Order in the Underground – How to Automate the Drawing of Metro Maps

Martin Nöllenburg and *Alexander Wolff*

Department of Computer Science

Karlsruhe University

GD 2005

Outline

1. Modeling the Metro Map Problem
   - What is a metro map?
   - Hard and soft constraints
2. Our Solution
   - Mixed-integer programming formulation
   - Experiments
   - Labeling
3. NP-Hardness
   - Rectilinear vs. octilinear drawing
   - Reduction from planar 3-SAT

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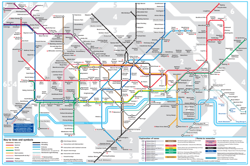
What is a Metro Map?



- schematic diagram for public transport
- visualizes lines and stations
- goal: ease navigation for passengers
  - "How do I get from A to B?"
  - "Where to get off and change trains?"
- distorts geometry and scale
- improves readability
- compromise between
  schematic road map $\leftrightarrow$ abstract graph

More Formally

The Metro Map Problem

Given: planar embedded graph $G = (V, E)$, $V \subset \mathbb{R}^2$,
line cover $\mathcal{L}$ of paths or cycles in G (the metro lines),

Goal: draw G and $\mathcal{L}$ nicely.

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- What is a nice drawing?
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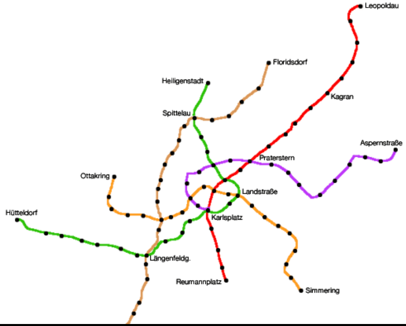
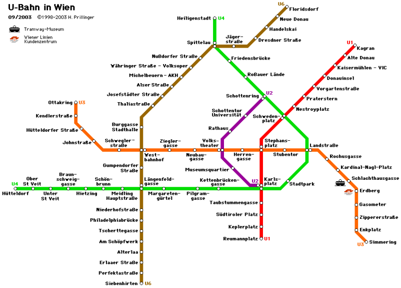
  - *hard* constraints – must be fulfilled,
  - *soft* constraints – should hold as tightly as possible.

Hard Constraints

(H1) preserve embedding of G





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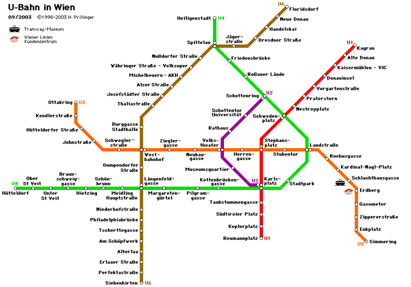

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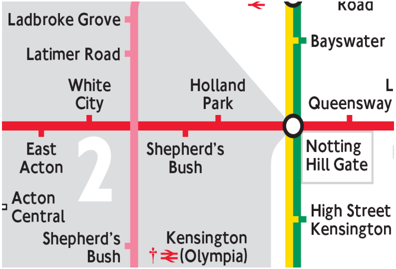
- (H4) keep vertices d_{min} away from non-incident edges

Soft Constraints

(S1) draw metro lines with few bends



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(S1) draw metro lines with few bends

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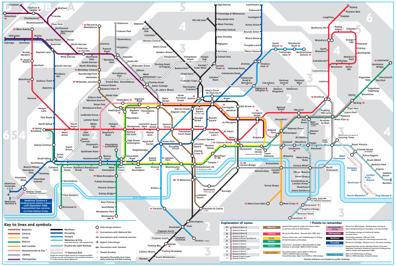
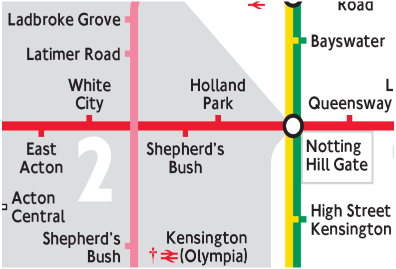


Soft Constraints

- (S1) draw metro lines with few bends
- (S2) keep total edge length small
- (S3) draw each octilinear edge similar to its geographical orientation: keep its relative position



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Theorem (Nöllenburg & Wolff GD'05)

The problem MetroMapLayout can be formulated as a MIP s.th.

linear constraints $\rightarrow$ hard constraints,
objective function $\rightarrow$ soft constraints.

Example: Octilinearity and Relative Position

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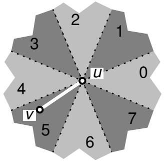

Sectors

– for each vtx. u partition plane into sectors 0–7

- here: $\mathrm{sec}(u, v) = 5$ (input)

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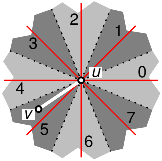

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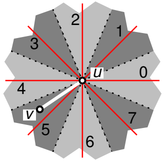

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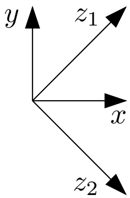


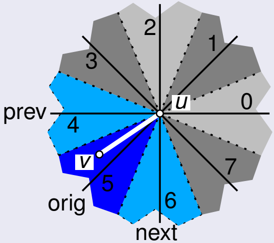
Coordinates

assign z_1- and z_2-coordinates to each vertex v:

- $z_1(v) = x(v) + y(v)$
- $z_2(v) = x(v) - y(v)$

Example: Octilinearity and Relative Position

Goal



Draw edge uv

- with minimum length ℓ_{uv}
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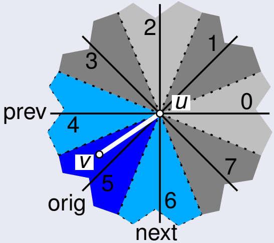

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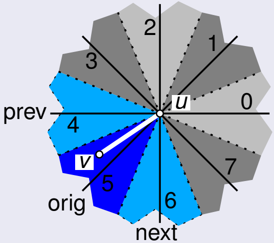

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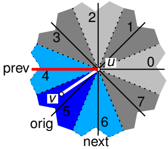
How to model this using linear constraints?

