Visualizing Multispecies Coalescent Trees: Drawing Gene Trees Inside Species Trees

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THE UNIVERSITY OF AUCKLAND Te Whare Wānanga o Tāmaki Makaurau NEW ZEALAND

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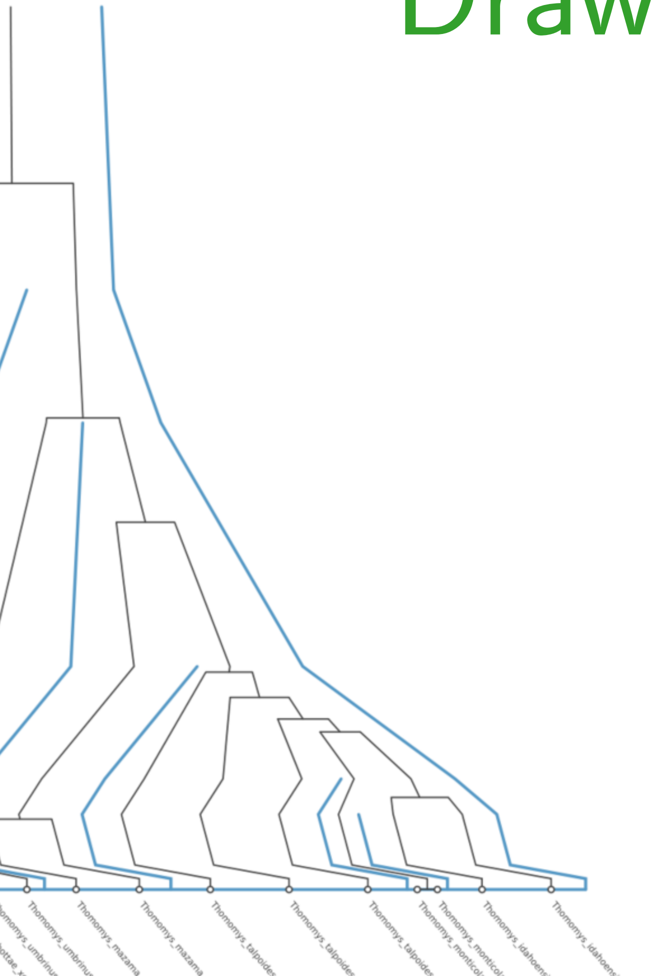
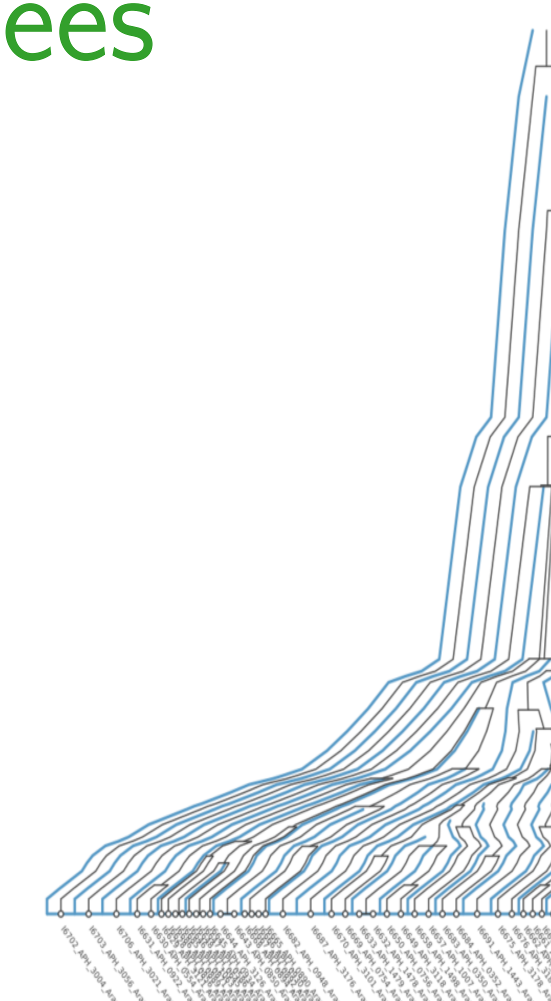
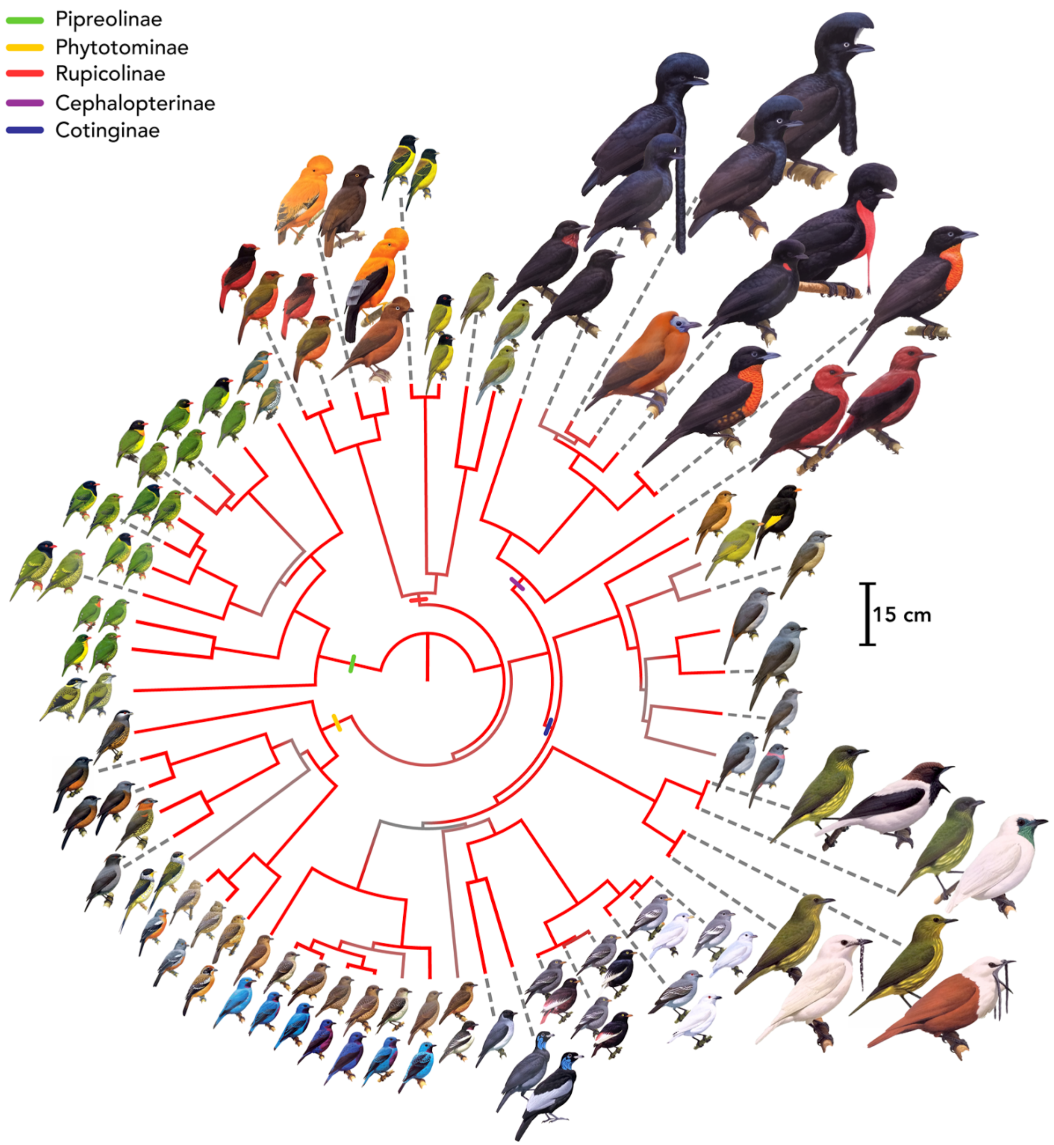
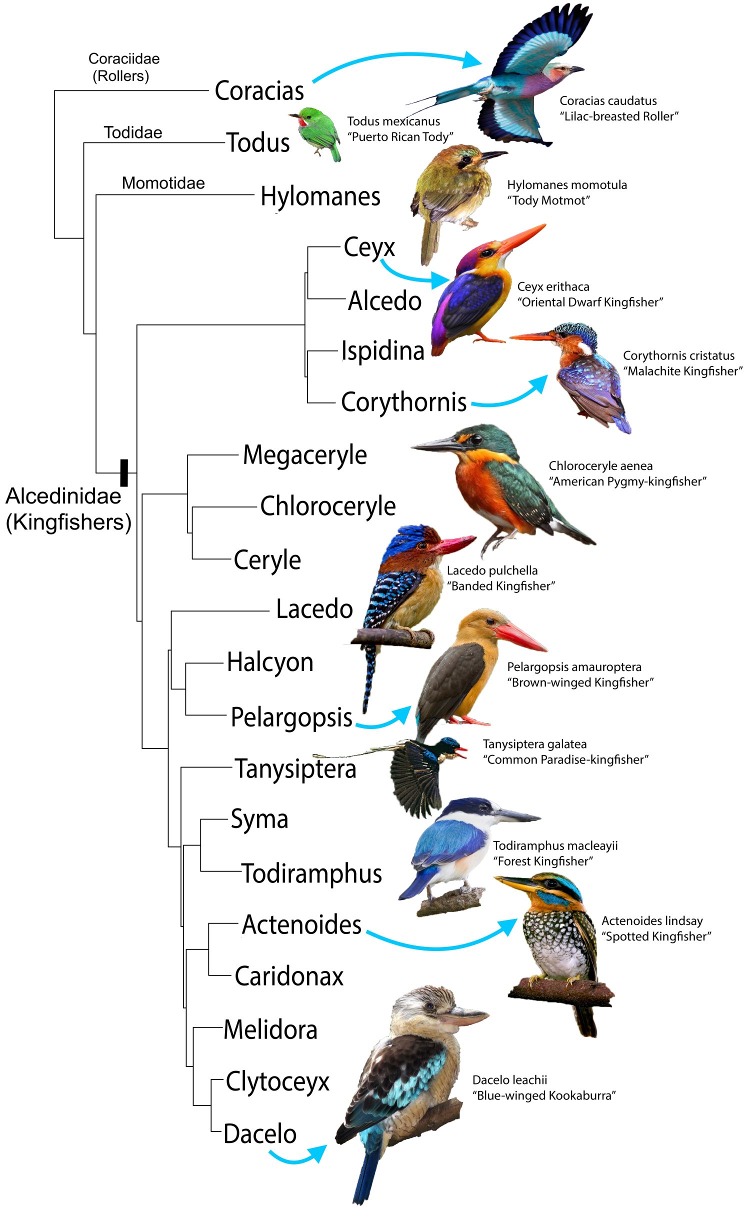




Phylogenetic trees

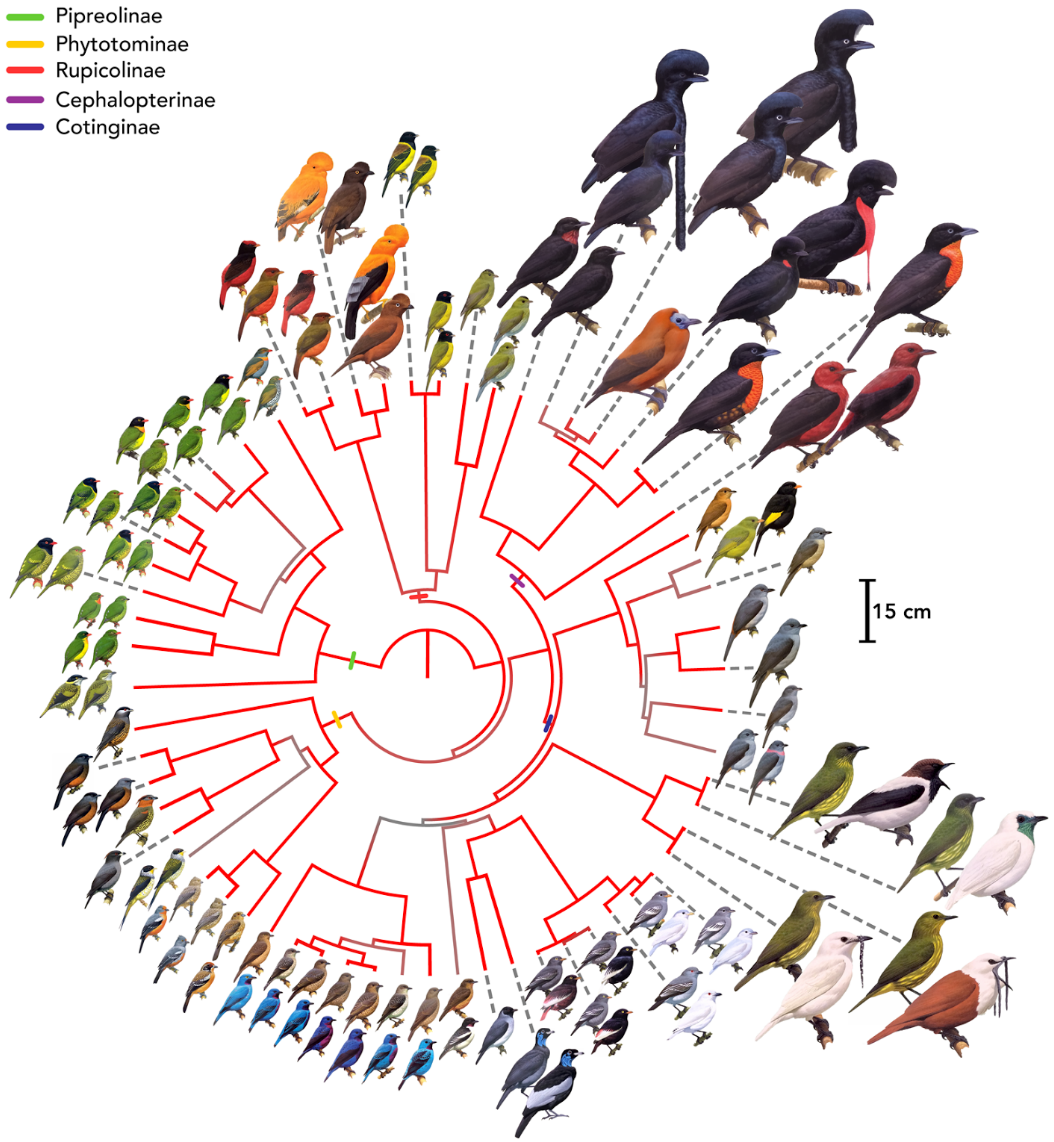


by Berv & Prum 2014



by Jenna McCullough 2016

Phylogenetic trees



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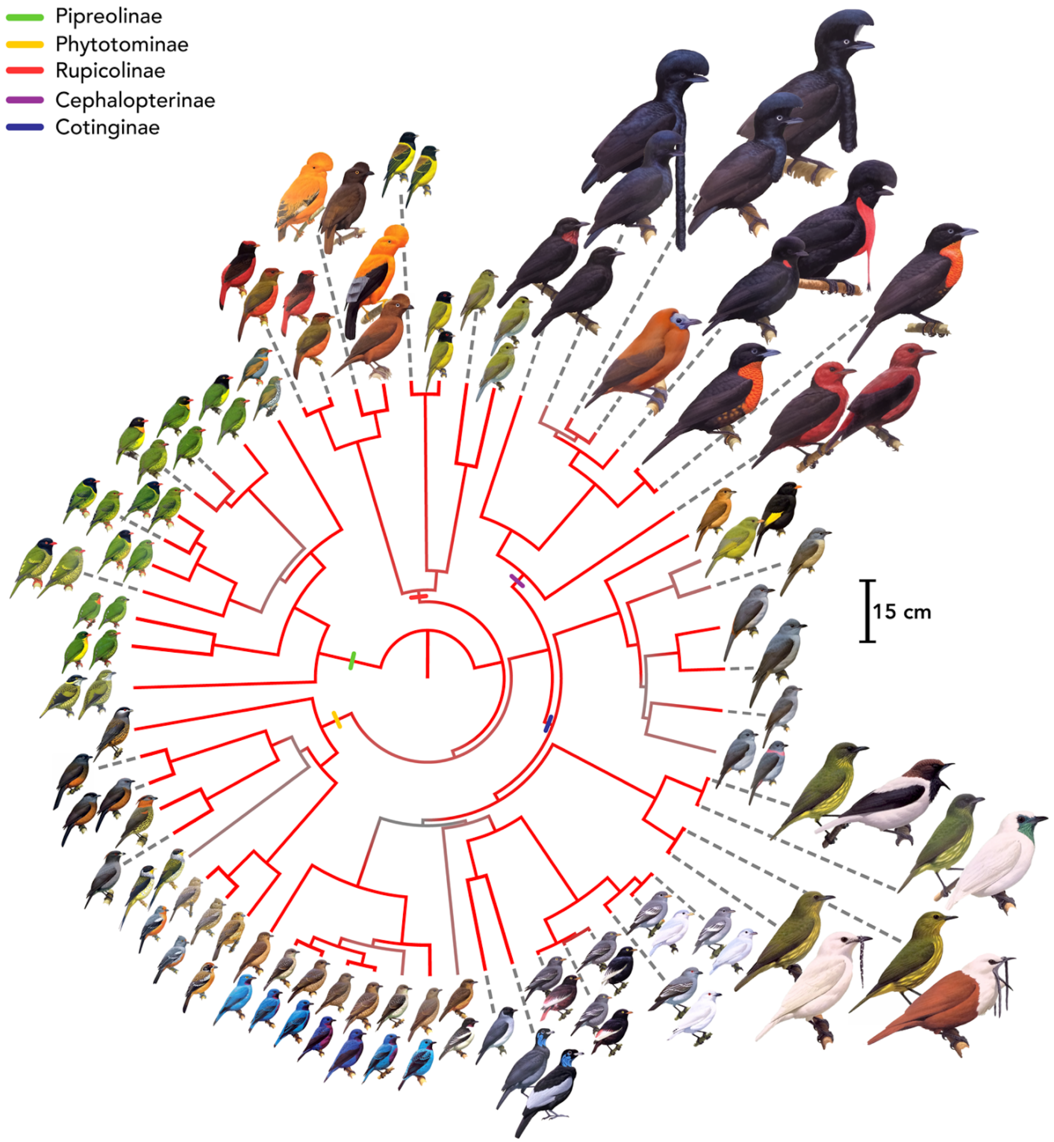
A **phylogenetic tree** T is a rooted, binary tree with labelled leaves and vertex heights.

Phylogenetic trees



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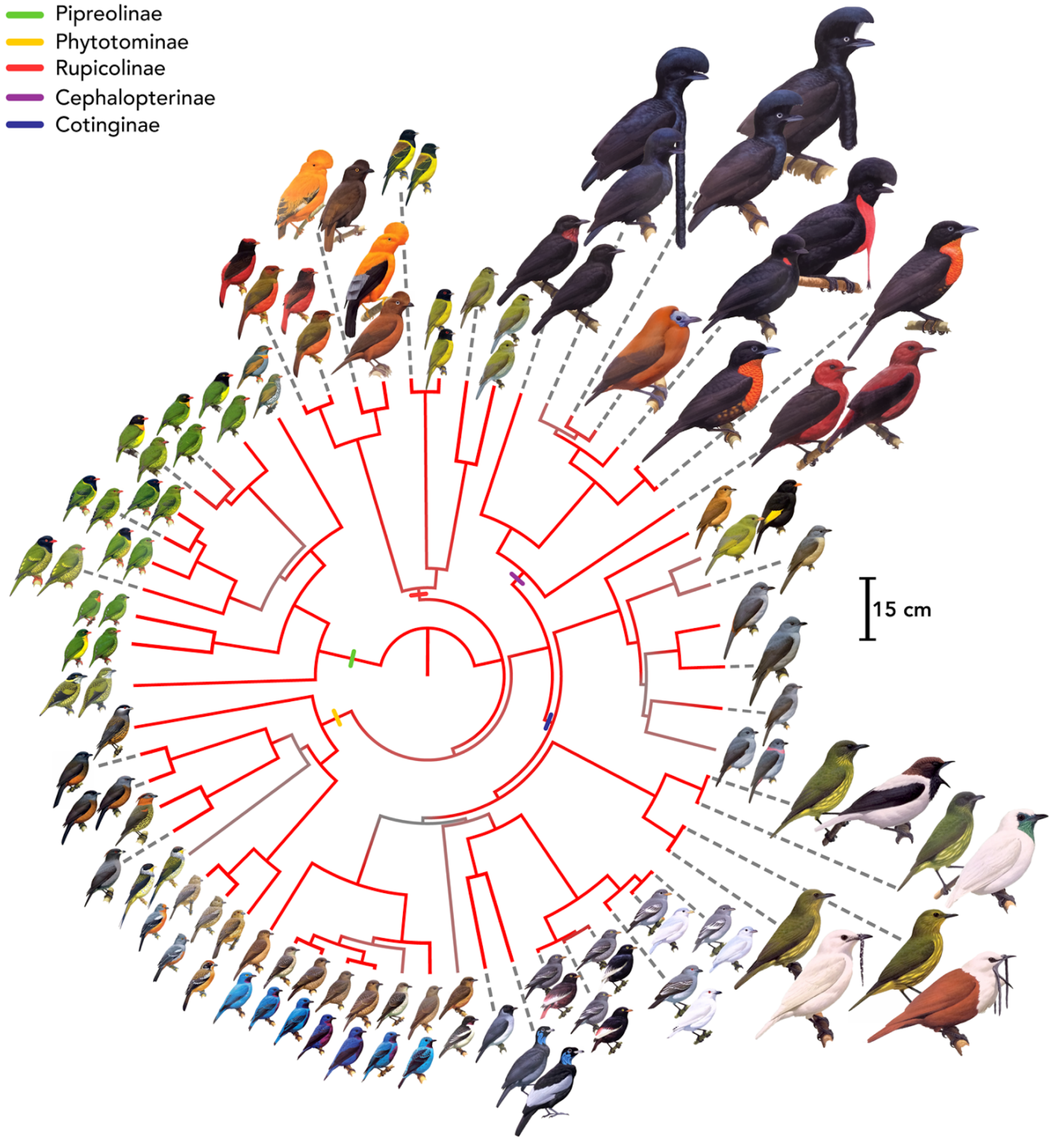
Assume all leaves have height 0.

Phylogenetic trees



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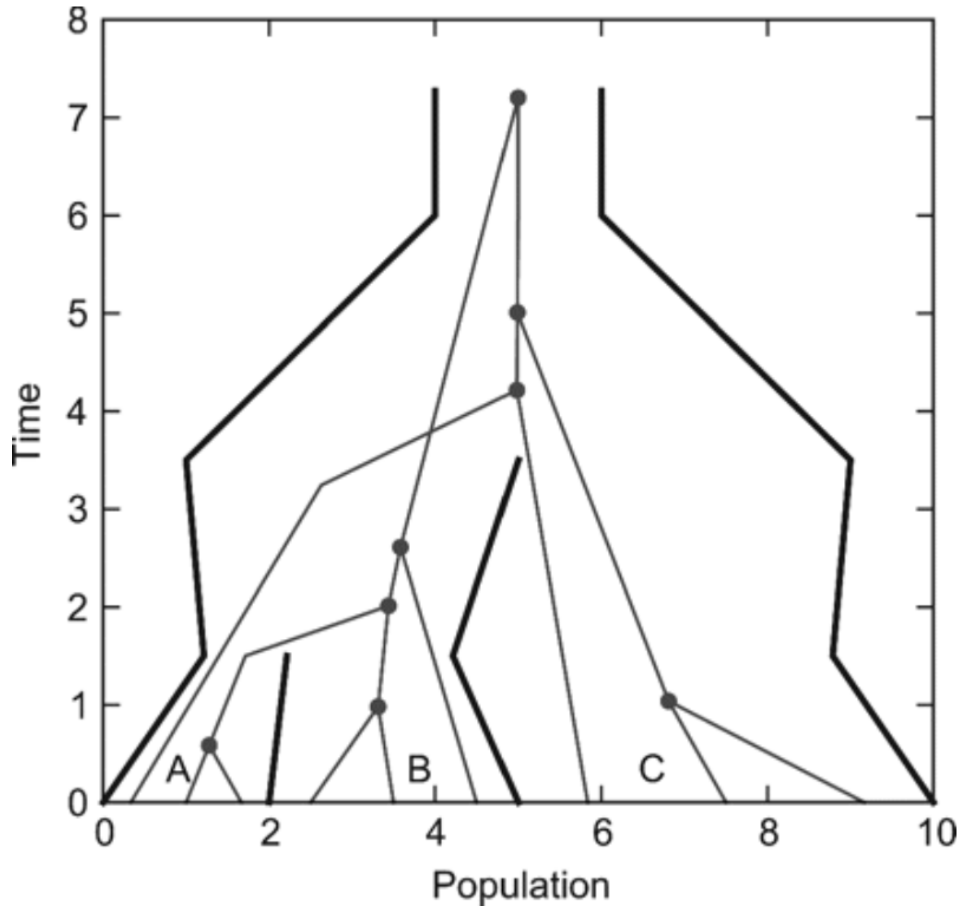
A **phylogenetic tree** T is a rooted, binary tree with labelled leaves and vertex heights.

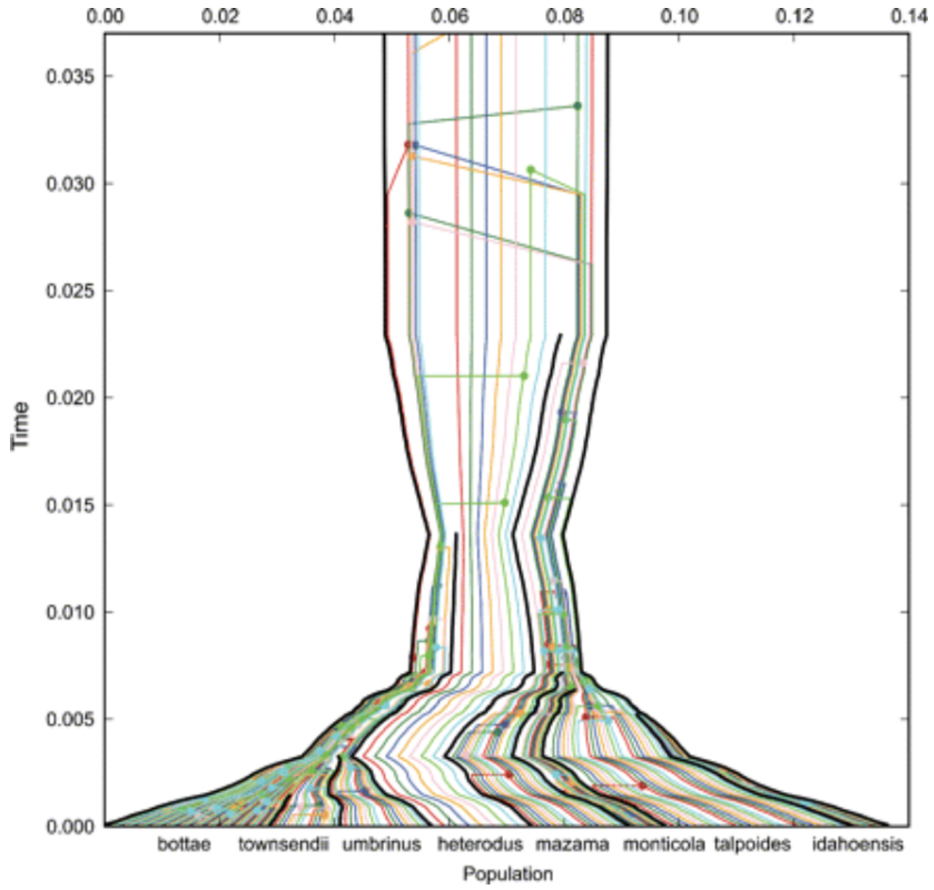
Assume all leaves have height 0.

Call T a **species/gene tree** if the leaves represent species/biological sequences.

Multispecies coalescent

- In practice, a species tree and its gene trees *differ*.





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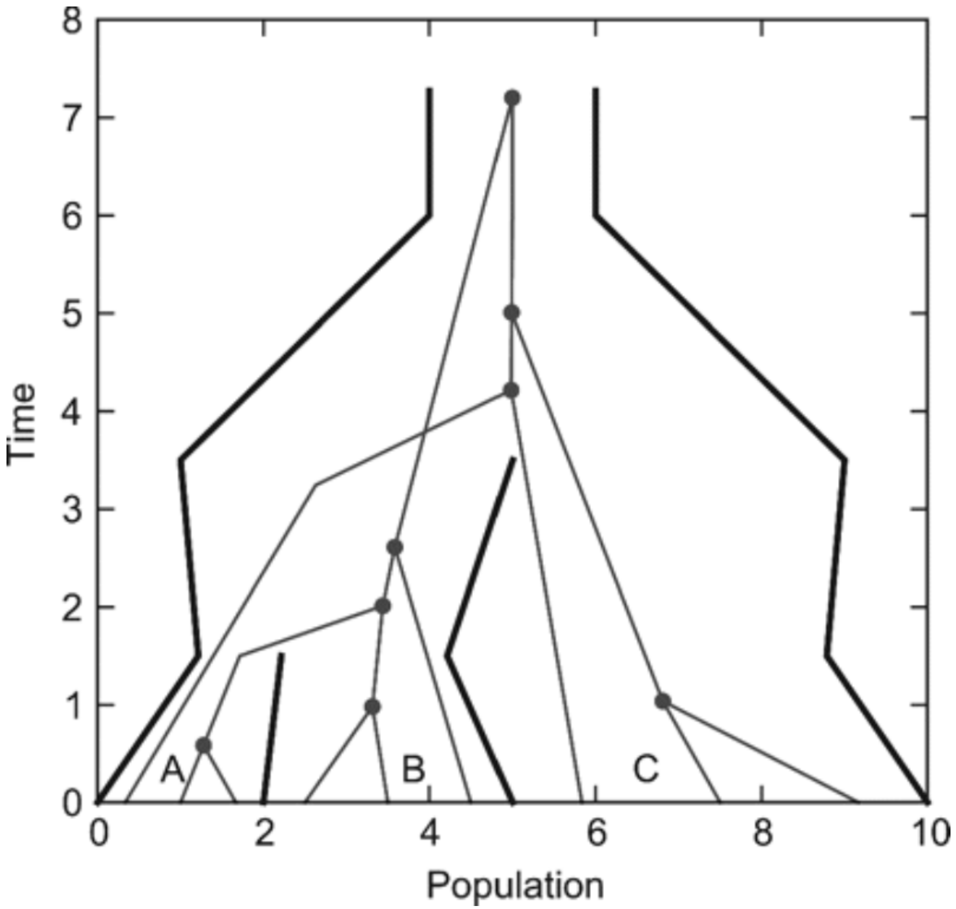

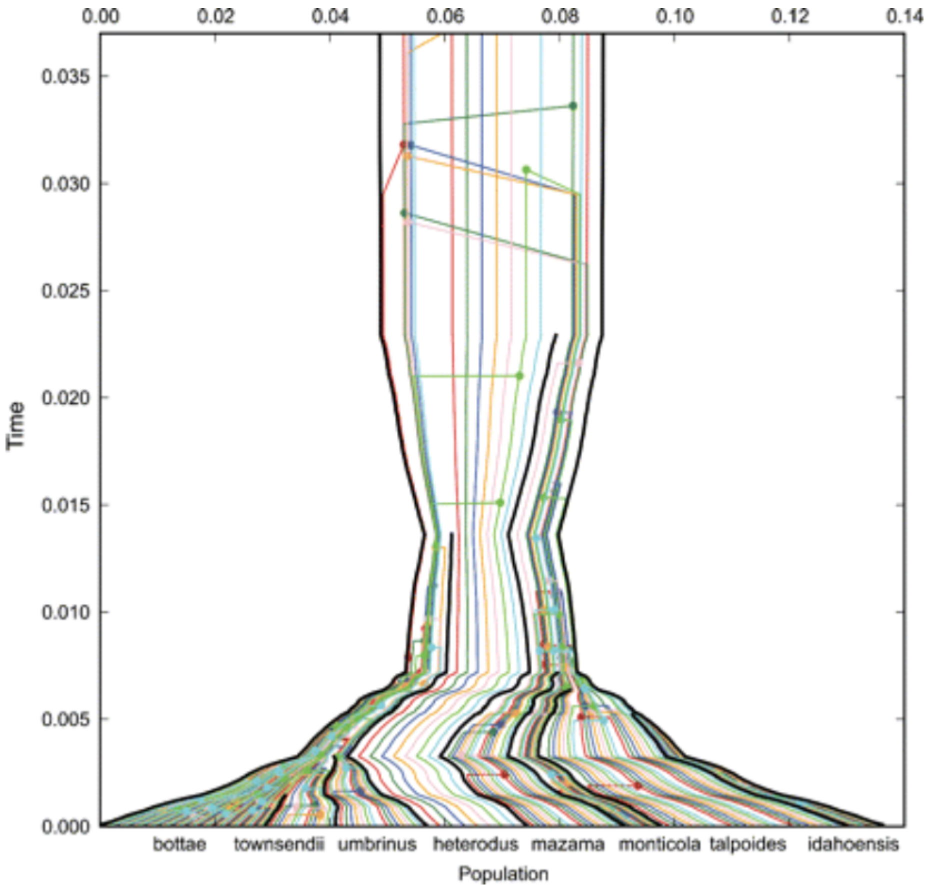

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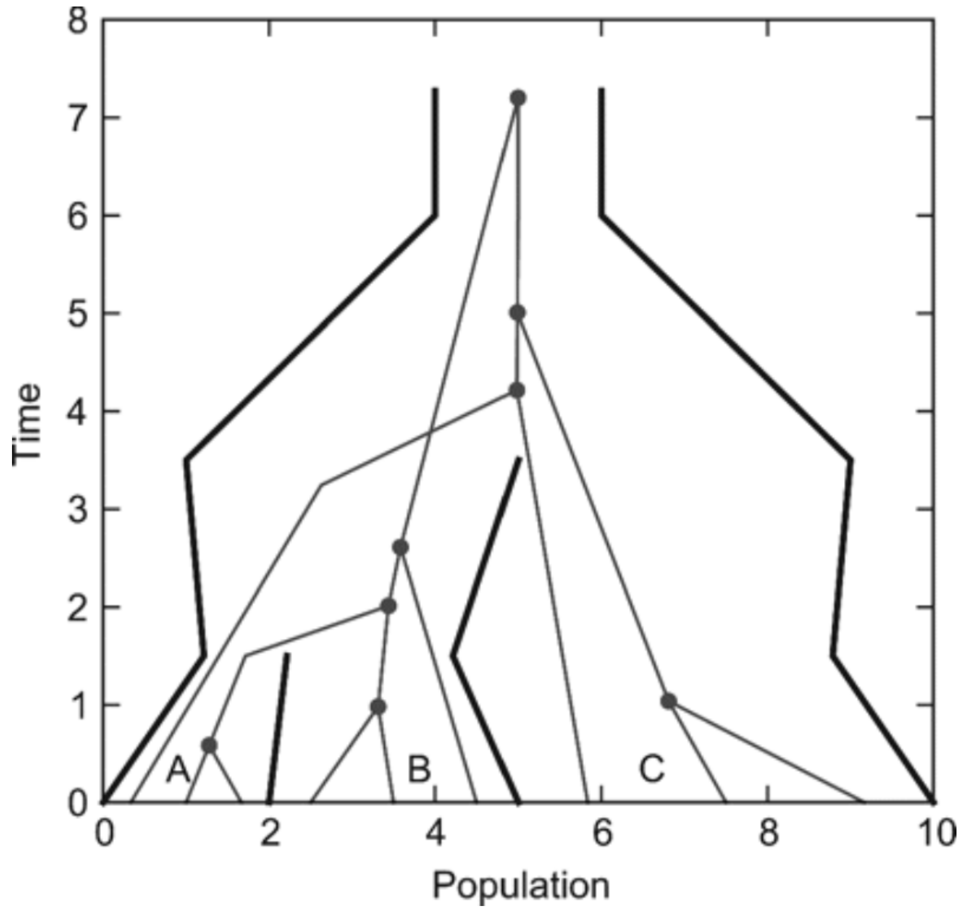

