# Approximate Center Points in Dense Point Sets

K. Verbarg

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*Department of Computer Science, University of Würzburg*
*Am Hubland, D–97074 Würzburg, Germany*
*Tel.: +49 931 888 5058, Fax: +49 931 888 4600*
*Email: verbarg@informatik.uni-wuerzburg.de*

**Abstract**

The notion of *center points* is one way to generalize the median of a finite ordered point set to higher dimensions. A set $P$ of $n$ points in d–dimensional euclidean space $\mathbb{E}^d$ is *$\delta$–dense*, if the ratio of the largest to the smallest distance between any two points of $P$ is bounded by $\delta n^{1/d}$, with some constant $\delta$. We describe a simple, yet efficient algorithm to compute an approximate center point for a $\delta$–dense set $P$ in time $\mathcal{O}(dn)$. The quality of the approximation depends on $\delta$ and the dimension $d$. We also present an iteration method with a linear rate of convergence, which computes an improved $1/(4(d-1)(2\delta)^d)$–center point.

**Keywords:** Computational geometry, cluster analysis, center point, density.

# 1 Introduction

## Center Points

The notion of center points is one way to generalize the median of finite ordered point sets to higher dimensions. Let $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$ be a set of points in the $d$–dimensional euclidean space. A point $c \in \mathbb{R}^d$ is a *$\beta$–center* ($\beta \in ]0, \frac{1}{2}]$) of $P$, if every hyperplane $h$ containing $c$ $\beta$–splits $P$, i.e. there are at least $\lceil n\beta \rceil$ points of $P$ in each closed halfspace determined by $h$. Since both of these halfspaces contain the points $P \cap h$, we do not have to pose a general position assumption to handle extreme configurations[1]. The set of $1/(d+1)$–centers is simply called the *center* of $P$.

It is shown in [Ede87] that a center always exists. But unlike the one–dimensional median, in general one can not expect to find points of $P$ in the center. Imagine, e.g., all points of $P$ distributed on the perimeter of a circle, then no point of $P$ belongs to the center. The center of 5 points is displayed in Figure 1.

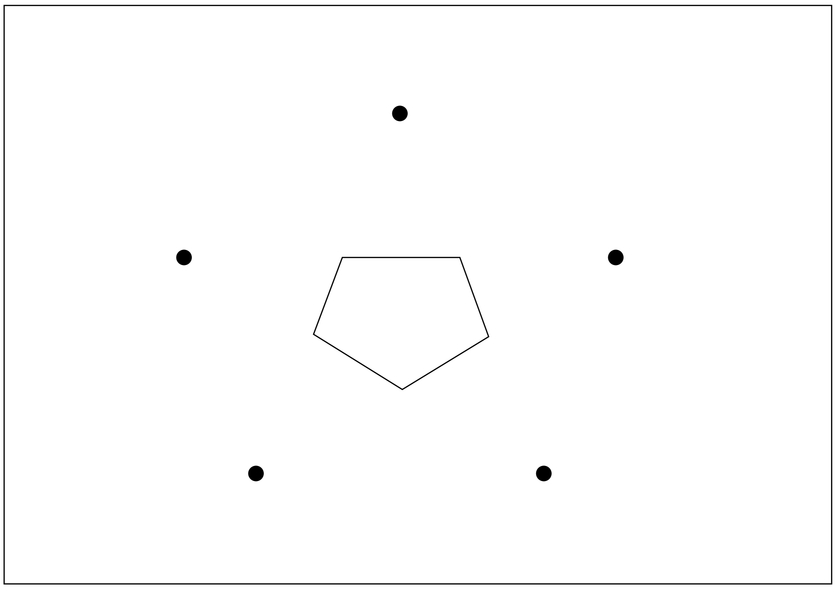

Figure 1: Center of 5 points distributed near a circle.

For the planar case ($d = 2$) a point in the center can be computed in linear time [JM94]. Using a randomized approach [Cea93], an approximate center can be computed via iterated radon points. A radon point of $P$ is a point in both convex hulls $\mathcal{CH}(P_1) \cap \mathcal{CH}(P_2)$, where $P_1$, $P_2$ form a partition of $P$. The algorithm computes in time subexponential in $d$ a $\beta$–center with $\beta \in \Omega(1/d^2)$ and error probability at most $1/n$.

## Dense Point Sets

To define density, the structure of a distance function is necessary. In this paper we always use $\mathbb{R}^d$ equipped with the euclidean metric $|.|$ and denote this euclidean space by $\mathbb{E}^d$.

A set of $n$ points $P$ in $\mathbb{E}^d$ is called *$\delta$–dense* ($\delta > 0$), if the ratio of the maximum to the minimum distance between two points in $P$ is at most $\delta n^{1/d}$. Arbitrary large dense sets can be constructed as the intersection of a grid with a disk. So it is equivalent to say that

[1]E.g. if all points of $P$ lie on the same line $l$, then $l$ 1–splits $P$.

the minimum distance is a constant and the maximum distance is bounded linearly by the diameter of $n$ as close as possible packed points (respecting the minimum distance). In $\mathbb{E}^2$, $\delta \approx 1.05$ is sufficient to admit arbitrary large $\delta$–dense sets.