Binary Variables

$$\alpha_{\text{prev}}(u, v) + \alpha_{\text{orig}}(u, v) + \alpha_{\text{next}}(u, v) = 1$$

Example: Octilinearity and Relative Position



Previous Sector

$$\begin{aligned} y(u) - y(v) &\leq M(1 - \alpha_{\text{prev}}(u, v)) \\ -y(u) + y(v) &\leq M(1 - \alpha_{\text{prev}}(u, v)) \\ x(u) - x(v) &\geq -M(1 - \alpha_{\text{prev}}(u, v)) + \ell_{uv} \end{aligned}$$

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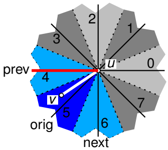

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How does this work?

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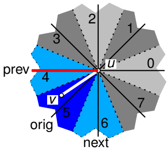

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Case 1: $\alpha_{\text{prev}}(u, v) = 0$

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y(u) - y(v) &\leq M \\
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x(u) - x(v) &\geq \ell_{uv} - M
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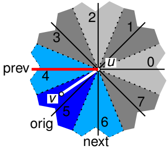

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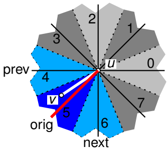

Original Sector

$$\begin{aligned} z_2(u) - z_2(v) &\leq M(1 - \alpha_{\mathrm{orig}}(u,v)) \\ -z_2(u) + z_2(v) &\leq M(1 - \alpha_{\mathrm{orig}}(u,v)) \\ z_1(u) - z_1(v) &\geq -M(1 - \alpha_{\mathrm{orig}}(u,v)) + 2\ell_{uv} \end{aligned}$$

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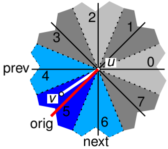

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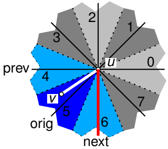

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- The above constraints enforce
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Results – Vienna

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Input	$\|V\|$	$\|E\|$	lines
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*) 29 seconds w/o proof of opt.

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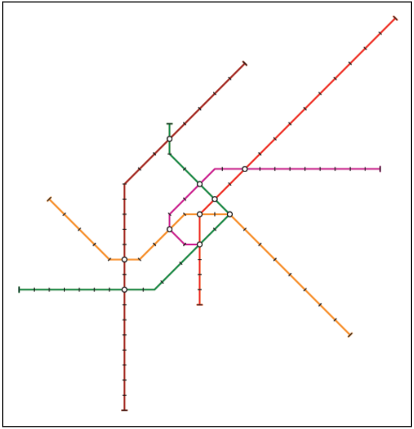
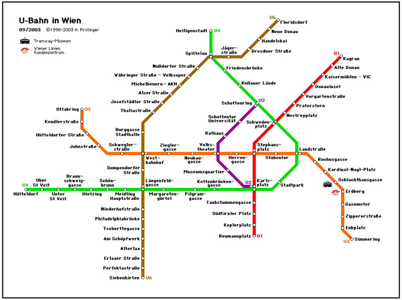


Results – Vienna

Official map



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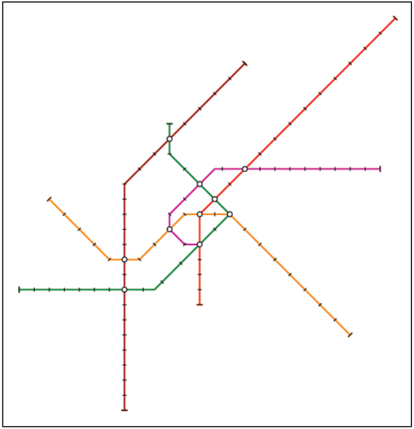
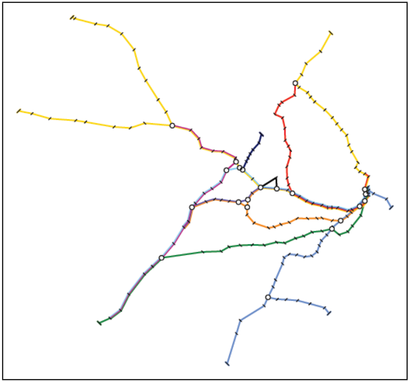


Results – Sydney

Input



Input	$\|V\|$	$\|E\|$	lines
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Results – Sydney

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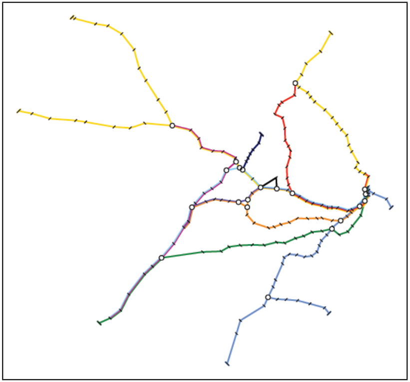

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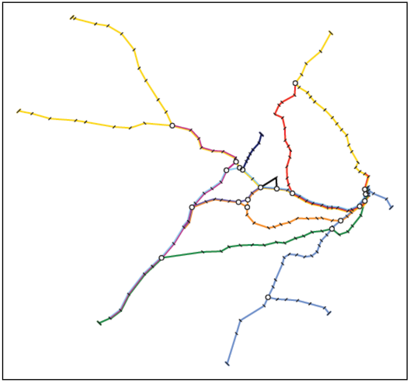

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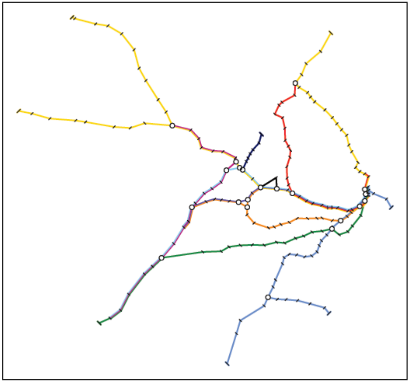

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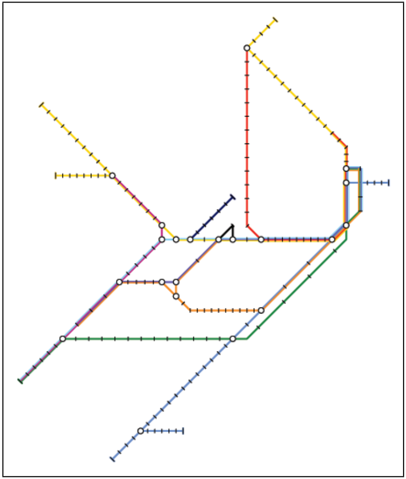