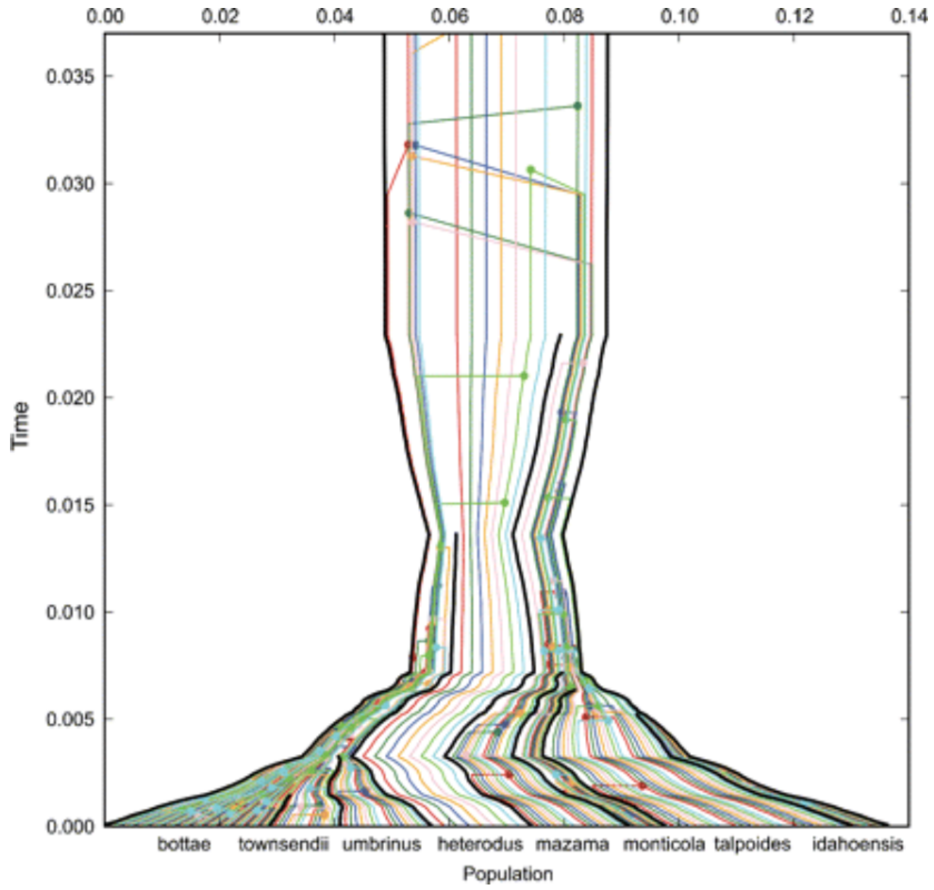

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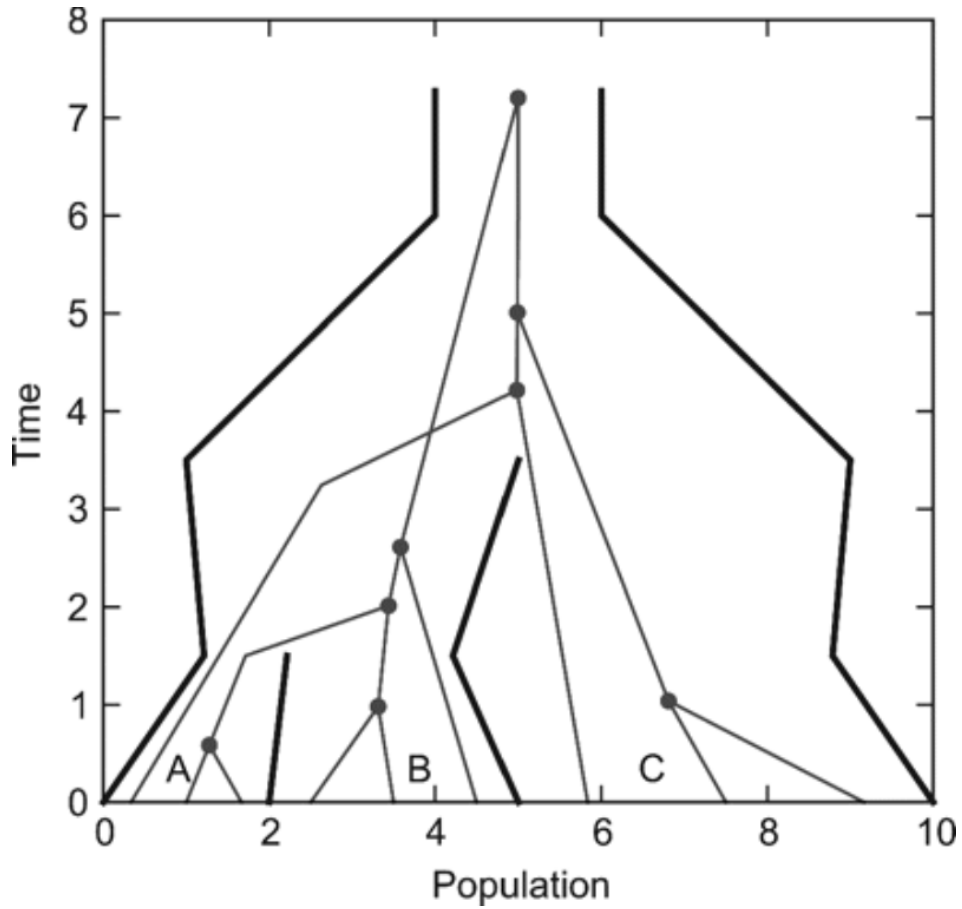
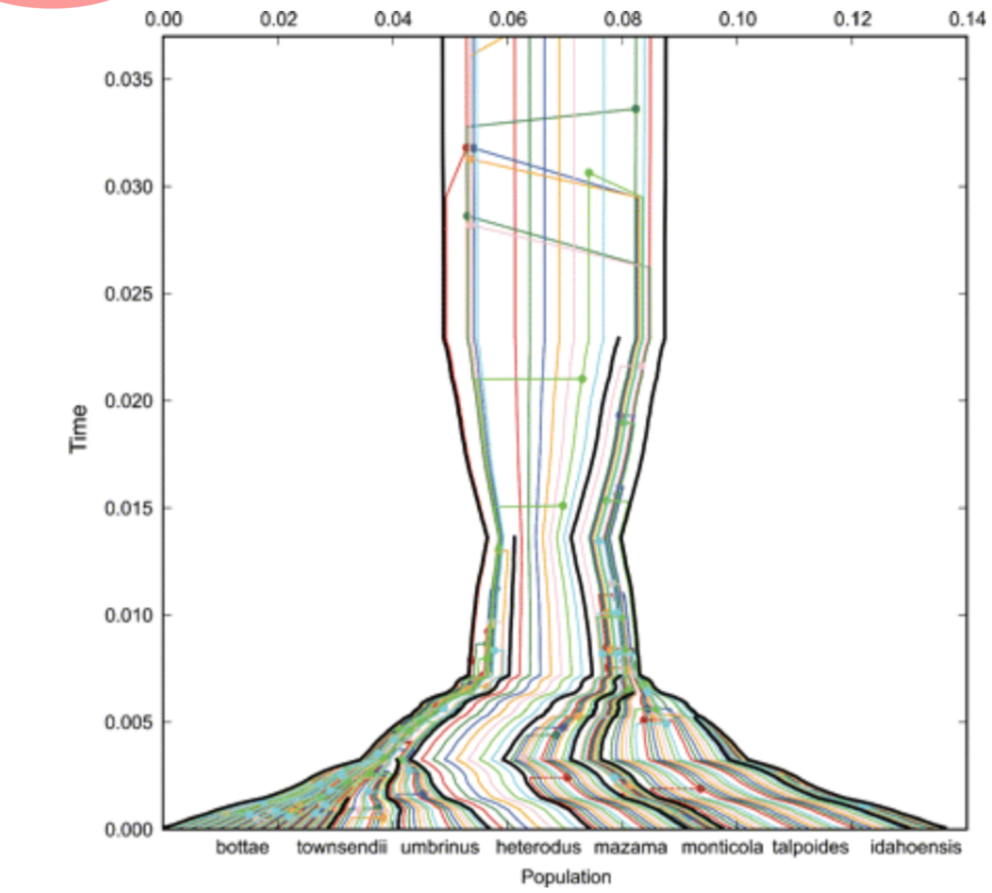


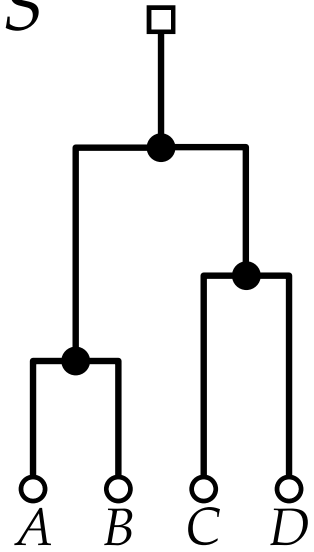


Multispecies coalescent trees

A **multispecies coalescent tree** $\langle S, T, \varphi \rangle$ consists of

Multispecies coalescent trees

Species tree S



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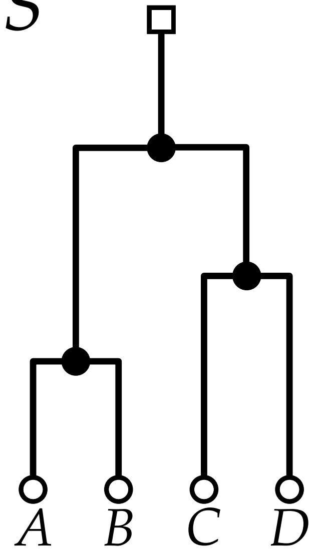
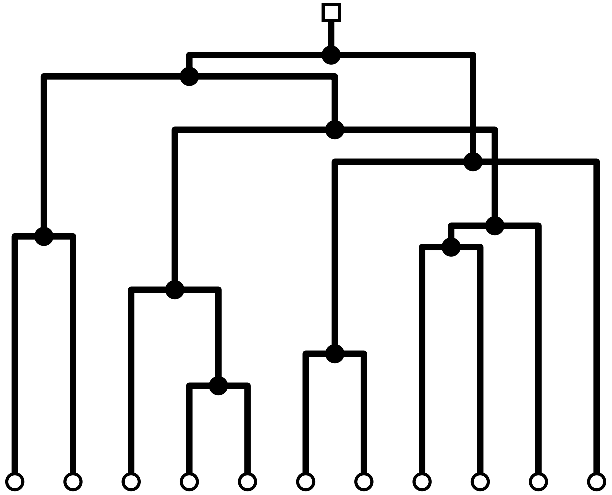


Gene tree T



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Multispecies coalescent trees

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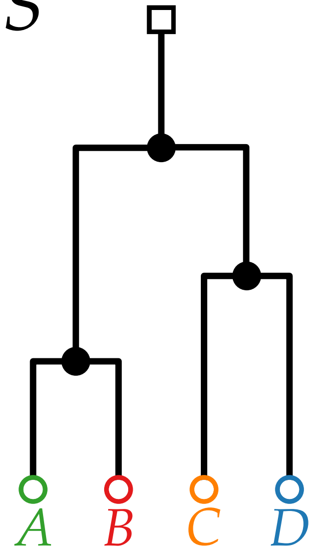

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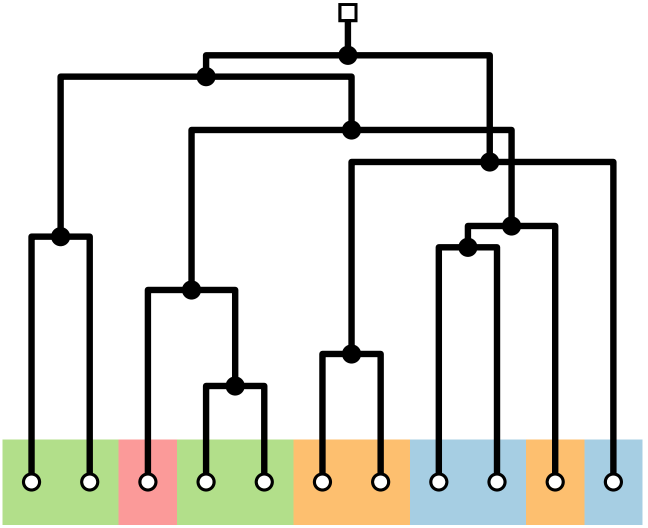

A **multispecies coalescent tree** $\langle S, T, \varphi \rangle$ consists of

- a species tree S,
- a gene tree T, and
- a mapping φ of the leaves of T (**genes**) to the leaves of S (**species**).

Multispecies coalescent trees

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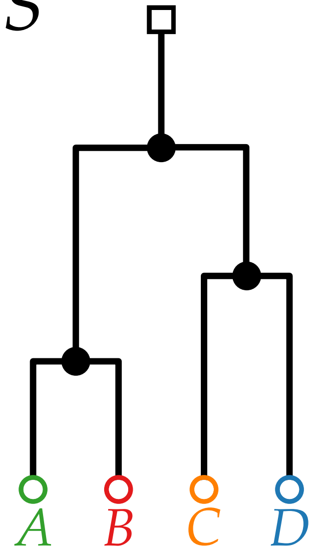

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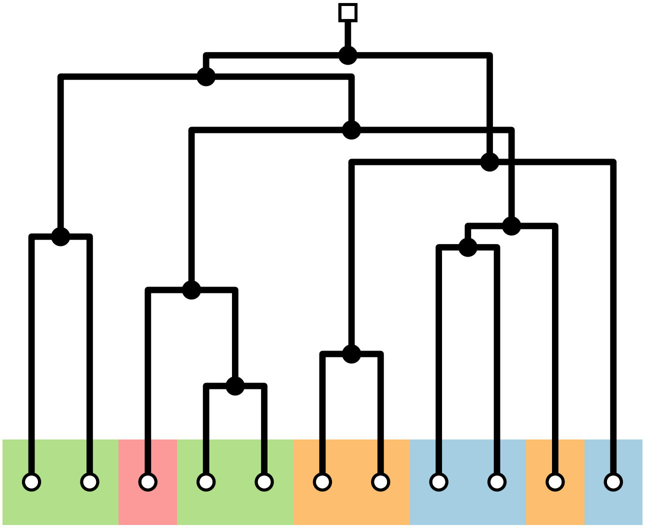

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Additionally:

- Effective population size – branch widths

Multispecies coalescent trees

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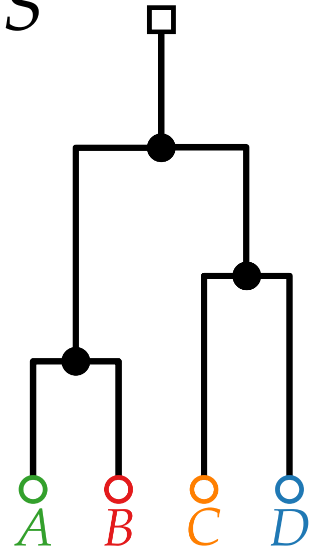

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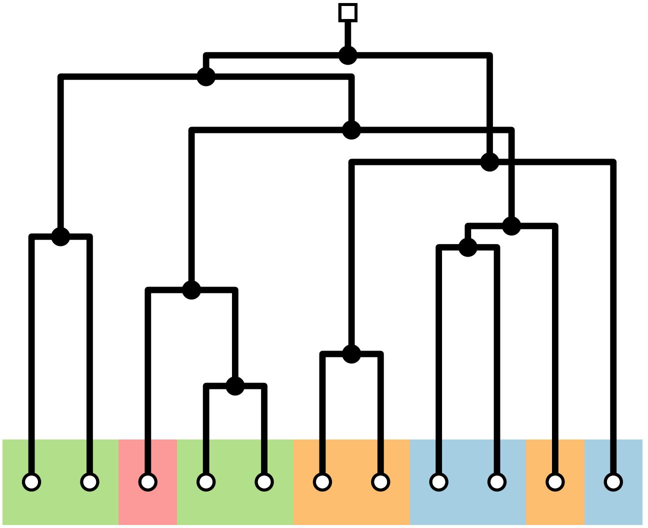

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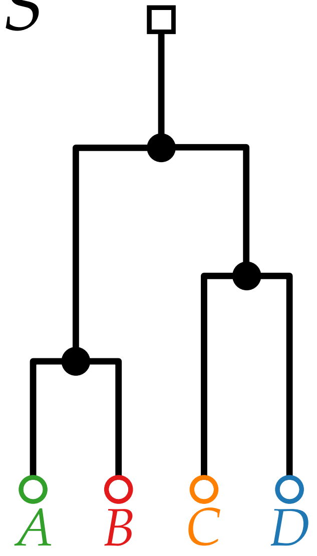
Drawing style

Species tree S



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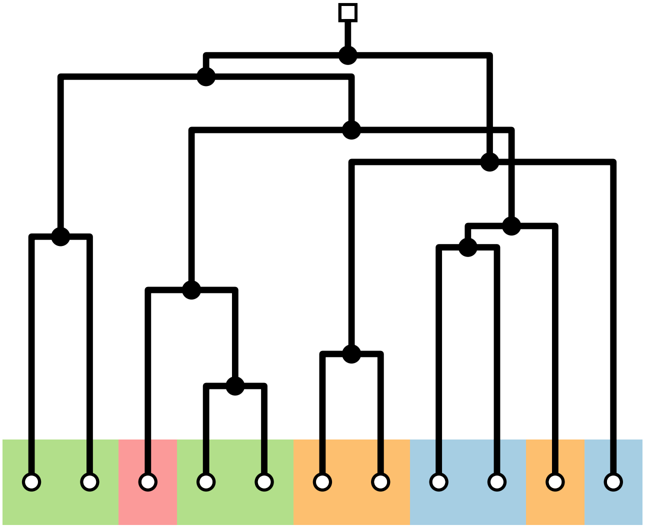

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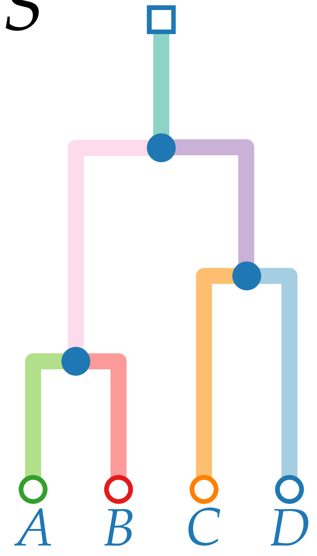

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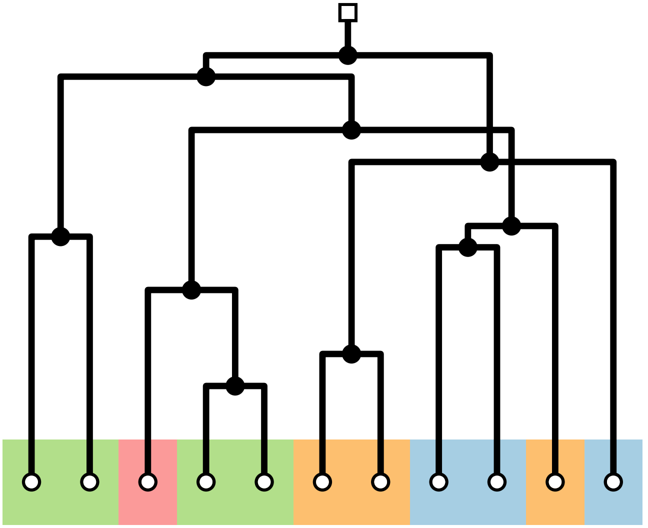

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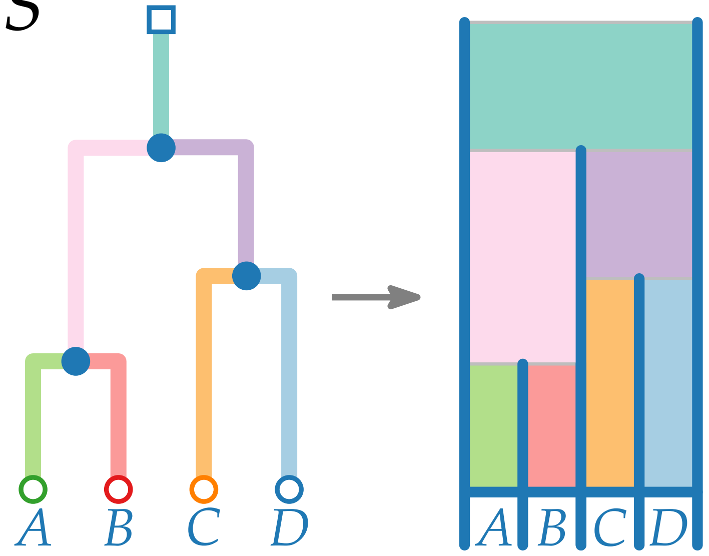

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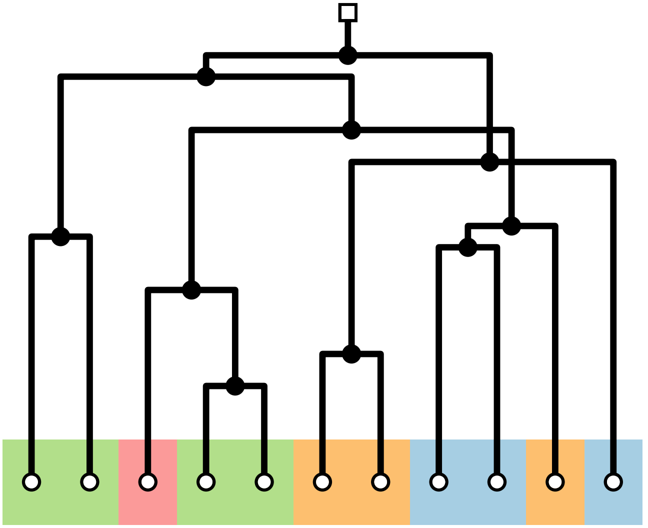

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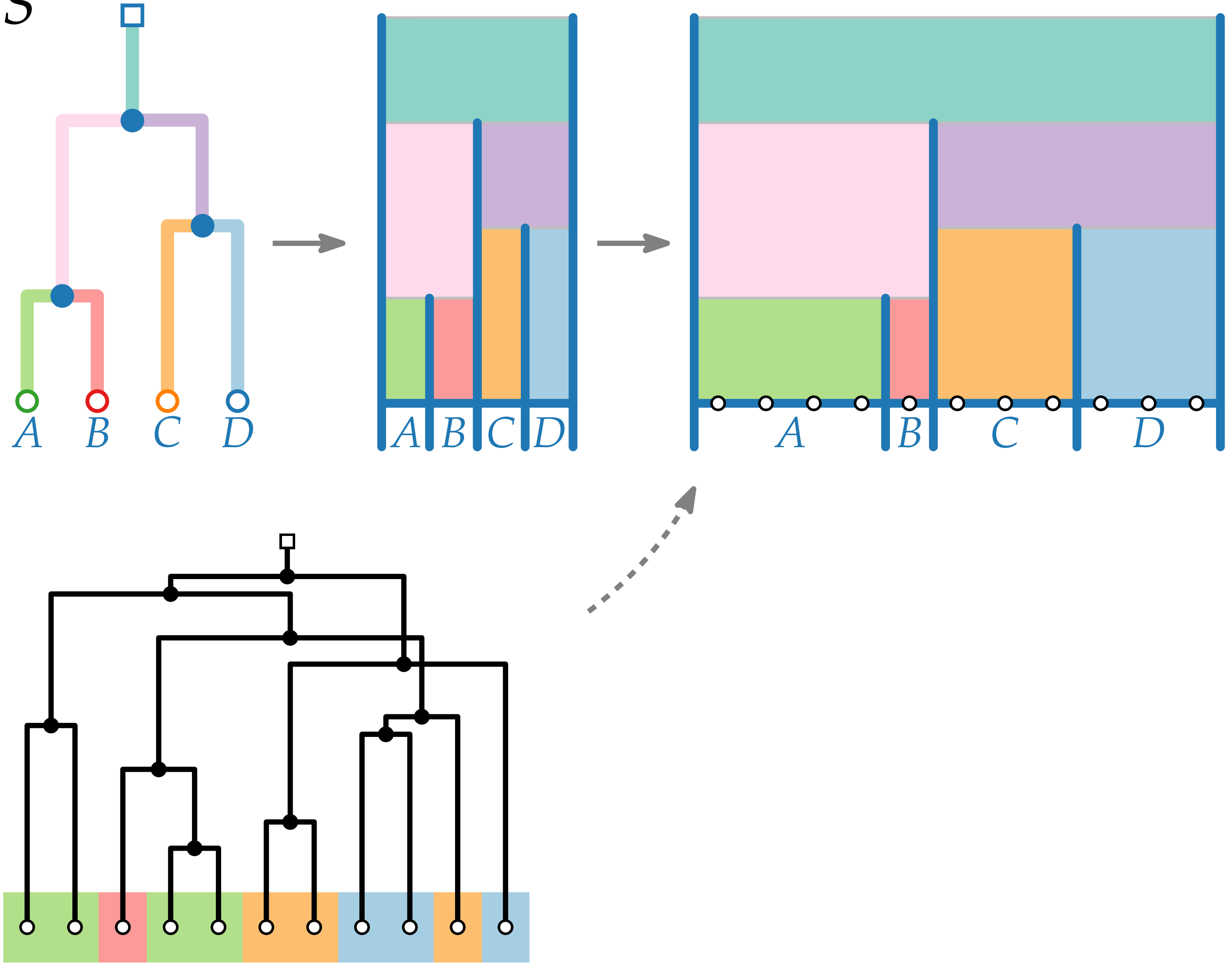

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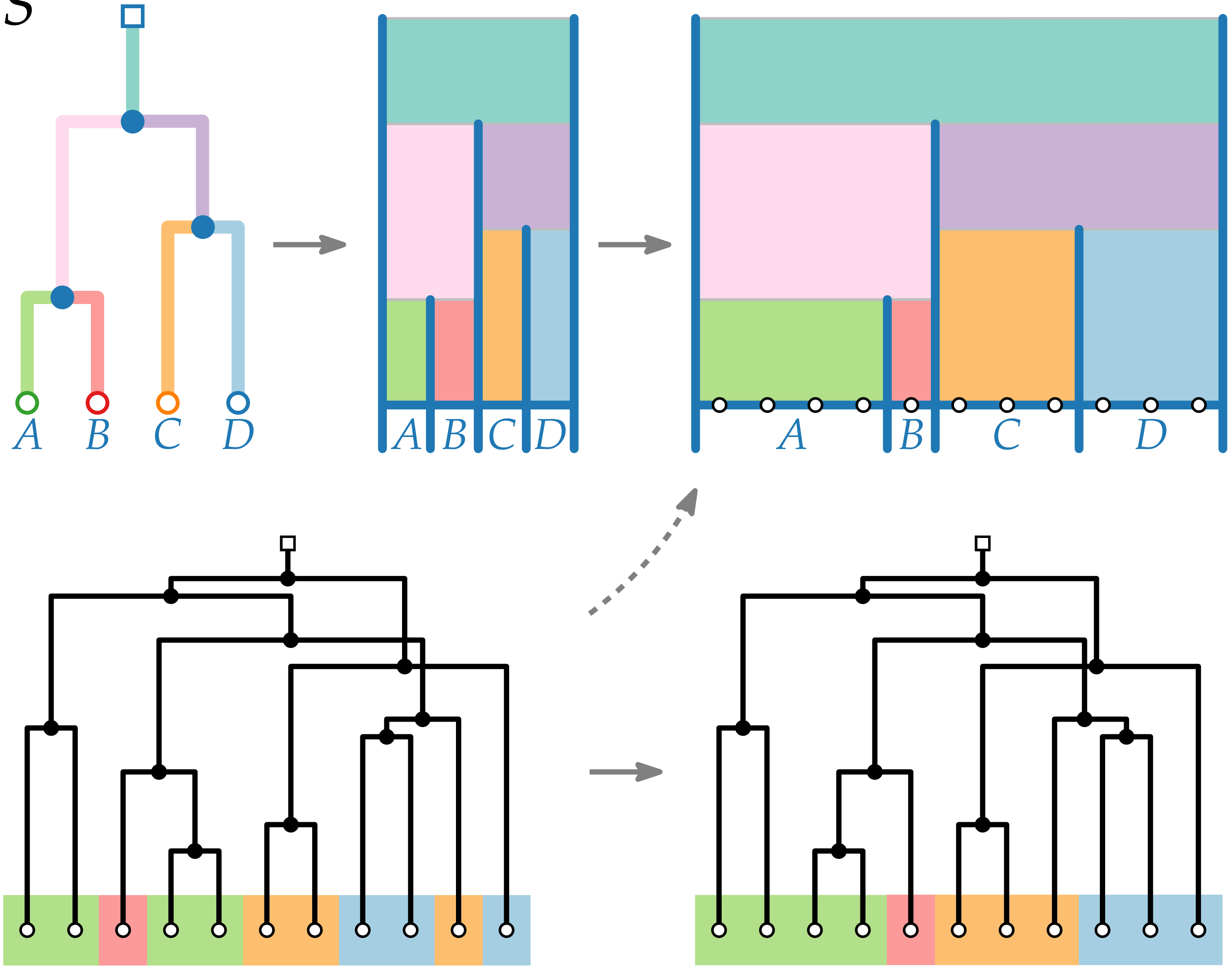

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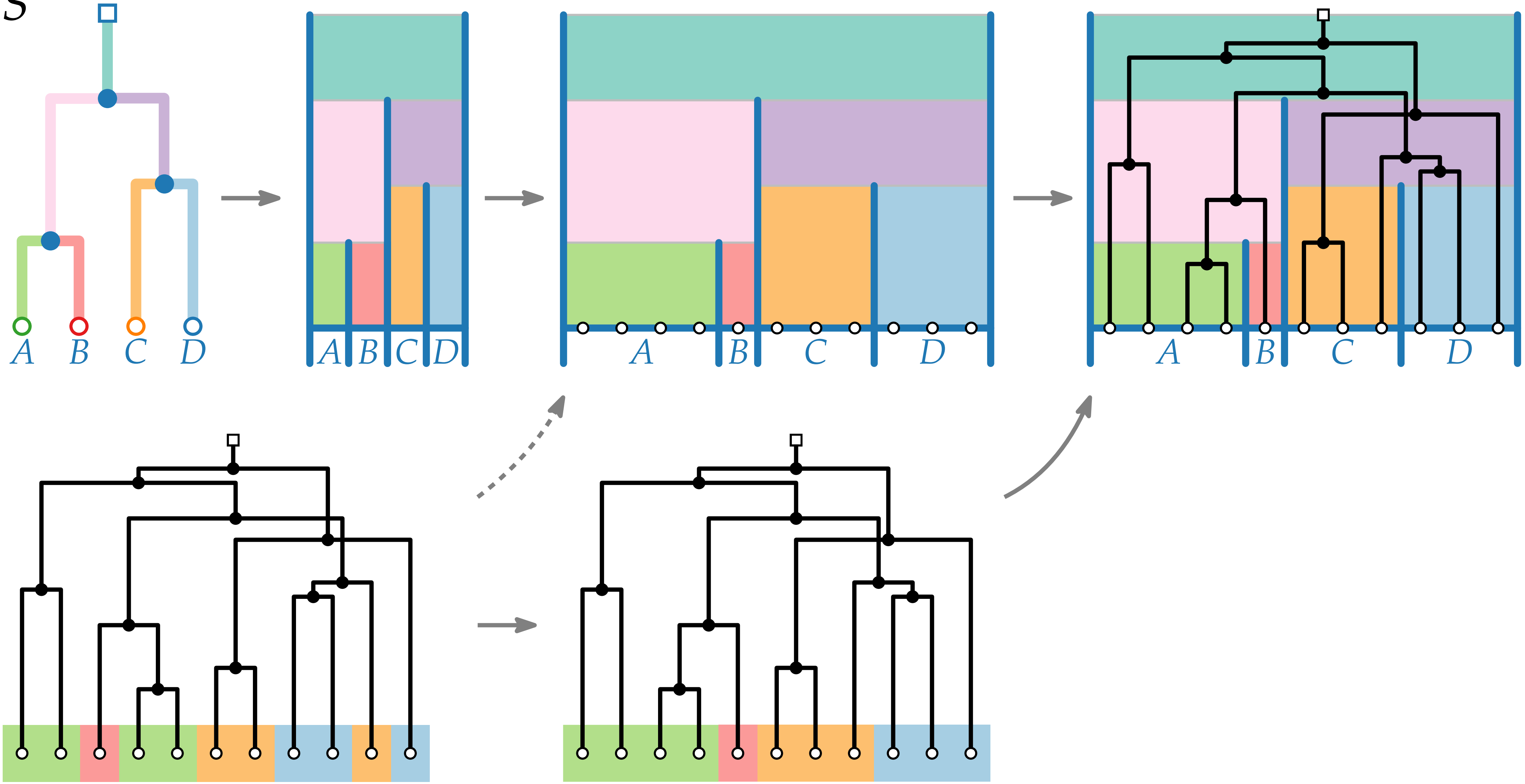