Without loss of generality, we assume that the minimum distance is $m := 1$. So the diameter of the $\delta$–dense set $P$ is at most $M := \delta n^{1/d}$.

Dense point sets are described in [EVW94]. They are relevant in practice, since in many interesting applications, e.g. motion planning in factories, density of the environment can be assumed. Contrary to this, in theoretical worst–case analysis often large distance ratios are constructed [Zir92, p. 107], sometimes exponential in the number of points, to obtain extremal complexity of an algorithm.

# 2 Algorithm

This section describes a simple algorithm to compute a center point. The quality of this approximation will be estimated in the following sections.

The Algorithm $A$ is simply to compute the *center of mass* or weighted average, i.e.:

$$A(P) := \frac{1}{n} \sum_{i=1}^{n} p_i. \tag{1}$$

The time effort to compute (1) is clearly $\mathcal{O}(dn)$.

If not all points of $P$ are on the same line, then $A(P)$ is in the interior of the convex hull of $P$, and it is therefore at least a $1/n$–center. But we will give a better estimation of the quality in the sequel. If we wanted to show that Algorithm A only computes a $1/n$–center in the worst–case, we would e.g. construct a sequence with an accumulation point or place one point of $P$ arbitrarily far away. This is not possible in dense point sets.

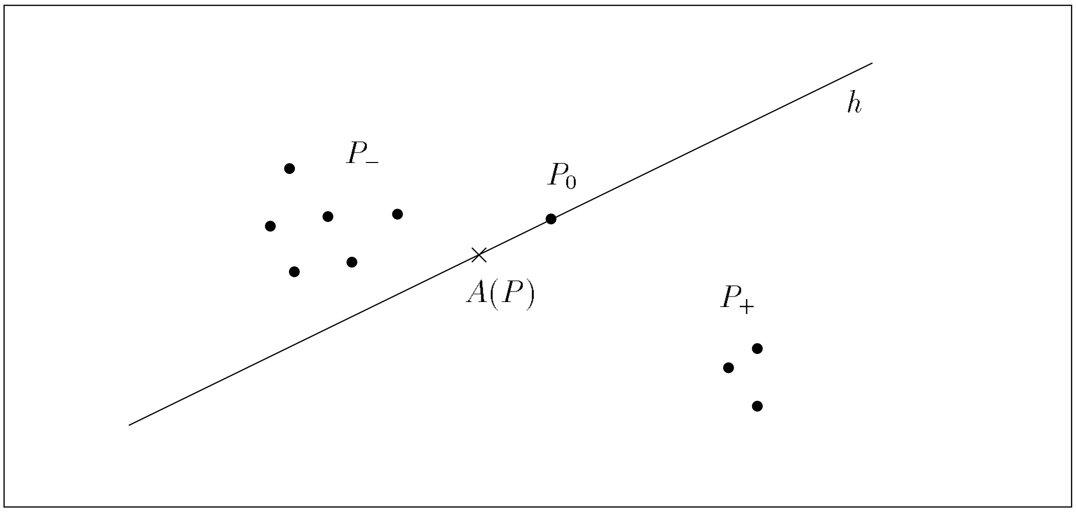


Figure 2: $h$ partitions $P$ into $P_-$, $P_0$ and $P_+$.

Let $h$ be any hyperplane containing $A(P)$. This hyperplane induces three subsets $P_+$, $P_-$ and $P_0$ which contain the points of $P$ in each of the two open halfspaces determined by $h$ and the points on $h$ itself, respectively (see Figure 2).

We now want to derive an estimation of the minimal cardinality of $P_+$ and $P_-$ for any $h$ containing $A(P)$. Assume that w.l.o.g. $A(P) = 0 \in \mathbb{E}^d$, then from (1) it follows that:

$$-\sum_{p \in P_-} p = \sum_{p \in P_+} p + \sum_{p \in P_0} p \tag{2}$$

In other words, without concerning the points in $P_0$, we can say that the weight of the points on either side is the same. We therefore have to estimate the relation between the weight and the cardinality of points on one side of $h$. This will be done in the next two sections.

For worst–case analysis we may assume that $P_0$ is empty: If it is not empty, then a small shift of $h$ gives a similar[2] configuration where the number of points in $P_-$ or $P_+$ is only decreasing. This results from the definition of the splitting ratio in Section 1 where points on $P_0$ are counted on both sides of the hyperplane.

# 3 Analysis in One Dimension

We first want to analyze the case $d = 1$. Then w.l.o.g. $m := \min_{x,y \in P} |x - y| = 1$ and so the diameter of $P$ is at most $M := \delta n \geq \max_{x,y \in P} |x - y|$.