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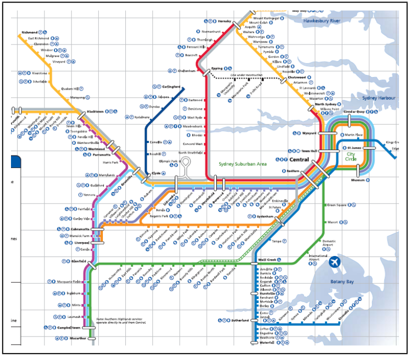

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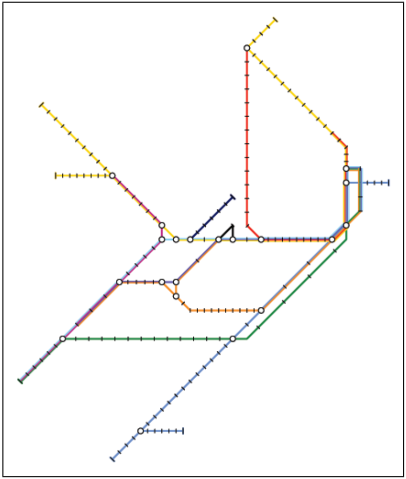
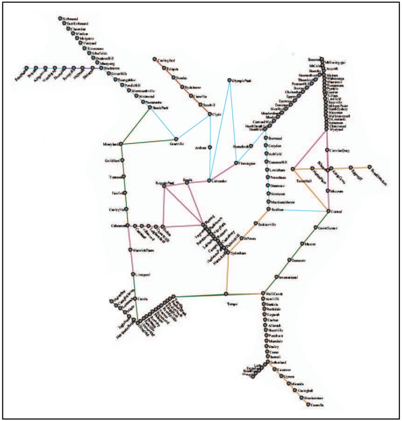
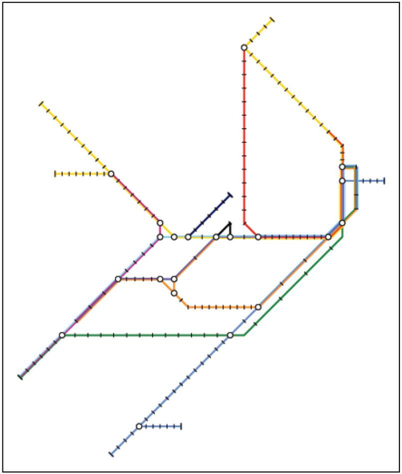


Sydney: Related Work

[Hong et al.: GD04] 7.6 sec.

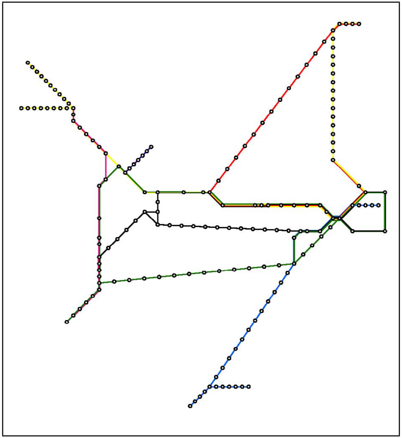


Output (22 min.)



Sydney: Related Work

[Stott & Rodgers: IV04] reduced: 4 min.



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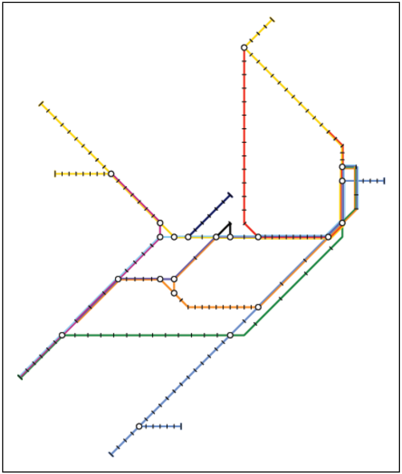
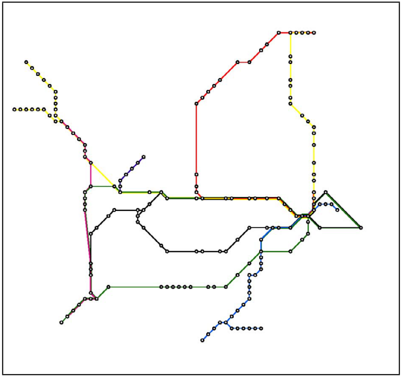


Sydney: Related Work

[Stott & Rodgers: IV04] normal: 28 min.