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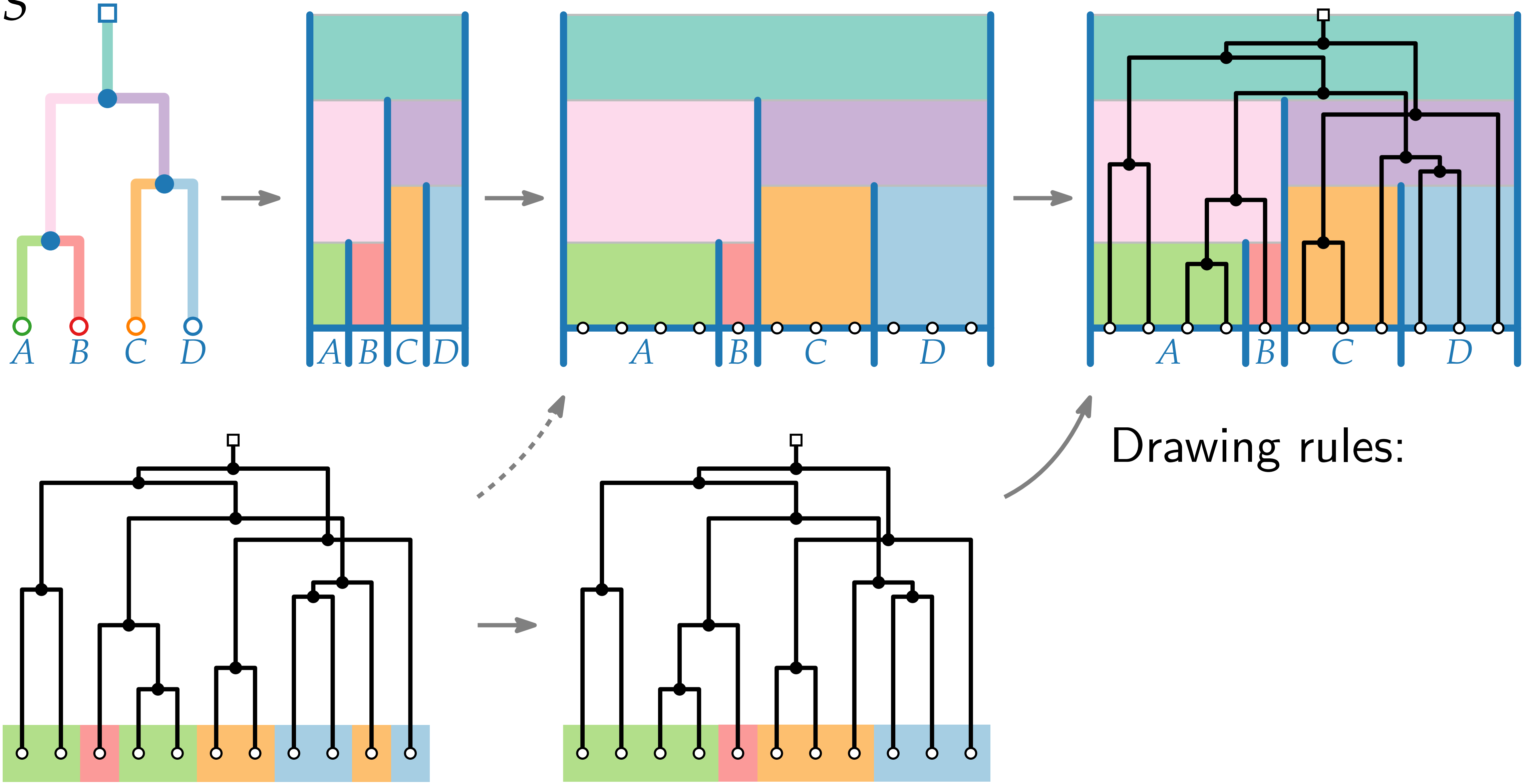
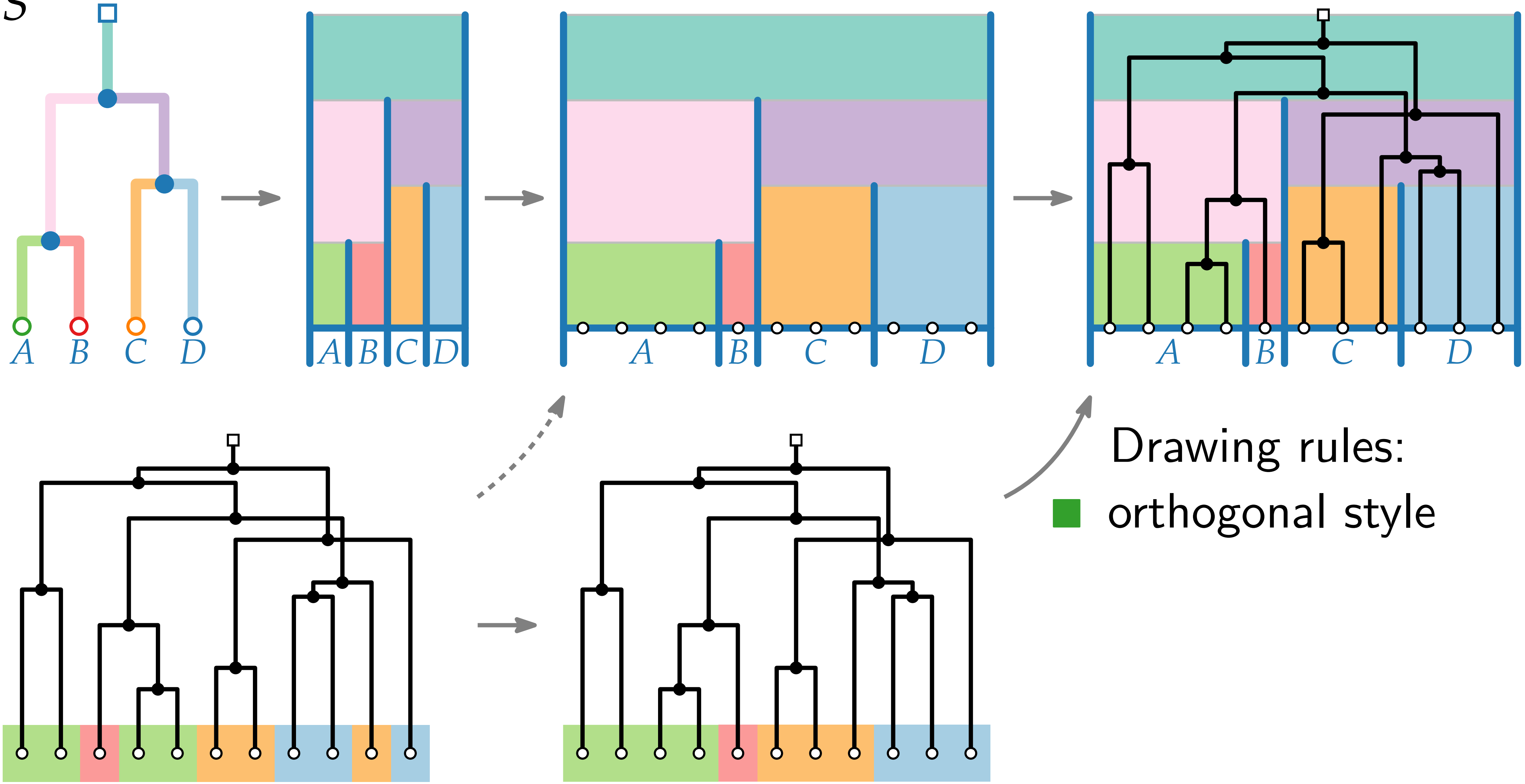


Drawing rules:

Drawing style

Species tree S

Gene tree T



Drawing rules:

🟩 orthogonal style

Drawing style

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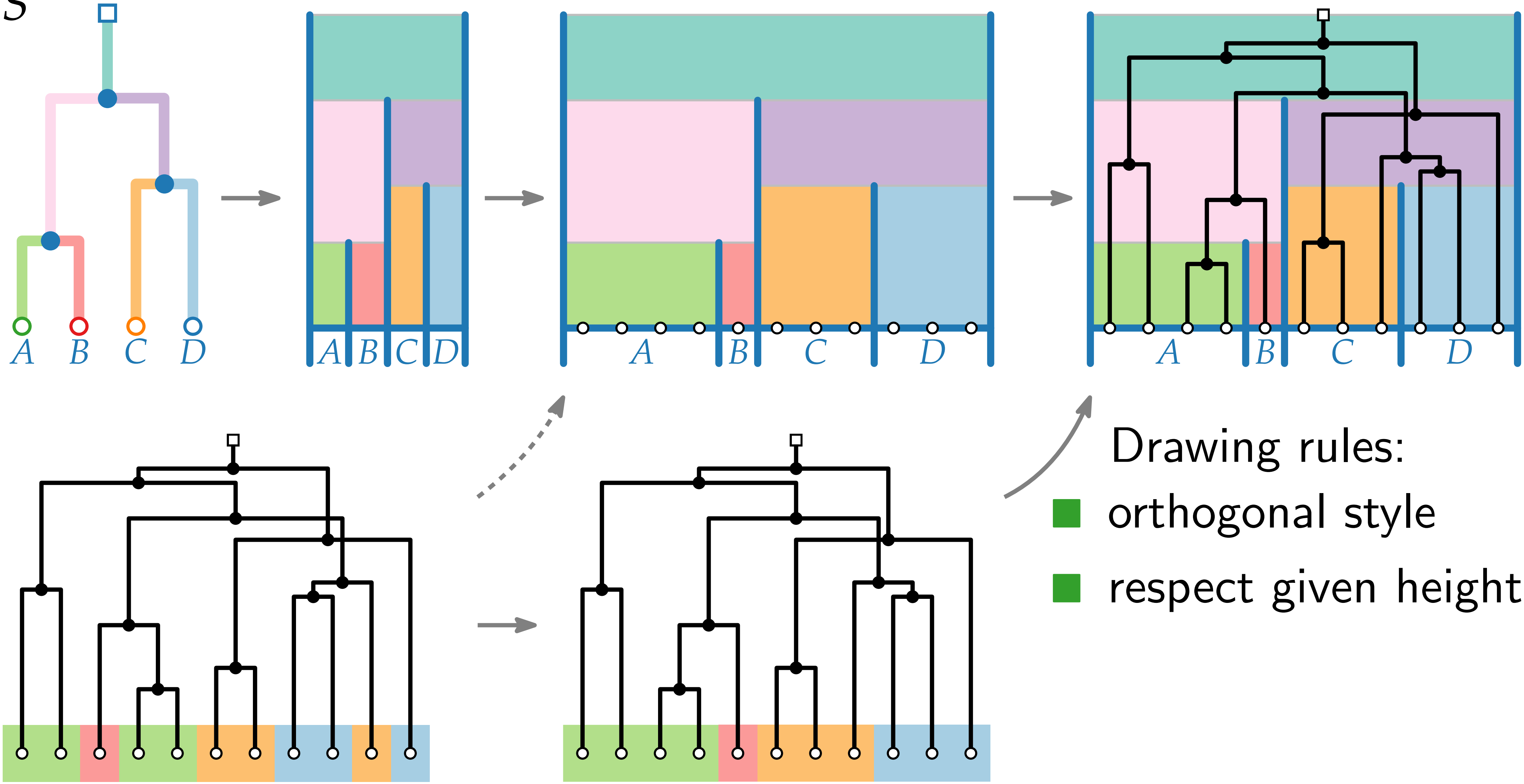
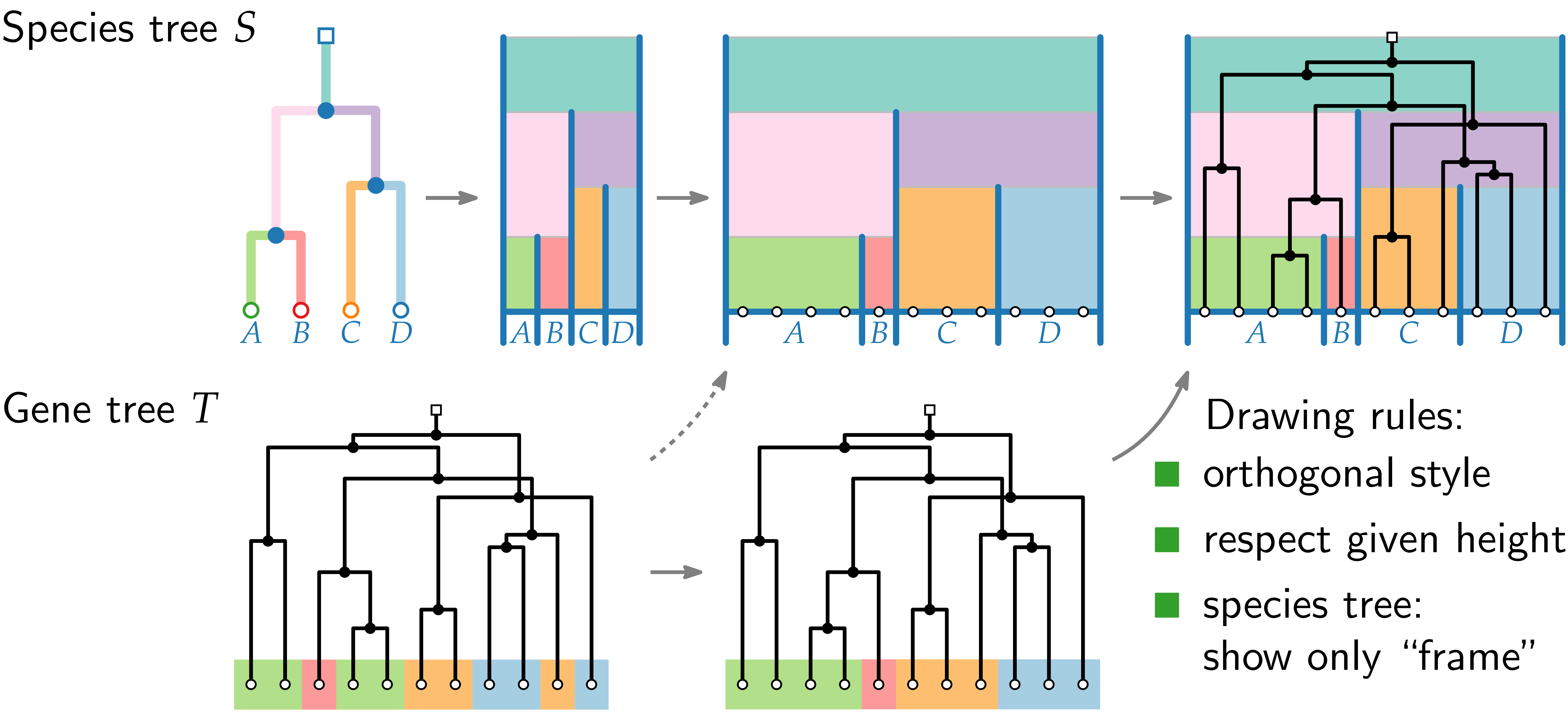


Drawing rules:

- 🟩 orthogonal style
- 🟩 respect given heights

Drawing style



Drawing rules:

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Drawing style

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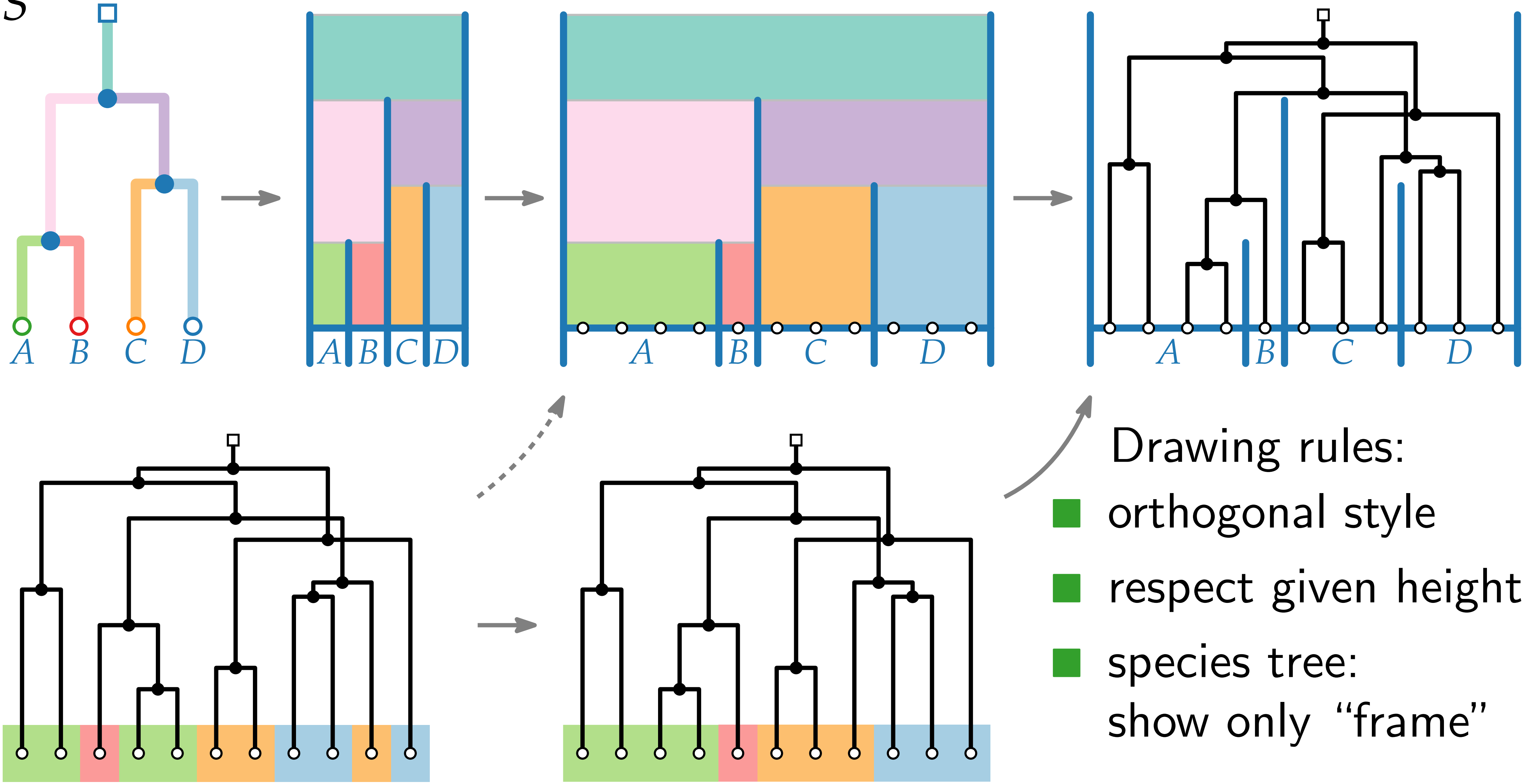
Gene tree T



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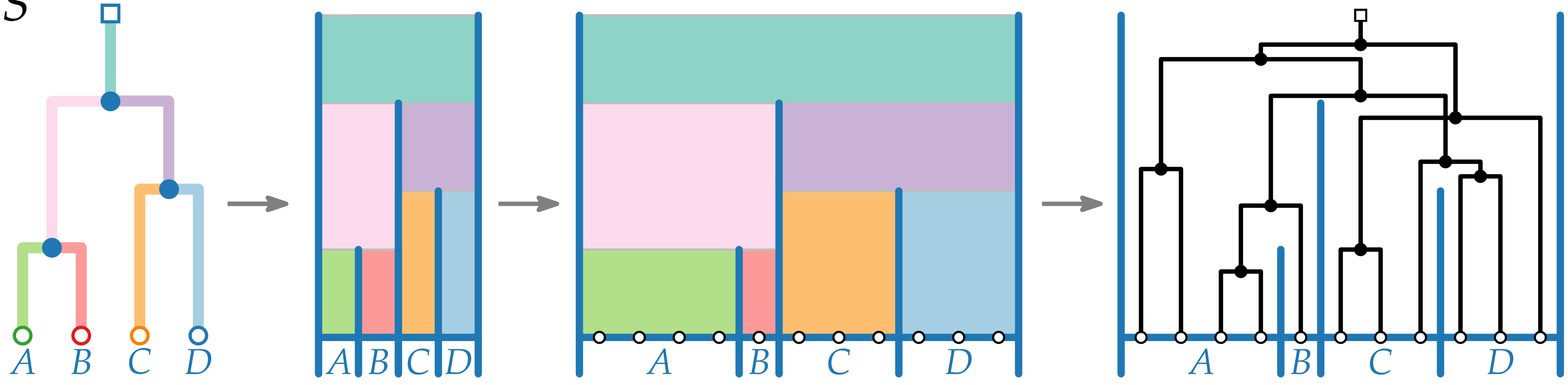

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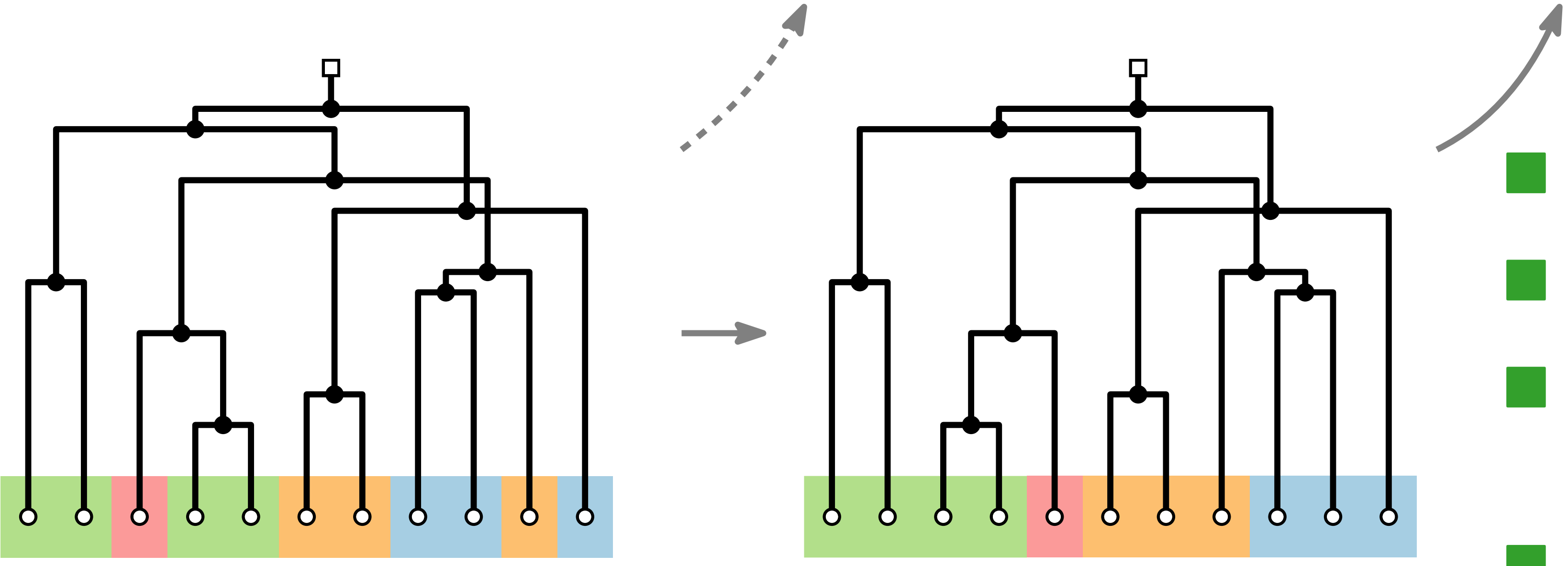

Drawing rules:

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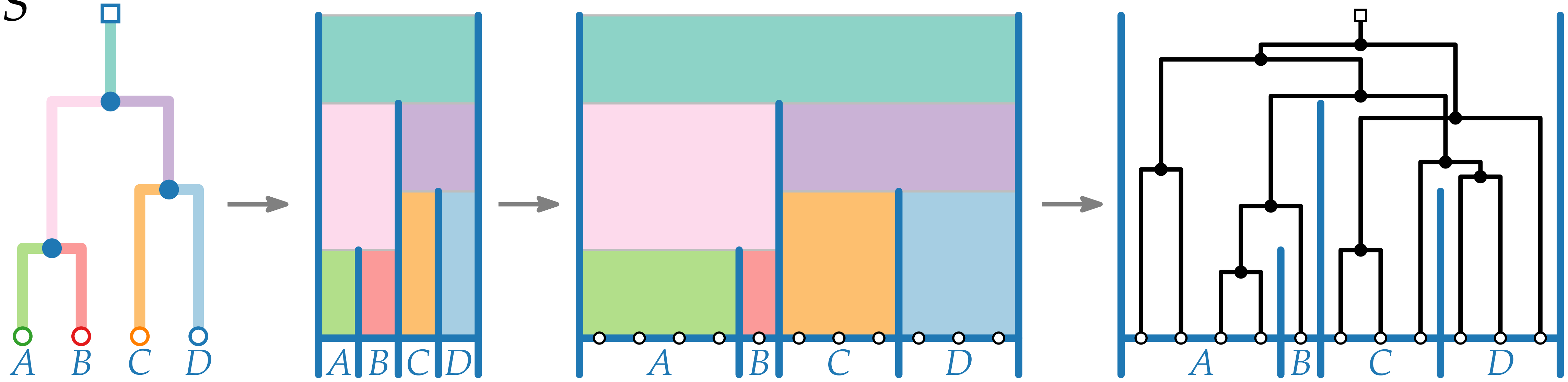
- gene tree: center vertices horizontally

Drawing style

Species tree S



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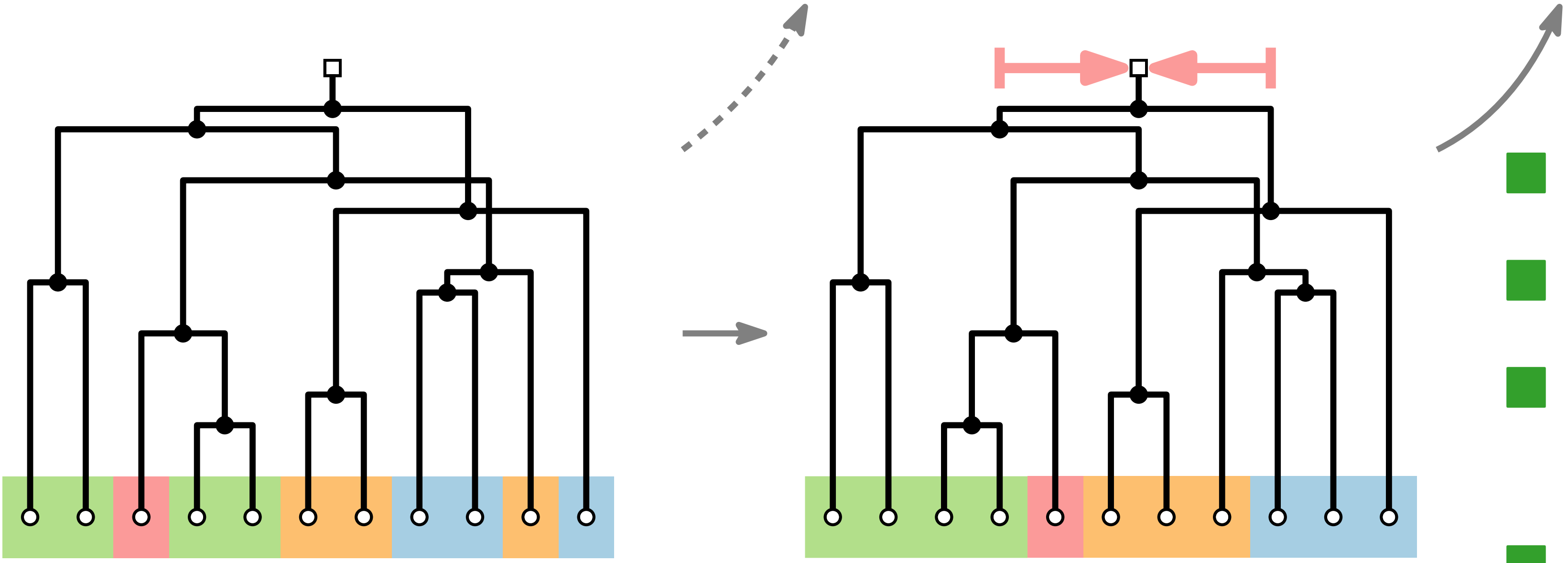

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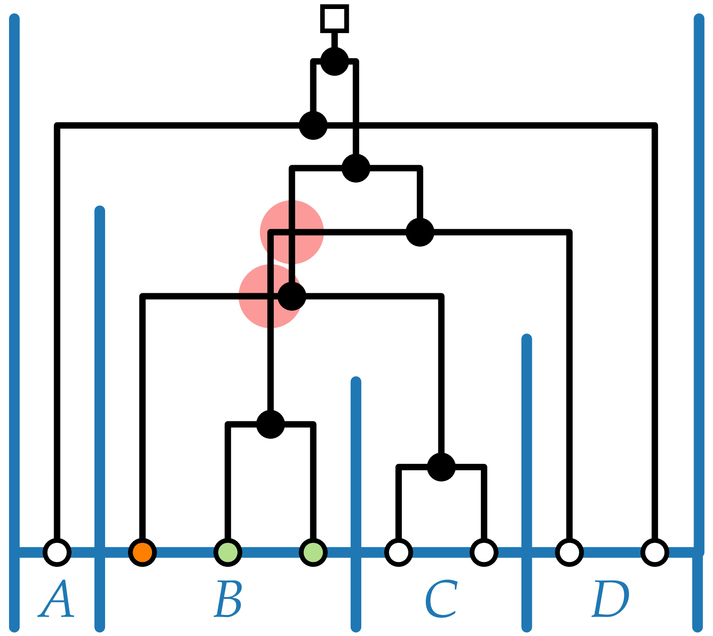
Crossing minimization

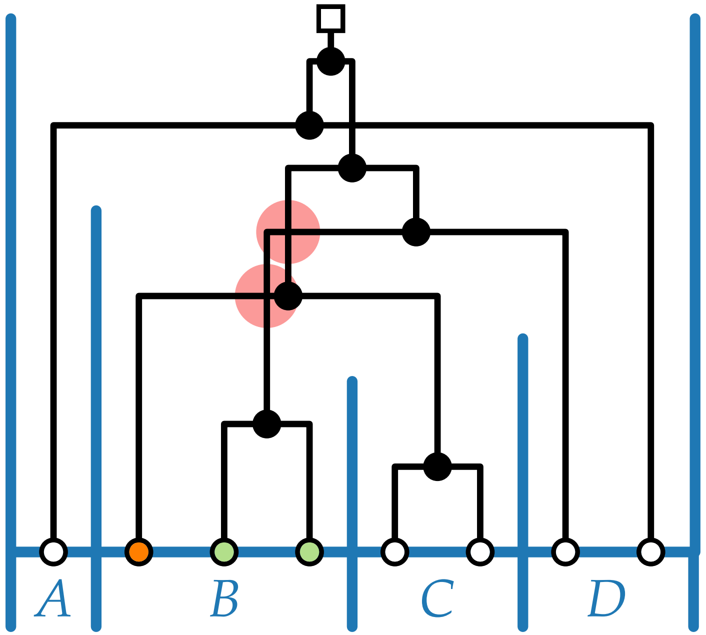
Minimize the number of crossings by. . .