From (2) we can see that:

$$\sum_{p \in P_-} |p| = \sum_{p \in P_+} |p|. \tag{3}$$

The worst–case for the partitioning ratio of $h$ is displayed in Figure 3. We have to place as many points as possible on one side of $A(P)$ subject to the constraints that the weights of points on both sides are the same and the diameter is at most $M$.

To achieve a maximum number of points with a minimum weight in $P_-$ we have to place them at $0, -1, -2, \ldots$, because the minimum distance $m$ is 1. We count the point at 0 only as a point in $P_-$ (compare to the argument above that $P_0$ is empty). In the sequel we will use the abbreviation $n_- := |P_-|$ and $n_+ := |P_+|$.

Remember that $A(P) = 0$ and $n = n_- + n_+$. So:

$$\begin{aligned}\sum_{p \in P_-} |p| &\geq 0 + 1 + 2 + \ldots + (n_- - 1) = \frac{n_-(n_- - 1)}{2} \\ &= \frac{1}{2}(n - n_+)(n - n_+ - 1) = \frac{1}{2}(n_+^2 + n_+(-2n + 1) + n^2 - n).\end{aligned} \tag{4}$$

[2] Only one point of $P$ has to be moved by arbitrarily small $\epsilon$.

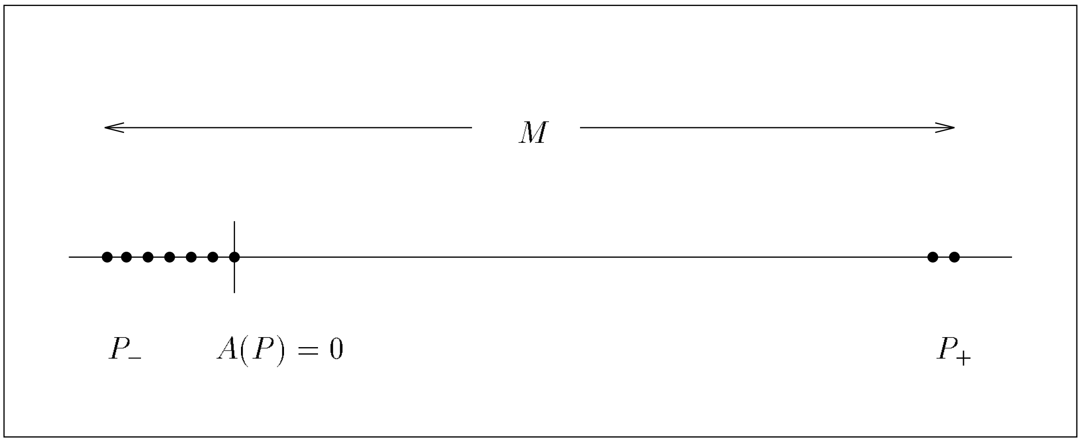


Figure 3: Worst–case configuration of points.

As we can see above, the diameter of $P_-$ is $n_- - 1$. So on the other side, since the diameter of $P$ is $M$, the farthest point can be at most at $M - n_- + 1$. With the same argument as before we have:

$$\begin{aligned}\sum_{p \in P_+} |p| &\leq (M - n_- + 1) + (M - n_-) + (M - n_- - 1) + \ldots + (M - n_- - n_+ + 2) \\ &= \frac{1}{2} n_+ (2(M - n_- - n_+ + 2) + n_+ - 1) \\ &= \frac{1}{2} n_+ (2M - 2n_- - n_+ + 3) \\ &= \frac{1}{2} n_+ (2M - 2n + n_+ + 3). \end{aligned} \tag{5}$$

If we put (3), (4) and (5) together, this yields:

$$\begin{aligned} \frac{1}{2}(n_+^2 + n_+(-2n+1) + n^2 - n) \leq \sum_{p \in P_+} |p| &= \sum_{p \in P_-} |p| \leq \frac{1}{2} n_+ (2M - 2n + n_+ + 3) \\ \iff n_+ + n^2 - n &\leq n_+(2M + 3) \\ \iff n_+ &\geq \frac{n^2 - n}{2(M+1)} = \frac{n-1}{2(\delta + 1/n)}. \end{aligned} \tag{6}$$

For large $n$ this tends to the following:

$$\lim_{n \to \infty} \frac{n_+}{n} \geq \frac{1}{2\delta}.$$

So the quality of the center of mass as a median can be expressed directly in terms of $\delta$.

We summarize the results of this section in the following

**Theorem 1** *For a $\delta$–dense point set $P = \{p_1, \ldots, p_n\} \subset \mathbb{E}$ the center point $A(P)$ is a $\frac{n-1}{2(\delta n+1)}$–center that can be computed in $\mathcal{O}(n)$ time.*

# 4 Extension to Higher Dimensions

We will now extend the previous approach to point sets in $\mathbb{E}^d$. Again, we assume $M = \delta n^{1/d}$, $m = 1$ and there are no points on $h$.

Consider any hyperplane $h$ containing $A(P)$ as shown in Figure 4. Since $m = 1$, every point $p \in P$ occupies at least a hypersphere $D_p$ with radius $1/2$ around $p$. And because of the bounded diameter, all points are contained in a hypercube with side length $M$.