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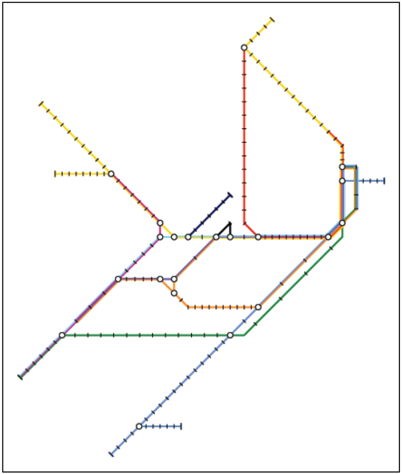
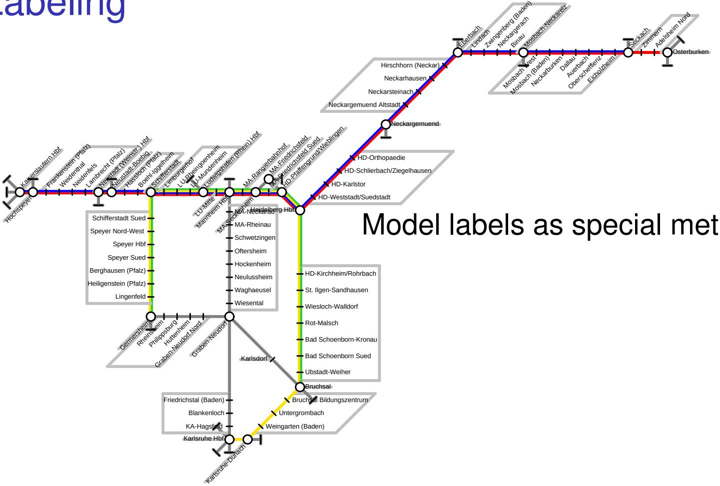
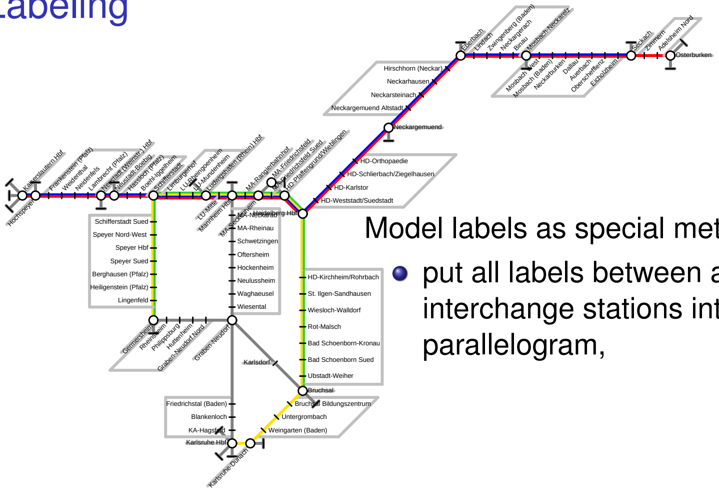


Labeling



Model labels as special metro lines:

Labeling



Model labels as special metro lines:

- put all labels between a pair of interchange stations into one parallelogram,

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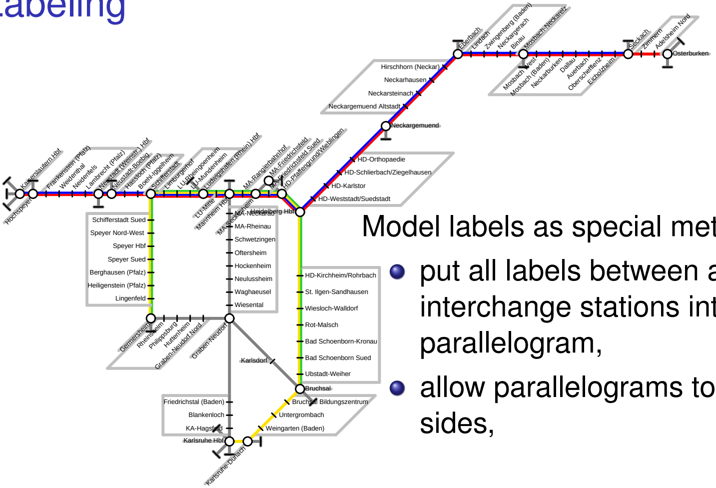

Model labels as special metro lines:

- put all labels between a pair of interchange stations into one parallelogram,
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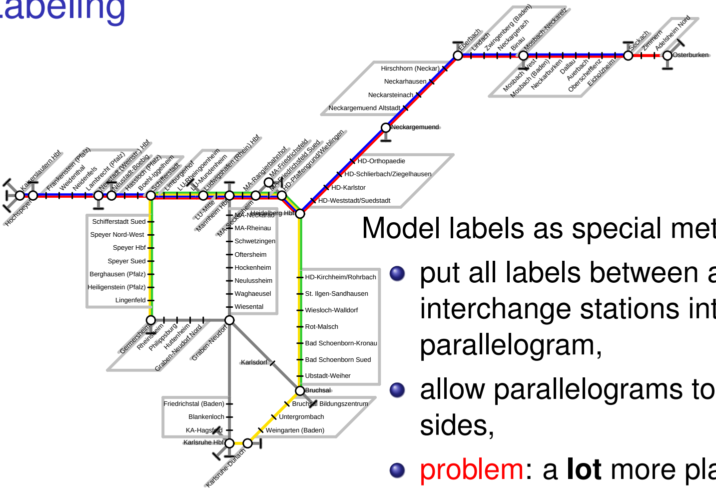
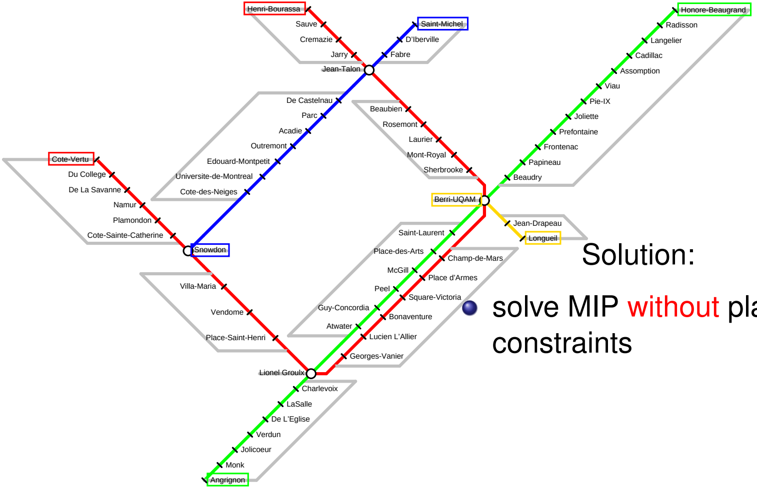


Model labels as special metro lines:

- put all labels between a pair of interchange stations into one parallelogram,
- allow parallelograms to change sides,
- problem: a **lot** more planarity constraints :-(

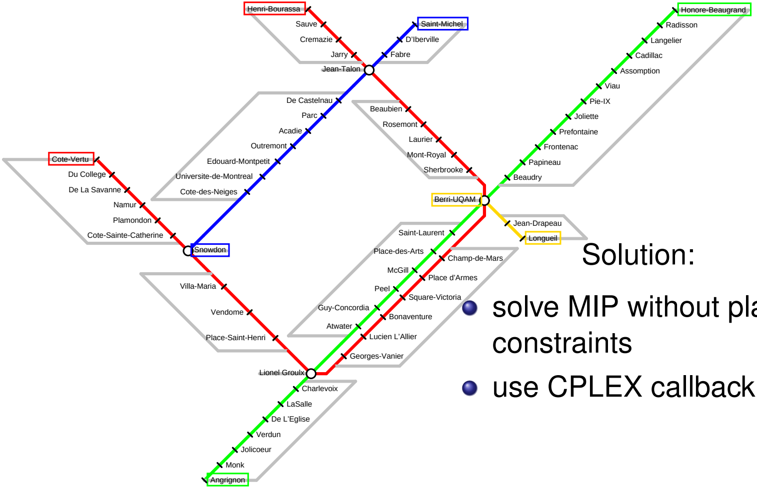
Labeling



Solution:

