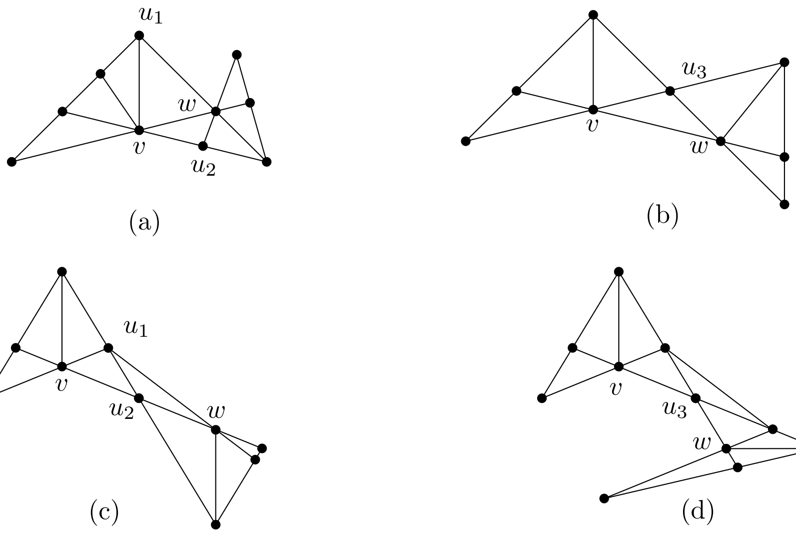


Fig. 3.6: Examples for the connection types: (a) strong Type A connection, (b) weak Type A connection, (c) strong Type B connection, (d) weak Type B connection

- The centres c_2 and c_3 are adjacent and have one vertex they both are adjacent to. They thus have a weak Type A connection.
- The centres c_3 and c_4 are not adjacent and have two vertices they both are adjacent to. Therefore they have a strong Type B connection.
- Between c_4 and c_5 there is no connection, because they don't have a vertex that is a common neighbour and they are not adjacent.
- The center c_6 shares two neighbours with c_5 and is also adjacent to c_5. Thus they have a strong Type A connection.

Since the weak dual graph H is a path, a center c can only be connected to at most two other centres. That is why we only considered some pairs of centres, namely the centres right before and after c, in the order of the path H.

3.3 Centres at the End of the Path

Our main proof for a lower bound will be performed via induction. In its inductive step we will compare two graphs, namely any maximal outerpath G with no strong Type A connection and the graph $G \setminus v$ where v is a vertex in G with degree 2. As stated in Corollary 2.7 one of the neighbours of v has degree 3. So, to cover all possible transitions from $G \setminus v$ to G, we only need to make a distinction of cases for the degree of the other neighbour of v. We already cover one case with Corollary 2.6 where v is adjacent to a vertex w with degree 4. Thus we now want to analyse the case where v is adjacent to a center c.

For the following section we want to analyse properties of minimum-segment drawings of a maximal outerplanar graph G where v is adjacent to a center c with $\deg(v) = 2$. To do so, we suppose that these properties do not apply for a minimum-segment drawing D of G. We then provide an algorithm to obtain a drawing D' of G for which these properties apply and argue why D' uses less segments then D. In this algorithm we always redraw $S(c)$. But we do not assure that the drawing D' is crossing-free as $S(c)$ might be causing crossings with edges that are not in $S(c)$ or even overlap with them.

Therefore we need a different concept of drawings for this analysis, that allows this kind of overlapping while keeping our results so far. This concept is rather theoretical.

Definition 3.11. Let $G = (V, E)$ be a maximal outerplanar graph. For each vertex $v \in V$ we consider the subgraph $S(v) = (U, F)$ with $V = \{w \in V \mid w \text{ is adjacent to } w\}$ and $F = \{wu \mid w, u \in U\} \cap E$. We call a straight-line drawing D of G *locally-crossing-free* if for every $v \in V$ the subgraph $S(v)$ is drawn crossing-free. It is considered *outerplanar* if for every $v \in V$ the subgraph $S(v)$ is drawn outerplanar.

To illustrate the concepts of locally-crossing-free drawings consider the drawings in Figure 3.7. Both drawings D and T are drawings of the same graph G. For illustration the same part of the graph G is coloured orange and its edges dashed. The drawing D is a straight-line crossing-free drawing as we know it. The drawing T however is not crossing-free, one crossing is marked blue. The drawing T however is locally-crossing-free, for each vertex v the subgraph $S(v)$ is crossing-free. One problem is that locally-crossing-free drawings allow edges to be on top of each other, also marked blue. To consider whether T is minimum-segment we still need to be able to analyse their number of segments. However given that we allow edges to be on top of each other, the original definition of segments is not sufficient any more.

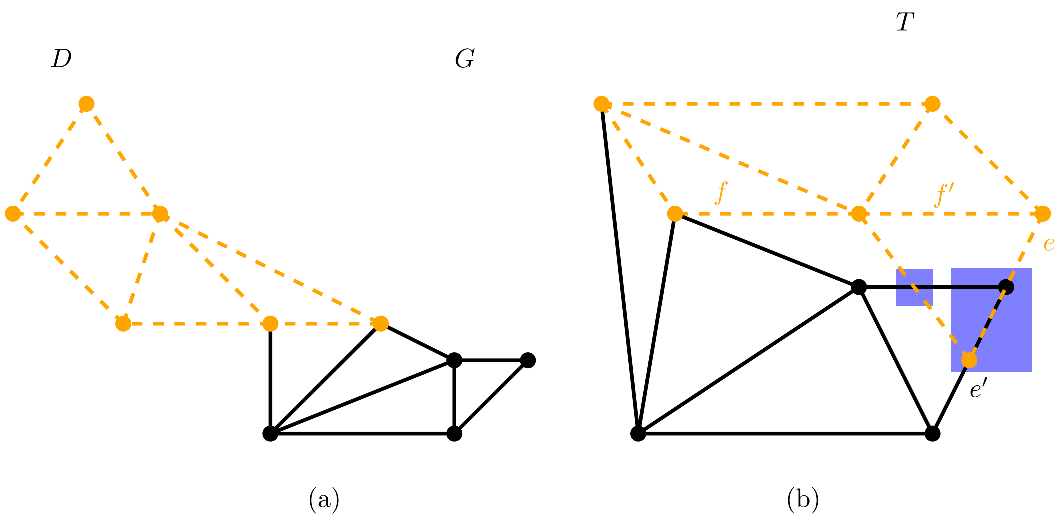


Fig. 3.7: Two drawings of the same maximal outerplanar graph G : (a) A straight-line crossing-free drawing D and (b) a straight-line triangle-invariant drawing T with crossing and overlapping edges.

Definition 3.12. Let $G = (V, E)$ be a maximal outerpath and D be a locally-crossing-free drawing of G. We call a maximal set S of edges a *segment* if they form a straight-line segment in D and if S is a path in G. We call a locally-crossing-free drawing D a minimum-segment drawing of G if it uses the least number of segments among all triangle-invariant drawings of G.

With this new definition consider the edges e and e' in T. They are the maximum set to form a straight-line segment, hence by the old definition this would be one segment. With with the new definition, however it is not any more: they are the only edges forming that straight-line segment and do not have a vertex they both are incident to. Hence they are not a path.

On the other hand the edges f and f' are on the same segment by new and old definition. They form a straight-line segment and they are a path as they have a vertex they are both incident to.

These concepts are crucial for the next sections, however we will not show any more drawings with overlapping edges. We will only consider the drawing of a part of G in which there are no crossings or overlapping. Note that a straight-line crossing-free drawing always is a locally-crossing-free drawing. Before we can go on, we need to evaluate which of the results so far apply for locally-crossing-free drawings.

For simplicity all drawings in the rest of this chapter are straight-line triangle-invariant drawings, unless stated otherwise. We start off with a theorem in which we only consider a specific case, namely that $\deg(c)$ is odd. We consider the odd case so that we can later draw conclusions for the even case.

Theorem 3.13. *Let $G = (V, E)$ be a maximal outerpath. Let $v \in V$ with degree 2 be adjacent to a center c with odd degree m. Let $S(c) = (U, F)$ be the c-centered subgraph of G. Let $u_1, \ldots, u_m$ be the sequence of vertices for the path $S(c) \setminus c$ with $v = u_m$. Let D be a minimum-segment drawing of G.*

(a) There is only one edge $e \in F_{inner}$ such that e does not share a segment with another $e' \in F_{inner}$.

(b) The edge $u_{(m+1)/2}u_{(m+1)/2+1}$ is either sharing a segment with u_2u_3 or $u_{m-1}u_m$.

Proof. For the sake of contradiction suppose D is a minimum-segment drawing such that either (a) or (b) does not hold. We will provide an algorithm to obtain a drawing D' for which (a) and (b) apply and then analyse why it uses less segments then D. For illustration Figure 3.8 (a) is a drawing of a graph G for which (b) does not apply. In our illustrations u_2u_3 is not sharing a segment with u_1u_2. Its sharing behaviour through u_2 does not have an impact on our algorithm. Hence it could be sharing with u_1u_2 or edges in $E \setminus F$ as well.