The projection of sum (2) onto a line normal to $h$ gives:

$$\sum_{p \in P_-} d(h,p) \;=\; \sum_{p \in P_+} d(h,p), \tag{7}$$

where $d(h,p)$ is the distance of $p$ to the hyperplane $h$ induced by the euclidean metric[3].

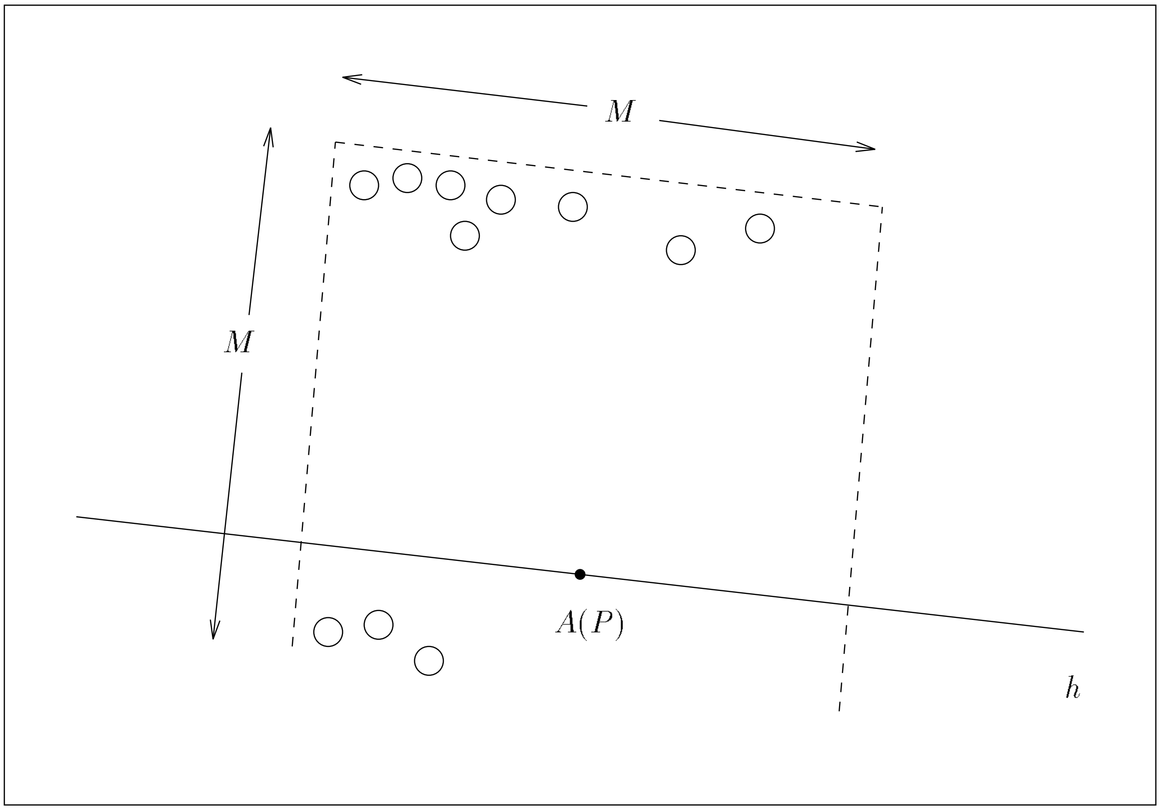


Figure 4: Configuration of points on one side of the hyperplane.

We inscribe a hypercube $C_p$ in $D_p$ with diagonal 1. Therefore the edge length of each $C_p$ is $1/\sqrt{d}$. We want to estimate the number of points for a given weight similar to the one–dimensional worst–case example. Since we place cubes $C_p$ instead of the larger hyperspheres $D_p$, the placement will only be more tight. But placing cubes is easy: The optimal placement of the cubes $C_p$ in the feasible cube $M^d$ is admitted in levels, cf. Figure 5.

[3] $d(h,p) = |(p - A(P)) \cdot \vec{n_h}|$, with $|\vec{n_h}| = 1$ the normal vector of $h$.