- solve MIP without planarity constraints

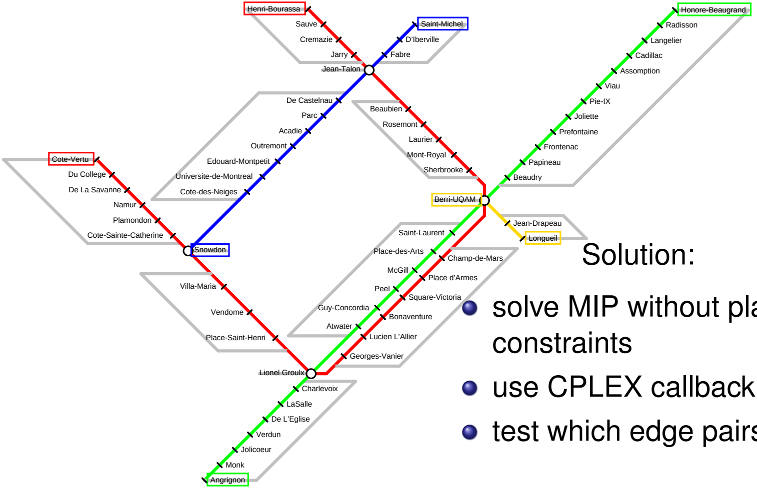
Labeling



Solution:

- solve MIP without planarity constraints
- use CPLEX callback fct.

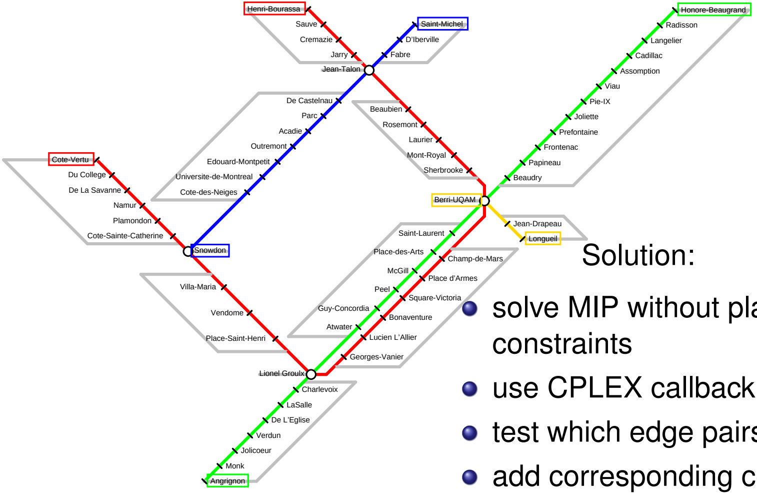
Labeling



Solution:

- solve MIP without planarity constraints
- use CPLEX callback fct.
- test which edge pairs cross

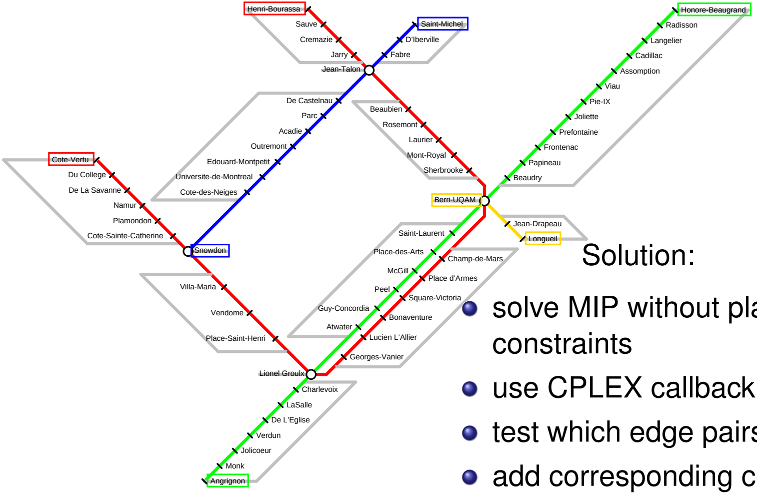
Labeling



Solution:

- solve MIP without planarity constraints
- use CPLEX callback fct.
- test which edge pairs cross
- add corresponding constr.

Labeling



Solution:

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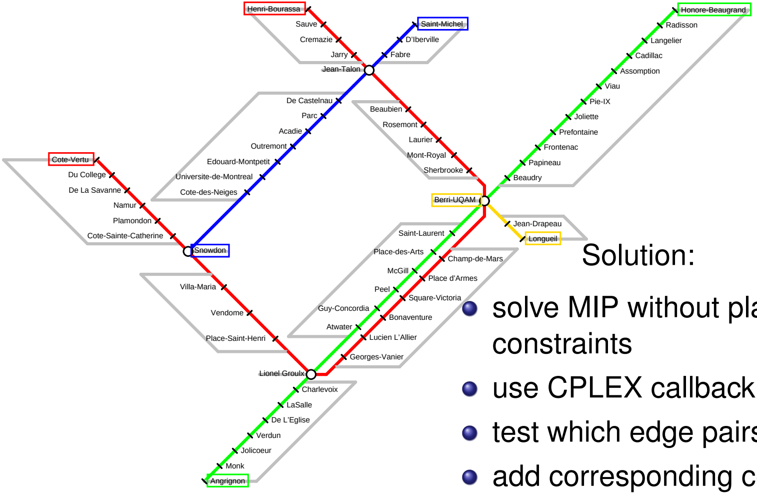
- use CPLEX callback fct.
- test which edge pairs cross
- add corresponding constr.
- continue to solve MIP

Labeling



Montréal: 17 min.