To gain a drawing D' we remove all vertices and edges from D that are in $S(c)$ except the vertices u_1, u_2, u_3, c and the edges between them. By doing so we obtain a drawing similar to the one in Figure 3.8 (b). We will use Theorem 3.6 to draw a centered outerpath with $\deg(c) - 1$ vertices. For the triangle mentioned in the theorem we consider $u'_1u'_2c$ with $u'_1 = u_2$ and $u'_2 = u_3$. By doing so we obtain the drawing in

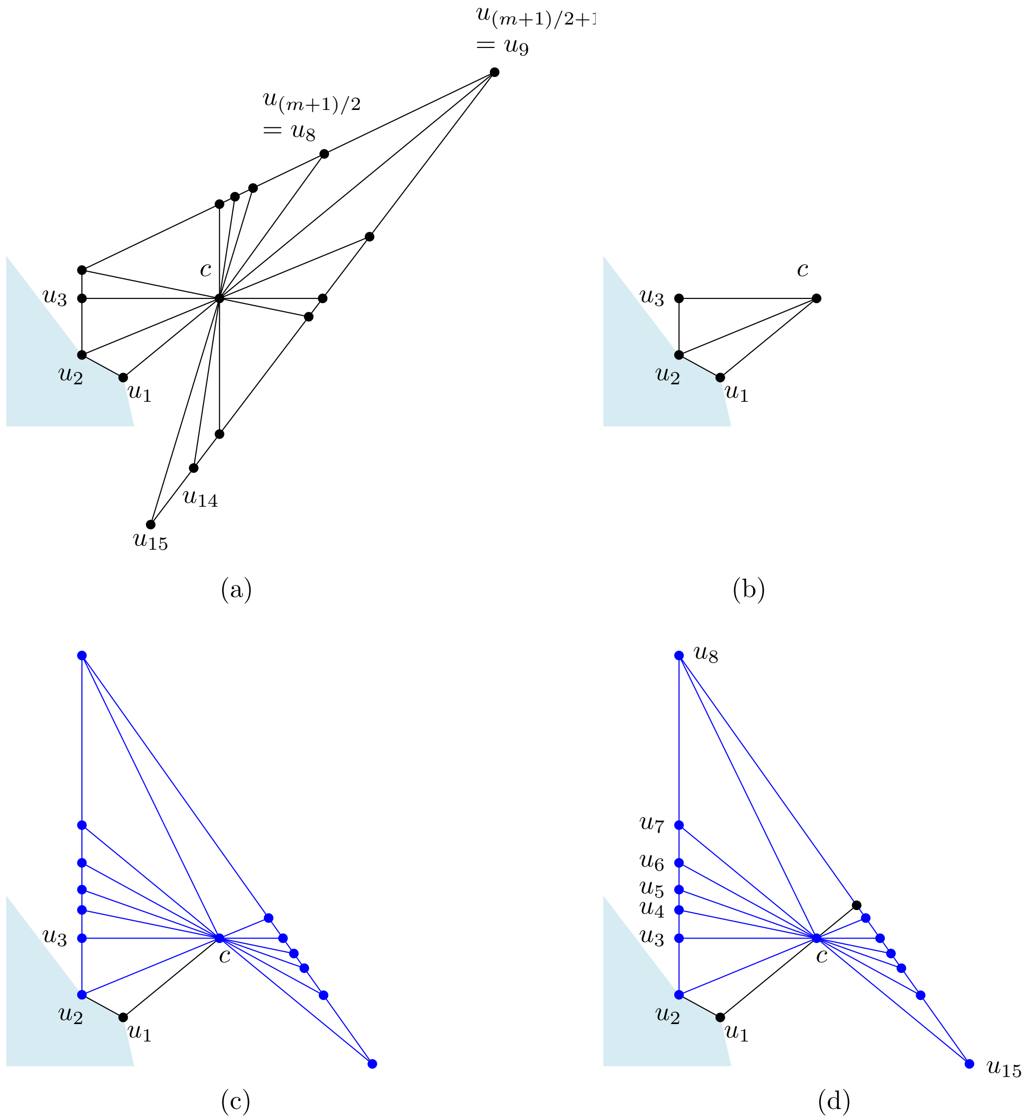


Fig. 3.8: Procedure to reduce D by one segment: (a) Initial case where $u_{(m+1)/2}u_{(m+1)/2+1}$ is neither sharing a segment with u_2u_3 or $u_{m-1}u_m$. (b) After removing all vertices and edges from D that are in $S(c)$ except the vertices u_1, u_2, u_3, c. (c) After applying Theorem 3.6. (d) New drawing D' with one segment less.

Figure 3.8 (c). The part drawn with Theorem 3.6 is coloured blue. We do not use the labelling of the vertices as in Theorem 3.6 because in our case the vertices $U \setminus c$ have different indices. We then extend u_1c such that it intersects with an outer edge and position another vertex on that intersection. The result is a drawing of a c-centered subgraph with $\deg(c)+1$ vertices, see Figure 3.8 (b). Given that we did not change any edges $\tilde{e} \in E \setminus F$ we know u_2 and u_1 still have the same neighbours as in D. Together with Lemma 3.2(c) we know that the drawing D' is a valid drawing for G. We can now label the vertices with $u_4, \dots u_{\deg(c)}$ in the proper order.

The new drawing is triangle-invariant as D was triangle invariant and the algorithm Theorem 3.6 provides a crossing-free drawing.

Both (a) and (b) hold for D'. We did not change the positions of u_1, u_2, u_3, c and therefore did not change the edges that are incident to u_1 and u_2. These are the only vertices incident to edges in $E \setminus F$. Hence for $i \in \{1, 2\}$ if any edges incident to u_i is sharing a segment with another edge incident to u_i in D, it still is in D'. Thus to compare D with D' in respect to the number of segments we only need to consider the segments we use to cover $S(c)$. We know that inner edges cannot share a segment with outer edges by Theorem 3.4. Thus we can analyse F_{inner} and F_{outer} independent from each other.

(a) Suppose (a) does not hold for D. Then there are at least two inner edges not sharing a segment with another inner edge in D. There are at least three such edges because of Theorem 3.4 and since $\deg(c)$ is odd. Therefore D uses at least $(\deg(c) - 3)/2 + 3 = (\deg(c) - 1)/2 + 4$ segments to cover F_{inner} which is one more than we use in D', a contradiction to D being a minimum-segment drawing.

(b) Suppose (b) does not hold for D. Then we know that $u_{(m+1)/2}u_{(m+1)/2+1}$ is neither sharing a segment with u_2u_3 nor $u_{m-1}u_m$. This means that there are at least 3 segments to cover $u_2u_3, \dots, u_{m-1}u_m$. That is one more segment than in D', a contradiction to D being a minimum-segment drawing.

□

Theorem 3.13 applies no matter if c has a connection or which type of connection the center c has. We consider Theorem 3.13 (a) for a strong and a weak Type A connection in Figure 3.9. In both cases we have marked the segment blue that covers

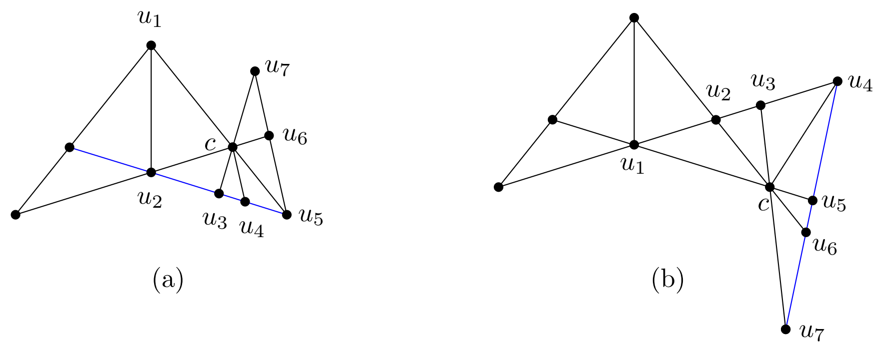


Fig. 3.9: Drawings of graph G with center c where $u_{(m+1)/2}u_{(m+1)/2+1}$ shares (a) with u_2u_3 and (b) with $u_{m-1}u_m$

.

$u_{(m+1)/2}u_{(m+1)/2+1}$. For our examples applies $m = 7$, thus $u_{(m+1)/2}u_{(m+1)/2+1} = u_4u_5$. In one case the edge shares with shares with u_6u_7 in the other with u_2u_3. Thus both of these drawings are possibly minimum-segment drawings.

3.4 Centres with no strong Type A Connection

When finding a lower bound the segment number of maximal outerpaths, the goal is to allow all kinds of connections in an outerpath. In this thesis we will only succeed for maximal outerpaths with no strong Type A connections. Nevertheless we want our results to be as general as possible. That is why in following theorems we only forbid the center c that again is adjacent to a vertex with degree 2 to have a strong Type A connection. By doing so, we make sure we can use this theorem, when analysing maximal outerplanar graphs which have strong Type A connections, but the center c has none.

The exclusion of c having no Type A connection is only implicit. If again $u_1, \dots u_{\deg(c)}$ is the path of $S(c) \setminus c$ the candidates c could have strong Type A connection with are u_2 and u_{m-1}. Both of them are not a center as we demand $\deg(u_2) = 4$ on the one hand and on the other hand we know $\deg(u_{m-1}) = 3$, with Corollary 2.7 and $\deg(u_m) = 2$.