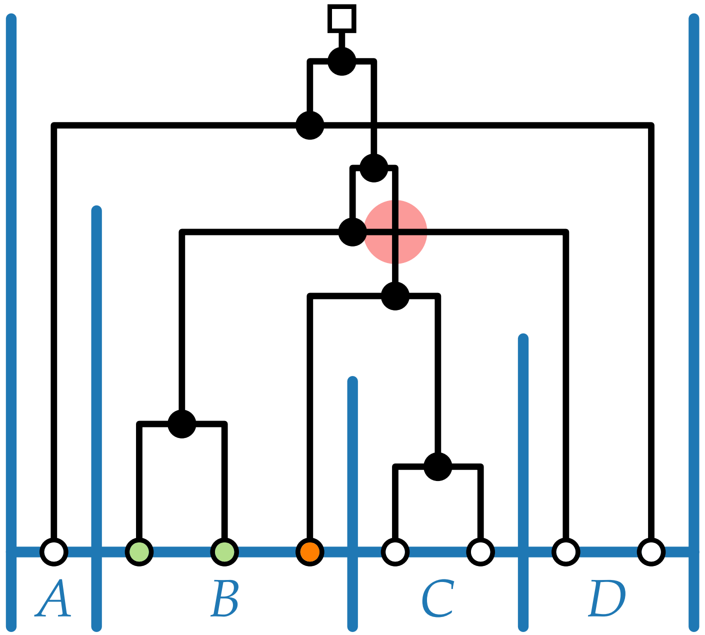


Crossing minimization

Minimize the number of crossings by. . .

changing gene order





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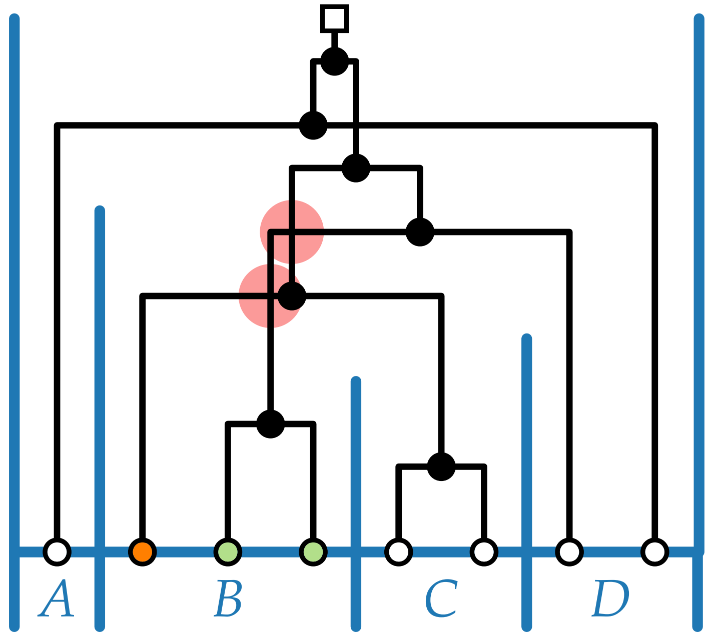

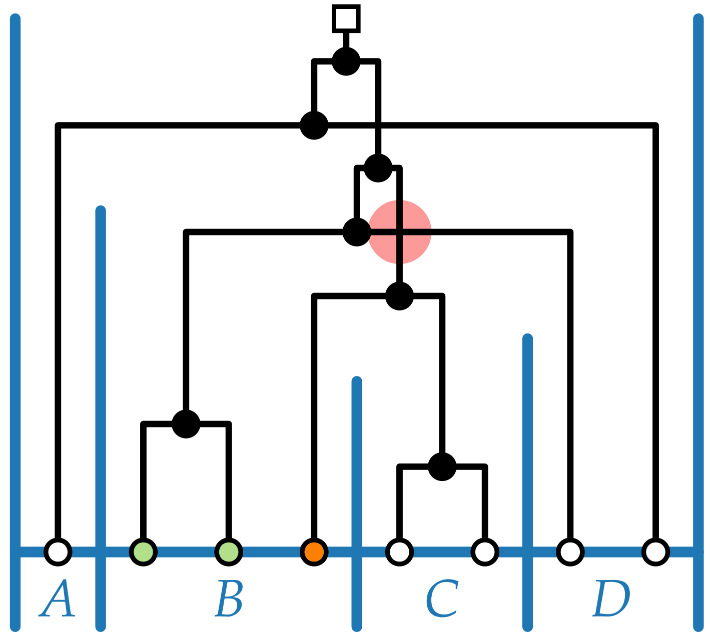

🟩 **fixed** species order

Crossing minimization

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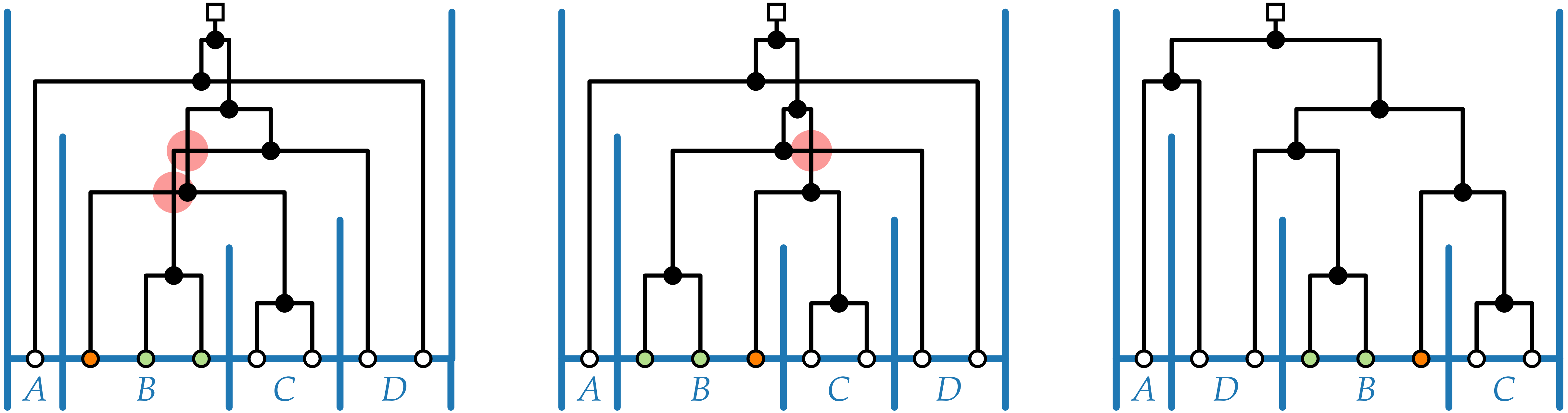
changing gene order + changing species order



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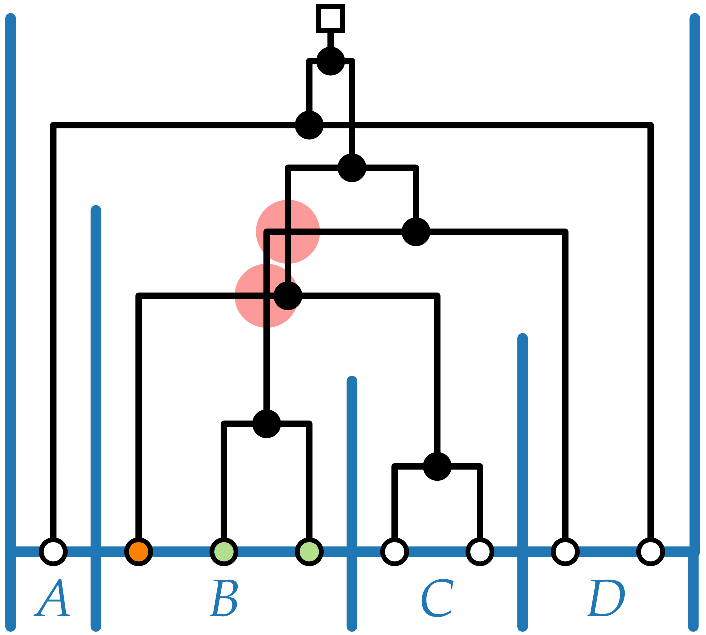

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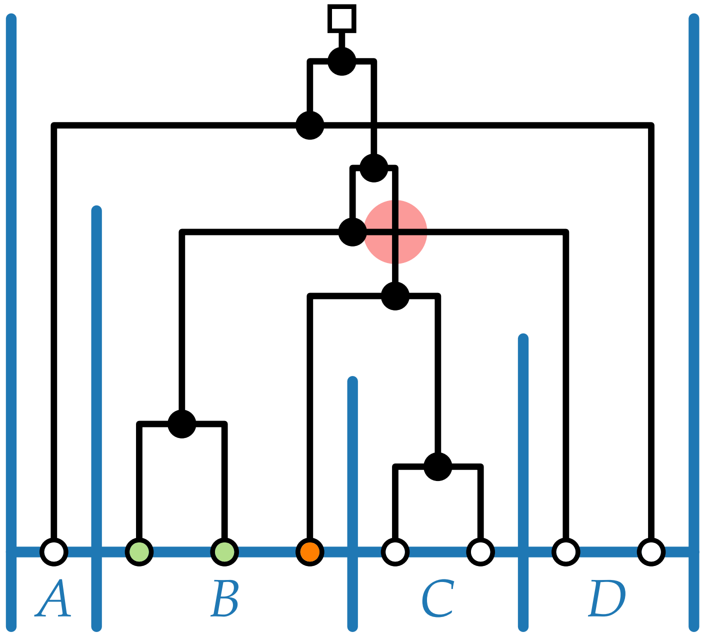

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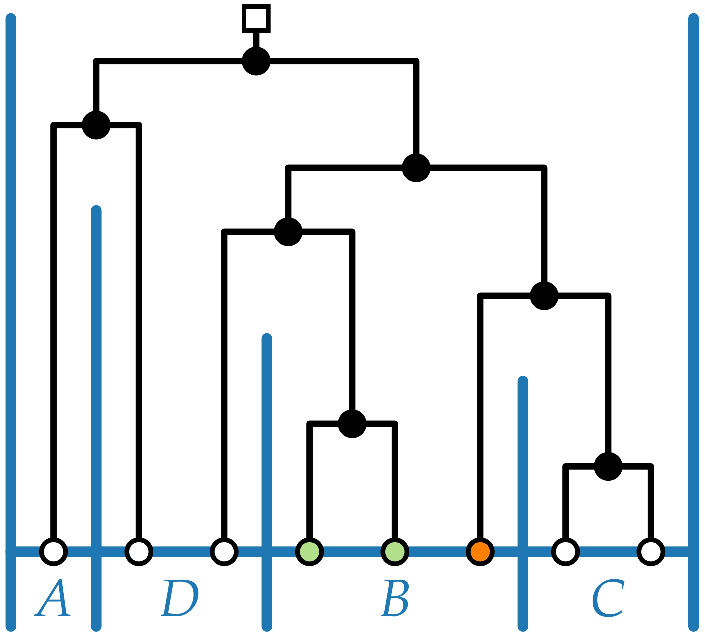

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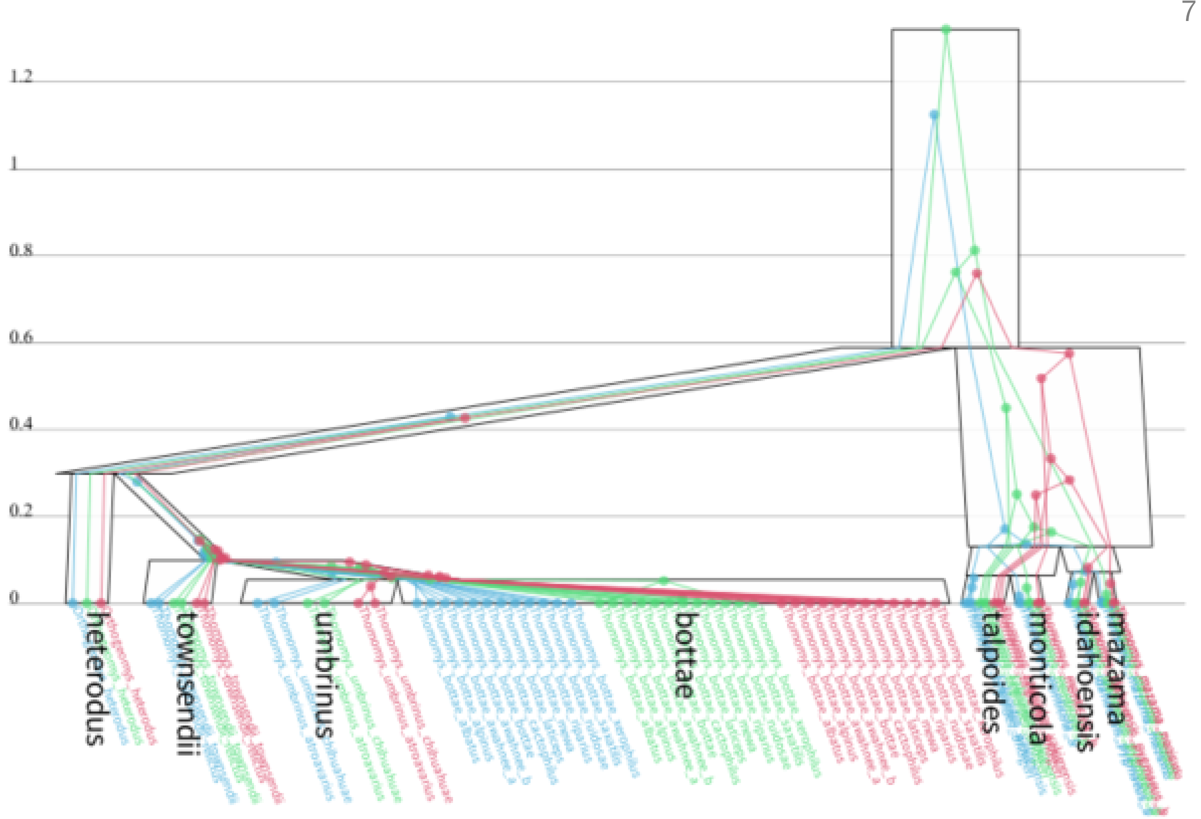
🟩 **variable** species order

Related work

- Tool *UglyTrees* by Douglas uses a different model of MSC trees.



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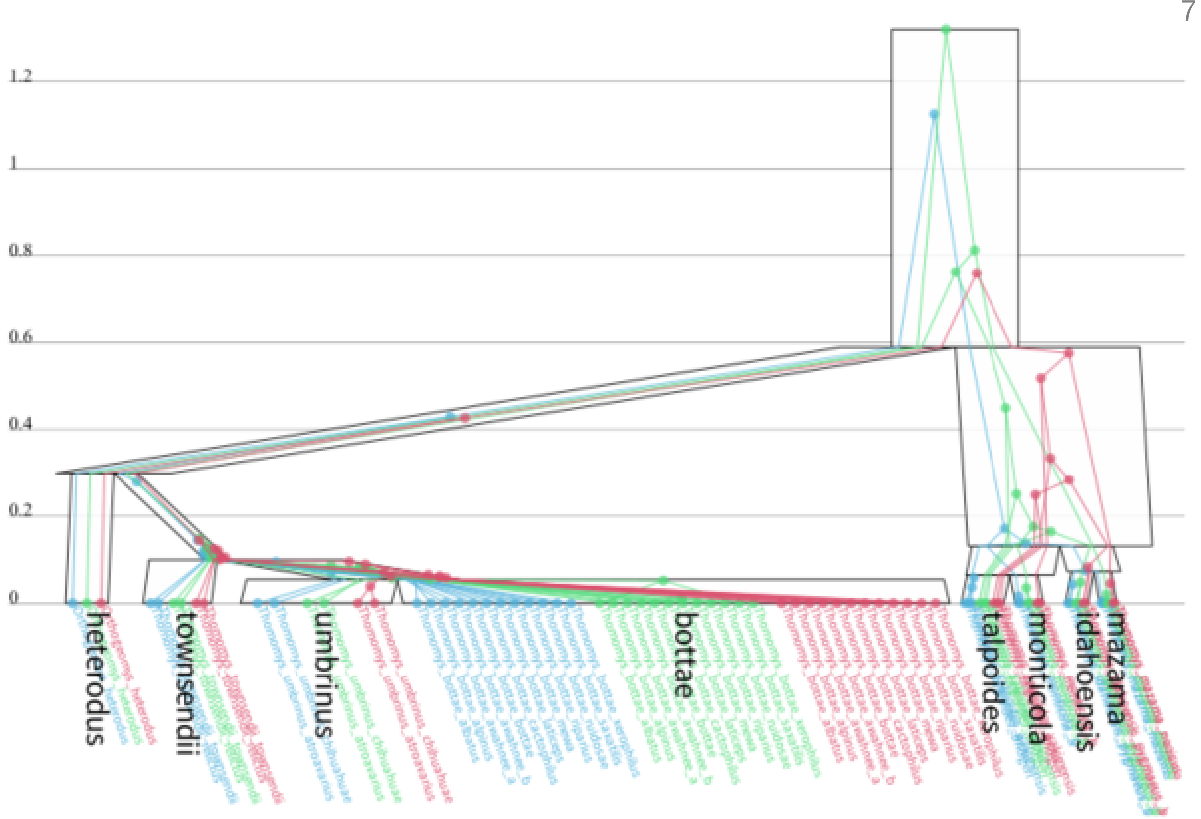
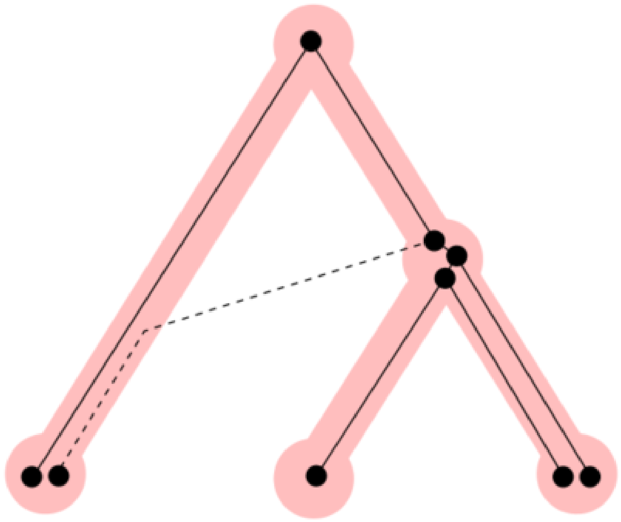
    - Pair of host and parasite tree
    - Parasite vertex heights not given
    - Host and parasite tree can cross





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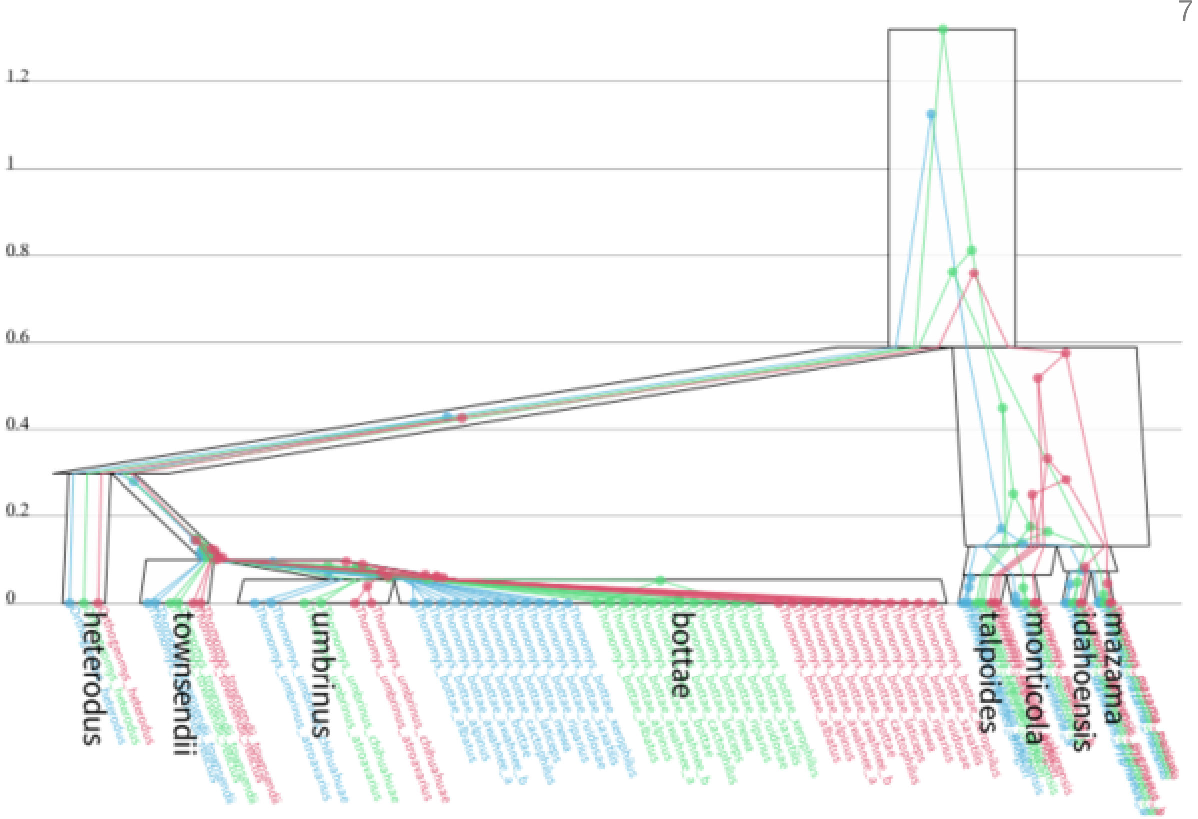
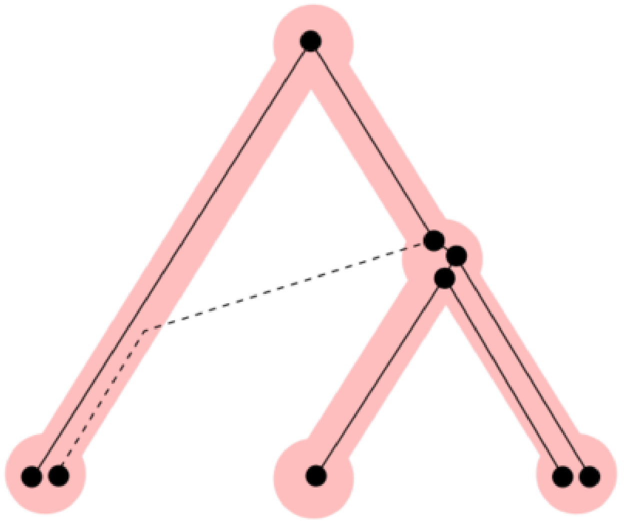
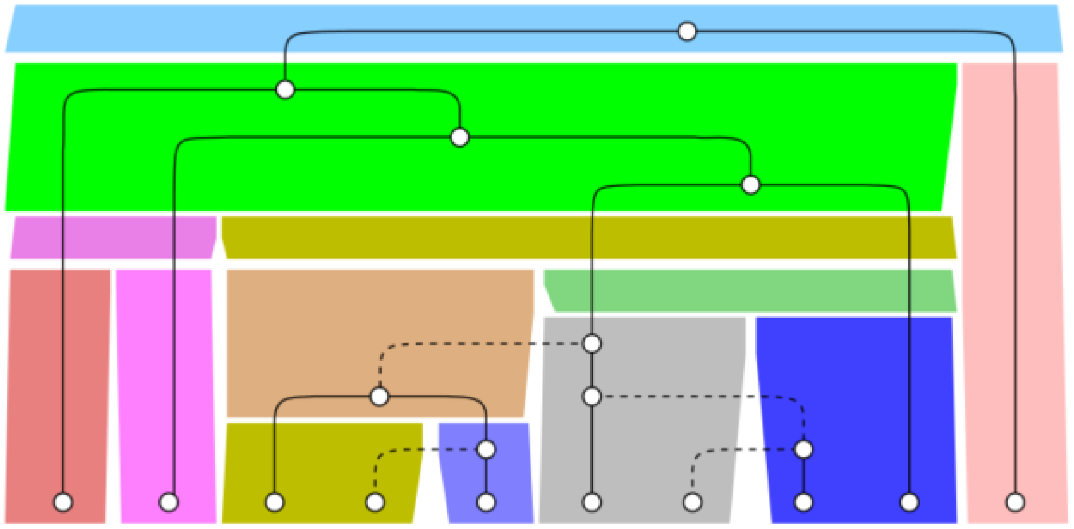
  - Space-filling approach by Calamoneri, Di Donato, Mariottini, Patrignani [TCS'20]







Our contribution

- We have defined drawing style for MSC trees.
- Planarity testing is easy.
- Crossing minimisation is NP-hard (for fixed & variable species order).
- We present two simple heuristics and an integer linear program (ILP).
- We do an experimental evaluation of the heuristics.

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Planar instances

variable species order

fixed species order

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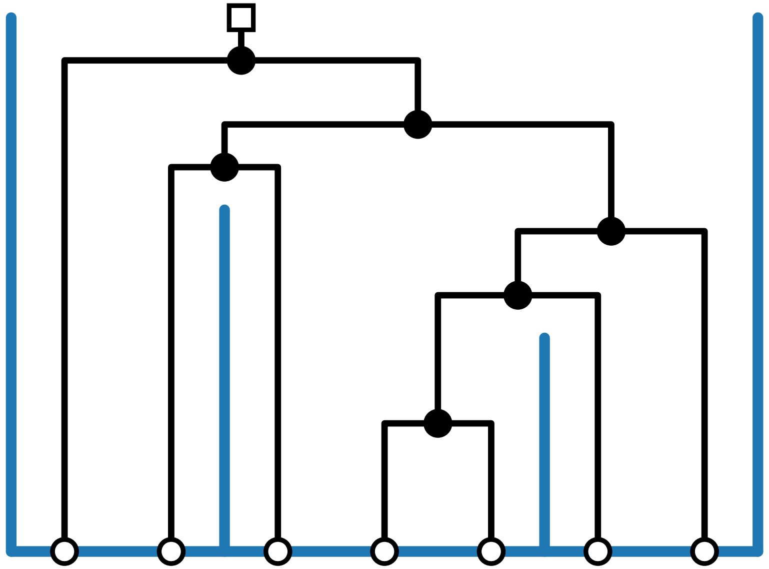
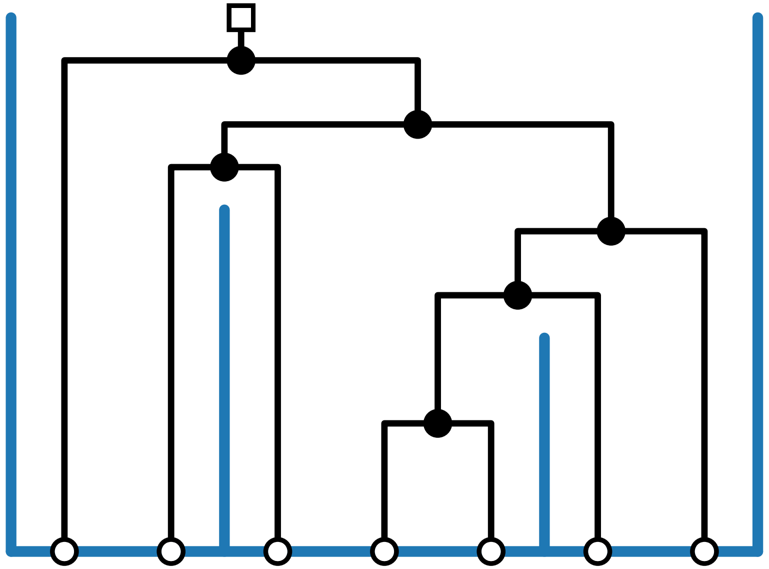
variable species order



fixed species order



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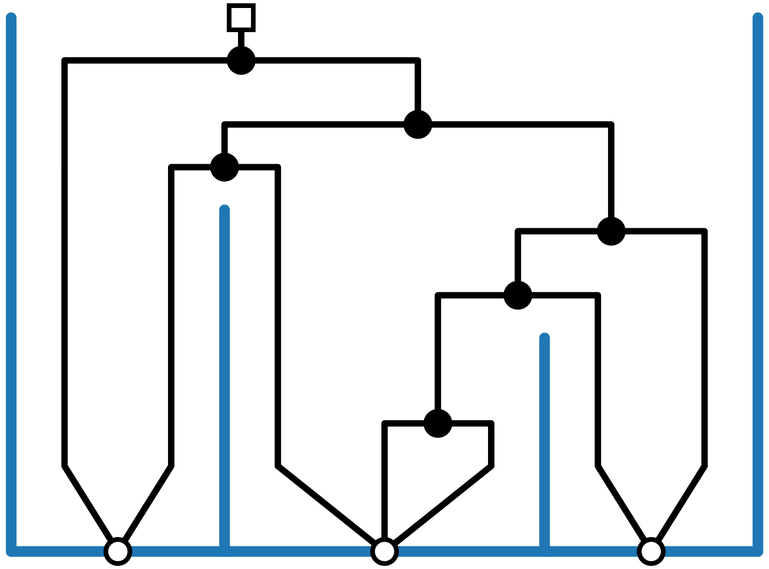

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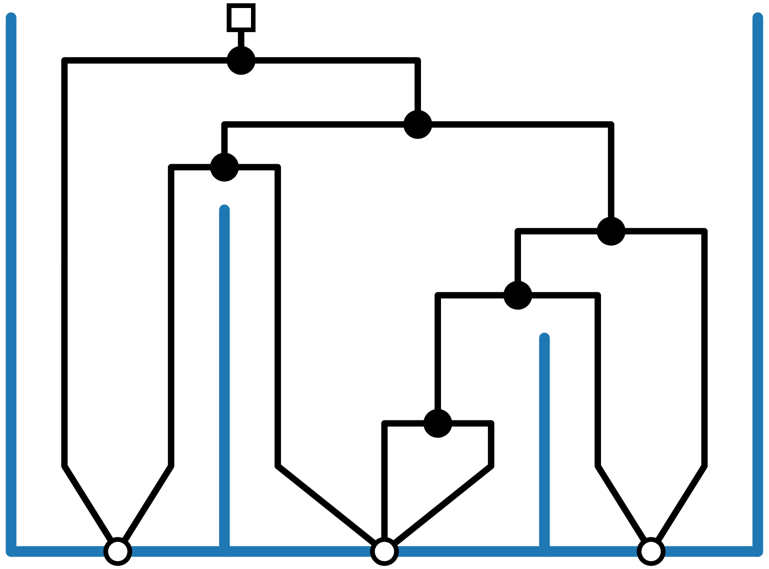
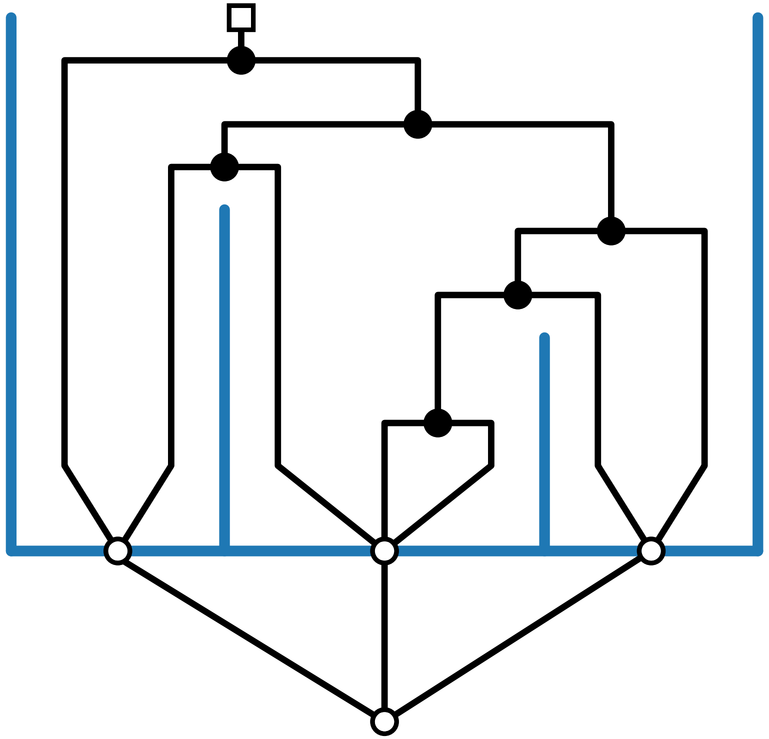
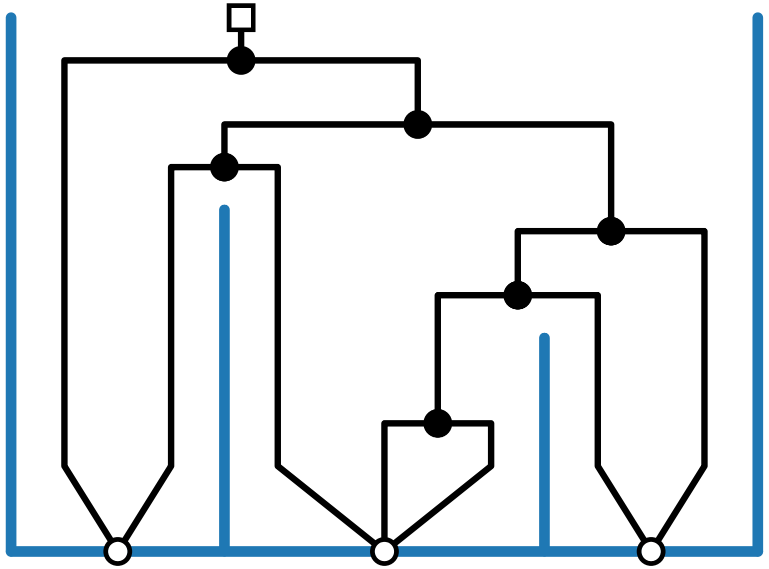


Planar instances

variable species order



fixed species order



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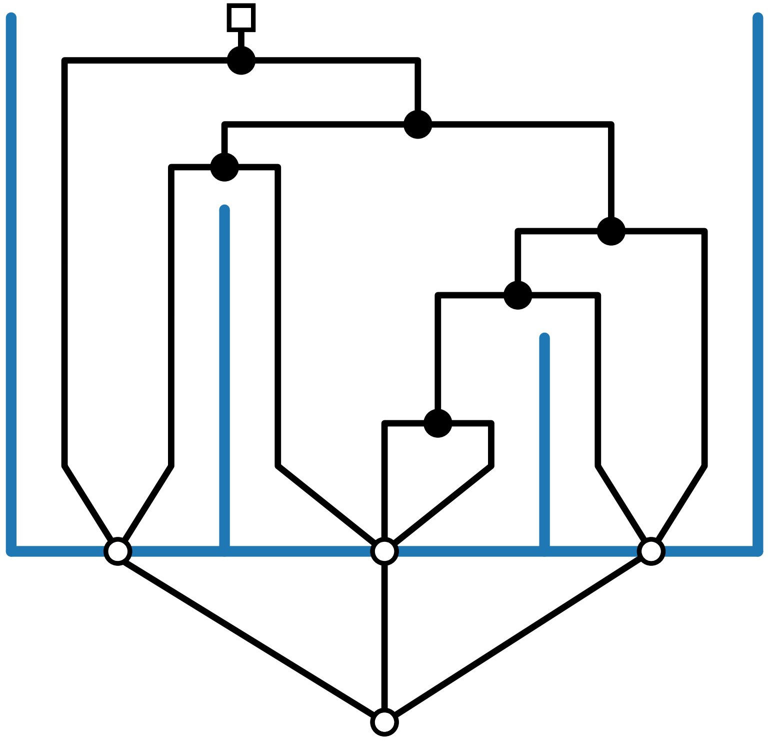

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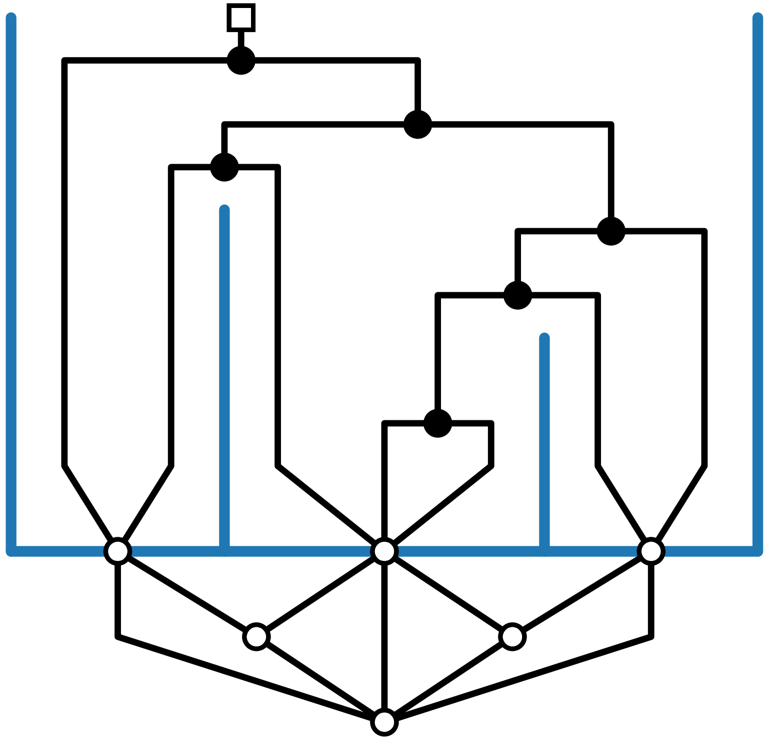