Solution:

- solve MIP without planarity constraints
- use CPLEX callback fct.
- test which edge pairs cross
- add corresponding constr.
- continue to solve MIP

Labeling

Wien: 1 day.

Labeling

Tokyo: $<$ 10 sec.

[Kameda & Imai: IEICE’03]

Outline

Another Problem

RectilinearGraphDrawing Decision Problem

Given a planar embedded graph G with max degree 4.
Is there a drawing of G that

- preserves the embedding,
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By reduction to a flow problem. □

Our Problem

MetroMapLayout Decision Problem

Given a planar embedded graph G with max degree 8.
Is there a drawing of G that

- preserves the embedding,
- uses straight-line edges,
- is octilinear?

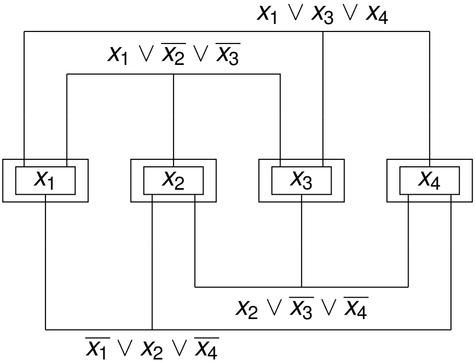
Theorem (Nöllenburg Master'sThesis'05)

MetroMapLayout *is NP-hard.*

Proof.

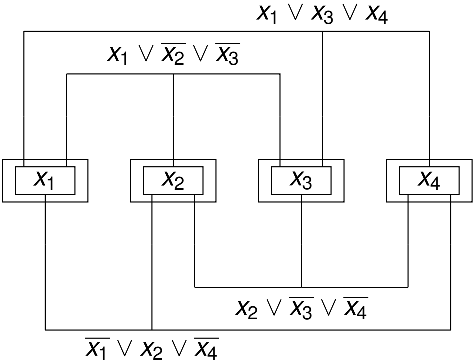
Reduction from Planar 3-Sat to MetroMapLayout. S □

Outline of the Reduction



Input: planar 3-SAT formula $\varphi = (x_1 \vee x_3 \vee x_4) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge \ldots$

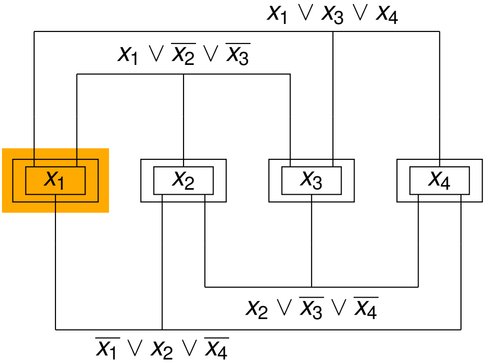
Outline of the Reduction



Input: planar 3-SAT formula $\varphi =$
$(x_1 \vee x_3 \vee x_4) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge \ldots$

Goal: planar embedded graph G_φ with:
G_φ has a metro map drawing $\Leftrightarrow$ φ satisfiable.

Outline of the Reduction



Input: planar 3-SAT formula $\varphi =$
$(x_1 \vee x_3 \vee x_4) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge \ldots$

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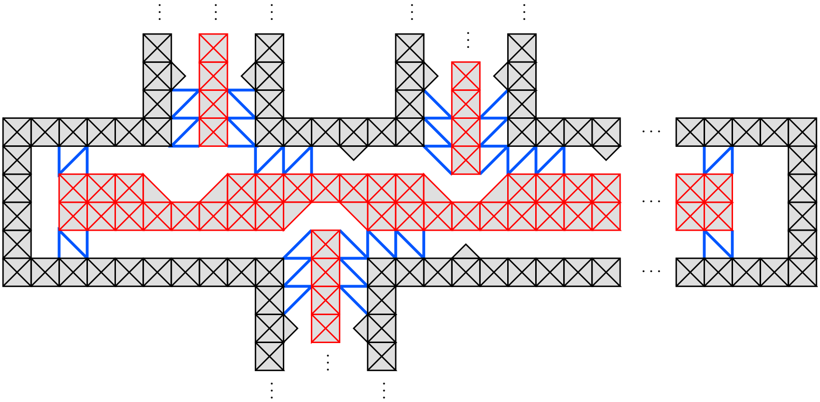
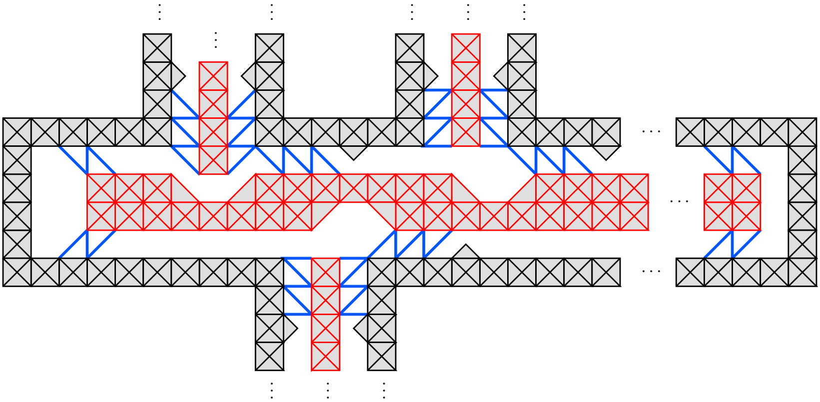
Variable Gadget