Lemma 3.14. Let $G = (V, E)$ be a maximal outerpath. Let $v \in V$ with degree 2 be adjacent to a center c with odd degree m. Let $S(c) = (U, F)$ be the c-centered subgraph of G. Let $u_1, \dots, u_m$ be the sequence of vertices for the path $S(c) \setminus c$ with $v = u_m$ and $\deg(u_2) \leq 4$. Let D be a minimum-segment drawing of G where $cu_{(m+1)/2} \in F_{\text{inner}}$ does not share a segment with another inner edge. Furthermore let every cu_i with $1 \leq i < (m + 1)/2$ share a segment with $cu_{(m+1)/2+i}$. Then there is only one segment which covers $u_{(m+1)/2}u_{(m+1)/2+1}, \dots, u_{m-1}u_m$.

Proof. For the sake of contradiction suppose in D there are two segments which cover $u_{(m+1)/2}u_{(m+1)/2+1}, \dots, u_{m-1}u_m$. Let us consider the vertex u_2. If $deg(u_2) = 3$, we know G is a centered outerpath. We know $u_{(m+1)/2}u_{(m+1)/2+1}$ cannot share a segment with u_1u_2 because with $cu_{(m+1)/2+1}$ and cu_1 sharing a segment that would imply that u_1u_2c are collinear and a triangle; a contradiction. Hence F_{outer} is covered by three segments. By Theorem 3.6 we know there is a drawing D' of a centered outerpath with $\deg(c)$ odd in which the sharing behaviour of the inner edges is the same, but the outer edges are covered by two segments. That is one less than in D; a contradiction to D being a minimum segment drawing.

If $\deg(u_2) = 4$ we consider the different edges adjacent to u_2 with which u_2u_3 may share a segment. Let $r \in V \setminus U$ be the vertex incident to u_2. Then u_2u_3 can share a segment with u_2u_1, u_2c, u_2r or with no other edge via u_2.

If $u_{(m+1)/2}u_{(m+1)/2+1}, \dots, u_{m-1}u_m$ is covered by more than two segments, then the edges $u_{(m+1)/2}u_{(m+1)/2+1}$ and $u_{m-1}u_m$ need to be on different segments. Let s be the segment covering $u_{(m+1)/2}u_{(m+1)/2+1}$. If s does not cover u_2u_3 and does not cover $u_{m-1}u_m$ it is not a minimum-segment drawing, see Theorem 3.13(b).

Hence suppose s is covering u_2u_3. For each of the following four cases we now want to find a contradiction.

Case 1: u_2u_3 shares a segment with u_2c. We know $S(c)$ is a centered outerpath with $u_2u_3 \in F_{\text{outer}}$ and $u_2c \in F_{\text{inner}}$. By Theorem 3.4 we know u_2u_3 and u_2c cannot share a segment.

Case 2: u_2u_3 shares a segment with u_2u_1. Since s is covering the edges u_2u_3 and $u_{(m+1)/2}u_{(m+1)/2+1}$ this case implies that $u_1, u_2, u_3, u_{m+1/2}, u_{m+1/2+1}$ are collinear. But we also know $c, u_1, u_{m+1/2+1}$ are collinear due to requirements for the inner edges. Hence the triangle u_1u_2c would be collinear; a contradiction to a triangle-invariant drawing.

Case 3: u_2u_3 shares a segment with no other edge via u_2. There are at least 3 segments which cover F_{outer} only one of which might also cover edges in $E \setminus F$, compare see Figure 3.10 (a). By removing all $F \setminus \{u_1c, u_2c, u_1u_2\}$ and $U \setminus \{c, u_1, u_2\}$ we obtain a drawing in which only the triangle u_1u_2c is left of $S(c)$, see Figure 3.10 (b). We now use Theorem 3.6 to gain the drawing D'. The new drawing D' still represents G.

The sharing behaviour of the edges in F_{inner} has not changed from D to D'. Now there are 2 segments which cover F_{outer} one of which might also cover edges in $E \setminus F$. The edges F_{outer} are covered by one less segment in D' than in D. Hence D' uses one segment less than D in total, implying that D was not a minimum-segment drawing to begin with.

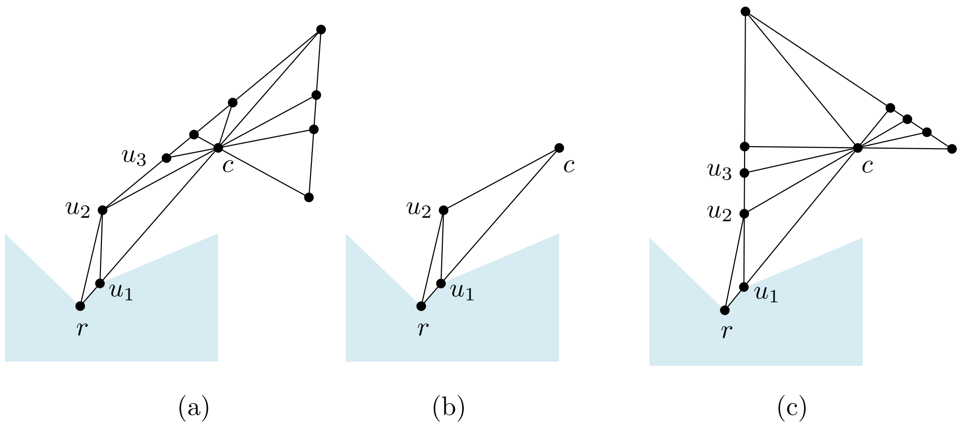


Fig. 3.10: Procedure to turn a drawing D where u_2u_3 is not sharing a segment through u_2 (a) into a drawing D' where u_2u_3 shares with u_2u_1

Case 4: u_2u_3 shares a segment with u_2r. Since $\deg(u_2) = 4$ and ru_2, u_2u_3 are sharing a segment with each other, the only other edges that could share a segment through u_2 are u_2c and u_1u_2. We know however that $u_1u_2 \in F_{\text{outer}}$ and $u_2c \in F_{\text{inner}}$ and with Theorem 3.4 (a) they cannot share a segment. Thus the segment covering u_2c and $cu_{(m+1)/2+2}$ is not covering another edge. We now want to reduce the drawing by this segment, for illustration it is coloured blue in our figures.

Without loss of generality the segment x which covers u_1u_2 is vertical, the center c is on the right half-plane of x and u_2 is above u_1, see Figure 3.11 (a).

We know that $u_{(m+1)/2+1}$ is an endpoint for the segment s and for the segment t which is covering $u_1c, cu_{(m+1)/2+1}$:

- **The vertex $u_{(m+1)/2+1}$ is an endpoint of t.** With Corollary 3.3 we know $\deg(u_{(m+1)/2+1}) = 3$ and that the other two edges e, e' incident to $u_{(m+1)/2+1}$ are in F_{outer}. Thus $cu_{(m+1)/2+1} \in F_{\text{inner}}$ cannot share a segment through $u_{(m+1)/2+1}$ due to Theorem 3.4 (a).

- **The vertex $u_{(m+1)/2+1}$ is an endpoint of s.** Note that s is covering u_2u_3 and $u_{(m+1)/2}u_{(m+1)/2+1}$. Suppose $u_{(m+1)/2+1}$ is not an endpoint for s, then s is covering another edge adjacent to $u_{(m+1)/2+1}$. With Theorem 3.4 (a) $u_{(m+1)/2}u_{(m+1)/2+1}$ cannot share a segment with $cu_{(m+1)/2+1}$. Therefore s has to cover $u_{(m+1)/2+1}u_{(m+1)/2+2}$. But we also know that $cu_{(m+1)/2+2}$ is sharing a segment with u_2c. Therefore u_2 and $u_{(m+1)/2+2}$ are on a segment with c and on a segment with u_3. This implies that u_2, u_3, c are collinear but also a triangle; a contradiction.

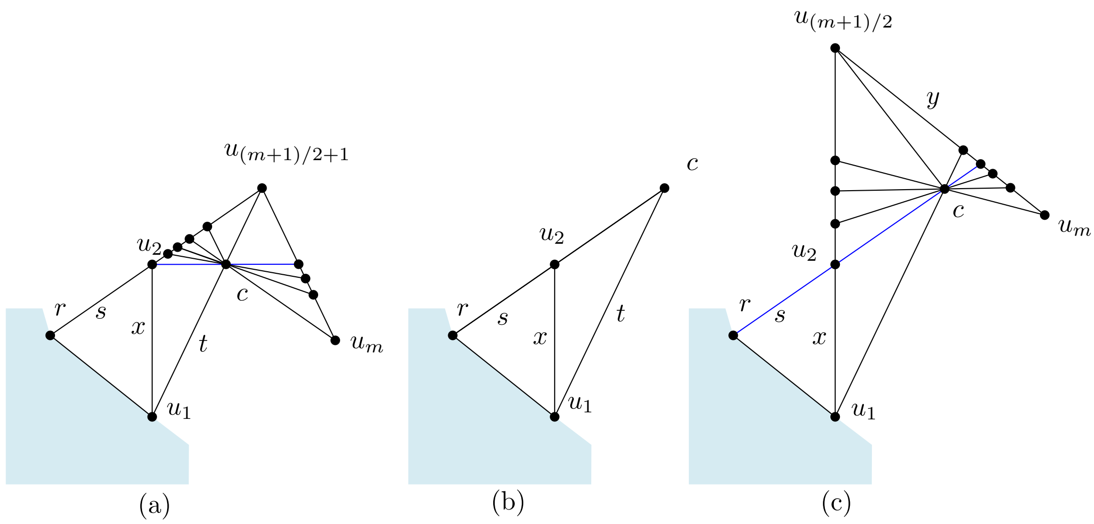


Fig. 3.11: Procedure to reduce D by one segment: (a) Initial case. (b) After removing all segments that cover inner edges except t. (c) The optimized drawing D'.