Planar instances

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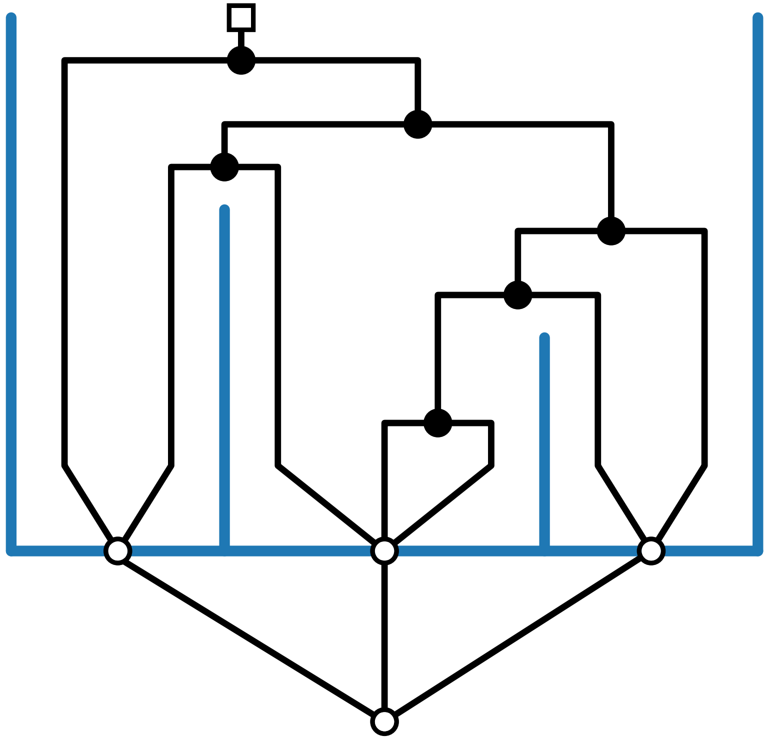
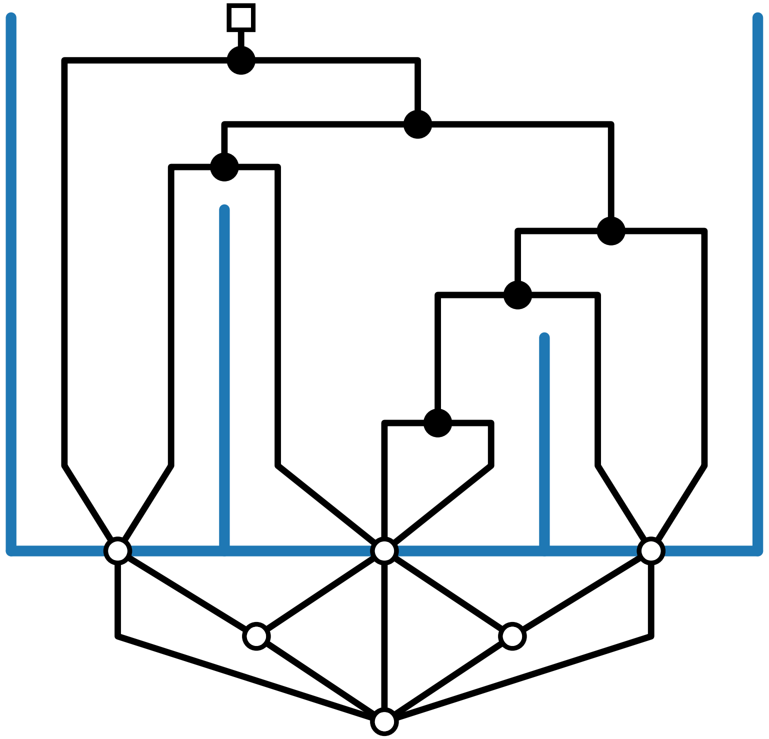


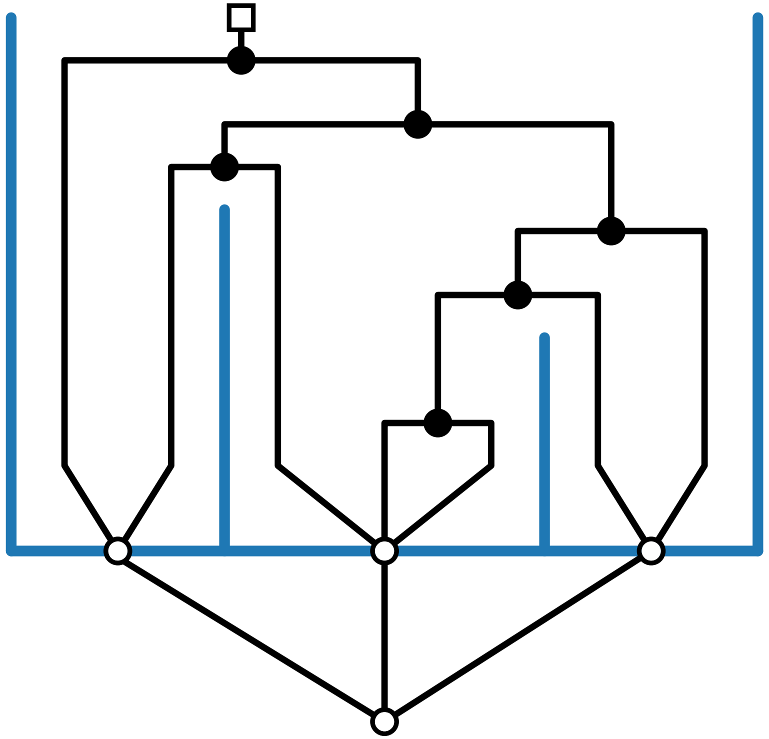
Upward planar!

fixed species order



Planar instances

variable species order



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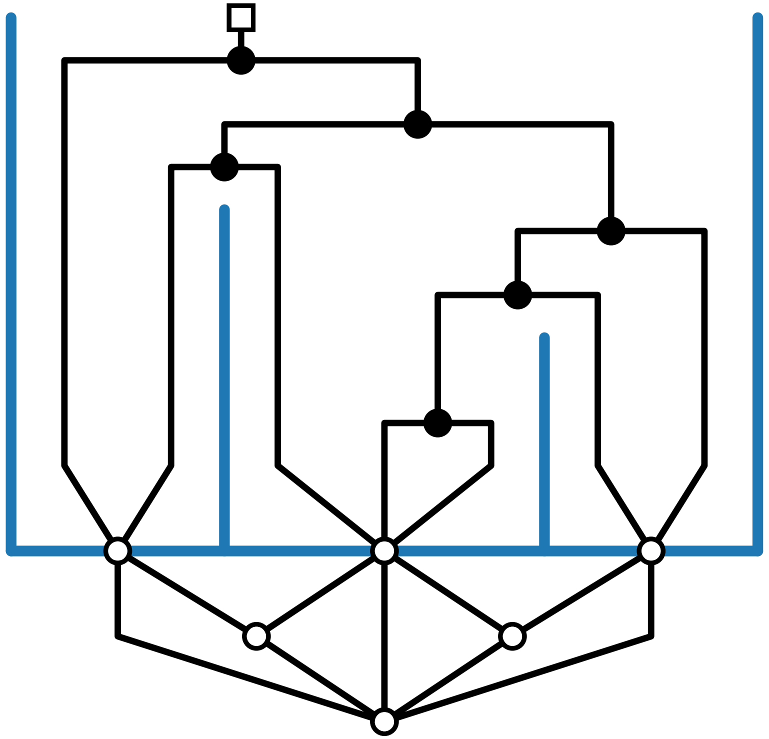

Upward planar! → Testing upward planarity of single-source digraphs can be done in linear time.

[Bertolazzi, Di Battista, Mannino, Tamassia: SIAM J. Comput. '98]

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NP-Hardness

Theorem.
Crossing minimisation for tree-in-tree drawings of MSC trees with *variable* species order is **NP-hard**.

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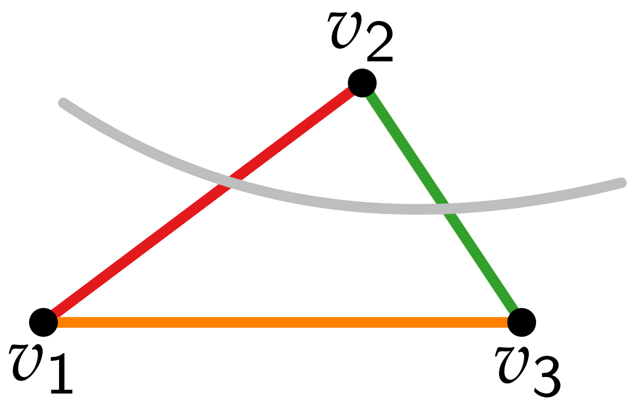
NP-Hardness with variable species order

Reduction from MAX-CUT



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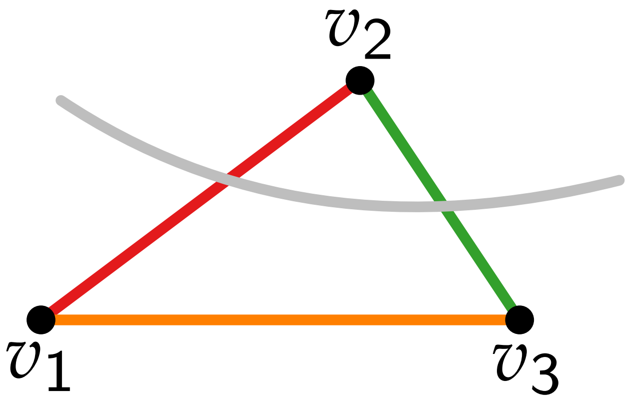

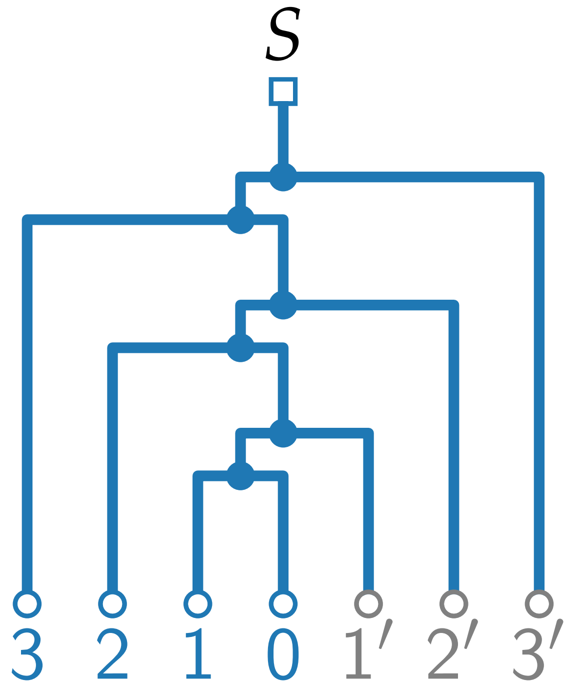

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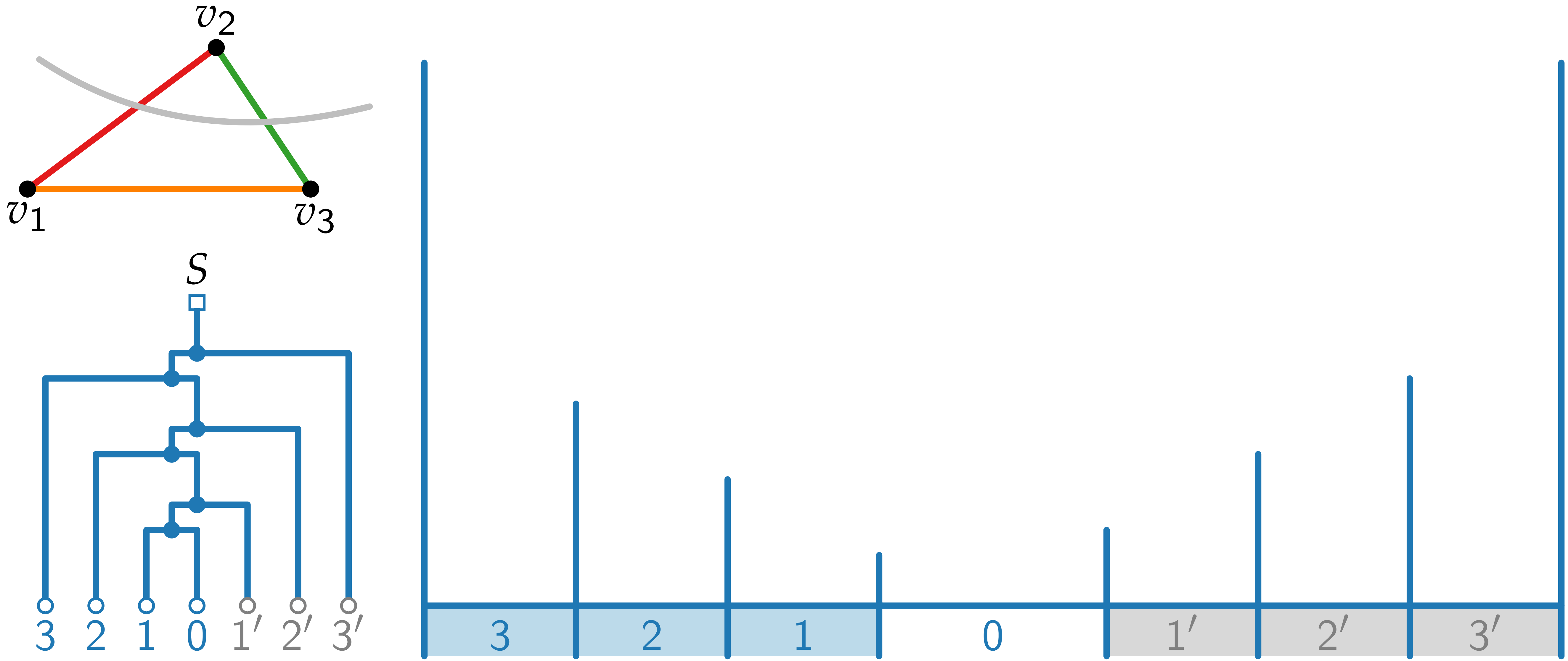

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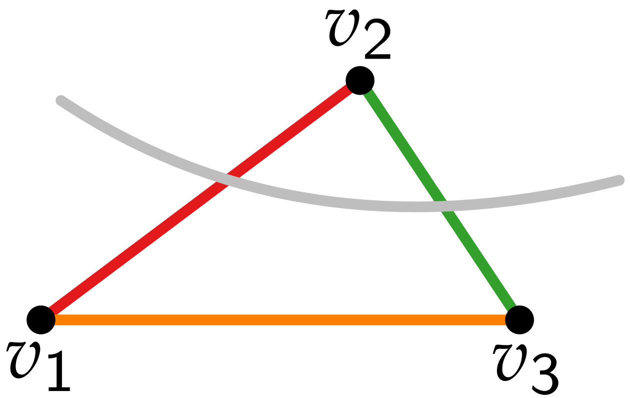

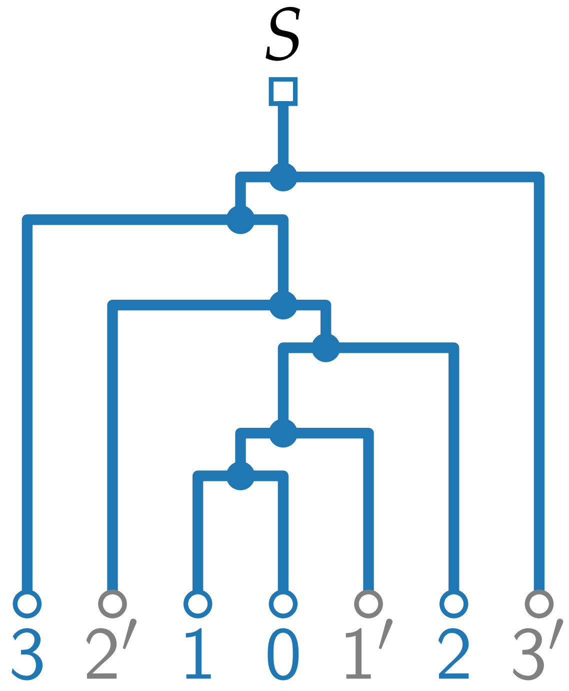

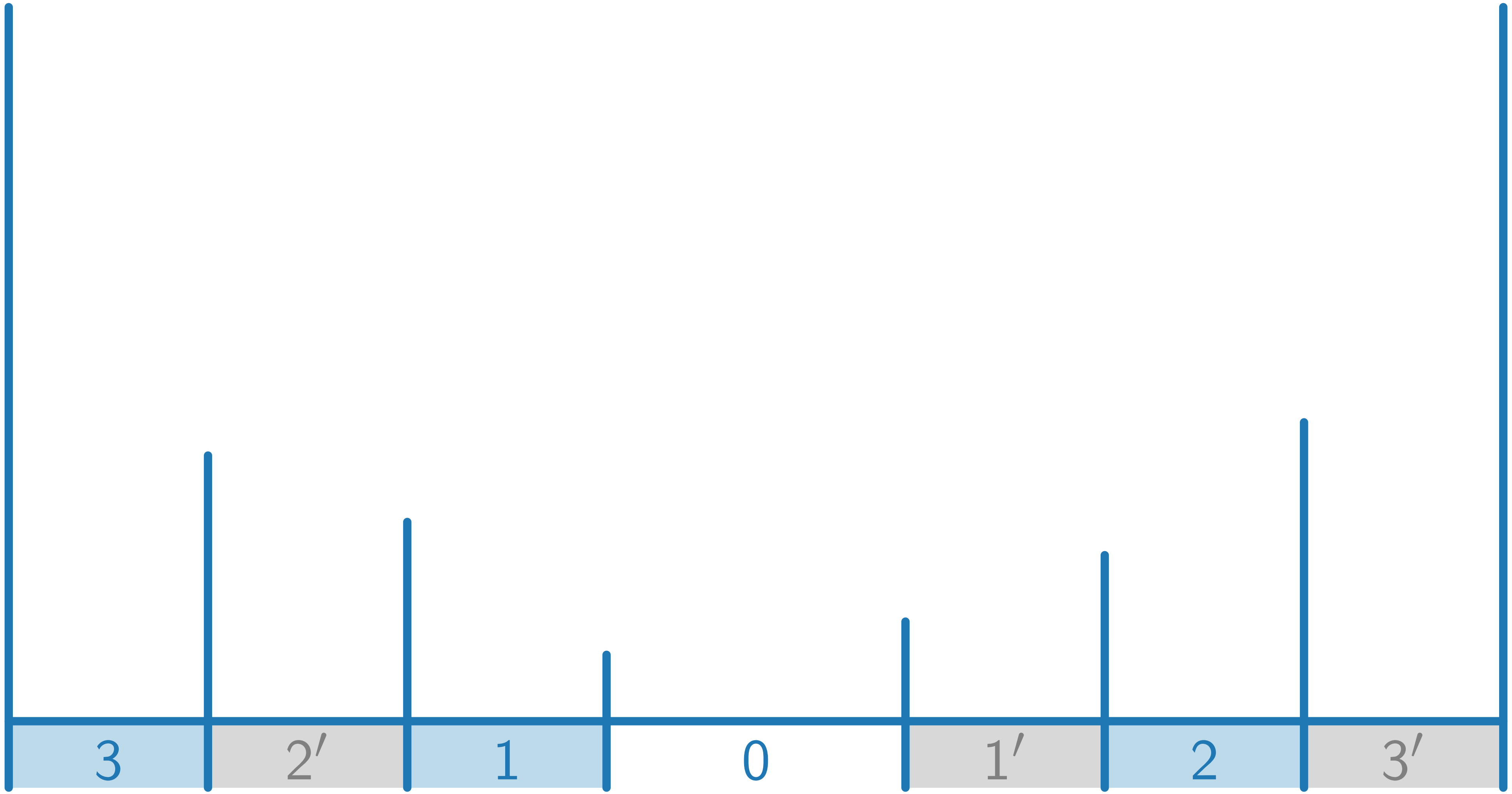

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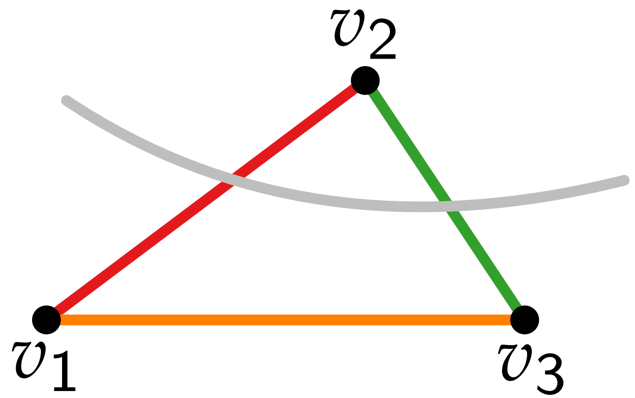

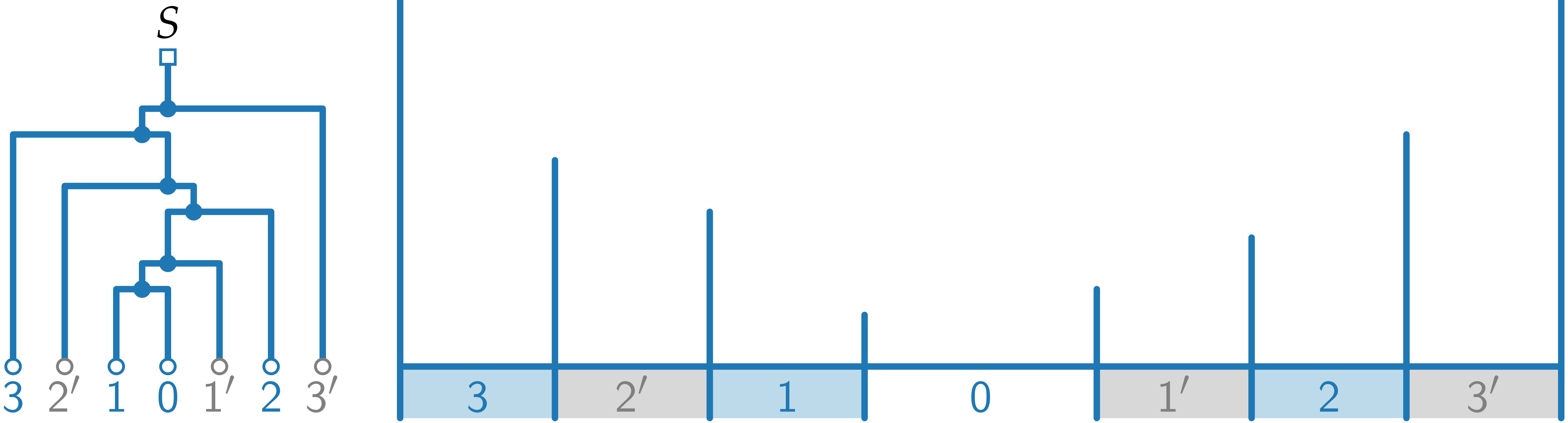

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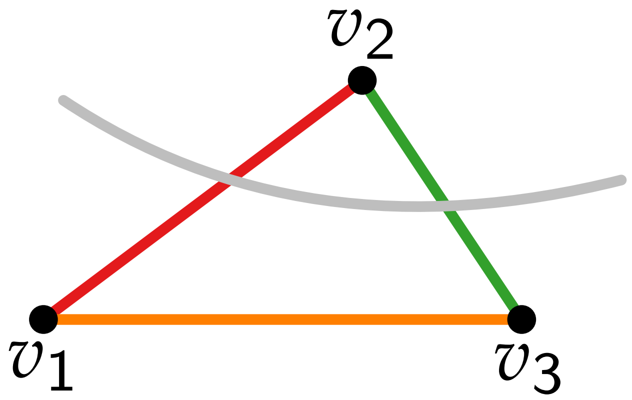

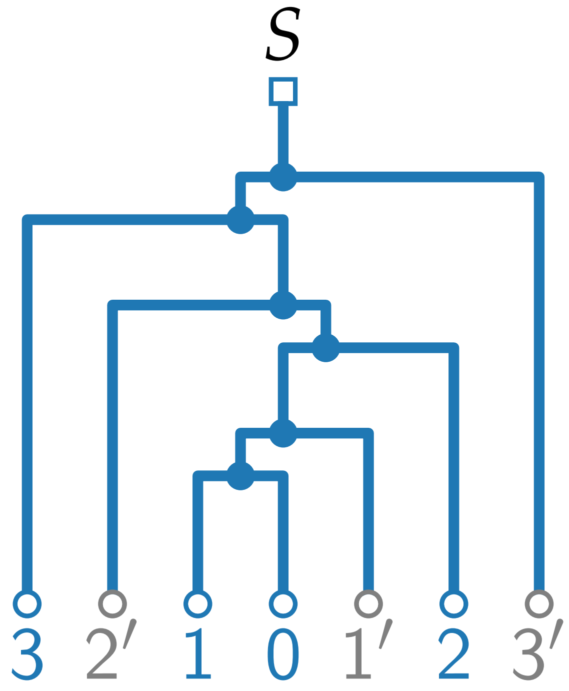

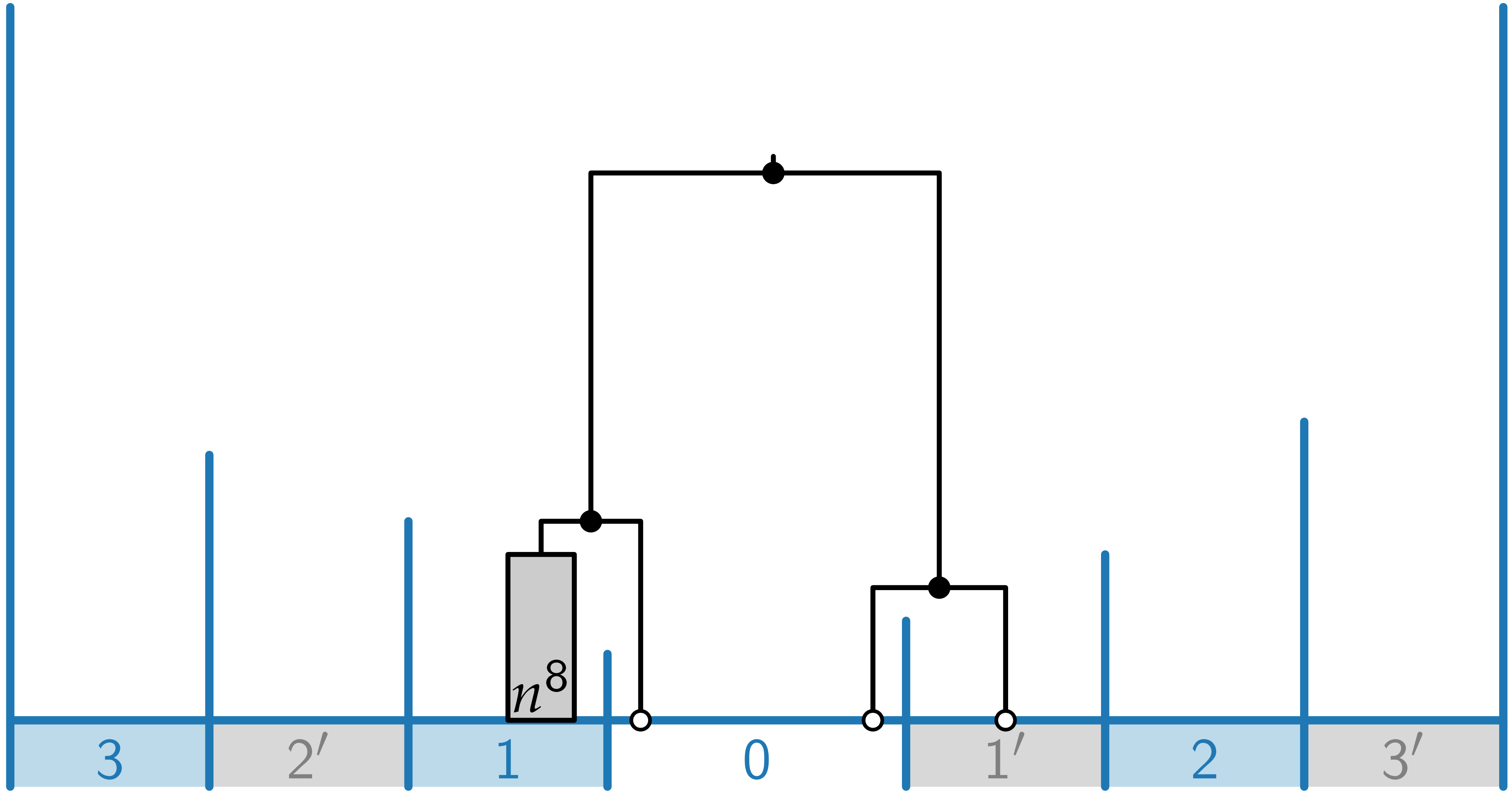

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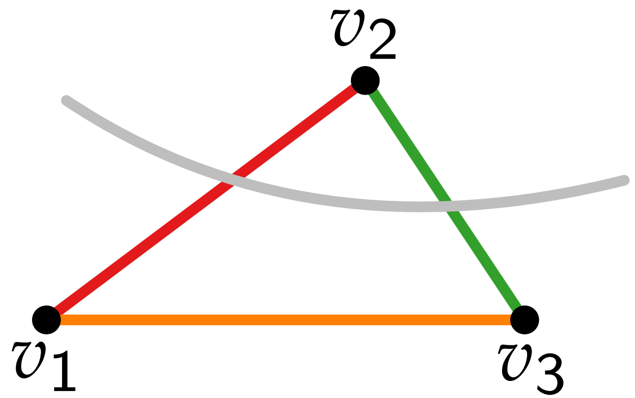

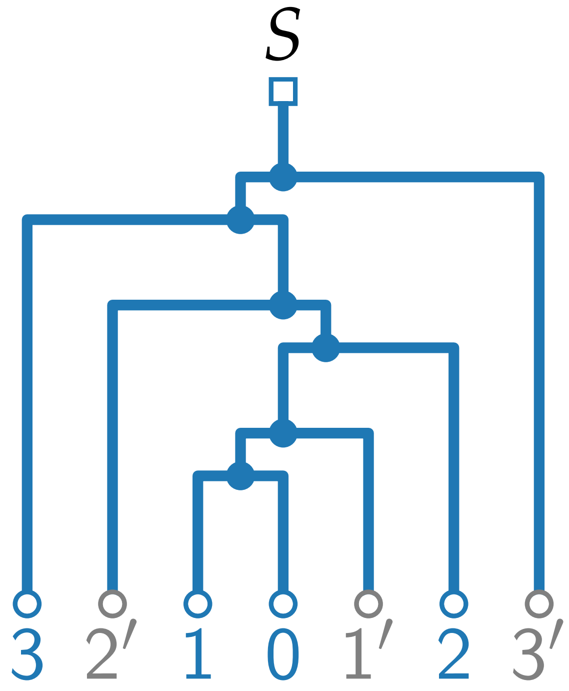

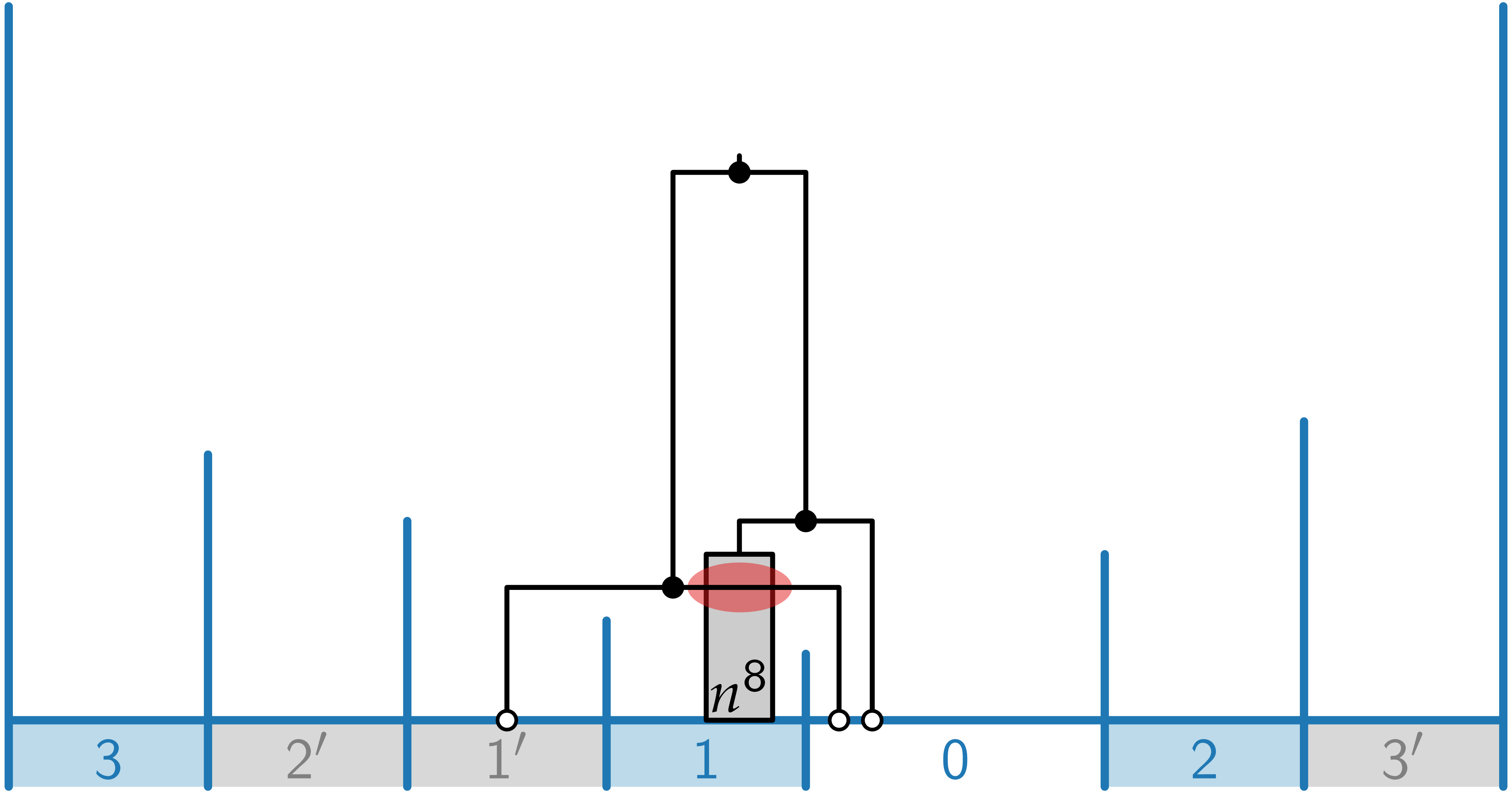

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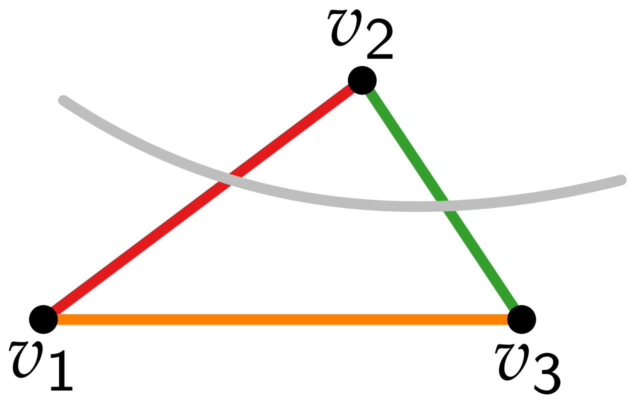