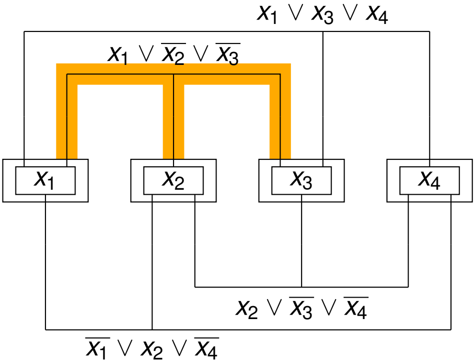
$x = \text{true}$

Variable Gadget



x = false

Outline of the Reduction



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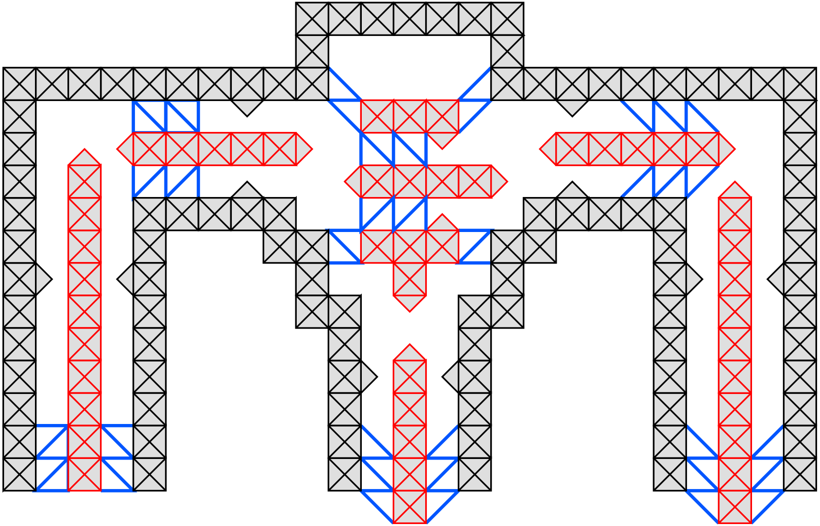
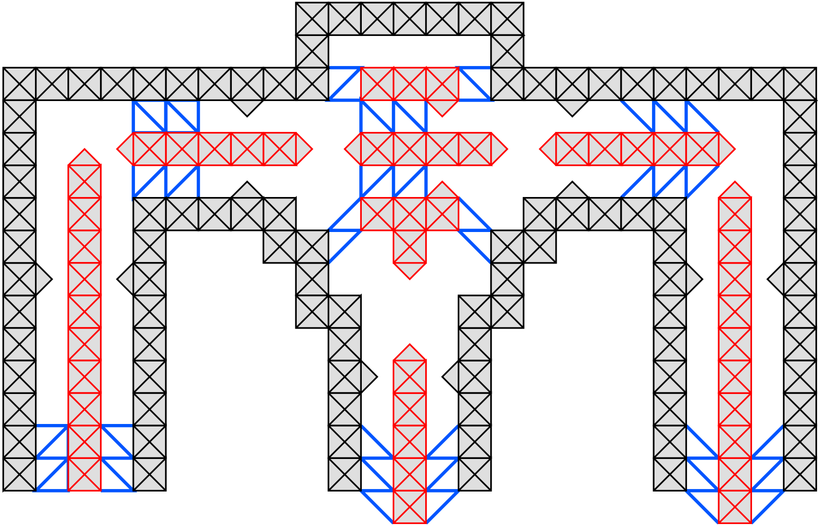
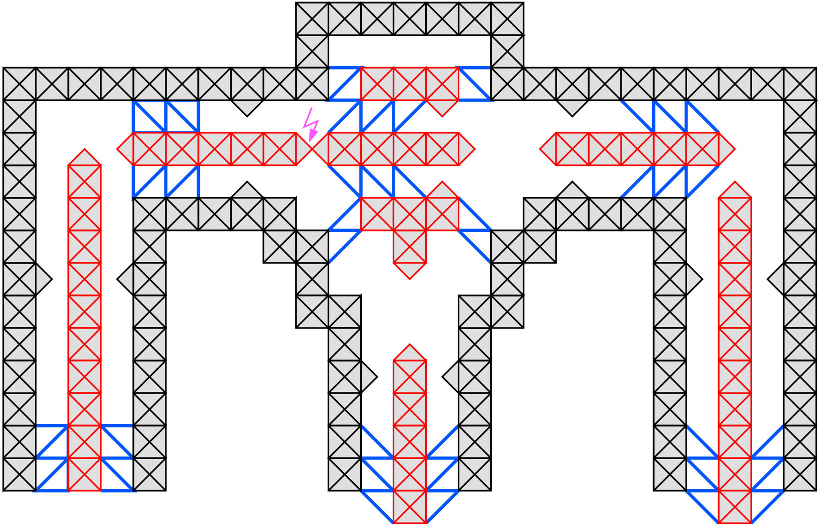
Clause Gadget