We remove every segment covering F_{inner} except t and position c on the intersection of s and t, see Figure 3.11 (b). We then use the method of Theorem 3.6 to obtain the drawing in Figure 3.11 (c). The new drawing D' still represents G with Lemma 3.2.

Let us now analyse the number of segments. Given that we only changed positions of edges in F, we only need to analyse F_{inner} and F_{outer}. We did however change the sharing behaviour of F with edges in $E \setminus F$.

- **Consider F_{outer}.** The edges $u_2u_3, \ldots, u_{(m+1)/2-1}u_{(m+1)/2}$ in D' are not on s like they are in D, but on x. Both segments already exist in D and D', which means we did not produce a new segment. In D the edges $u_{(m+1)/2+1}u_{(m+1)/2+2}, \ldots, u_{m-1}u_m$ are all on at least one segment which is not covering any edge in $E \setminus F$. In D' the edges $u_{(m+1)/2}u_{(m+1)/2+1}, \ldots, u_{m-1}u_m$ are all on the same segment which is not covering any edge in $E \setminus F$. Therefore no segment was added to cover F_{outer}.

- **Consider F_{inner}.** In D as well as in D all inner edges except $cu_{(m+1)/2}$ share a segment with one other inner edge. Also in both drawings applies that for

$3 \leq i \leq (m+1)/2$ the segments covering $u_i c$ and $u_{i+(m+1)/2}c$ cannot cover edges in $E \setminus F$. The only segments that can are t covering $u_1 c$ and $u_{1+(m+1)/2}$ and the segment covering $u_2 c$ and $u_{2+(m+1)/2}$. We did not change the segment t, but in D' the segment covering $u_2 c$ and $u_{2+(m+1)/2}$ also covers an edge in $E \setminus F$, namely $u_2 r$. In D however $u_2 c$ and $u_{2+(m+1)/2}$ were on their own segment. Hence D' uses one segment less than D.

In conclusion D' uses one segment less than D.

□

We used the fact that $\deg(u_2) = 4$ at the very beginning of the prove, while analysing which pairs of edges incident to u_2 can share a segment. This argumentation obviously is not valid as soon as $\deg(u_2) = 5$. Consider the drawing in Figure 3.12. The center c now

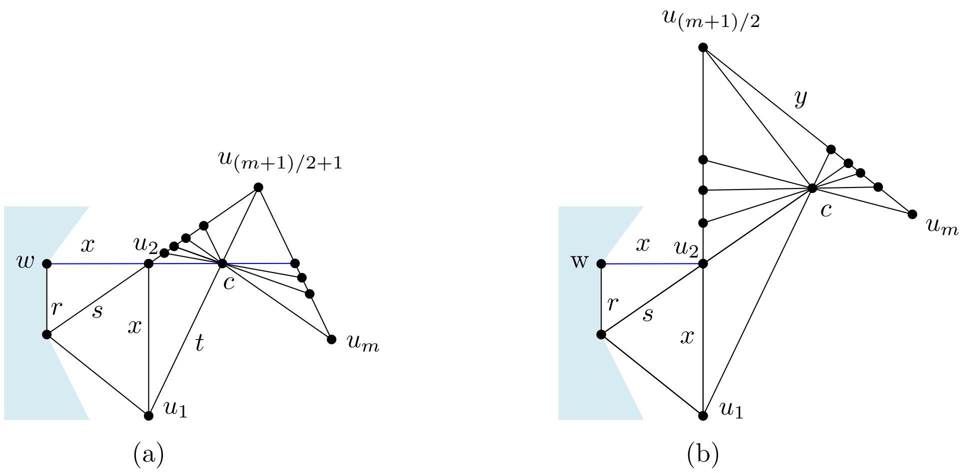


Fig. 3.12: A graph in which c has a strong Type A connection with u_2 (a) and thus the argumentation in Lemma 3.14 fails.

has a strong Type A connection with u_2. The segment x covering $u_2 c$ and $cu_{(m+1)/2+2}$ now is covering another edge, namely wu_2. Thus the provided procedure above will not remove a segment in this case, as you can see in Figure 3.12 (b).

3.5 Outerpaths with no strong Type A Connection

We want our lower bound to be as sharp as possible. Hence we need to consider more than just the number of vertices. Next to the connections and the connection types, which we already introduced, we will need to keep the vertices in mind that are not in a c-centered subgraph. Lemma 3.15 compares the already mentioned G and $G \setminus v$ in terms of the number of these vertices.

Additionally we use our results, namely Lemma 3.14 and Corollary 2.6, to compare the number of segments needed at least to draw G in comparison with the one needed for $G \setminus v$.

Lemma 3.15. Let $G = (V, E)$ be maximal outerplanar graph with no strong Type A connections. Let $v \in V$ have $\deg(v) = 2$. Suppose there exists a lower bound $\mathfrak{s}'$ for segments of a drawing of $G' = G \setminus v$. Let U' be the number vertices that are neither a center nor adjacent to one in G'. Let U be defined analogously for G and let D be a drawing of G.

1. If v is adjacent to a center, then $|U| = |U'|+1$ and D needs at least $\mathfrak{s}'+1$ segments.

2. Let v be adjacent to a center c with degree 5. D needs at least $\mathfrak{s}'$ segments.
   (a) If c has no connection to another center, then $|U| = |U'| - 5$
   (b) If c has a weak Type B connection to another center, then $|U| = |U'| - 4$
   (c) If c has a strong Type B connection to another center, then $|U| = |U'| - 3$.
   (d) If c has a weak Type A connection to another center, then $|U| = |U'| - 2$.
3. Let v be adjacent to a center c with degree at least 6. Then $|U| = |U'|$.
   (a) If $\deg(c)$ is odd, then D needs at least $\mathfrak{s}'$ segments.
   (b) If $\deg(c)$ is even, then D needs at least $\mathfrak{s}' + 1$ segments.

Proof. Let $G' = (V', E')$.

1. If c has no connection to another center, we have the same situation as in Corollary 2.6. The proof for it follows analogously for triangle-invariant drawings. Hence D needs at least $\mathfrak{s}' + 1$ segments.

   As v is not connected to a center nor a center as $\deg(v) = 2$, we know $v \in U \setminus U'$. Thus $|U| = |U'| + 1$.

2. Let us first consider the number of segments. We know that D always contains a drawing of G' and thus needs at least $\mathfrak{s}'$ segments.

   We now consider U. We know c is no center in G', because $\deg(c) = 4$ in G'. So, to figure out how U has changed in comparison to U' we need to analyse c and the five vertices adjacent to c. To be precise, we need to look at $U' \cap W$ for $S(c) = (W, F)$. We know $v \in W \setminus U'$ and $|W| = 6$, therefore $|U' \cap W| \leq 5$. We illustrate all cases in Figure 3.13, the set $U' \cap W$ is coloured blue.

   (a) If c has no connection to another center, all vertices in W c are only adjacent to one center in G, hence not connected to a center in G'. We therefore know $|U' \cap W| = 5$ and

   $$|U| = |U' \setminus (U' \cap W)| = |U'| - |U' \cap W| = |U'| - 5.$$

   .

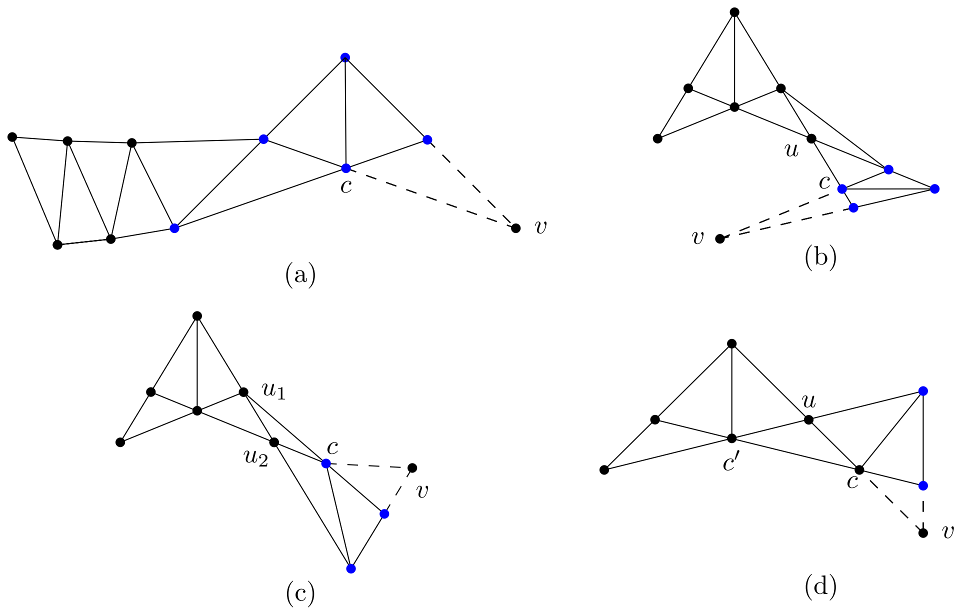


Fig. 3.13: Illustration of G with the dashed edges in comparison to $G \setminus v$, without the dashed edges. The vertex v is adjacent to a center c that has (a) no connection to another center, (b) a weak Type B connection to another center, (c) a strong Type B connection to another center, (d) a weak Type A connection to another center,.