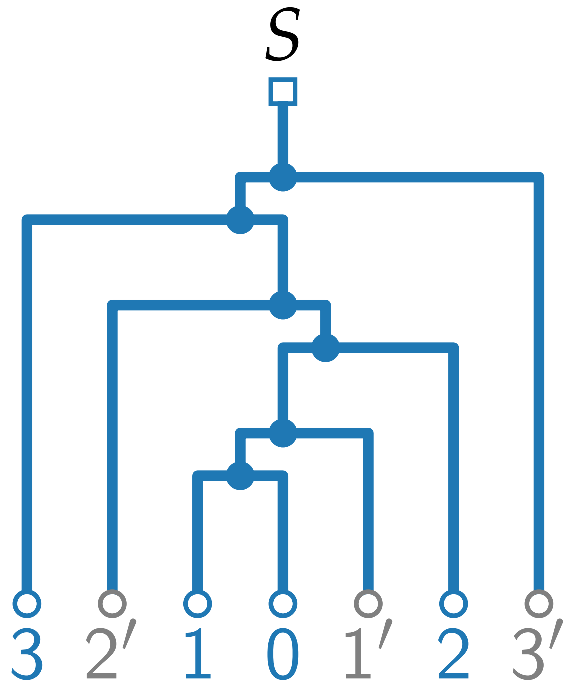

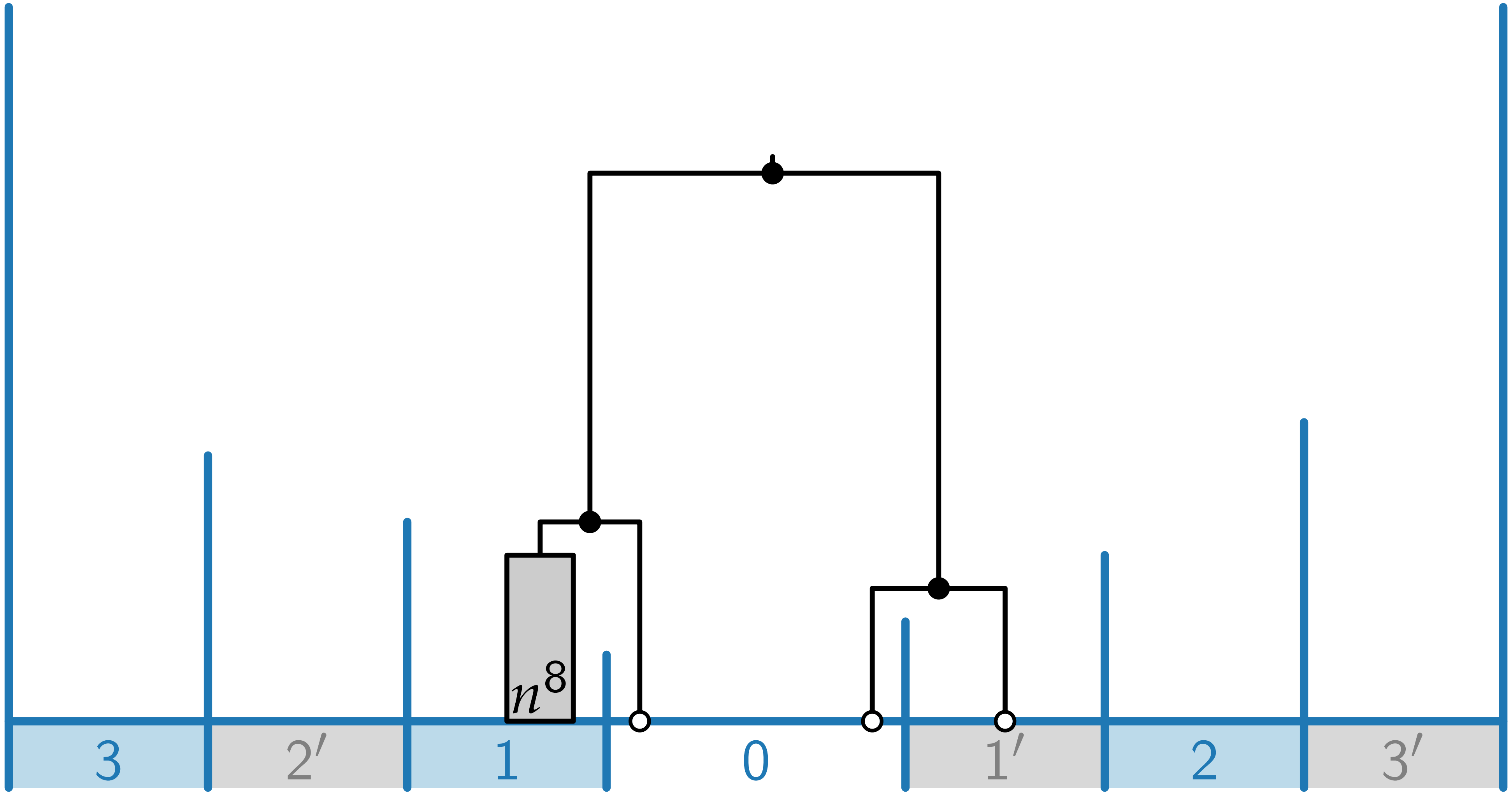

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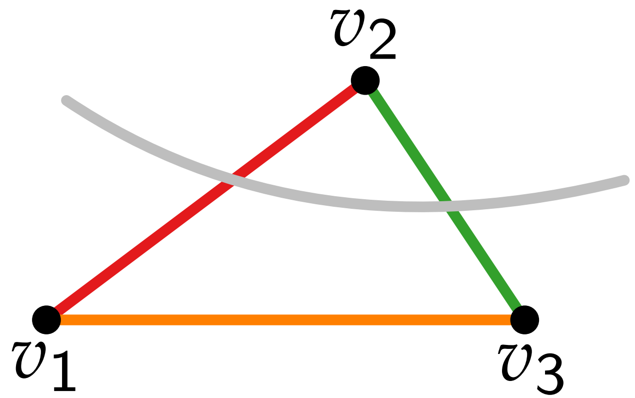

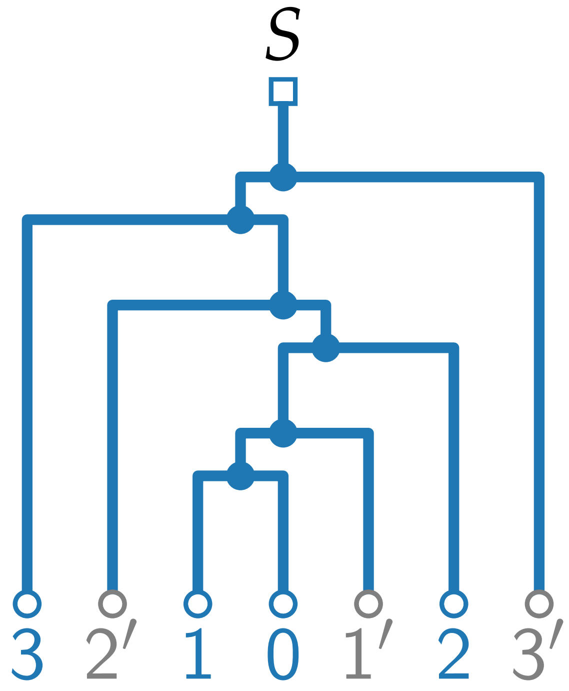

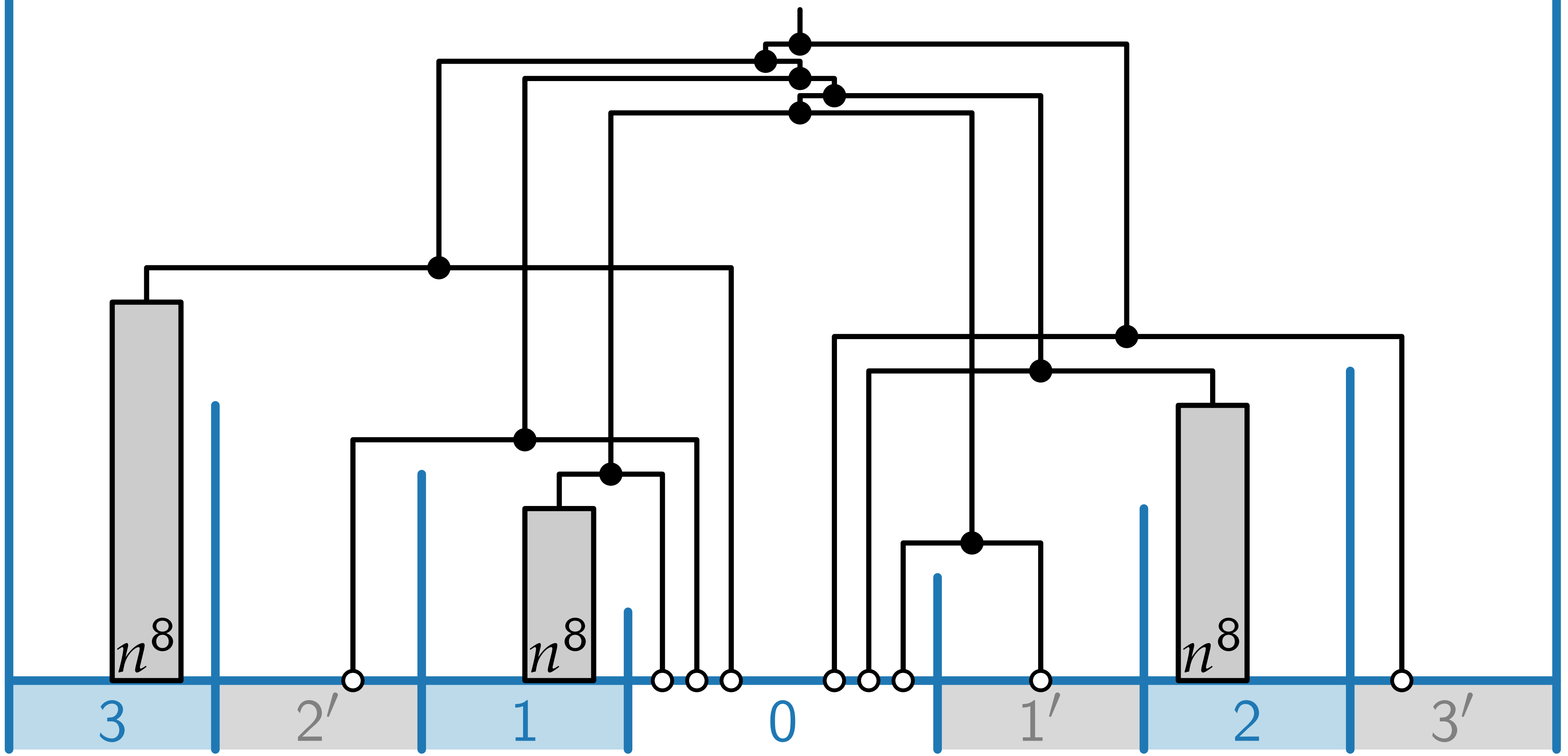

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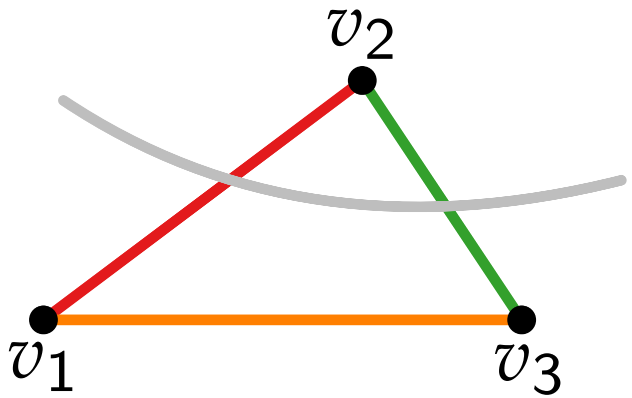

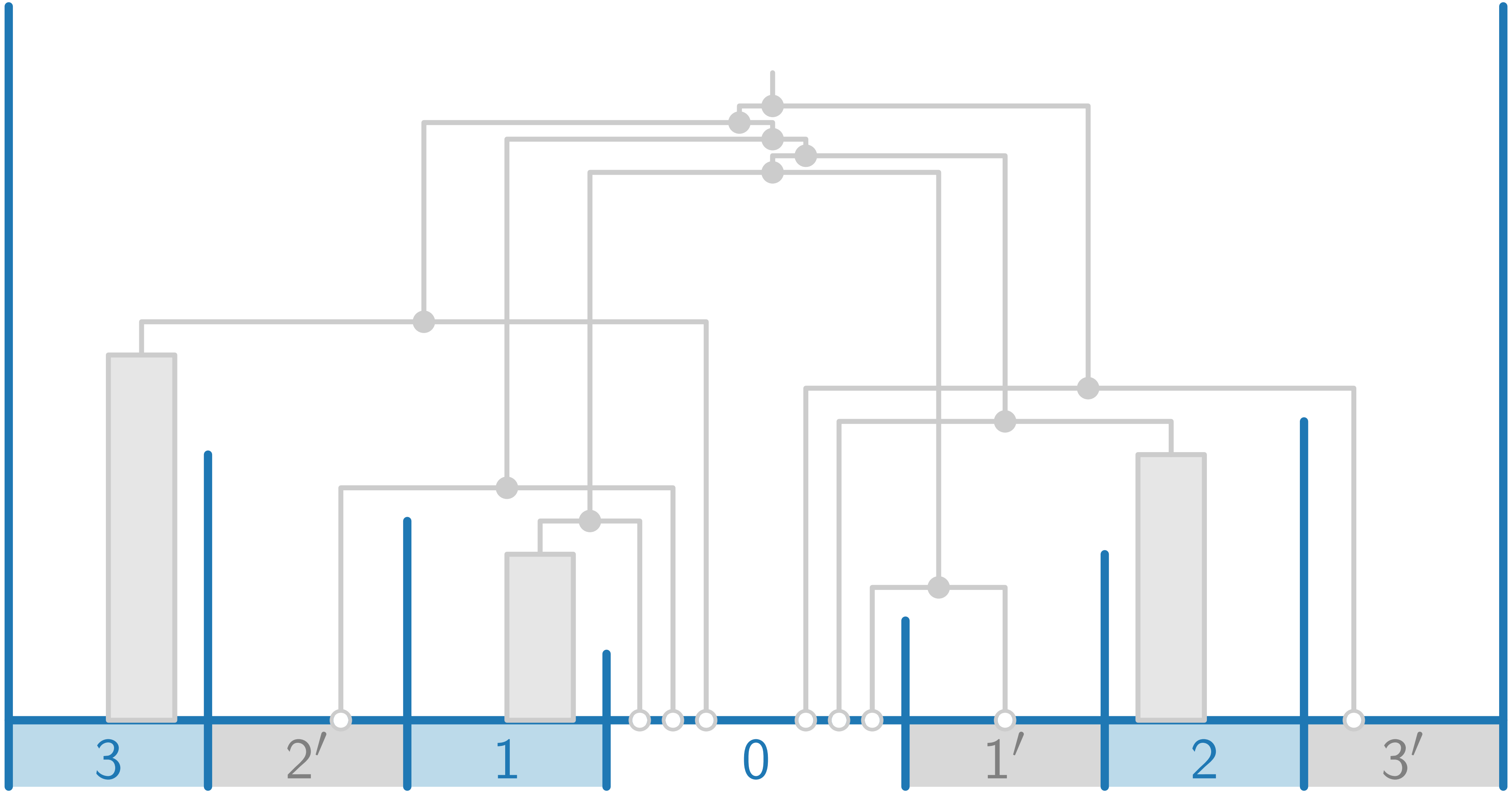

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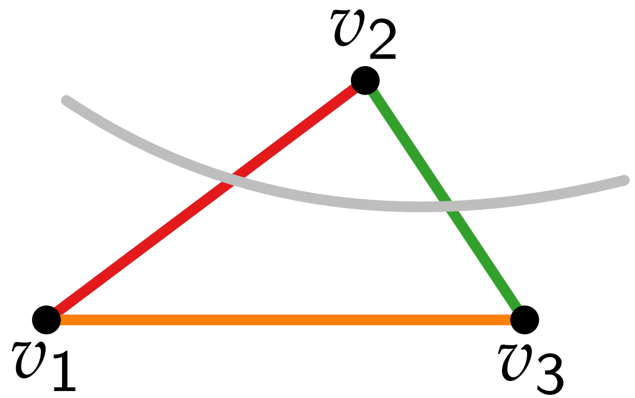

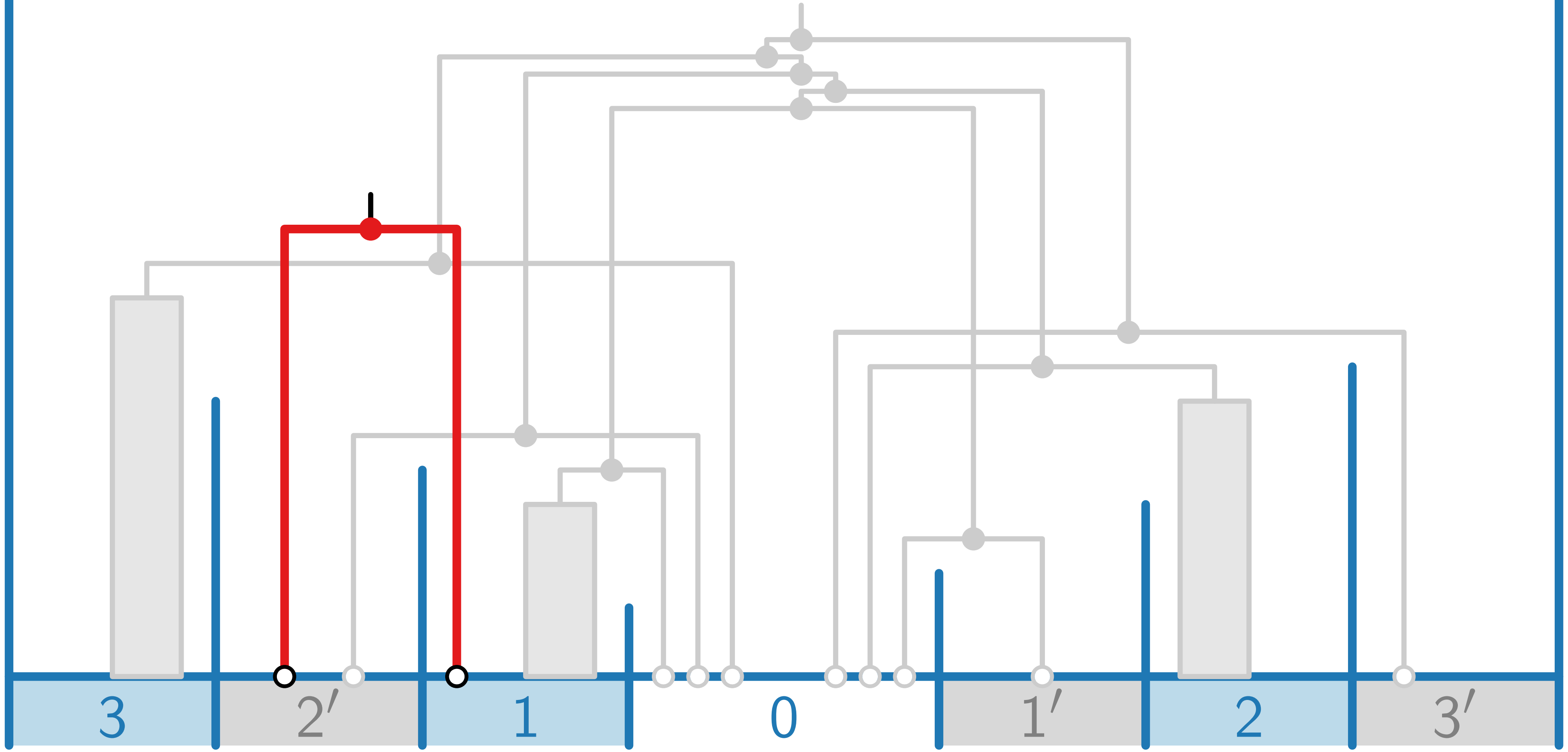

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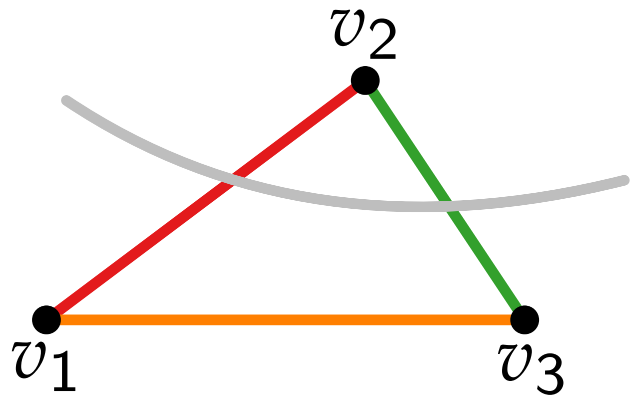

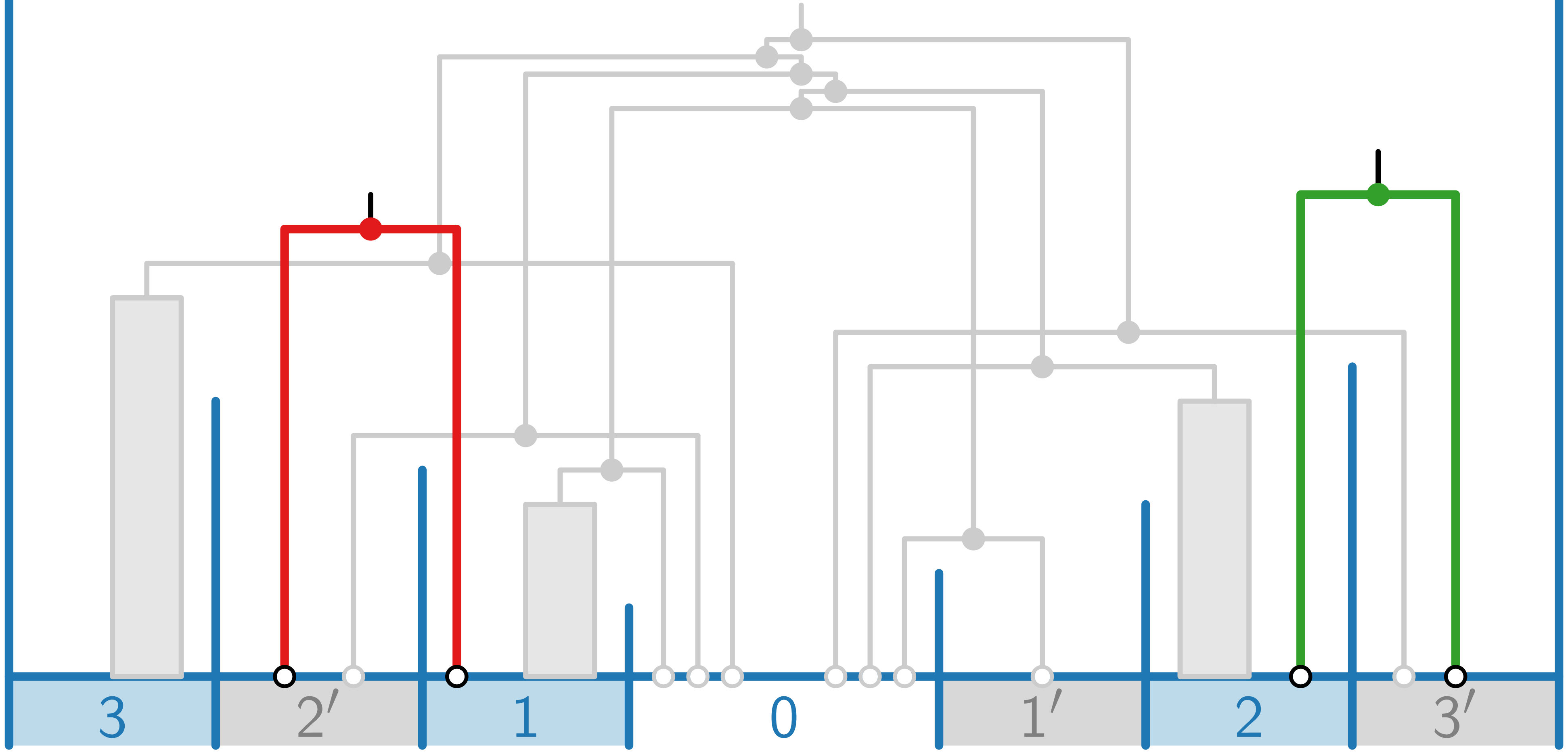

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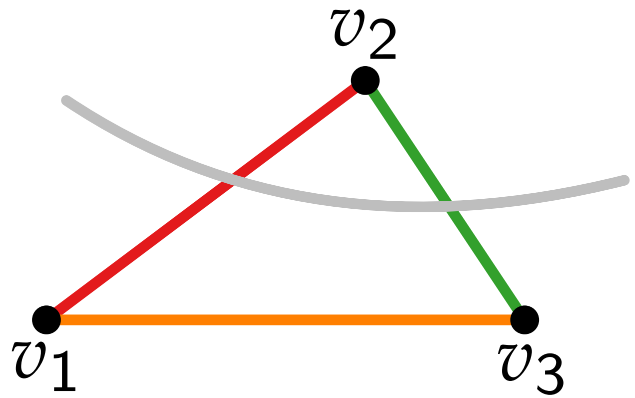

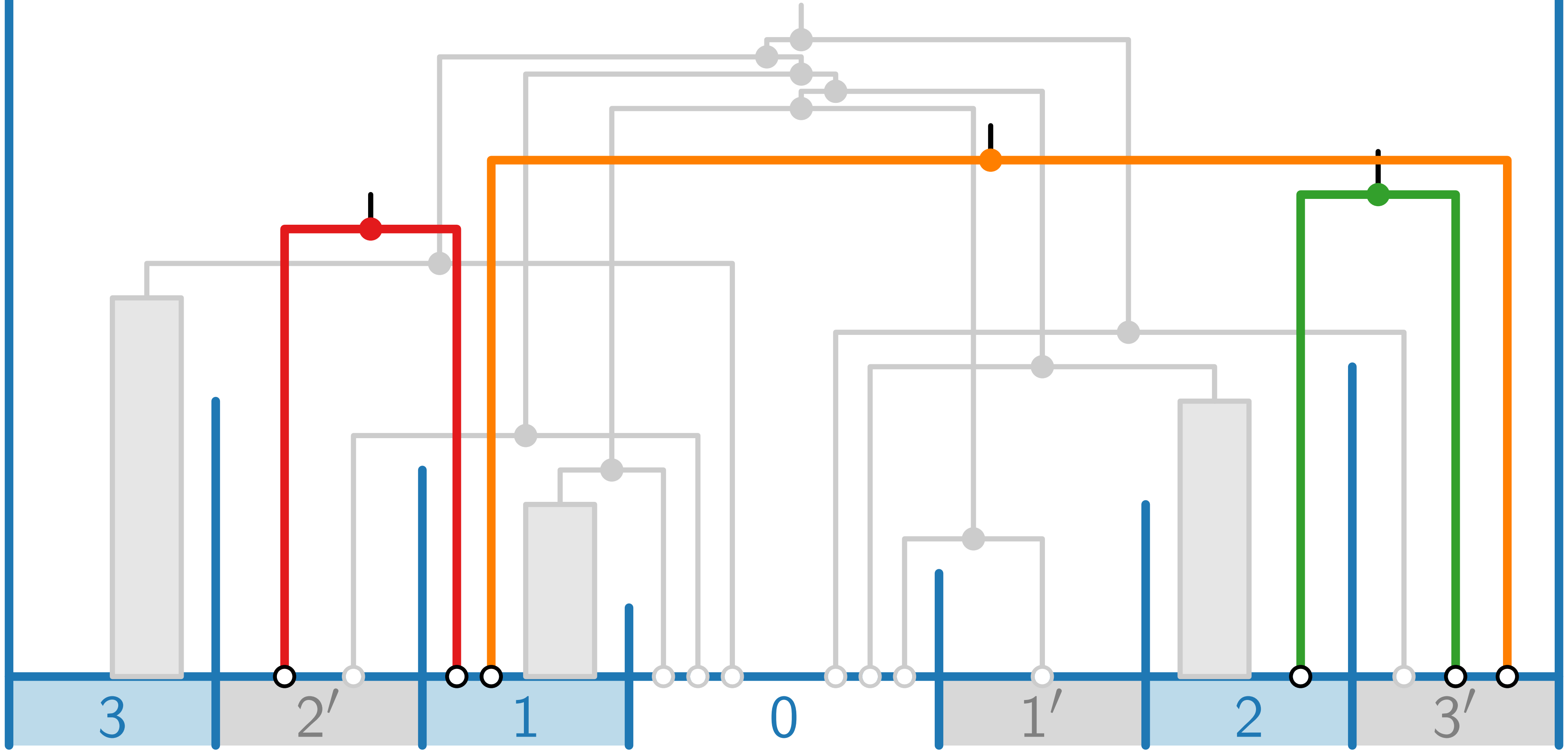

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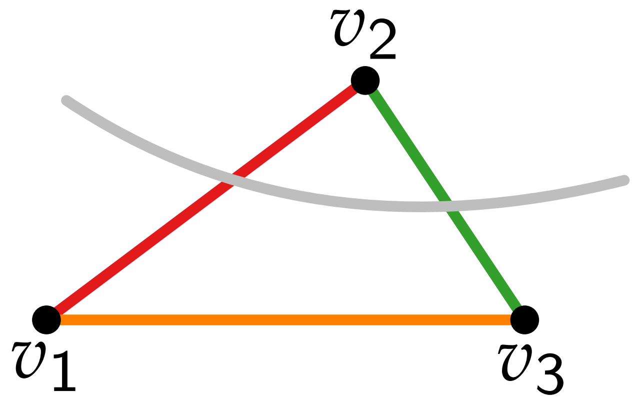

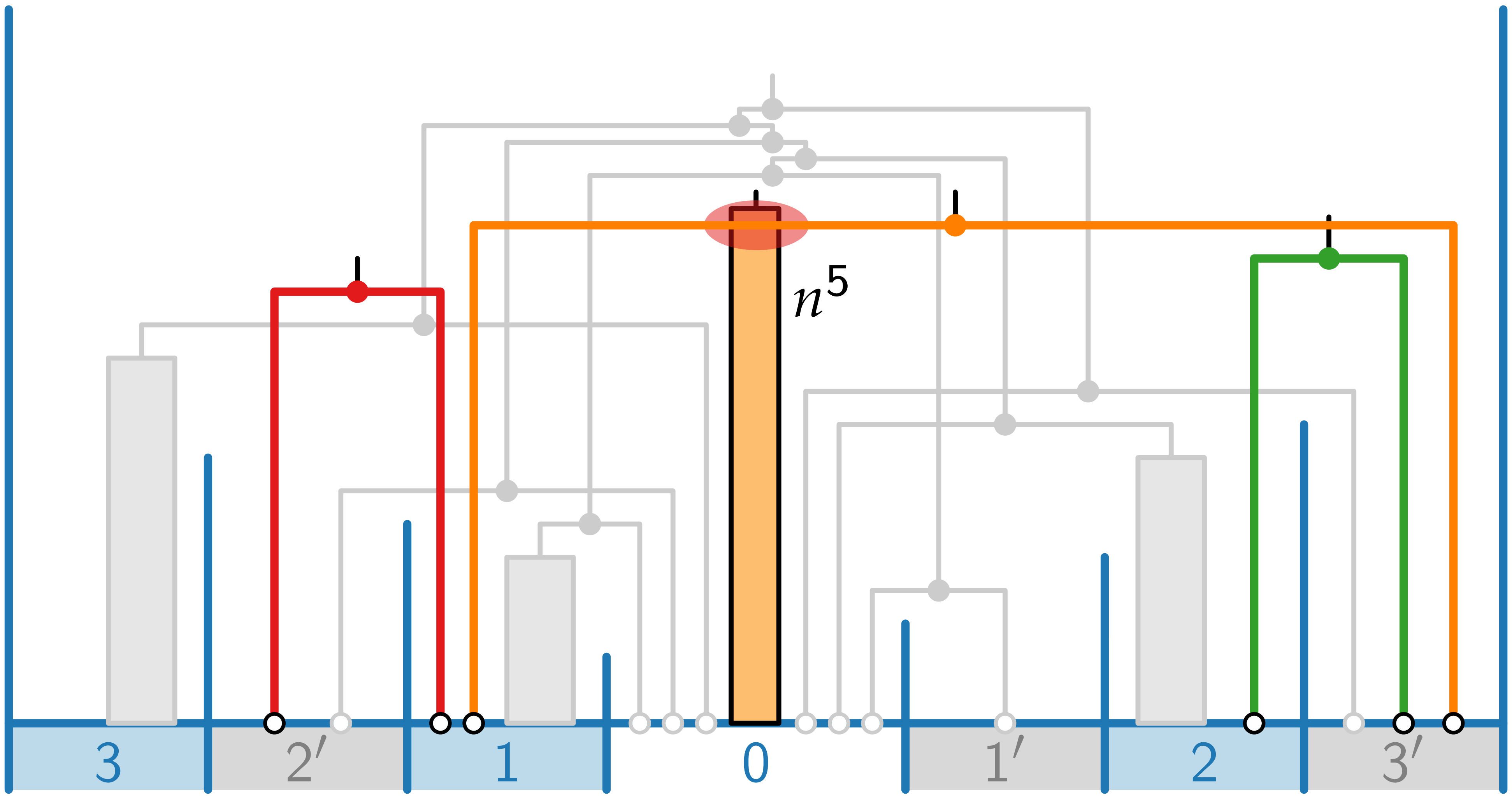

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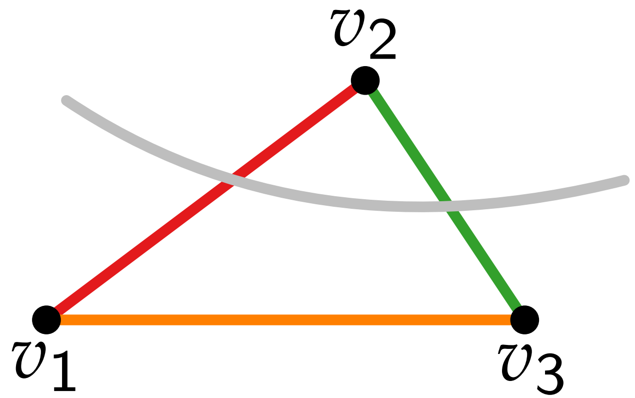