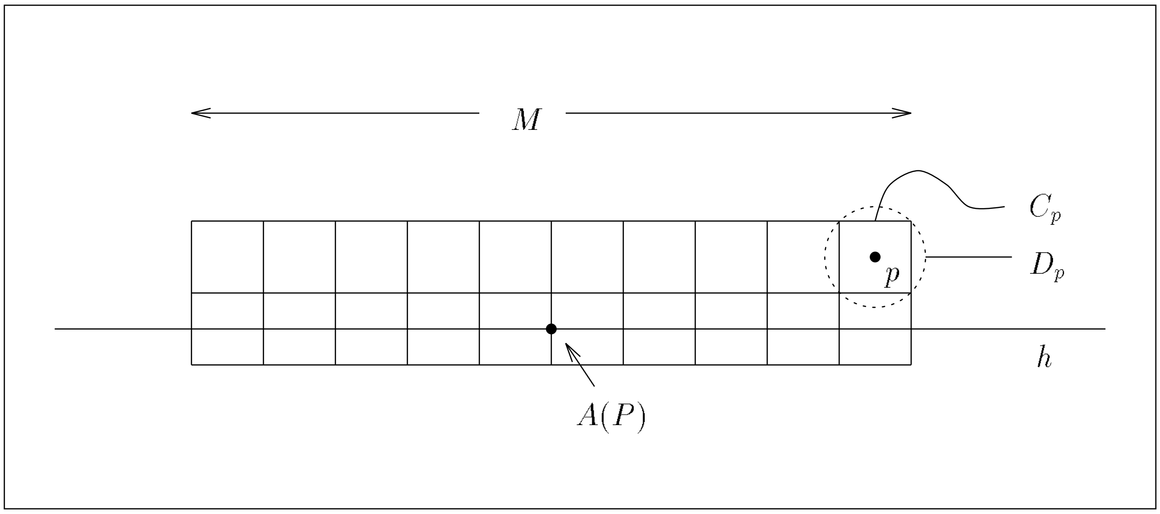


Figure 5: Placement of the cubes $C_p$ in levels.

Thus the number of cubes $K$ in one of these levels is at most:

$$K \;:=\; \frac{M^{d-1}}{\left(\frac{1}{\sqrt{d}}\right)^{d-1}} \;=\; (M\sqrt{d})^{d-1}.$$

And the minimum number of levels for $n_- = |P_-|$ points is then at least[4]:

$$L_- \;:=\; \frac{n_-}{K}$$

If we replace $n_-$ by $L_-$, and $n$ by $N$, respectively, we get a similar equation as in (4) with an additional factor of $K/\sqrt{d}$[5]:

$$\begin{aligned} \sum_{p \in P_-} |p| \;&\geq\; 0 \cdot K + \frac{1}{\sqrt{d}}K + \frac{2}{\sqrt{d}}K + \ldots + \frac{(L_- - 1)}{\sqrt{d}}K \\ &=\; \frac{K}{\sqrt{d}}\Big(0 + 1 + 2 + \ldots + (L_- - 1)\Big) \end{aligned} \tag{8}$$

In fact, there may be less than $K$ points per level. But this inequality is still sufficient, because we estimated not more than $n_-$ points in $P_-$ (see definition of $L_-$) yet with perhaps a too small weight: Using the factor $K$ we placed too many points in one level, so that some points are counted in a smaller level and therefore with less weight.

With $N := n/K$ we obtain:

$$L_+ \;=\; \frac{n_+}{K} \;=\; \frac{n - n_-}{K} \;=\; \frac{n}{K} - L_- \;=\; N - L_-.$$

[4]Remind that $L_-$ is not an integer: The number of levels is larger than $L_-$.
[5]$K$ objects per level and a level has got width $\sqrt{d}$.

According to (5) we get the estimation:

$$\sum_{p \in P_+} |p| \;\leq\; \frac{K}{\sqrt{d}}\Big((M - L_- + 1) + (M - L_-) + \ldots + (M - L_- - L_+ + 2)\Big). \tag{9}$$

With the same argument as above, although there may be more than $L_+$ levels, we estimated all $n_+$ points in $P_+$ yet with perhaps a too large weight.

Together, with the same transformations like in (6), we have:

$$n_+ \cdot \frac{1}{K} \;=\; L_+ \;\geq\; \frac{N^2 - N}{2(M+1)}.$$

And therefore:

$$n_+ \;\geq\; \frac{n^2 - nK}{2K(M+1)} \;=\; \frac{n^2 - nM^{d-1}\lambda}{2\lambda(M^d + 1)} \;=\; \frac{n - n^{\frac{d-1}{d}}\delta^{d-1}\lambda}{2\lambda(\delta^d + \frac{1}{n})},$$

if we set $\lambda := d^{(d-1)/2}$. $\lambda$ is a penalty factor dependent on the dimension.

For large $n$ this tends to:

$$\lim_{n \to \infty} \frac{n_+}{n} \;\geq\; \frac{1}{2\delta^d\lambda}.$$

**Theorem 2** *For a $\delta$–dense point set $P = \{p_1, \ldots, p_n\} \subset \mathbb{E}^d$ the centerpoint $A(P)$ is a $\frac{n-K}{2K(M+1)}$–center that can be computed in $\mathcal{O}(dn)$ time.*

# 5 Improvement of the Algorithm

The problem of Algorithm $A$ is that for large values of $\delta$, points which are far away from the center are counted more than nearer ones. Since we would like count the cardinality of points, we will add a weight factor to each point according to its distance to the center. The more far away a point is the smaller this factor will be. But since the exact center is not known, we have to start with the first approximation computed with algorithm $A$ as the center.