(b) If c has weak Type B connection to another center, one vertex $u \in W \setminus c$ is adjacent to two centres, all other vertices in $W \setminus c$ are only adjacent to one center in G. Therefore u is adjacent to a center in G' and thus $u \notin U'$. This implies $|U' \cap W| = 4$ and so

$$|U| = |U'| - |U' \cap W| = |U'| - 4.$$

(c) If c has strong Type B connection to another center, two vertices $u_1, u_2 \in W \setminus c$ are adjacent to two centres, all other vertices in $W \setminus c$ are only adjacent to one center in G. Therefore u_1 and u_2 are adjacent to a center in G' and thus $u_1, u_2 \notin U'$. Hence we conclude $|U' \cap W| = 3$ and

$$|U| = |U'| - |U' \cap W| = |U'| - 3.$$

(d) If c has weak Type A connection to another center, one vertex u is adjacent to two centres c and c', and c' is adjacent to c. Therefore u and c are adjacent to the center c' which is a center in G'. Thus $u, c, c' \notin U'$. Hence we know $|U' \cap W| = 2$ and so

$$|U| = |U'| - |U' \cap W| = |U'| - 2.$$

3. Let us first consider U. Since v is connected to a center c, we know $v \notin U$. Furthermore we know c was already a center in G' hence $U = U'$. Let us now consider the different cases for the number of segments.

(a) If $\deg(c)$ is odd, we know that D always contains a drawing of G' and thus needs at least $\mathfrak{s}'$ segments.

(b) If $\deg(c)$ is even, for the sake of contradiction suppose D needs less than $\mathfrak{s}'+1$ segments. Then it needs exactly $\mathfrak{s}'$ segments because a drawing of G always contains a drawing of G'. Let D be an minimum-segment drawing of G which uses $\mathfrak{s}'$ segments. And let D' be the drawing of G' obtained by removing v and its adjacent edges from D. Then we know D' uses at most $\mathfrak{s}'$ segments and is therefore an minimum-segment drawing for G'.

To find a contradiction we compare the segments needed to draw D in comparison to D'. To do so we consider $S(c) = (X, F)$, the c-centered subgraph of G and $S'(c) = (X', F')$ analogously defined for G'.

Suppose there is a $w \in F_{\text{inner}}$ which is not sharing a segment with another edge in F_{inner} in D. Then there are at least two edges in F_{inner} that are not paired with another edge in F_{inner}, because $|F_{\text{inner}}|$ is even. Therefore if D uses $\mathfrak{s}'$ segments, we know there are at leasttwo edges in F'_{inner} in the drawing D' which do not share a segment with another edge in F'_{inner}. By Theorem 3.13 (a) D' then cannot be a minimum-segment drawing; a contradiction.

Now suppose all edges in F_{inner} are sharing a segment with another edge in F_{inner} in D. Let $u_1, \ldots, u_m$ be the sequence for the path of $S(c) \setminus c$ where $u_{\deg(c)} = v$ and $m = \deg(c)$ in G. Thanks to Observation 3.8 we know $u_m c$ is sharing a segment t with $cu_{m/2}$. Thus $u_{m/2}u_{m/2+1}$ and $u_{m-1}u_m$ cannot be on the same segment, because otherwise the vertices u_{m-1}, u_m, c would be collinear in D and a triangle in G; a contradiction to a triangle-invariant drawing.

If D uses $\mathfrak{s}$ segments we know $u_{m-1}u_m$ needs to share at least a segment with $u_{m-2}u_{m-1}$. Otherwise $u_{m-1}u_m$ would be a segment on its own, which does not exist in D'. Therefore $u_{m-2}u_{m-1}$ and $u_{m/2}u_{m/2+1}$ need to be on different segments. So there has to be an minimum-segment drawing of G' where $u_{m-2}u_{m-1}$ and $u_{m/2}u_{m/2+1}$ are on different segments. Furthermore in that drawing $cu_{m/2}$ is not sharing a segment with another edge in F'_{inner} but for every cu_i with $1 \leq i < m/2$ share a segment with $cu_{m/2+i}$. This contradicts Lemma 3.14. Thus D needs at least $\mathfrak{s} + 1$ segments.

□

Now we only need to apply this Lemma in the inductive step to prove the following formula.

Theorem 3.16. *Let $G = (V, E)$ be a maximal outerpath which has no strong Type A connections with centres $c_1, \ldots, c_k$. Let a_{weak} be the number of weak Type A connections, let b_{strong} be the number of strong Type B connections and b_{weak} the number of weak Type B connections. Let $U \subseteq V$ be the vertices, that are neither a center nor adjacent to one.*

Then the number of segments s of a crossing-free straight-line drawing of G is bound by

$$s \geq |U| + 3k - 3a_{\text{weak}} - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{k} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor . \tag{3.1}$$

Proof. We prove this theorem via induction over $n = |V|$. We show that the provided bound as a lower bound for the number of segments of a triangle-invariant drawing D.

Initial case. For $n = 3$ know we the degree for all $v \in V$ is 3. Therefore there are no centres in G and thus no connection types. Thus, $U = V$ and we get $s \geq 3 = |V| = |U|$. Given that a maximal outerpath with 3 vertices is a triangle, that needs 3 segments, this lower bound is true.

Induction hypothesis. Let G be n-vertex maximal outerpath which has no strong Type A connection and D a triangle-invariant drawing of G. Then the given formula for the lower bound holds true for the number of segments in D.

Inductive step. Let $G = (V, E)$ be a maximal outerpath with $n + 1$ vertices and no strong Type A connection. Since G is maximal there exists a $v \in V$ with degree 2. We know $G' = G \setminus v = (V', E')$ is a n-vertex maximal outerpath with no strong Type A connection. We define $a'_{\text{weak}}, b'_{\text{strong}}, b'_{\text{weak}}, U', k'$ analogously for G'. Let c'_i be the centres in G' for $1 \leq i \leq k'$ and D' a drawing of G. With the induction hypothesis we then know that D' needs at least $\mathfrak{s}' = |U'| + 3k' - 3a'_{\text{weak}} - 2b'_{\text{strong}} - b'_{\text{weak}} + \sum_{i=1}^{k'} \left\lfloor \frac{\deg(c'_i)}{2} \right\rfloor$ segments.

We consider the following cases:

1. The vertex v is not adjacent to a center. Thanks to Lemma 3.15.1 we know that D needs at least $\mathfrak{s}' + 1$ segments and that $|U| = |U'| + 1$. Furthermore G has the same centres as G' and nothing has changed about their types, therefore:

$$\begin{aligned} s &\geq 1 + \mathfrak{s}' \\ &= 1 + |U'| + 3k' - 3a'_{\text{weak}} - 2b'_{\text{strong}} - b'_{\text{weak}} + \sum_{i=1}^{k'} \left\lfloor \frac{\deg(c'_i)}{2} \right\rfloor \\ &= |U| + 3k - 3a_{\text{weak}} - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{k} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor . \end{aligned}$$

   So, in this case, the inductive step is done.