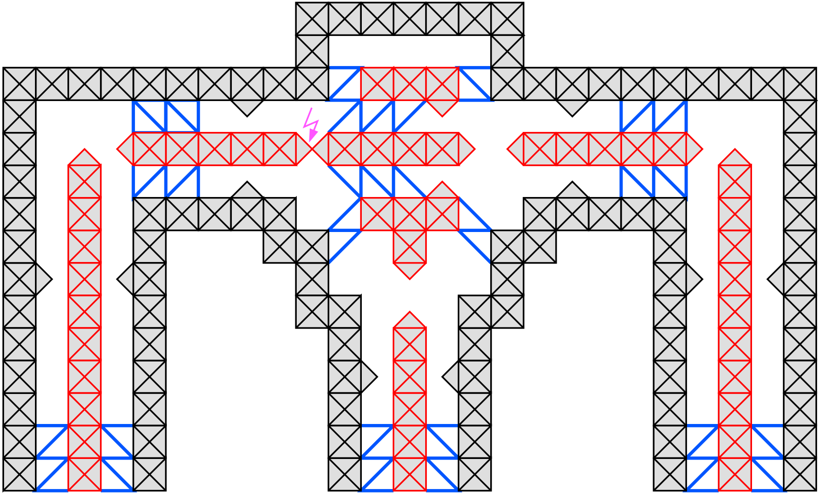
Clause Gadget



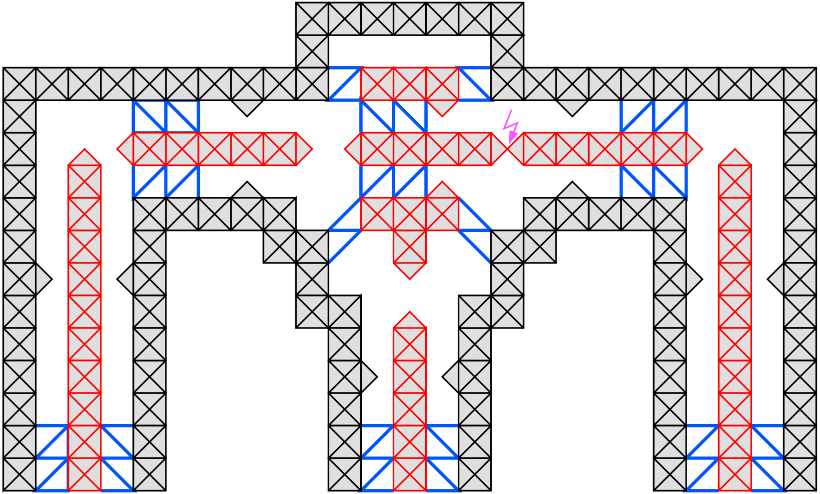
Clause Gadget



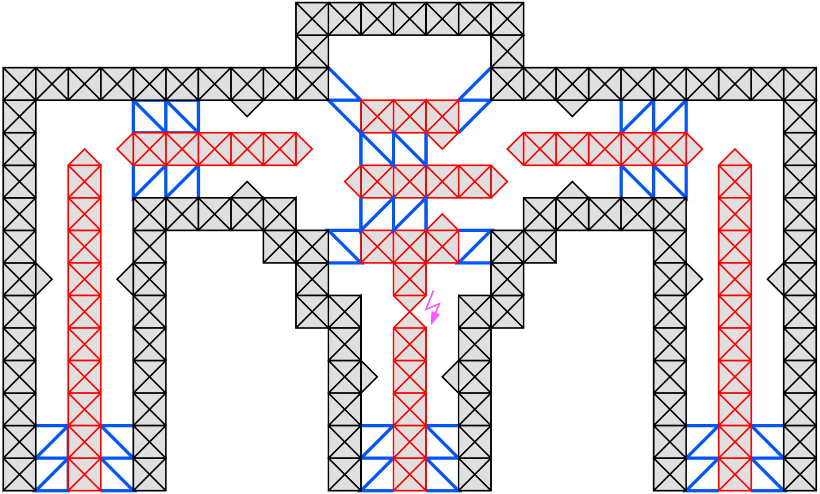
Clause Gadget



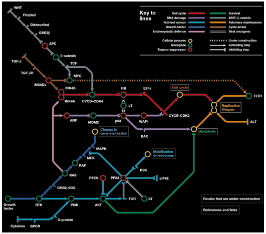
Clause Gadget

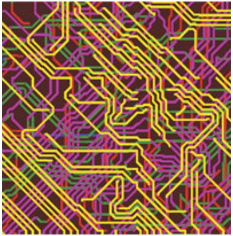


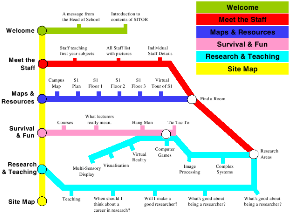
Clause Gadget

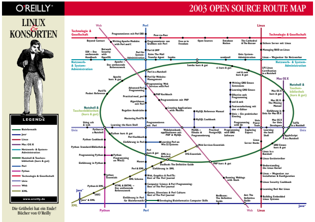


Other applications

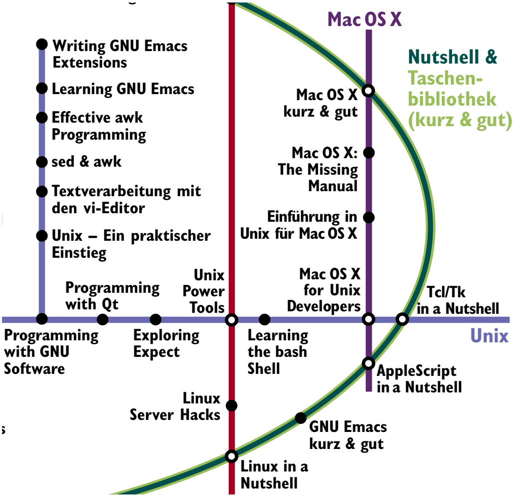




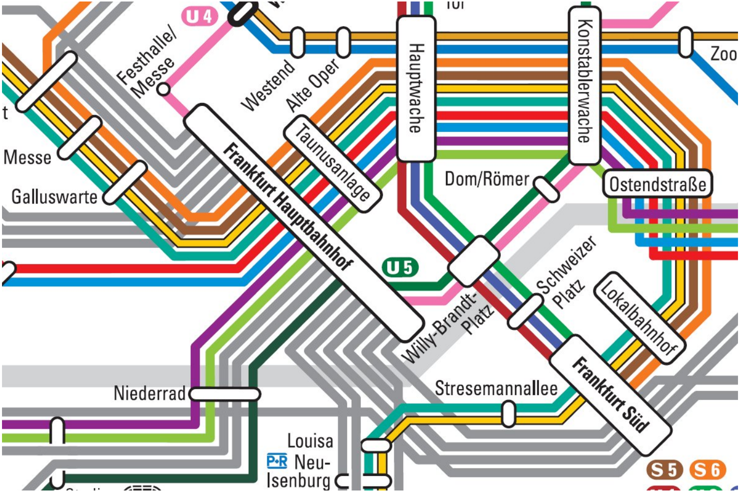




Clipping


Mac OS X
Nutshell &
Taschen-
bibliothek
(kurz & gut)
Writing GNU Emacs Extensions
Learning GNU Emacs
Effective awk Programming
sed & awk
Textverarbeitung mit den vi-Editor
Unix – Ein praktischer Einstieg
Mac OS X kurz & gut
Mac OS X: The Missing Manual
Einführung in Unix für Mac OS X
Mac OS X for Unix Developers
Programming with Qt
Unix Power Tools
Tcl/Tk in a Nutshell
Unix
Programming with GNU Software
Exploring Expect
Learning the bash Shell
AppleScript in a Nutshell
Linux Server Hacks
GNU Emacs kurz & gut
Linux in a Nutshell

To do: rectangular stations & multi-edges



Summary (metro maps)

- METROMAPLAYOUT is NP-hard.
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- Our MIP can draw *any* kind of sketch “nicely”.
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To Do

- rectangular stations
- multi-edges
- user interaction (e.g., fixing certain edge directions)