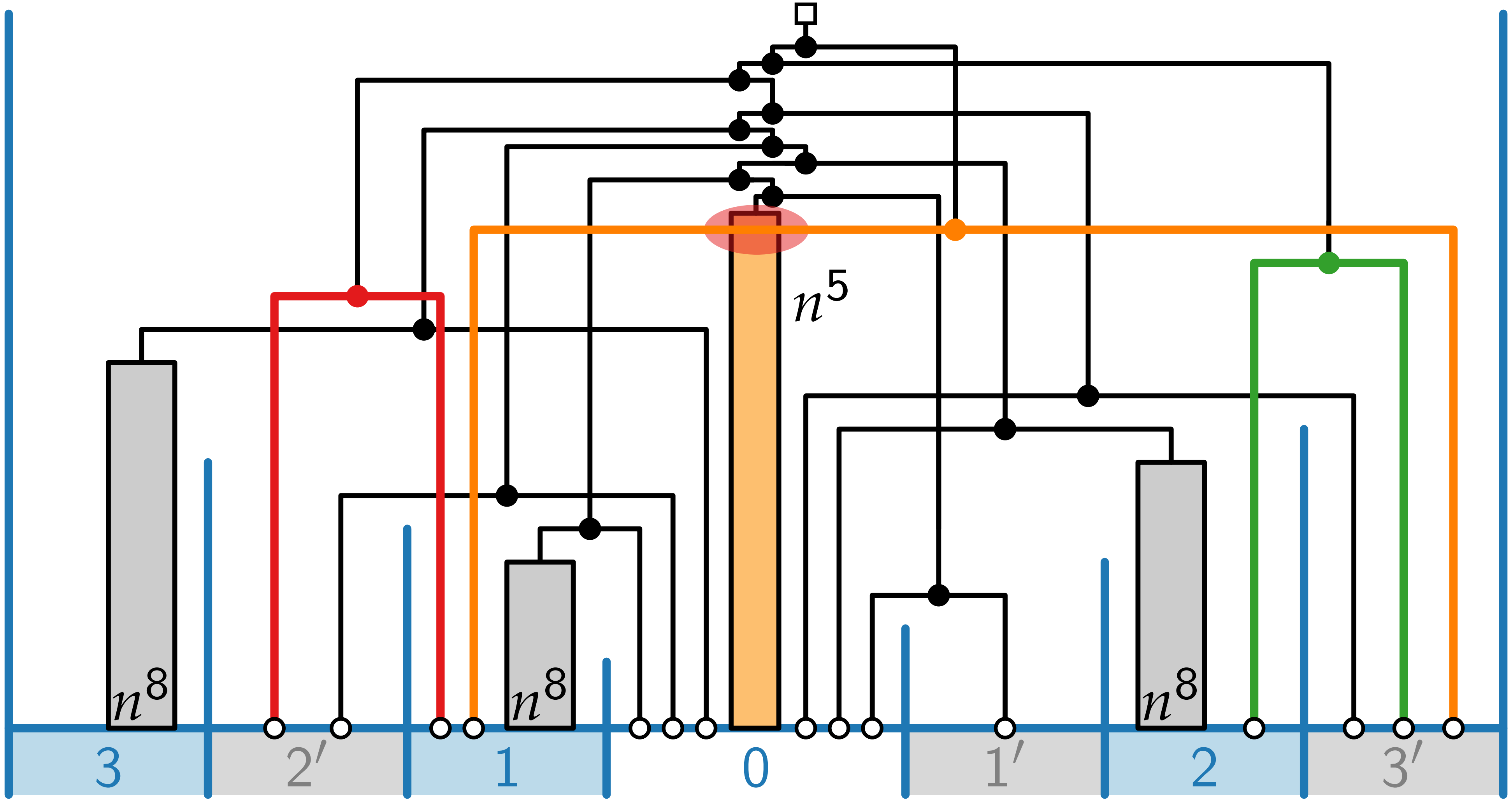

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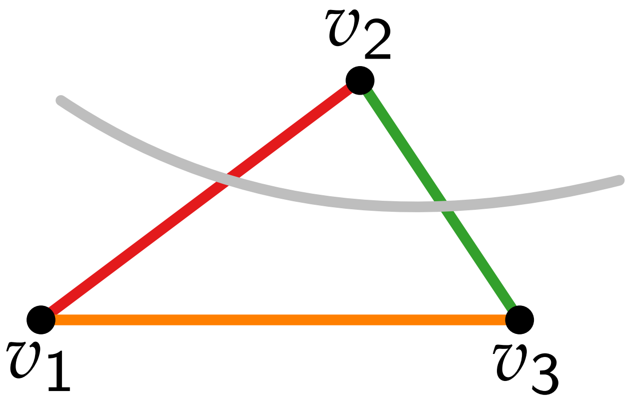
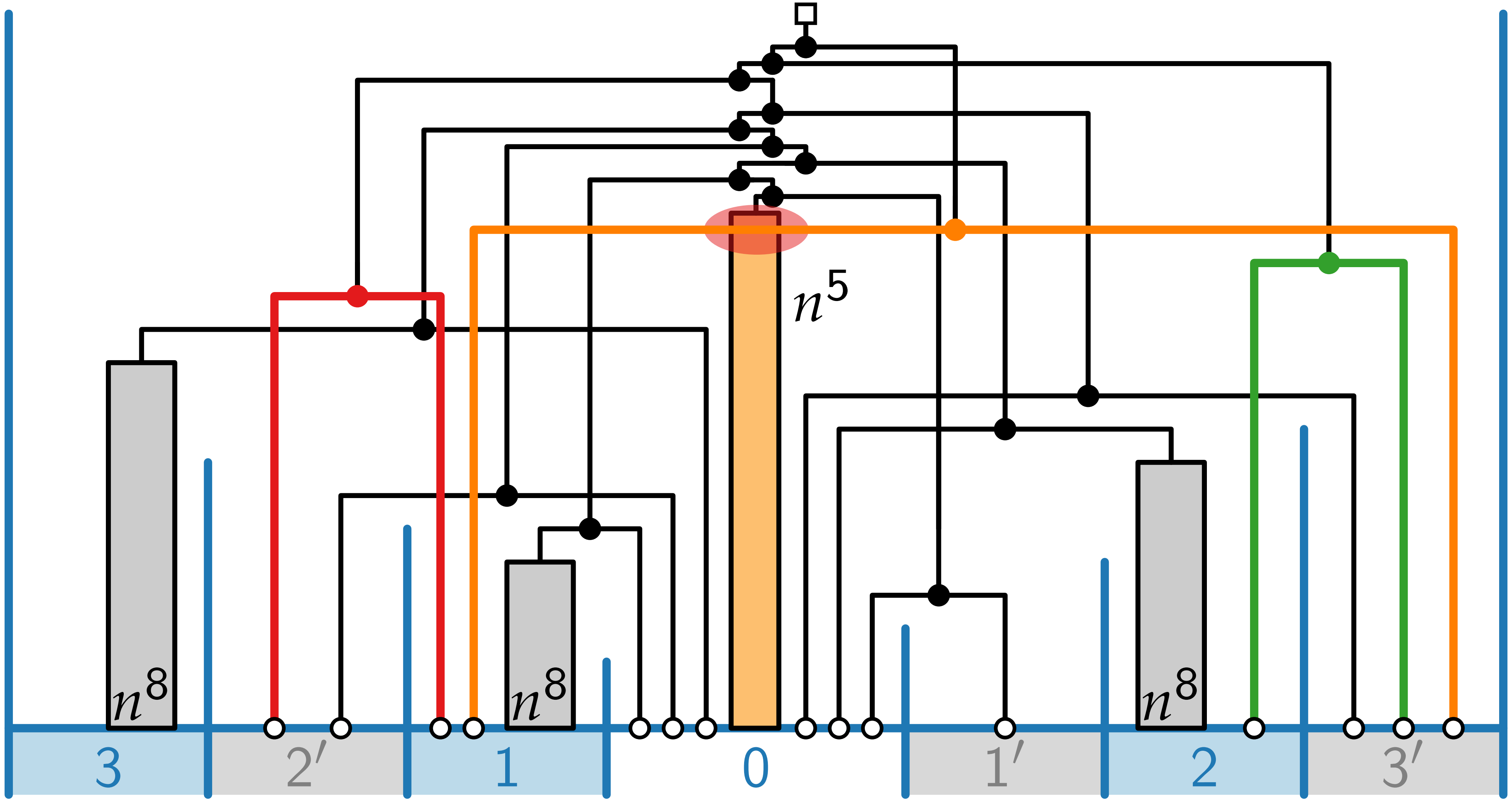


Done! :-)



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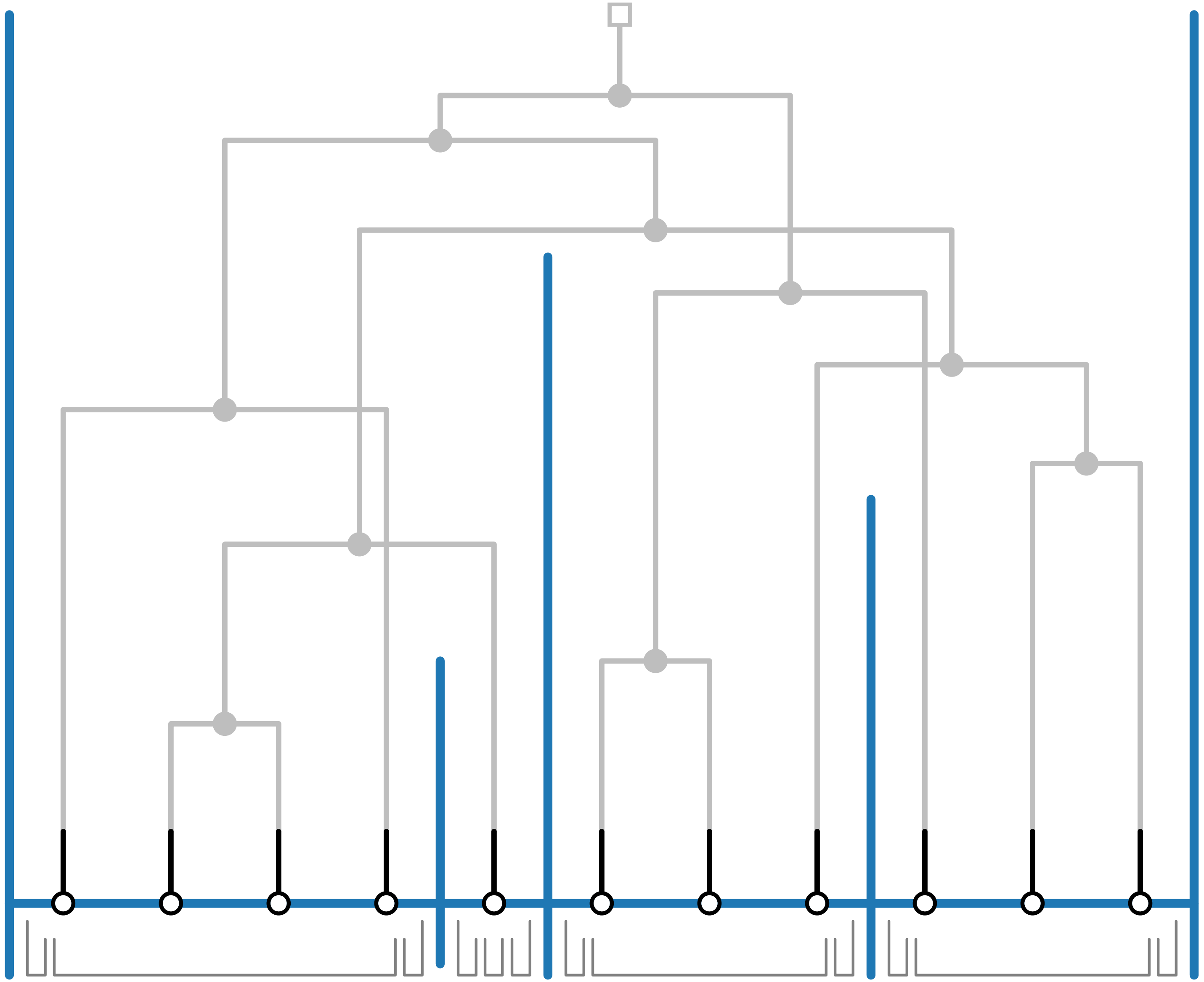
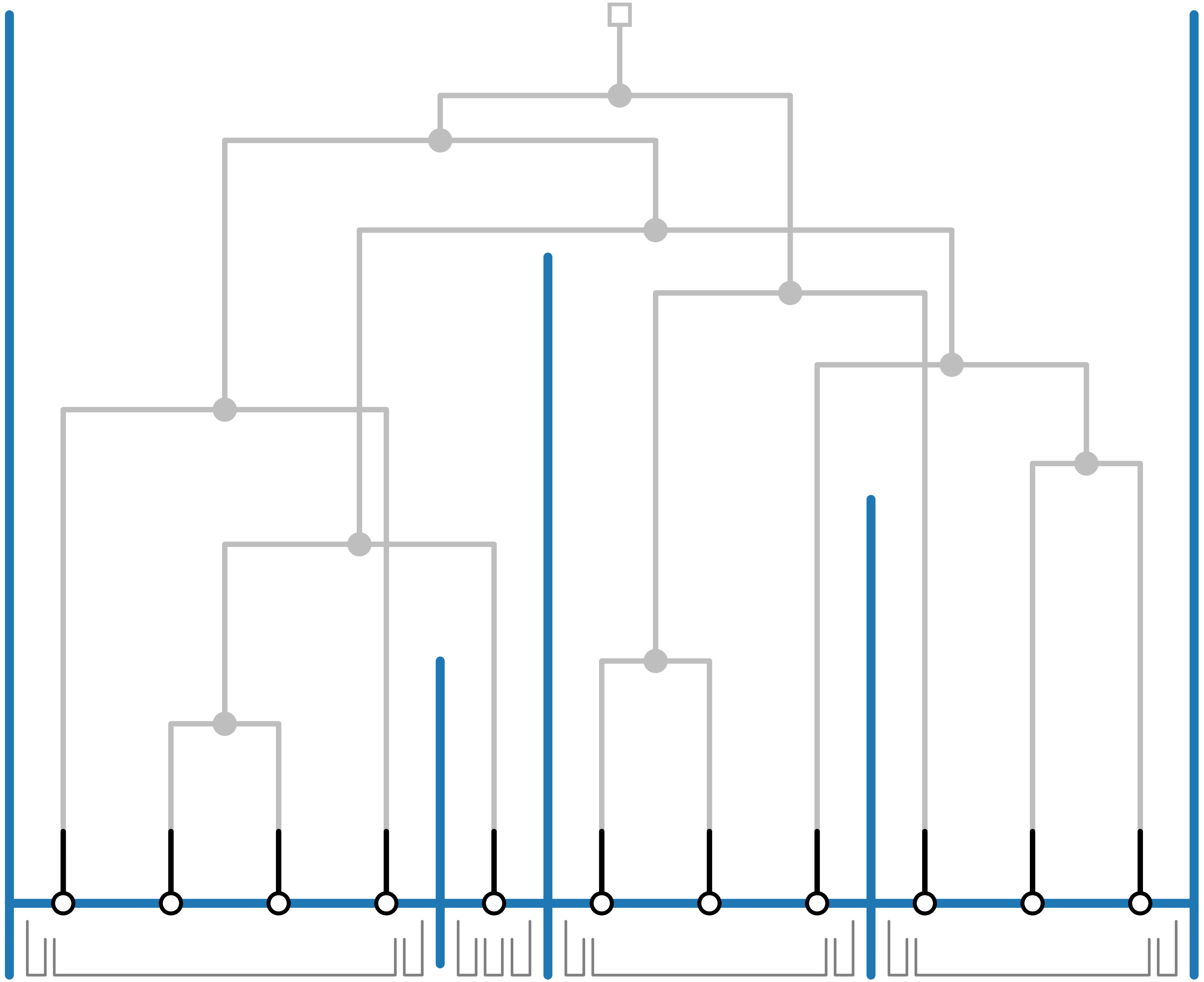
Heuristic for fixed species order

- bottom-up



Heuristic for fixed species order

- bottom-up
- left stack, mid bucket, and right stack for each species



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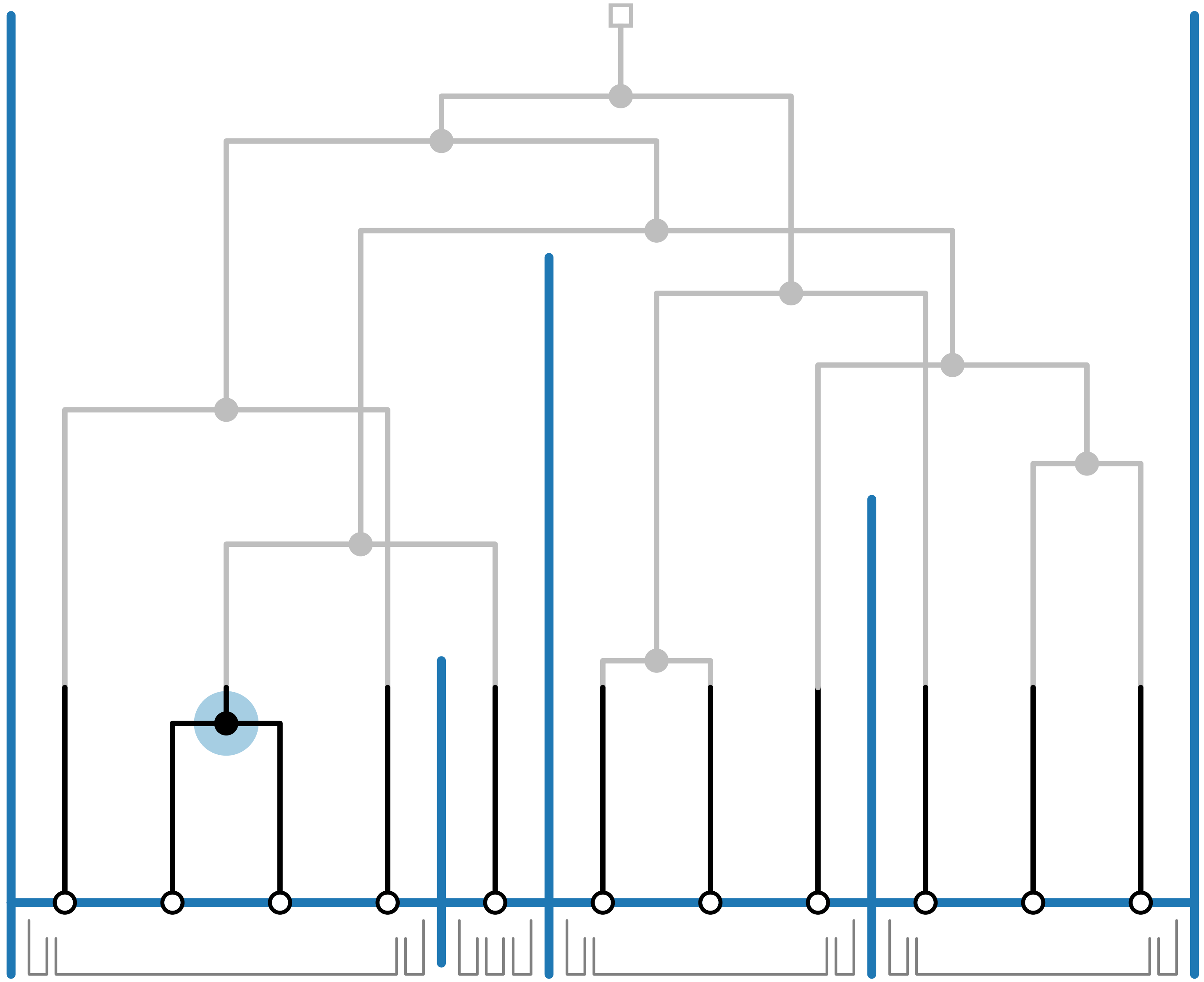

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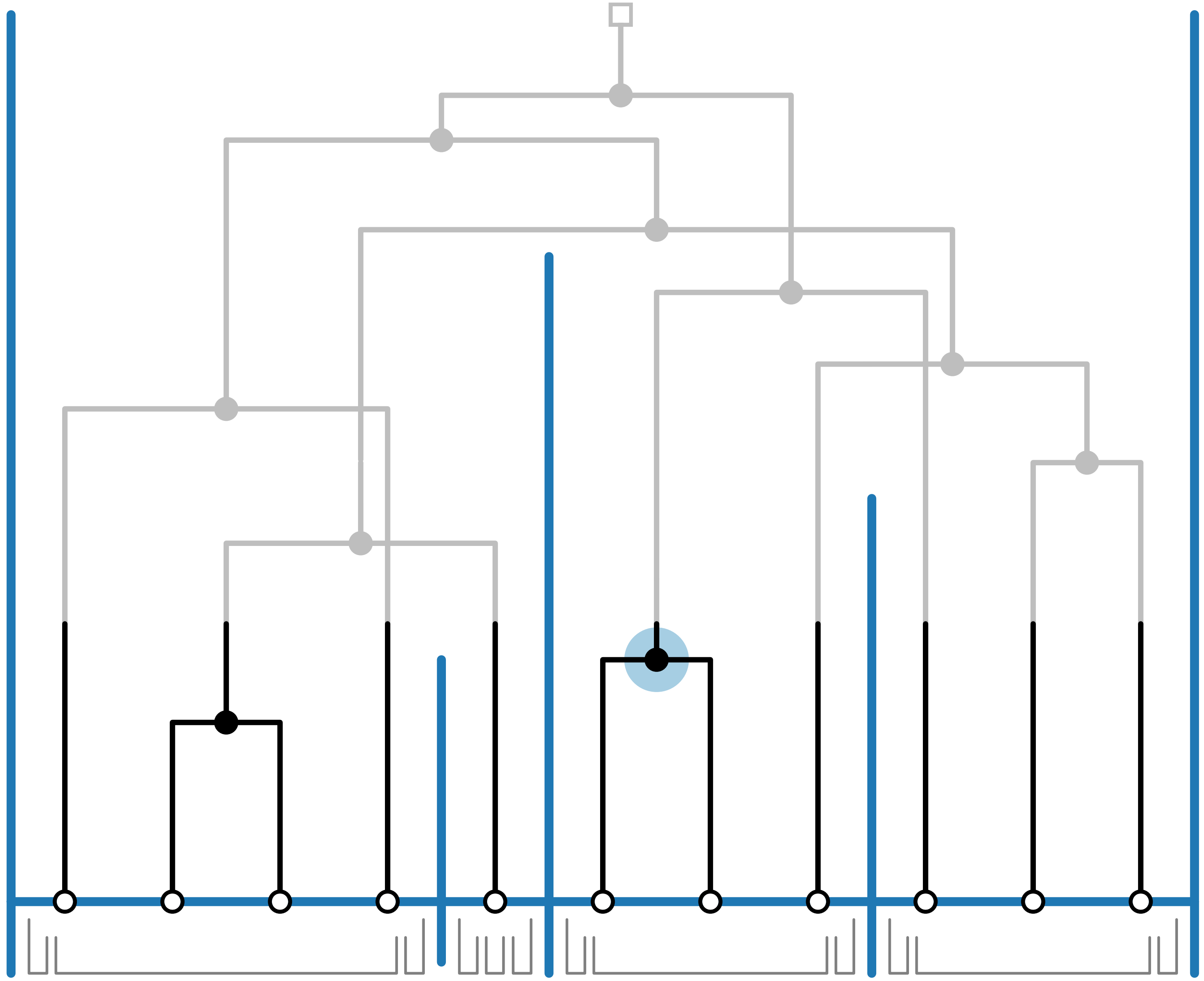

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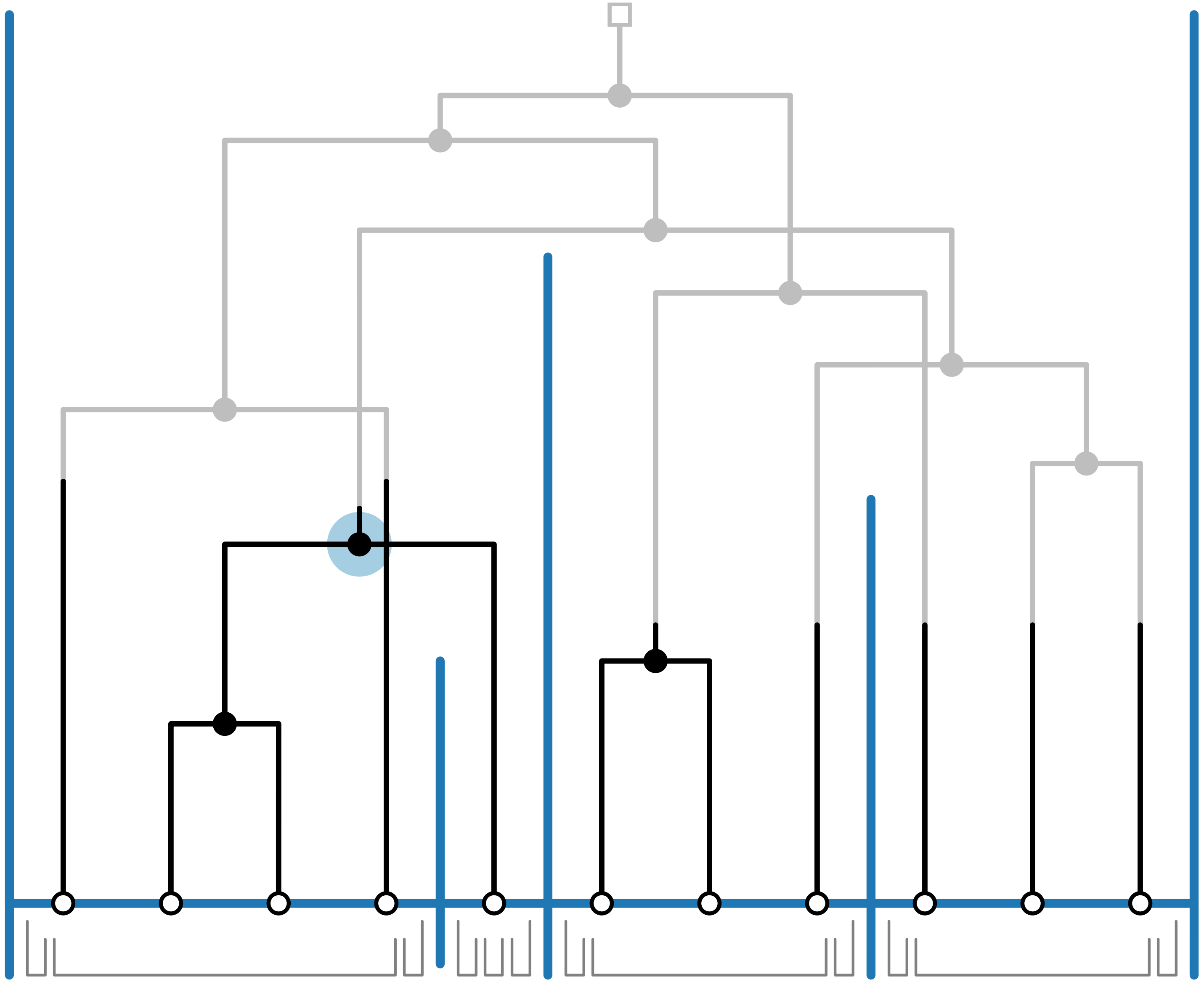

Heuristic for fixed species order

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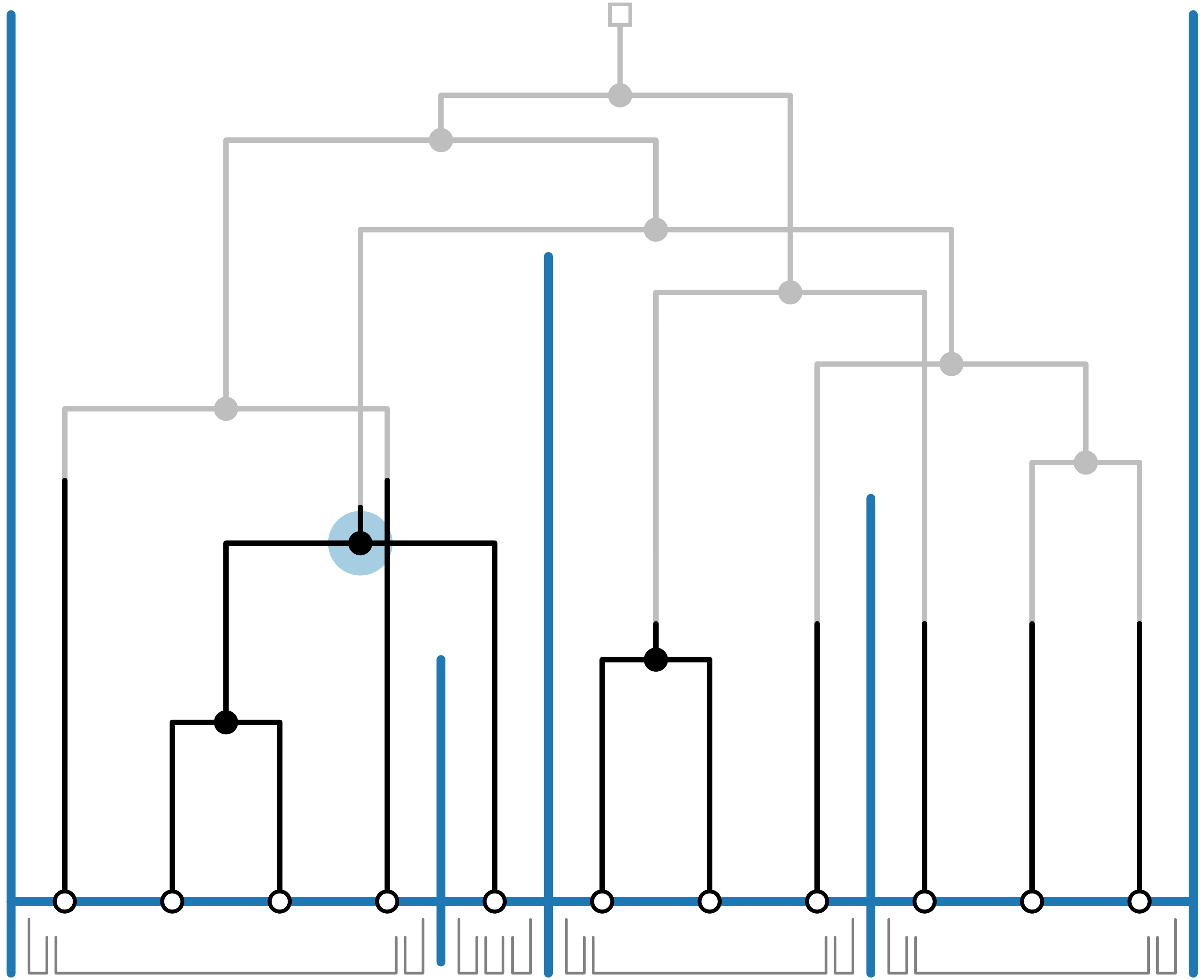
- left stack, mid bucket, and right stack for each species
- move subtrees together



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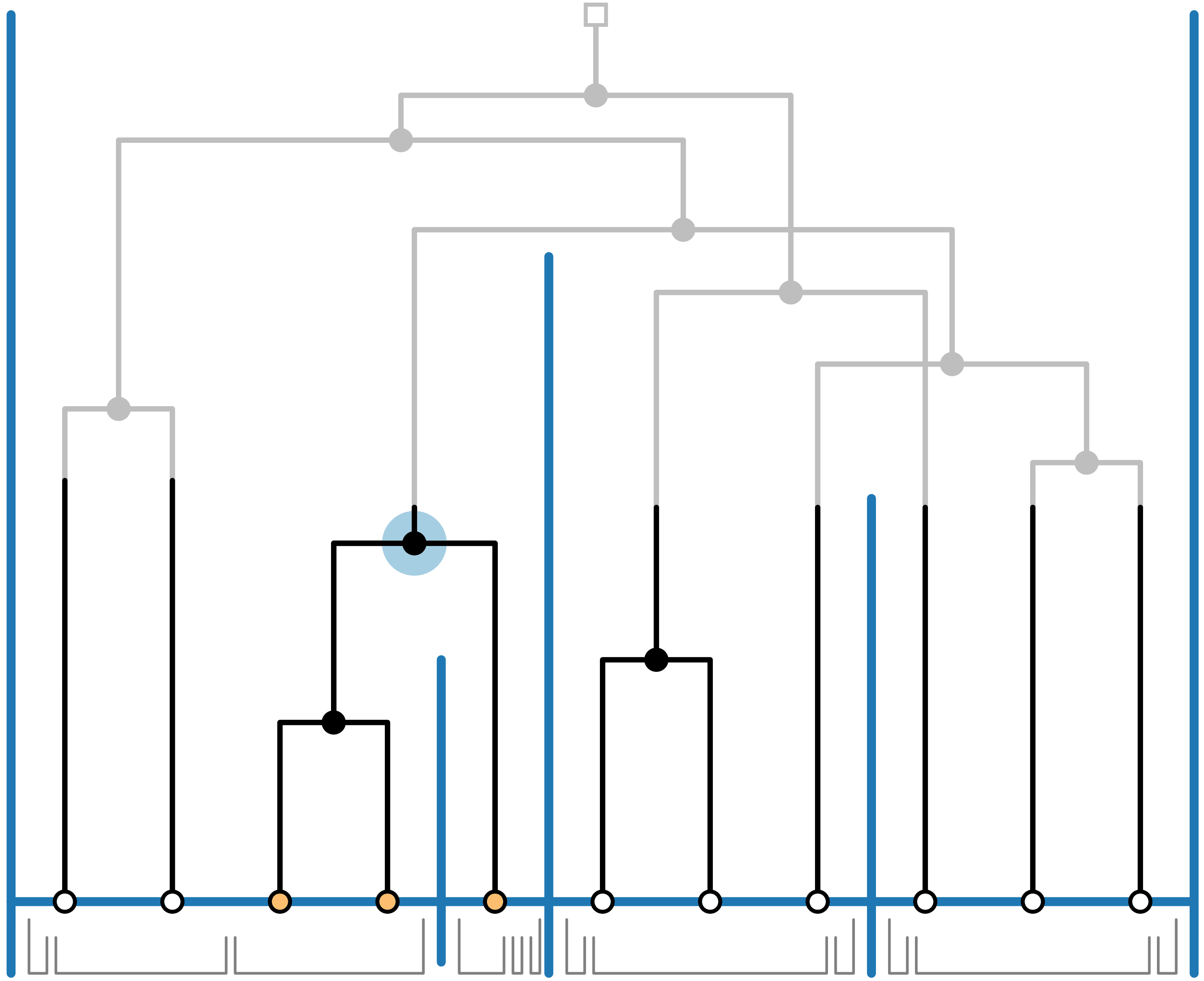