2. The vertex v is adjacent to a center c with degree 5. Thanks to Lemma 3.15.2 we know that D needs at least $\mathfrak{s}'$ segments. Since c is not a center in G', the number of centres increases, thus $k = k' + 1$ and $c_k = c$. All other centres remain the same. Note that $\left\lfloor \frac{\deg(c_k)}{2} \right\rfloor = 2$. We will now consider the different connections c can have:

(a) If c has no connection to another center, we know with Lemma 3.15.2 (a) that $|U| = |U'| - 5$. Nothing has changed about the amount of connection types. We therefore obtain:

$$
\begin{aligned}
s &\geq \mathfrak{s}' \\
&= |U'| + 3k' - 3a'_{\text{weak}} - 2b'_{\text{strong}} - b'_{\text{weak}} + \sum_{i=1}^{k'} \left\lfloor \frac{\deg(c'_i)}{2} \right\rfloor \\
&= |U| + 5 + 3(k-1) - 3a_{\text{weak}} - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{k-1} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor \\
&= |U| + \left\lfloor \frac{\deg(c_k)}{2} \right\rfloor + 3 + 3k - 3 - 3a_{\text{weak}} - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{k-1} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor \\
&= |U| + 3k - 3a_{\text{weak}} - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{k} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor .
\end{aligned}
$$

(b) If c has a weak Type B connection to another center, by Lemma 3.15.2 (b) we know that $|U| = |U'| - 4$. In this case, we know $b_{\text{weak}} = b'_{\text{weak}} + 1$, but nothing has changed about the other connection types. We can conclude:

$$
\begin{aligned}
s &\geq \mathfrak{s}' \\
&= |U'| + 3k' - 3a'_{\text{weak}} - 2b'_{\text{strong}} - b'_{\text{weak}} + \sum_{i=1}^{k'} \left\lfloor \frac{\deg(c'_i)}{2} \right\rfloor \\
&= |U| + 4 + 3(k-1) - 3a_{\text{weak}} - 2b_{\text{strong}} - (b_{\text{weak}} - 1) + \sum_{i=1}^{k-1} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor \\
&= |U| + \left\lfloor \frac{\deg(c_k)}{2} \right\rfloor + 3k - 3a_{\text{weak}} - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{k-1} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor \\
&= |U| + 3k - 3a_{\text{weak}} - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{k} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor .
\end{aligned}
$$

(c) If c has a strong Type B connection to another center, by Lemma 3.15.2 (c) we know that $|U| = |U'| - 3$. In this case, we know $b_{\text{strong}} = b'_{\text{strong}} + 1$, but nothing has changed about the other connection types. We can conclude:

$$
\begin{aligned}
s &\geq \mathfrak{s}' \\
&= |U'| + 3k' - 3a'_{\text{weak}} - 2b'_{\text{strong}} - b'_{\text{weak}} + \sum_{i=1}^{k'} \left\lfloor \frac{\deg(c'_i)}{2} \right\rfloor \\
&= |U| + 3 + 3(k-1) - 3a_{\text{weak}} - 2(b_{\text{strong}} - 1) - b_{\text{weak}} + \sum_{i=1}^{k-1} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor \\
&= |U| + 3k - 3a_{\text{weak}} - 2b_{\text{strong}} + \left\lfloor \frac{\deg(c_k)}{2} \right\rfloor - b_{\text{weak}} + \sum_{i=1}^{k-1} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor \\
&= |U| + 3k - 3a_{\text{weak}} - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{k} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor .
\end{aligned}
$$

(d) If c has a weak Type A connection to another center, analogously to the cases before we can conclude with Lemma 3.15.2 (d) that $|U| = |U'| - 2$ and $a_{\text{weak}} = a'_{\text{weak}} + 1$. We obtain:

$$
\begin{aligned}
s &\geq \mathfrak{s}' \\
&= |U'| + 3k' - 3a'_{\text{weak}} - 2b'_{\text{strong}} - b'_{\text{weak}} + \sum_{i=1}^{k'} \left\lfloor \frac{\deg(c'_i)}{2} \right\rfloor \\
&= |U| + 2 + 3(k-1) - 3(a_{\text{weak}} - 1) - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{k-1} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor \\
&= |U| + \left\lfloor \frac{\deg(c_k)}{2} \right\rfloor + 3k - 3a_{\text{weak}} - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{k-1} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor \\
&= |U| + 3k - 3a_{\text{weak}} - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{k} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor .
\end{aligned}
$$

3. The vertex v is adjacent to a center c with a degree of at least 6. Thanks to Lemma 3.15.3 we know $|U| = |U'|$. Furthermore we know c is a center in G' as well, thus $c \in \{c'_1, \ldots, c'_{k'}\}$. Without loss generality, we can assume $c = c'_{k'}$. Given that c was already a center in G', nothing has changed about the connection types and $k' = k$. Since c_k has a different degree in G' as in G, we consider $c'_k \in G'$ and $c_k \in G$. Note that $\deg(c_k) = \deg(c'_k) + 1$. We consider the following two cases:

   (a) If $\deg(c_k)$ is odd, we know with Lemma 3.15.3 (a) that D needs at least $\mathfrak{s}'$ segments. Furthermore we know $\left\lfloor \frac{\deg(c'_k)}{2} \right\rfloor = \left\lfloor \frac{\deg(c_k)-1}{2} \right\rfloor = \left\lfloor \frac{\deg(c_k)}{2} \right\rfloor$. We thus obtain

$$\begin{aligned}
s &\geq \mathfrak{s}' \\
&= |U'| + 3k' - 3a'_{\text{weak}} - 2b'_{\text{strong}} - b'_{\text{weak}} + \sum_{i=1}^{k'} \left\lfloor \frac{\deg(c'_i)}{2} \right\rfloor \\
&= |U| + 3k - 3a_{\text{weak}} - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{k} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor .
\end{aligned}$$

(b) If $\deg(c_k)$ is even, we know with Lemma 3.15.3 (b) that D needs at least $\mathfrak{s}'+1$ segments. Furthermore we know $\left\lfloor \frac{\deg(c'_k)}{2} \right\rfloor = \left\lfloor \frac{\deg(c_k)-1}{2} \right\rfloor = \left\lfloor \frac{\deg(c_k)}{2} \right\rfloor - 1$. We thus obtain

$$\begin{aligned}
s &\geq 1 + \mathfrak{s}' \\
&= 1 + |U'| + 3k' - 3a'_{\text{weak}} - 2b'_{\text{strong}} - b'_{\text{weak}} + \sum_{i=1}^{k'} \left\lfloor \frac{\deg(c'_i)}{2} \right\rfloor \\
&= |U| + 3k - 3a_{\text{weak}} - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{k} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor .
\end{aligned}$$

With Corollary 2.7 we know that one neighbour of v has degree of 3. Thus the case distinction above for the other neighbour of v covers all possible transitions between G and G'.

With this inductive proof the lower bound is true for triangle-invariant drawings. The fact that crossing-free straight-line drawings are triangle-invariant proves the theorem.

□

Let us first compare this universal lower bound with the bounds we already know.

In case of Theorem 2.5 we have a n-vertex graph $G = (V, E)$ whose maximum degree is at most 4. Thus we do not have any centres, leaving us with $V = U$. We therefore obtain in Equation (3.1) a lower bound of $\mathfrak{s} = |U| + 0 = |V| = n$ which is the same bound we obtained in Theorem 2.5.

In case of centered outerpaths, let us consider a n-vertex centered outerpath $G = (V, E)$ with center c_1. With its definition we know $\deg(c_1) = n - 1$ and $U = \emptyset$ because all vertices $V \setminus c_1$ are adjacent to c_1. Furthermore, since there is only one center, there are no connection types. With Equation (3.1) we therefore obtain

$$\mathfrak{s} = 3k - 0 + 0 + \sum_{i=1}^{1} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor = 3 + \left\lfloor \frac{\deg(c_1)}{2} \right\rfloor = 3 + \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil + 2.$$

This is the same lower bound as proven in Theorem 3.5.

For a more general case let us revisit the graph G_2 in Figure 1.1 (c). It has one center with degree 6 and two edges that are not connected to a center. Given that there is only one center, there are no connections we need to consider. We thus obtain as a lower bound: $\mathfrak{s} = |U| + 3 - 0 + \sum_{i=1}^{k} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor = 2 + 3 + \left\lfloor \frac{6}{2} \right\rfloor = 8$. Therefore the drawing in Figure 1.1 (c) is a minimum-segment drawing of G_2.

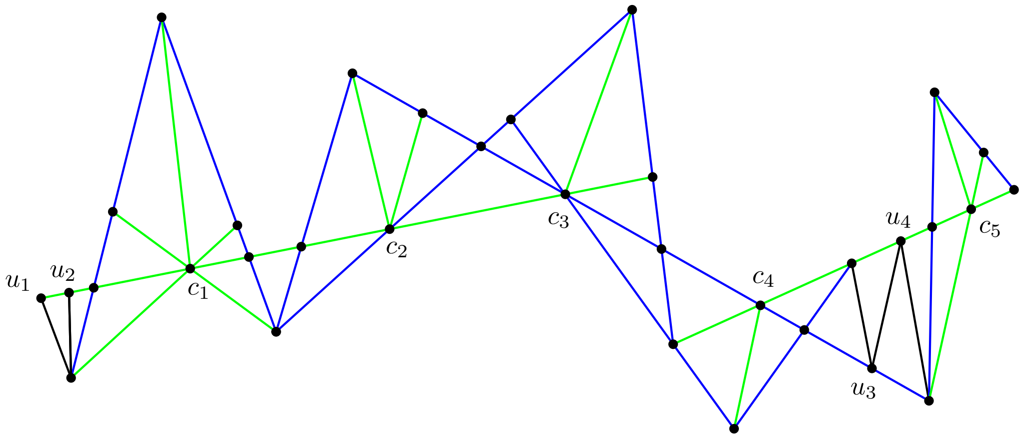


Fig. 3.14: A graph P as adaptation of the graph in Figure 3.5. For any center c_i, the segments covering the outer edges of the c_i-centered subgraph are coloured blue. The remaining segments are divided in two groups: Those covering inner edges are coloured green and the rest black.