Therefore we get the following iteration:

$$A_0(P) := A(P), \quad A_{i+1}(P) := \phi(A_i(P)) \tag{10}$$

$$\phi(x) := \frac{\sum_{j=1}^{n} w_j p_j}{\sum_{j=1}^{n} w_j}, \quad w_j := \begin{cases} 1, & \text{if } p_j = x \\ 1/|p_j - x|, & \text{if } p_j \neq x \end{cases}, \quad x \in \mathbb{E}^d. \tag{11}$$

This iteration is related to the problem of minimizing the function $f$, defined by:

$$f(x) = \sum_{j=1}^{n} |x - p_j|. \tag{12}$$

The problem of minimizing (12) occurs in facility layout problems, cf. [FMW92, page 199] and is often called Fermat–Weber problem. The function $f$ is convex, implying that any local minimum of $f$ is also a global minimum. Moreover, it can be shown easily that $f$ is strictly convex, provided that the $p_j$ are not collinear. A local minimum $x^*$ of $f$ must satisfy the necessary first order conditions:

$$\operatorname{grad} f(x^*) = \sum_{j=1}^{n} \frac{x^* - p_j}{|x^* - p_j|} = 0. \tag{13}$$

Unfortunately, $f$ is not differentiable for the finite number of points $x \in P$. To this end, we assume[6] that $x \notin P$. Then, (13) is equivalent to:

$$\begin{aligned} \sum_{j=1}^{n} \frac{x}{|x-p_j|} &= \sum_{j=1}^{n} \frac{p_j}{|x-p_j|} \\ \Leftrightarrow \qquad x &= \sum_{j=1}^{n} \frac{p_j}{|x-p_j|} \cdot \frac{1}{\sum_{j=1}^{n} \frac{1}{|x-p_j|}} \end{aligned} \tag{14}$$

For $x \notin P$ this leads exactly to iteration (11), which is a steepest decent algorithm to minimize (12). It is known as Weiszfeld algorithm and converges linearly to the minimum [Kat74, BL92]. The task to accelerate the speed of convergence received also much attention, cf. [Dre92].

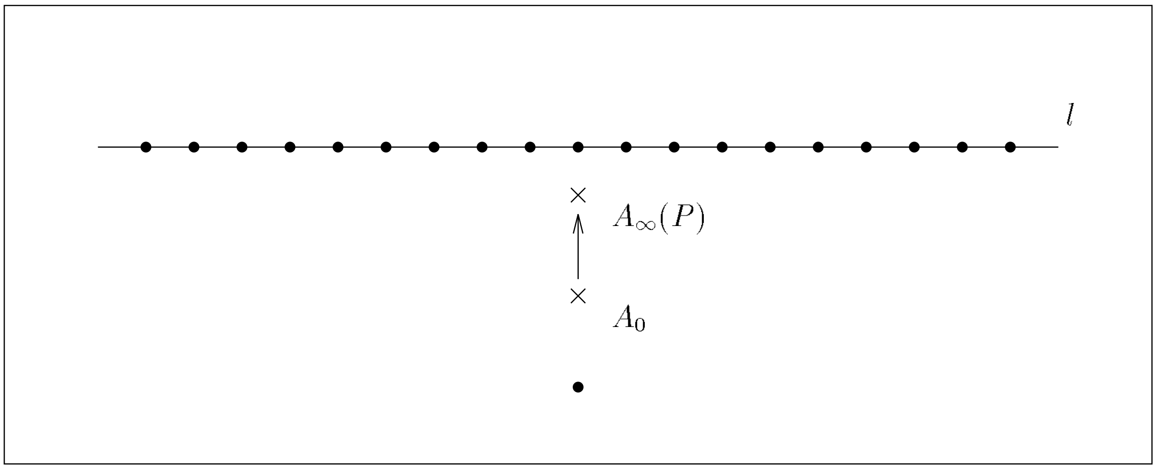


Figure 6: Sequence $A_i$ does not converge to the center.

We now want to investigate the quality of this minimum as a center point. The minimum of (12) is a fixed point of $\phi$. We assume w.l.o.g. that for a fixed point $\phi(x^*) = x^*$ we have $x^* = 0$ and so $\phi(0) = 0$. Then the iteration (11) yields:

$$\sum_{j=1}^{n} \frac{p_j}{|p_j|} = 0. \tag{15}$$

[6] We may use a small perturbation of the distance function.

In one dimension $d = 1$ this is the desired median, since $\sum_{p \in P_+} 1 - \sum_{p \in P_-} 1 = 0$ and therefore $|P_+| = |P_-|$.

But unlike the one–dimensional case, for $d > 1$ sequence $A_i$ does not always tend to the exact center: Imagine all points of $P$ be distributed on a line $l$ except one point placed properly outside $l$ like in Figure 6. Then the exact center is on $l$, but $A_i(P)$ is not on $l$ for any $i$. Like before, we have to postulate density to obtain an estimation of the quality of the point of convergence.