We will now consider a similar drawing to the one in Figure 3.5 as an example for a more complicated maximal outerpath. Since we do not know, how to handle strong Type A connections, we remove three vertices adjacent to c_6. This way there is no more strong Type A connection in the graph, see Figure 3.14. To count the segments more easily we group them. We have 10 blue, 11 green and 5 black segments. Thus this drawing has $10 + 11 + 5 = 26$ segments.

Tab. 3.1: Centres and their degrees of graph P in Figure 3.14

i	1	2	3	4	5
$\deg(c_i)$	7	6	7	5	5
$\lfloor \deg(c_i)/2 \rfloor$	3	3	3	2	2

Now let us analyse the structure of P with the terms as in Theorem 3.16. There are 4 vertices in $U = \{u_1, u_2, u_3, u_4\}$. There are 5 centres, thus $k = 5$. In comparison to Figure 3.5 the connection types only have changed the way that there is no more strong Type A connection. With the analysis in Section 3.2 we obtain $a_{\text{weak}} = b_{\text{strong}} = b_{\text{weak}} = 1$. The centres, their degrees and their rounded fractures are listed in Table 3.1. As a lower bound for the number of segments we therefore get with Equation (3.1):

$$s \geq \mathfrak{s} = |U| + 3 \cdot 5 - 3a_{\text{weak}} - 2b_{\text{strong}} - b_{\text{weak}} + \sum_{i=1}^{5} \left\lfloor \frac{\deg(c_i)}{2} \right\rfloor$$
$$= 4 + 15 - 3 - 2 - 1 + 13 = 26$$

In conclusion we know the drawing in Figure 3.14 is a minimum-segment drawing.

4 Lower Bound Constant

The objective to find a lower bound for the segment number can be viewed from different perspectives. One can analyse the total number of segments in the drawing as a function of some other characteristic of the graph, e. g. the number of vertices with a degree of at least five as seen in Theorem 3.16. As we have seen for maximal outerpaths, these functions can turn out to be very specific. Another way to find a lower bound is to consider a constant $0 < c < 1$, such that every n-vertex maximal outerplanar graph needs at least nc segments to be drawn. We call this c the *lower bound constant.* If such a c existed for maximal outerplanar graphs, it would be easier to give an approximation for the number of segments needed.

4.1 Segment-Vertex-Ratio

To analyse the lower bound constant it is helpful to consider the ratio of segments over the vertices:

Definition 4.1. Let G be a n-vertex graph, and D be a drawing of G with s segments. Then we call

$$r_{\mathrm{sv}}(D) := \frac{s}{n} \tag{4.1}$$

the *segment-vertex-ratio of D* or *sv-ratio of D.*

Consider a minimum segment drawing D of a maximal outerplanar graph. This graph then needs $nr_{\mathrm{sv}}(D) = s$ segments to be drawn. We then know $c \leq r_{\mathrm{sv}}(D)$. Therefore a lower bound for the sv-ratio is equivalent to the lower bound constant c.

For a better understanding of the sv-ratio, we consider the class we defined earlier as centered outerpaths. For simplicity, we only consider n-vertex centered outerpaths G_n with n being even. Let D_n be the drawing of G_n obtained with the algorithm Theorem 3.6. We then know $r_{\mathrm{sv}}(D_n) = (n/2+2)/n = 1/2 + 2/n$. In Figure 4.1 you can see the problem with the sv-ratio: Since there are several vertices on one segment, the sv-ratio does not tell you anything about the number of segments a vertex is on. For a better illustration, we therefore introduce another ratio:

Definition 4.2. Let $G = (V, E)$ be a graph, and D be a drawing of G with s segments. Then we call

$$r_{\mathrm{es}}(D) := \frac{|E|}{s} \tag{4.2}$$

the *edge-segment-ratio of D* or *es-ratio of D.*

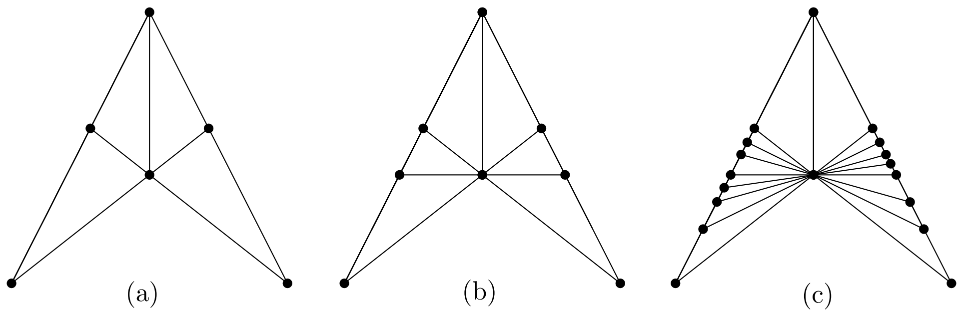


Fig. 4.1: Centered outerpaths with n vertices with
(a) $n = 6$ and the ratios $r_{\text{sv}}(D_n) = 5/6$ and $r_{\text{es}}(D_n) = 1.8$,
(b) $n = 8$ and the ratios $r_{\text{sv}}(D_n) = 3/4$ and $r_{\text{es}}(D_n) = 13/6 \approx 2.17$,
(c) $n = 18$ and the ratios $r_{\text{sv}}(D_n) = 11/18$ and $r_{\text{es}}(D_n) = 3$

The es-ratio has the advantage, that each edge is exactly on one segment. Therefore the ratio tells us, how many edges on average are on a segment for the given graph. Since for maximal outerplanar graphs the number of edges is dependent only on the number of vertices, compare Observation 2.3, the es-ratio and the sv-ratio are directly related. We will therefore consider both the es-ratio and the sv-ratio in the following chapter.

As example, let us revisit the centered outerpaths in Figure 4.1. With Theorem 3.5 and Observation 2.3 we obtain: $r_{\text{es}}(D_n) = (2n-3)/(n/2+2)$ with D_n being a minimum-segment drawing of G_n. Thus for D_6 the es-ratio is 1.8 which tells us that the average number of edges a segment covers is almost two; a conclusion, we can easily draw from the drawing as all except one segment cover two edges. We have a similar situation for D_8. Of course the es-ratio has the problem every average has: If the variance is high, we do not gain a lot information about the amount of edges each segment covers from the average any more, as you can see with D_{18}. The two outer segments cover so many edges that the average is three, even though most of the segments only cover two edges. Despite this fact, the es-ratio still helps us compare two graph drawings intuitively.

4.2 Asymptotic upper Bound

Given the research on segment numbers of maximal outerplanar graphs and its bounds presented in Chapter 2 we know the sv-ratio of an minimum-segment drawing D of a maximal outerplanar graph is bound by $1 \geq r_{\text{sv}}(D) > 0$. Analogously with Observation 2.3 the es-ratio is at least $(2n-3)/n = 2 - 3/n$ if G has n vertices. While an example for the worst case $r_{\text{sv}}(D) = 1$ is already given in Theorem 2.5, there are only few attempts to find a possible lower bound for the sv-ratio, in other words to find the lower bound constant. So far Park and Wolff [PW20] found a drawing D of an maximal outerplanar graph with $r_{\text{sv}}(D) = 13/29$, see Figure 4.2. Therefore c is bound by $13/29 \geq c \geq 0$. This drawing has a es-ratio of $r_{\text{es}}(D) = 55/13 \approx 4.23$. Thus the question now is if we can fit more than an average of 4.23 edges on a segment.

Based on the graph in Figure 4.2, we define a sequence of graphs which serves as new

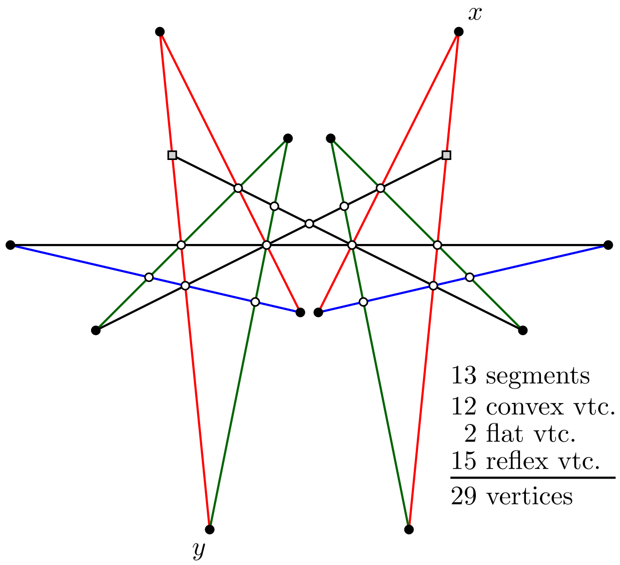


Fig. 4.2: Graph with sv-ratio $r_{\text{sv}}(G) = 13/29$ and es-ratio $r_{\text{es}}(G) = 55/13$

asymptotic upper bound for the lower bound constant c. The sv-ratio of the drawings of this graph sequence converges to $3/7 \leq 13/29$ and the es-ratio converges to $14/3 > 55/13$. The main idea of this sequence is to combine the graph in Figure 4.2 with it self repeatedly. The formal definition is done in 5 steps:

Step 1: First element of the sequence. We define G_0 and its drawing D_0 by Figure 4.2. Furthermore we define y, x as the points displayed in Figure 4.2.

Step 2: Combine G_0 with G_0. Let us consider the drawings D_0 and an exact copy D_0'. We turn D_0' by 180° and put the two points y and y' on top of each other. As a result we obtain the graph given in Figure 4.3 (a). It has 57 vertices and 24 segments.

Step 3: Adding missing edge e. This graph is not yet maximal, because a maximal outerplanar graph has $2n - 3 = 2 \cdot 57 - 3 = 111$ edges. In this case we have twice as many vertices as G_0 has, so we get $2 \cdot (2 \cdot 29 - 3) = 110 < 111$ vertices. The missing edge e can be added without any crossings as displayed in Figure 4.3 (b). We define the resulting graph as G_1 and its drawing as D_1. It has 25 segments and therefore the drawing D_1 has a sv-ratio of $25/57 < 13/29$ and a es-ratio of $111/25 = 4.44$.

Step 4: Combine G_1 with G_0. Let us consider the point z in Figure 4.3 (b). We combine the drawing D_1 with D_0 by putting z on top of x and adding an edge which makes the new drawing a maximal outplanar graph. By doing so we get the graph in Figure 4.3 (c), which we now define as our G_2. The drawing D_2 of this graph has 37 segments and 85 vertices. For D_2 the sv-ratio therefore is $37/85 < 25/57$ and the es-ratio $167/37 \approx 4.51$.

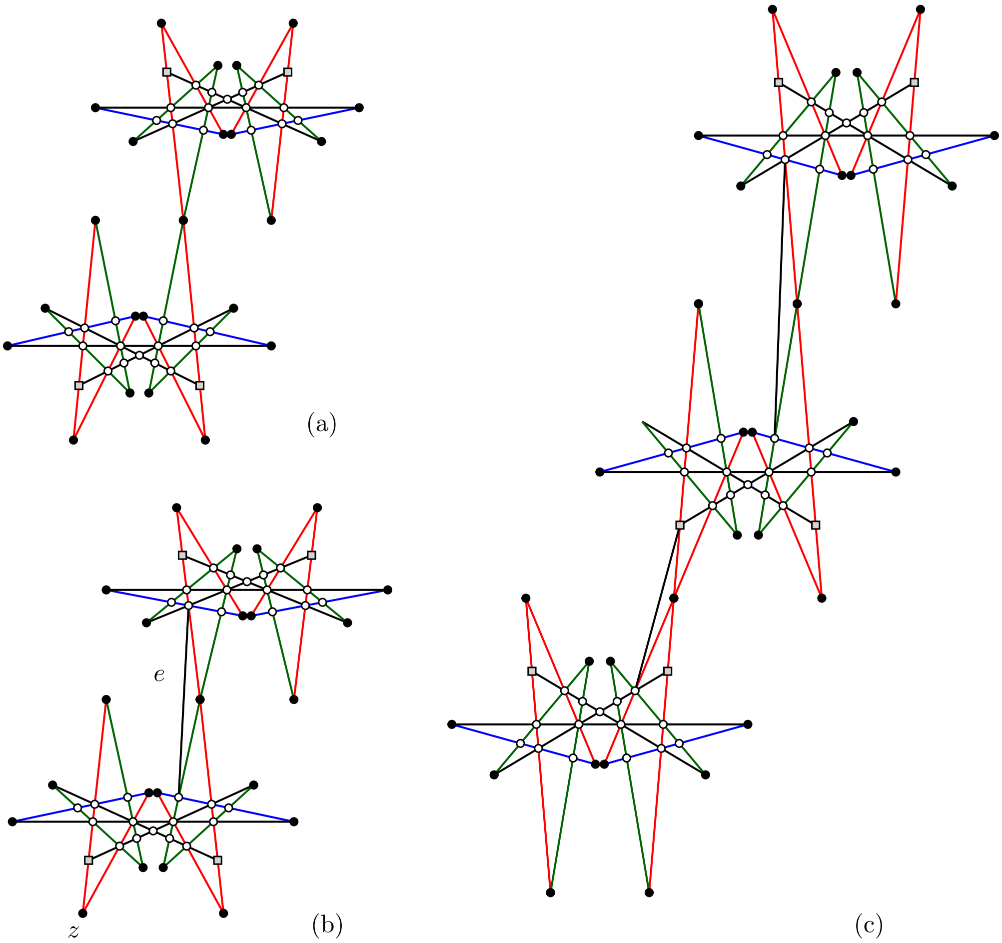


Fig. 4.3: Construction of the graph sequence G_i: (a) Combining G_0 with itself, (b) Illustration of G_1 with the edge e, (c) Illustration of G_2

Step 5: Combine G_i with G_0.

- If i is odd, the left lowest vertex in the drawing of G_i is on the intersection of two red segments. We then combine G_i with G_0 analogously to step 4.
- If i is even, the left lowest vertex in the drawing of G_i is on the intersection of one green and one red segment. Then we combine G_i with G_0 analogously to step 2 and 3.

Analysis. For the given sequence of graphs G_i and their drawing D_i the number of segments is given by $12i + 13$ and the number of vertices by $28i + 29$. We therefore have a sv-ratio of

$$r_{\mathrm{sv}}(D_i) = \frac{12i + 13}{28i + 29}. \tag{4.3}$$

The limit of this sequence is

$$\lim_{i \to \infty} r_{\mathrm{sv}}(D_i) = \frac{12}{28} = \frac{3}{7}. \tag{4.4}$$

With Observation 2.3 the es-ratio of the drawing D_i is given by

$$r_{\mathrm{es}}(D_i) = \frac{(28i + 29) \cdot 2 - 3}{12i + 13} = \frac{56i + 55}{12i + 13}. \tag{4.5}$$

with its limit

$$\lim_{i \to \infty} r_{\mathrm{es}}(D_i) = \frac{56}{12} = \frac{14}{3} \approx 4.67. \tag{4.6}$$

Given this sequence, we therefore know the constant lower bound c is bound by $\frac{3}{7} \geq c > 0$. Note that the proposed method to define a sequence can be applied to any finite maximal outerplanar graph.

5 Conclusion and Outlook

Summary. In this thesis, we discussed the segment number of maximal outerplanar graphs from different perspectives. For its upper and existential lower bound the results and proofs of Dujmović et al.[DESW07] were presented. We then turned to maximal outerpaths and defined connections types. We prove a lower bound for the segment number of maximal outerpaths with no strong Type A connection. In the end we defined the lower bound constant and a graph sequence that provides a asymptotic upper bound for the lower bound constant.

Maximal Outerpaths: Upper Bound. Due to time constraints it was not possible to show that the provided lower bound in Theorem 3.16 is the segment number of the given graph. My approach would be to divide the graph into blocks, show that for each of them there is a crossing-free drawing that matches the lower bound and then combine them. One block type can be subgraphs of G such that all vertices in L are neither a center nor adjacent to one in G. The other blocks can be grouped by the connection types of Definition 3.10: e.g. a subgraph L_2 of G such that there are only weak Type A connections or L_3 that only allows strong Type B connections. To avoid crossings I would suggest some kind of monotony: Consider $x(v)$ as x-value of a vertex v in $\mathbb{R}^2$. We could then define a path $w_1, \ldots, w_h$ through each block such that for a drawing D the monotony of $x(w_i)$ proves it is crossing-free. As seen in Figure 3.6 (b) and (d) some connections let the drawing curve. Thus for each block and for the union of blocks one has to make sure that despite this curvature the monotony is still possible. The centres themselves can serve as link between the center-based blocks. Given that a center c can have two connections, it can be in L_3 as well as in L_2. A c-centred subgraph $S(c)$ can be drawn with an adjustment of the algorithm in Theorem 3.6. This algorithm still has two degrees of freedom: The position of the vertex $u_{(m+1)/2}$ and the position of the vertices on the segment between u_2 and $u_{(m+1)/2}$ have only few constraints. They can be used to assure the monotony. An open question is how to link center-based blocks with blocks that have no centres.

Maximal Outerpaths: Lower Bound. Another open task is to find a lower bound for the segment number of maximal outerplanar graphs that also allow strong Type A connections. In the inductive step from $G' = G \backslash v$ to G with $\deg(v) = 2$ the critical point was to show that G needs one more segment than G'. We made a difference between v being adjacent to a center c that has no strong Type A connection and v not being adjacent to a center. The proof for both cases can be traced down to the crucial fact that there was vertex w with degree 4 or 2. This limits the possibilities of how edges incident to w can share a segment through w. In one case w was adjacent to v, in the other adjacent to c. Hence a possible approach for the case that v is adjacent to a center c that has a strong Type A connection is the following: Let $c_1, \dots, c_m$ be the maximal sequence in G such that c_i has strong Type A connection with c_{i+1} and $c = c_1$. Then c_m is adjacent to a vertex with degree 2 or 4. We then again need to analyse the outer and inner edges of these centres.

Lower Bound Constant.

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Erklärung

Hiermit versichere ich die vorliegende Abschlussarbeit selbstständig verfasst zu haben, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt zu haben, und die Arbeit bisher oder gleichzeitig keiner anderen Prüfungsbehörde unter Erlangung eines akademischen Grades vorgelegt zu haben.

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