Again we assume that $h$ is any hyperplane containing the fixed point $x^* = 0$. $h$ is defined by $c^{\mathrm{T}} x = 0$, $|c| = 1$. $h$ divides $P$ in $P_-$, $P_0$ and $P_+$ respectively. W.l.o.g. we may assume that $P_0 = \emptyset$. Density provides $m = 1$ and $M = \delta n^{\frac{1}{d}}$ like in the previous section. This situation is displayed in Figure 7.

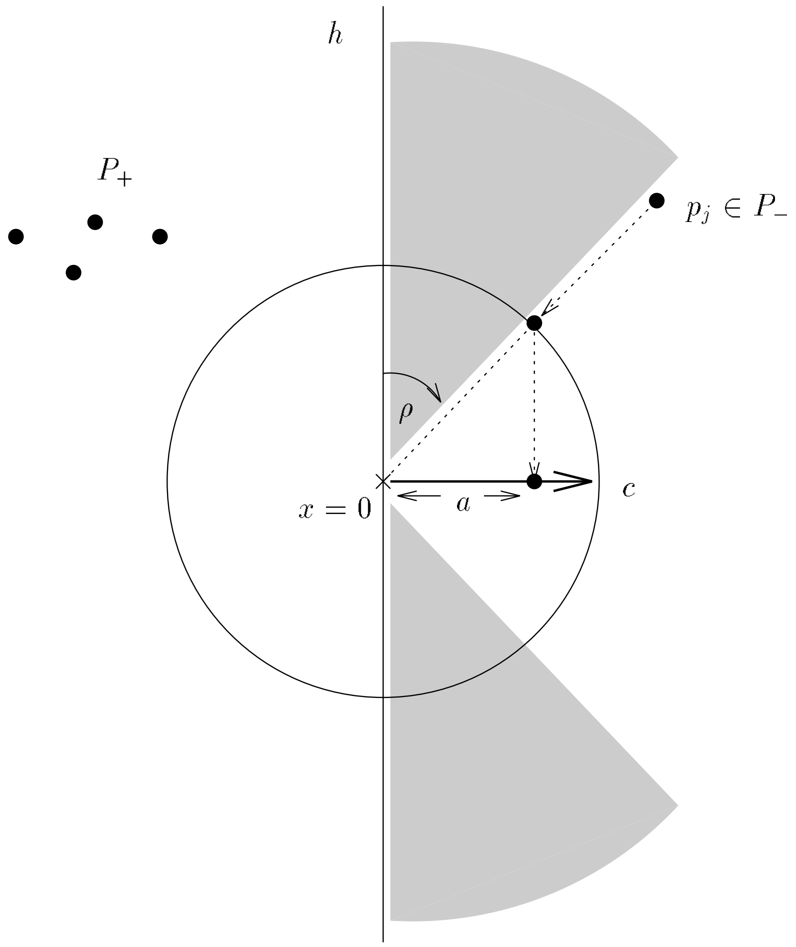


Figure 7: Estimation of sequence $A_i$.

Equality (15) leads to:

$$\left|\sum_{p\in P_-} c^{\mathrm{T}} \frac{p_j}{|p_j|}\right| = \left|\sum_{p\in P_+} c^{\mathrm{T}} \frac{p_j}{|p_j|}\right|. \tag{16}$$

This can be read as the projection of each point onto the unit sphere, followed by a projection onto the line determined by $c$.

We will now estimate the maximum number of points of $P_-$ with a weight (related to (16)) of at most $0 \le a \le 1$. These points are part of the shaded region in Figure 7 which is a part of the ball with radius $M$, described by the parameter angle $\sin\rho = a$. The volume of this part relatively to the volume of the whole ball (by integration over multi–dimensional polar coordinates) is given by:

$$\frac{\int\limits_{\pi/2-\rho}^{\pi/2} \sin^{d-2}\varphi d\varphi}{\int\limits_{0}^{\pi} \sin^{d-2}\varphi d\varphi} \le \frac{\int\limits_{\pi/2-\rho}^{\pi/2} 1 d\varphi}{2\int\limits_{0}^{\pi/2} (\frac{2}{\pi}\varphi)^{d-2} d\varphi} = \frac{\rho}{2\left(\frac{2}{\pi}\right)^{d-2}\frac{1}{d-1}\left(\frac{\pi}{2}\right)^{d-1}} = \frac{(d-1)\rho}{\pi}.$$

With this volume we can estimate $|P_-^a|$:

$$\begin{aligned} |P_-^a| \le \frac{\text{volume of shaded region}}{\text{volume of } D_p} &\le \frac{(d-1)\frac{1}{\pi}(\arcsin a)\omega_d M^d}{\omega_d(\frac{1}{2})^d} \\ \Leftrightarrow \quad a \ge \sin\frac{|P_-^a|\pi}{(d-1)(2\delta)^d n} &\ge \frac{2|P_-^a|}{(d-1)(2\delta)^d n}. \end{aligned} \tag{17}$$

Due to the fact that $\delta \ge 1$ and $d \ge 1$, the argument of the sin–function above is in the range of $[0, \frac{\pi}{2}[$ and therefore $\sin x \ge \frac{2}{\pi}x$.

Since we have a linear relation between $a$ and $|P_-^a|$ in (17), we may assume for simplicity that the weight of points in $P_-$ is at least $\frac{1}{2}a$ on the average, with $a$ chosen such that $|P_-| = |P_-^a|$ in the sequel. Then with equation (16) and (17) we conclude:

$$\begin{aligned} |P_+| \ge \left|\sum_{p\in P_+} c^{\mathrm{T}} \frac{p_j}{|p_j|}\right| = \left|\sum_{p\in P_-} c^{\mathrm{T}} \frac{p_j}{|p_j|}\right| &\ge |P_-| \cdot \frac{1}{2}a \ge \frac{1}{2}|P_-|\frac{2|P_-|}{(d-1)(2\delta)^d n}. \\ \Leftrightarrow \quad |P_+| &\ge \frac{|P_-|^2}{(d-1)(2\delta)^d n}. \end{aligned}$$

$P_-$ is the larger partition in our worst–case example. With $\frac{|P_-|}{n} \ge \frac{1}{2}$ we finally have:

$$\frac{|P_+|}{n} \ge \frac{1}{4(d-1)(2\delta)^d}. \tag{18}$$

We have established the following improved bound for the iteration $A_i(P)$:

**Theorem 3** *For a $\delta$–dense point set $P = \{p_1, \dots, p_n\} \subset \mathbb{E}^d$ the sequence $A_i(P)$ converges to a $1/(4(d-1)(2\delta)^d)$–center with a linear rate of convergence.*

In practice, the algorithm will be aborted after a finite number of iterations $k$. So the question arises, whether the quality of the computed center $A_k$ which is sufficiently near the fixed point of the sequence $A_i$, can be estimated too. Let $h_k$ be the hyperplane containing $A_k$ establishing the worst partitioning factor. Let $h_\infty$ be the parallel of $h_k$ containing the fixed point $A_\infty$ as displayed in Figure 8. Now, the splitting factor of $h_\infty$

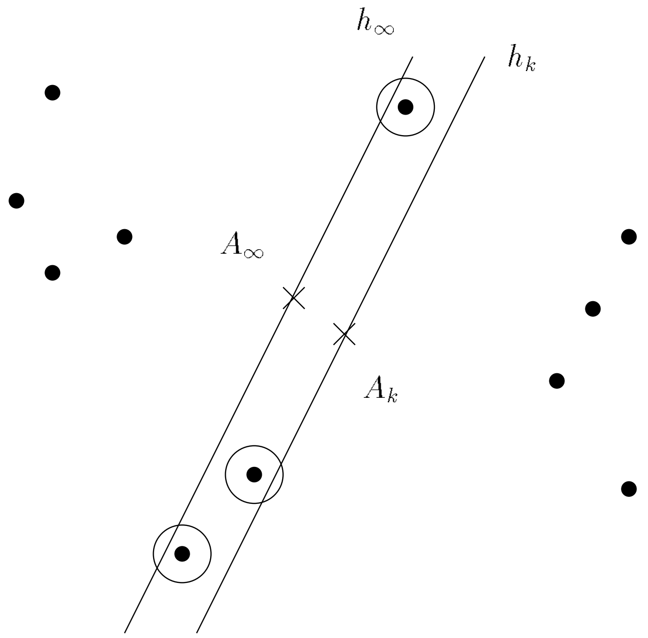


Figure 8: Estimation of the quality of points near the fixed point of the iteration.

can be estimated by Theorem 3. While sweeping the volume from $h_\infty$ to $h_k$ due to density only a restricted number of points of $P$ can be passed. Therefore also the partitioning factor of $h_k$ or any other hyperplane containing $A_k$ can be easily estimated. We omit the exact formulas here.

# 6 Conclusion

The concept of *density* is relatively new, although dense sets appeared already earlier. As we have seen, it is a strong but often realistic supposition. It enables to investigate algorithms with otherwise bad worst–case behavior due to artificially constructed extremal point sets. Moreover the analysis under a density restriction gives some insight to the behavior of an algorithm, if density is not fulfilled. Other questions for finite point sets may be interesting (or sometimes trivial) as a question for dense point sets. The concept of density is related to the concept of fatness [OS94] and uniformly distributed sets. The requirements on density can be weakened, because e.g. few points with small distance do not disturb the obtained results.

Applying this concept of density, we are able to improve the approximations for *center points* in arbitrary dimensions. We have examined experimentally the quality and performance of our center point algorithms [OH95] to others and obtained reasonable results. The implementation of the presented algorithms is very easy, whereas the efficient implementation of radon points in higher dimensions is not obvious.

Center points can be applied in cluster algorithms. We especially examine the benefits of center points for the *monotonous bisector tree* [NVZ93]. In this context we use them to retrieve good initial points for the spatial index and to repeatedly compute a partitioning with guaranteed balancing factor.

# 7 Acknowledgment

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