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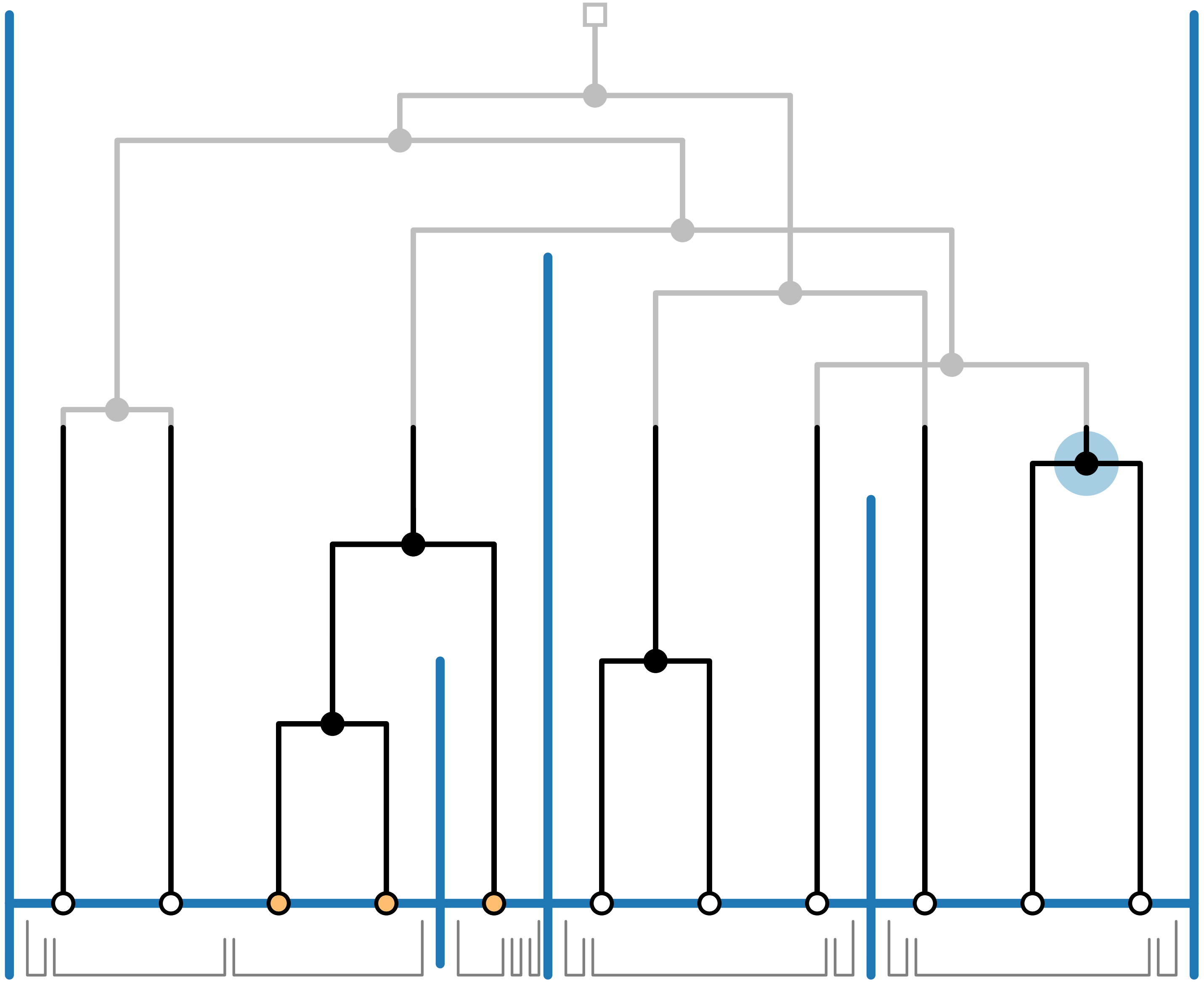

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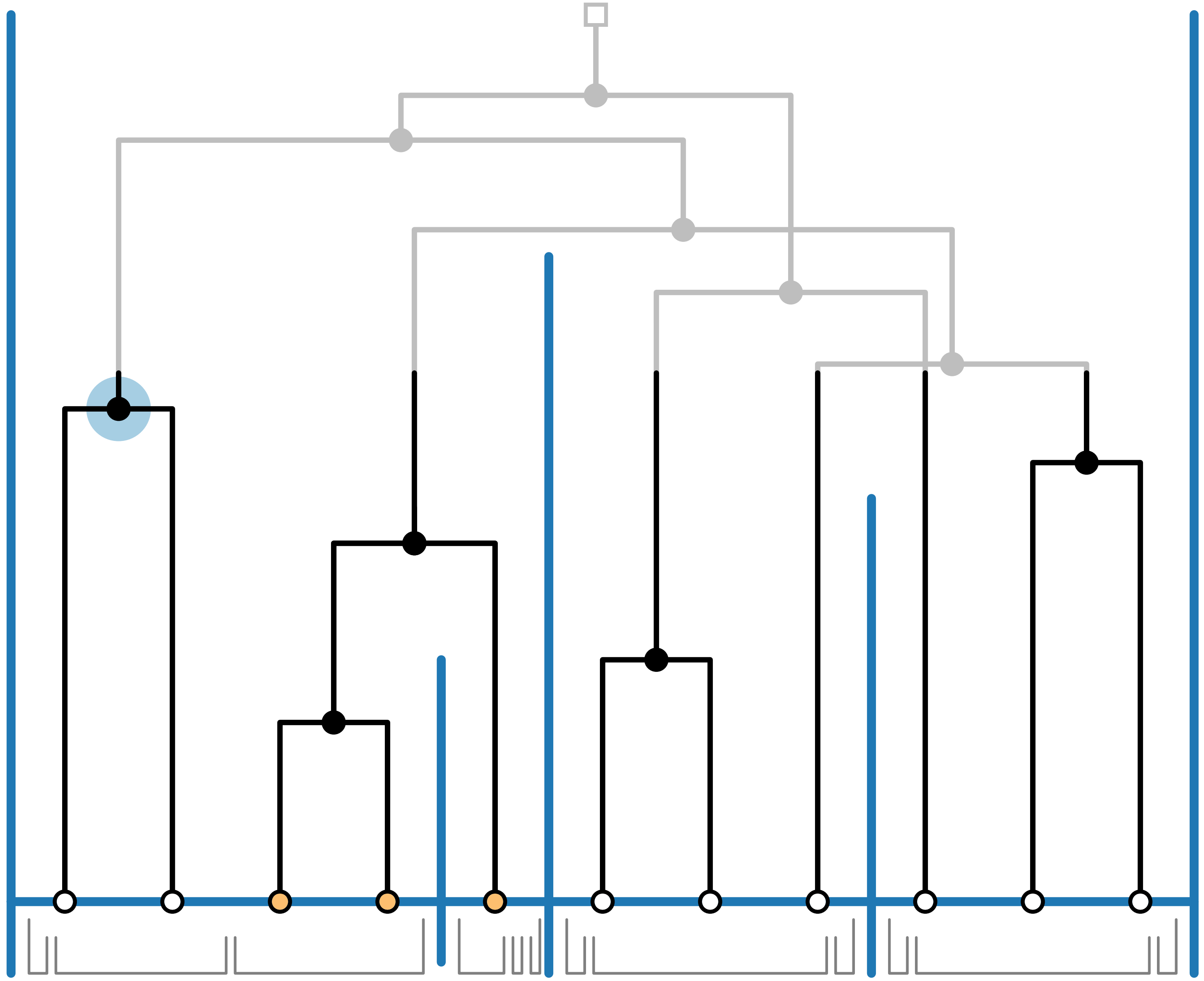

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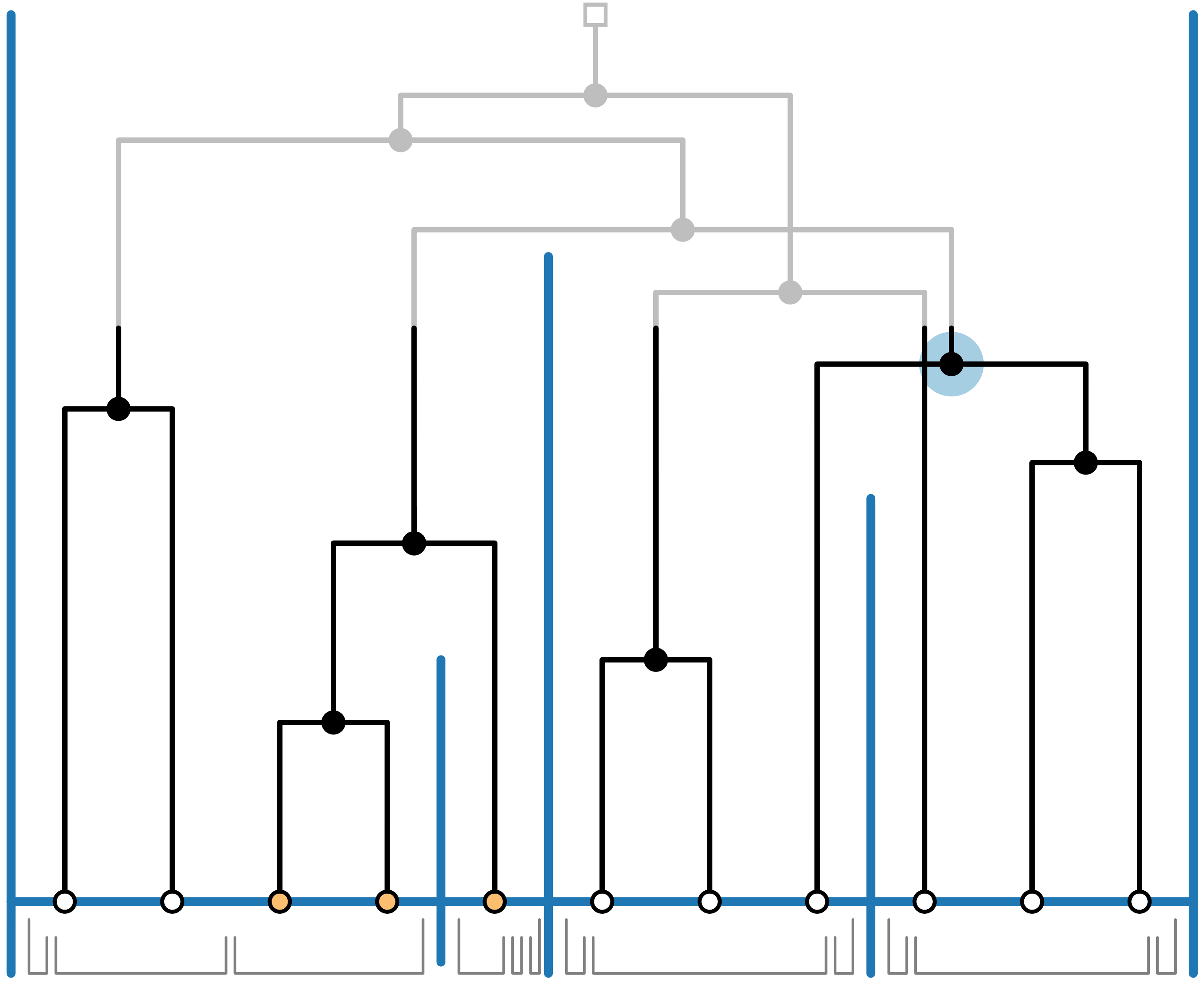

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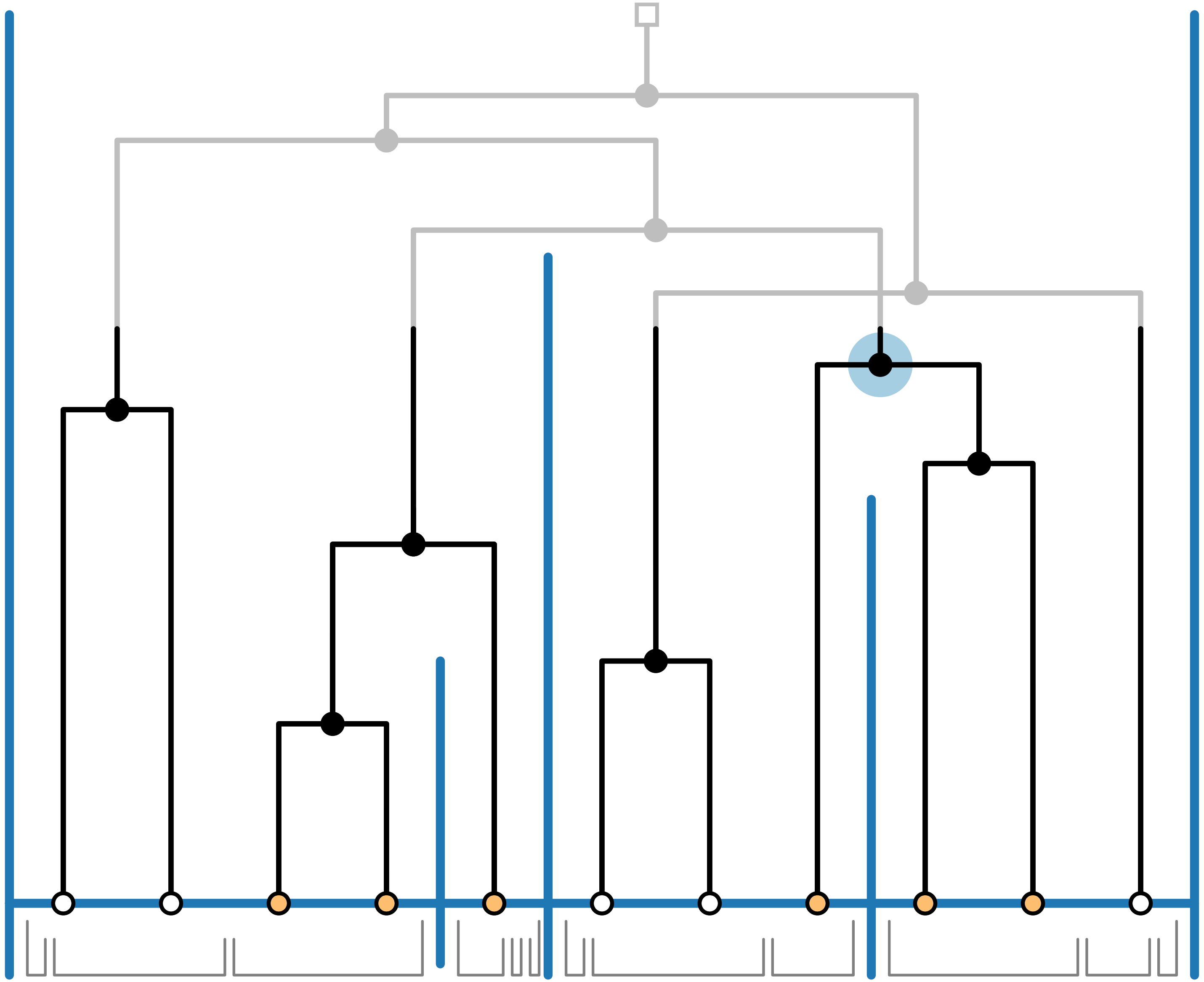

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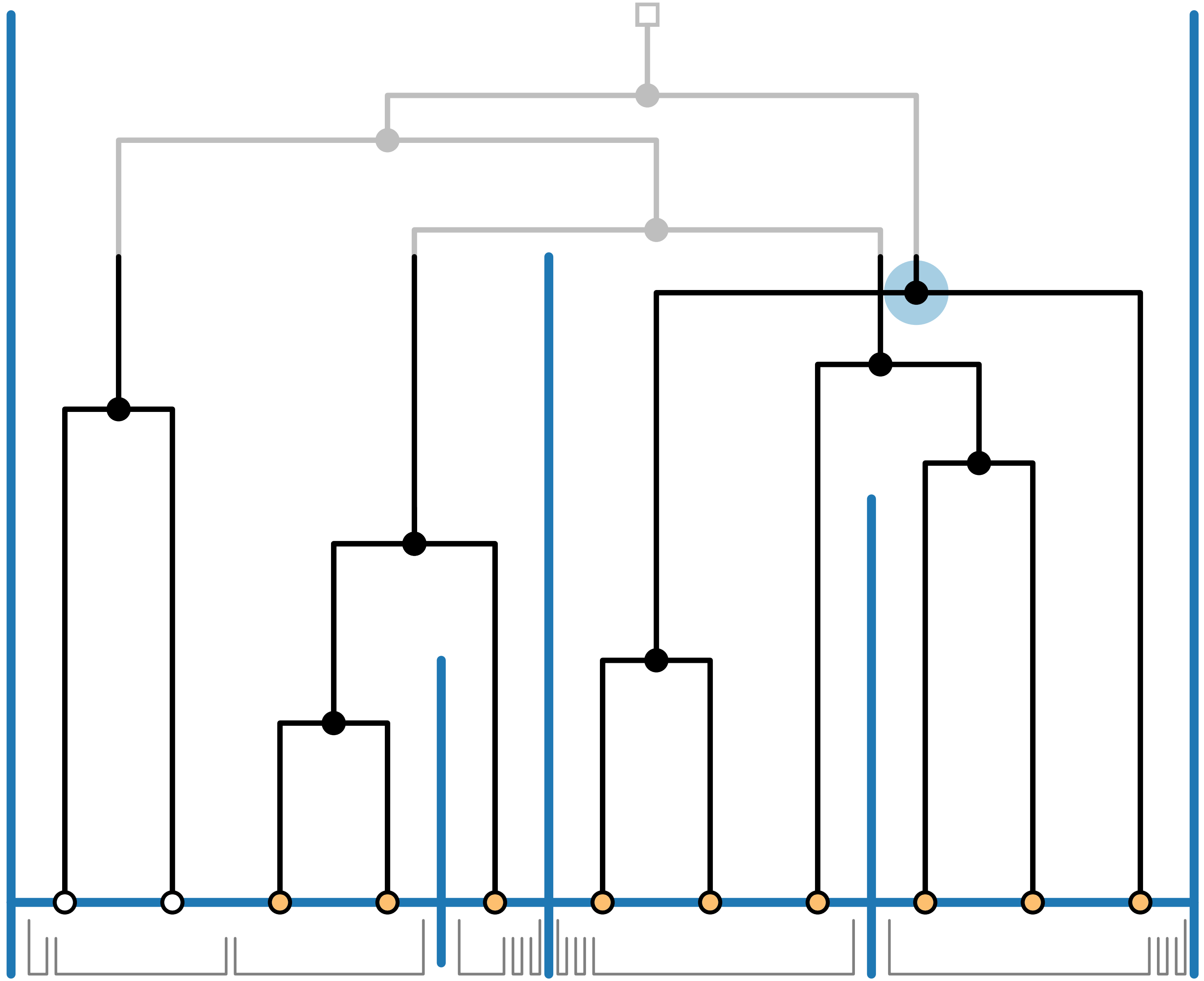

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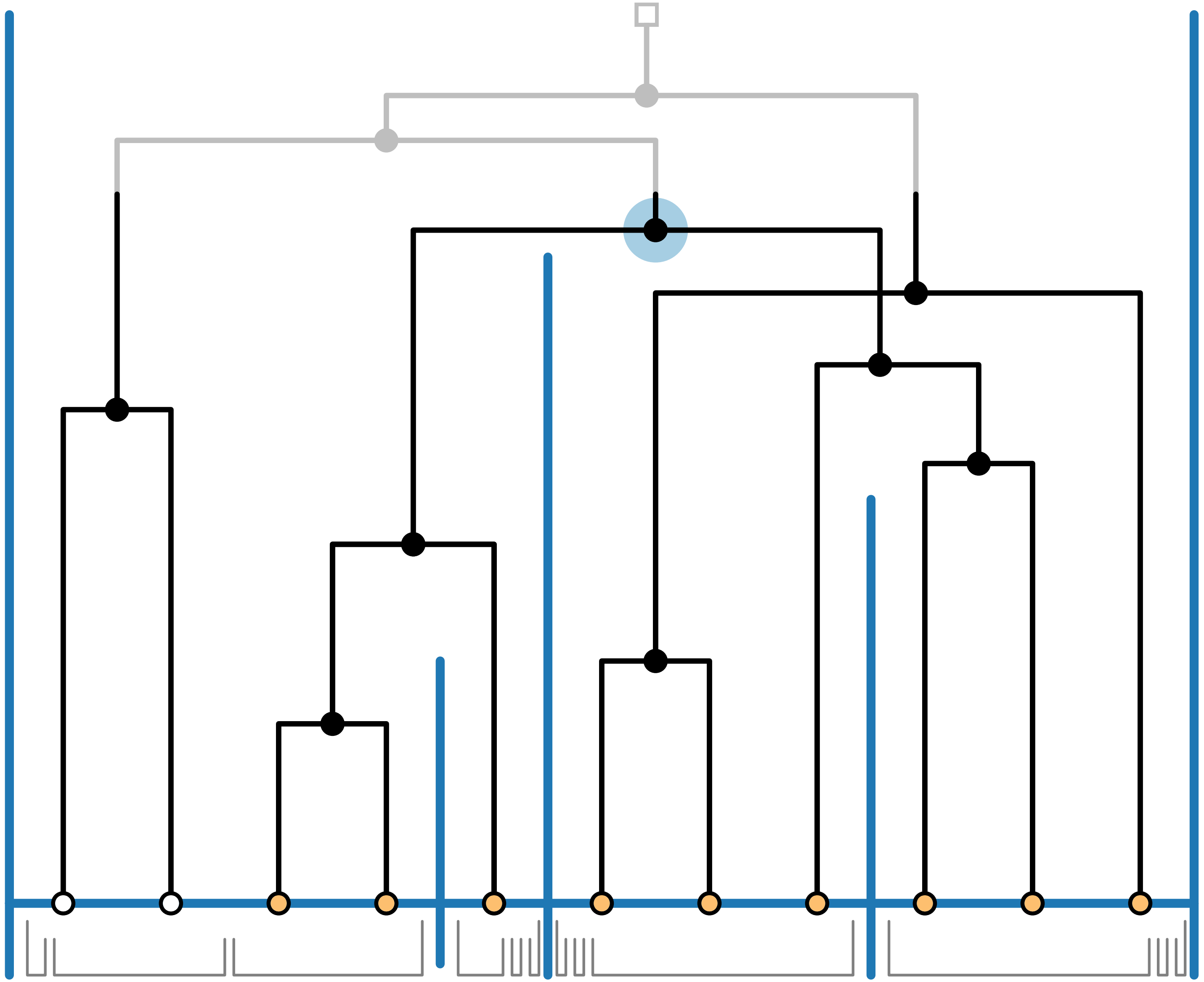

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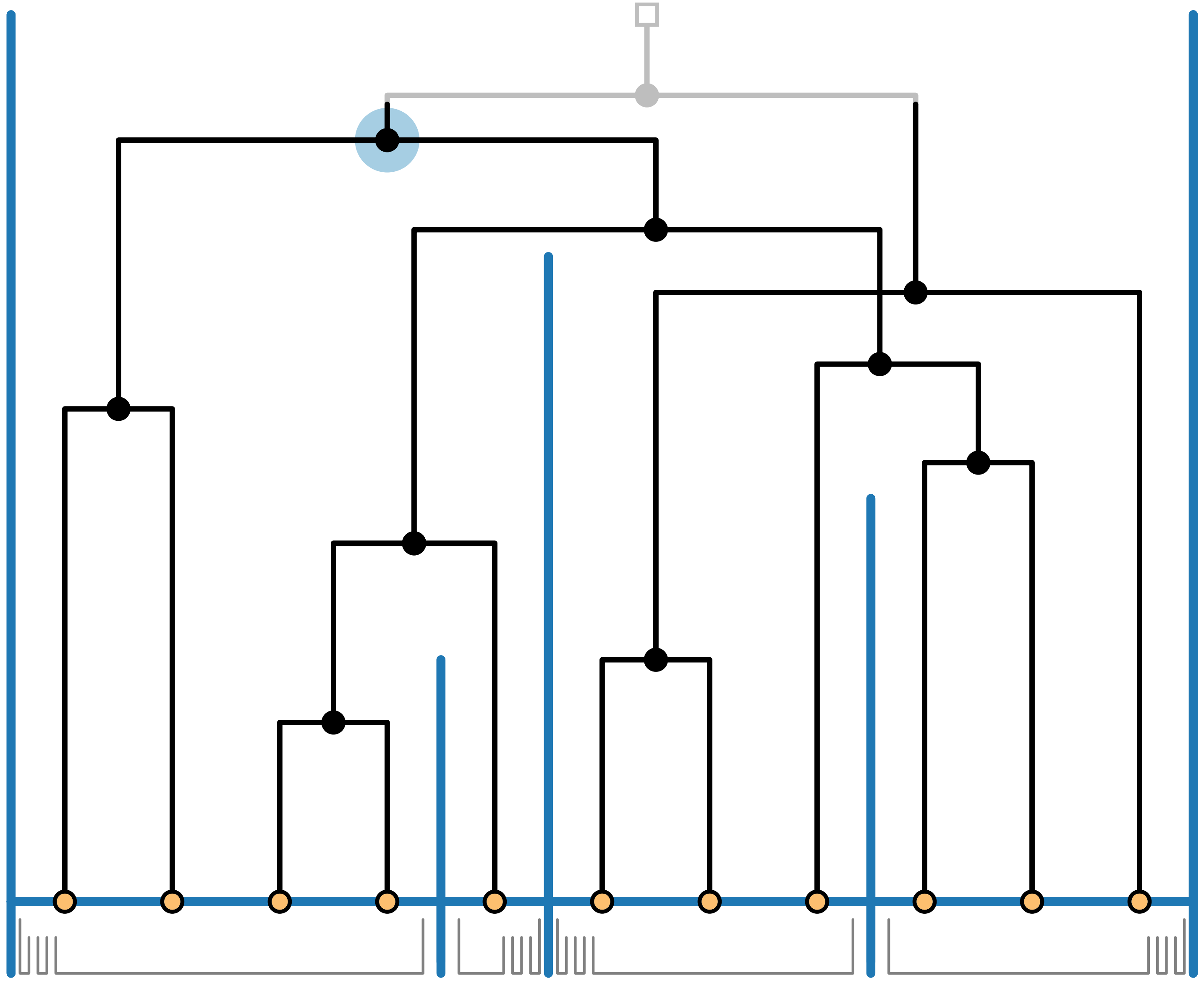

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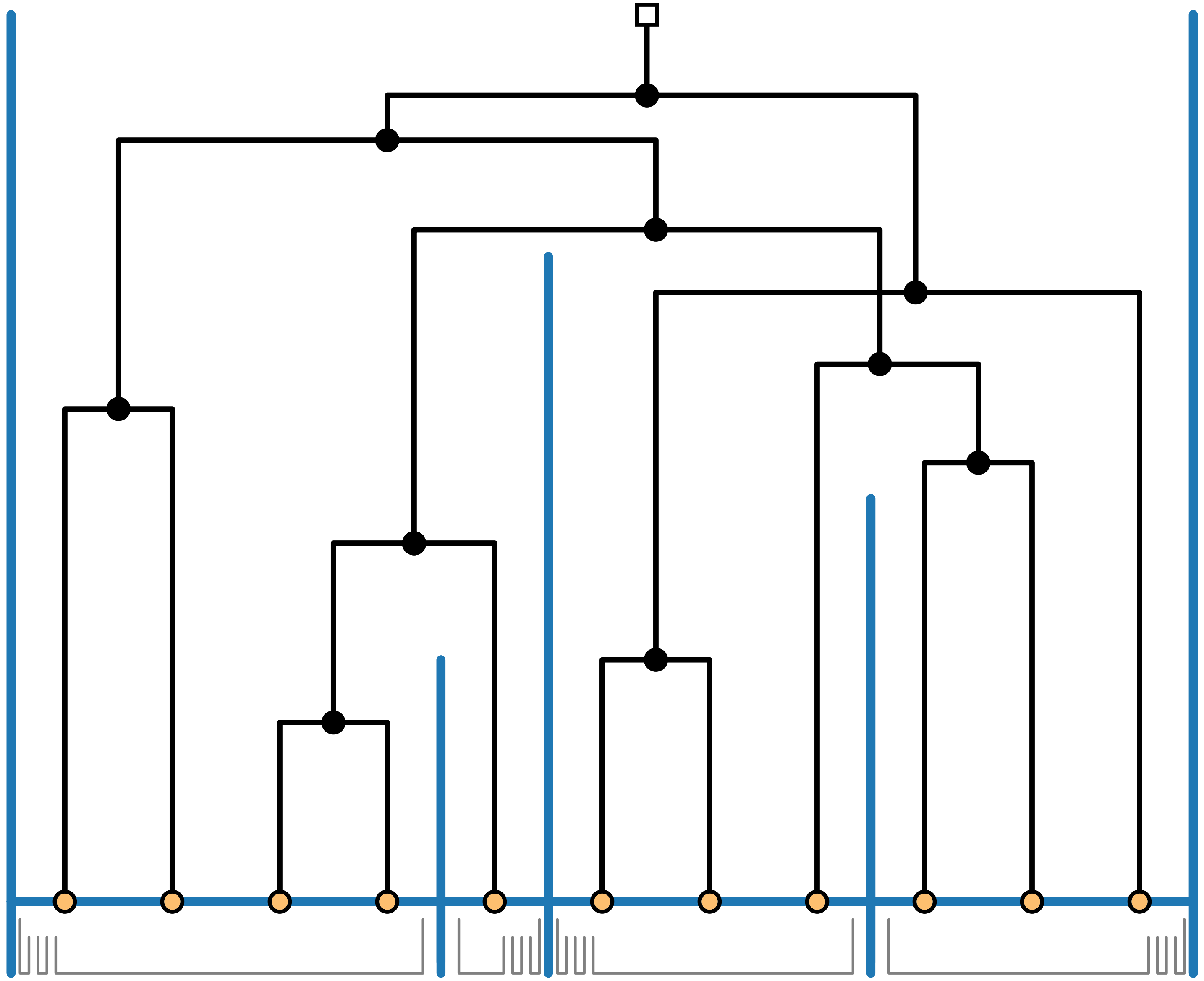

Heuristic for fixed species order

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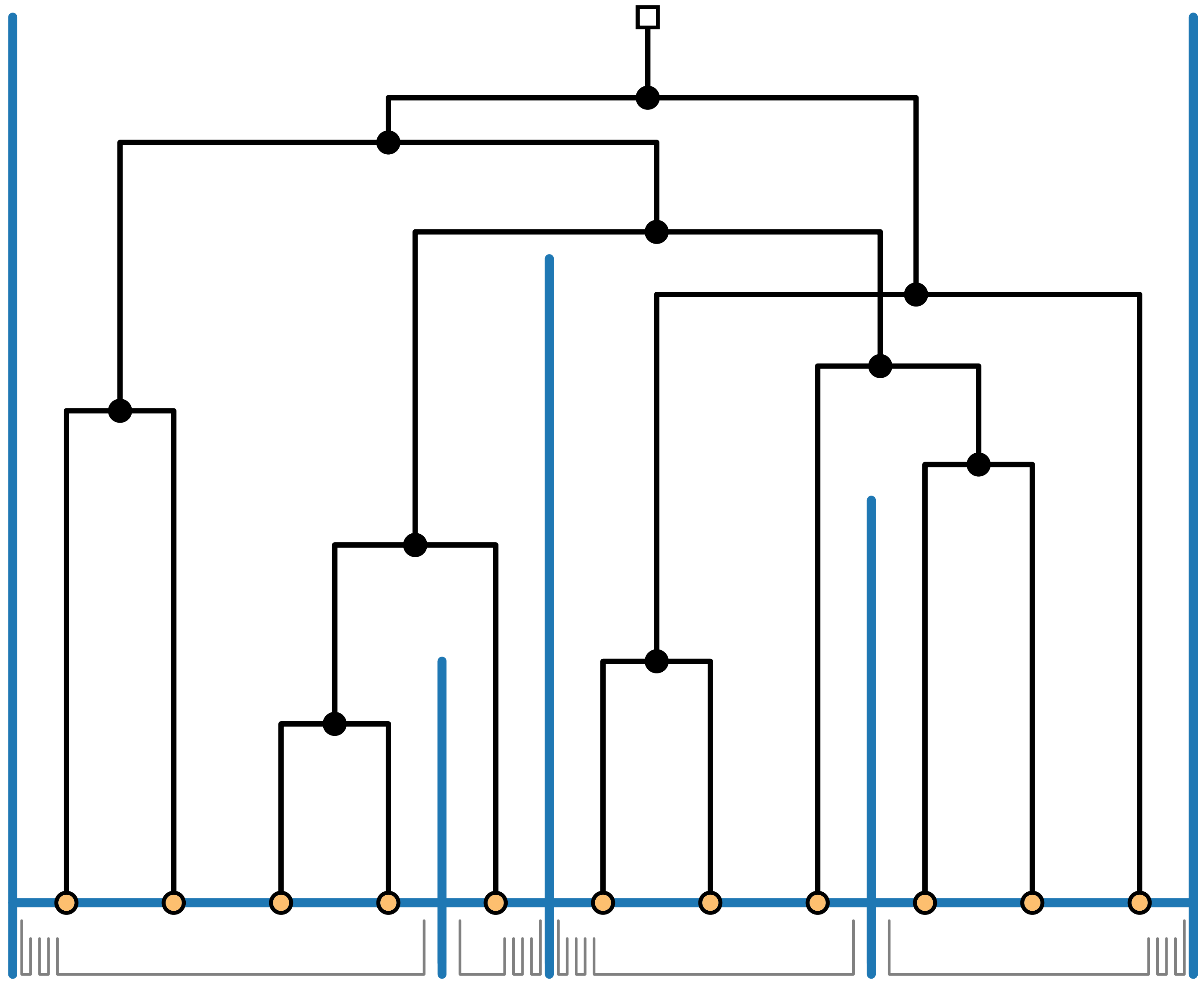
Done! :-)



Heuristic for fixed species order

- bottom-up
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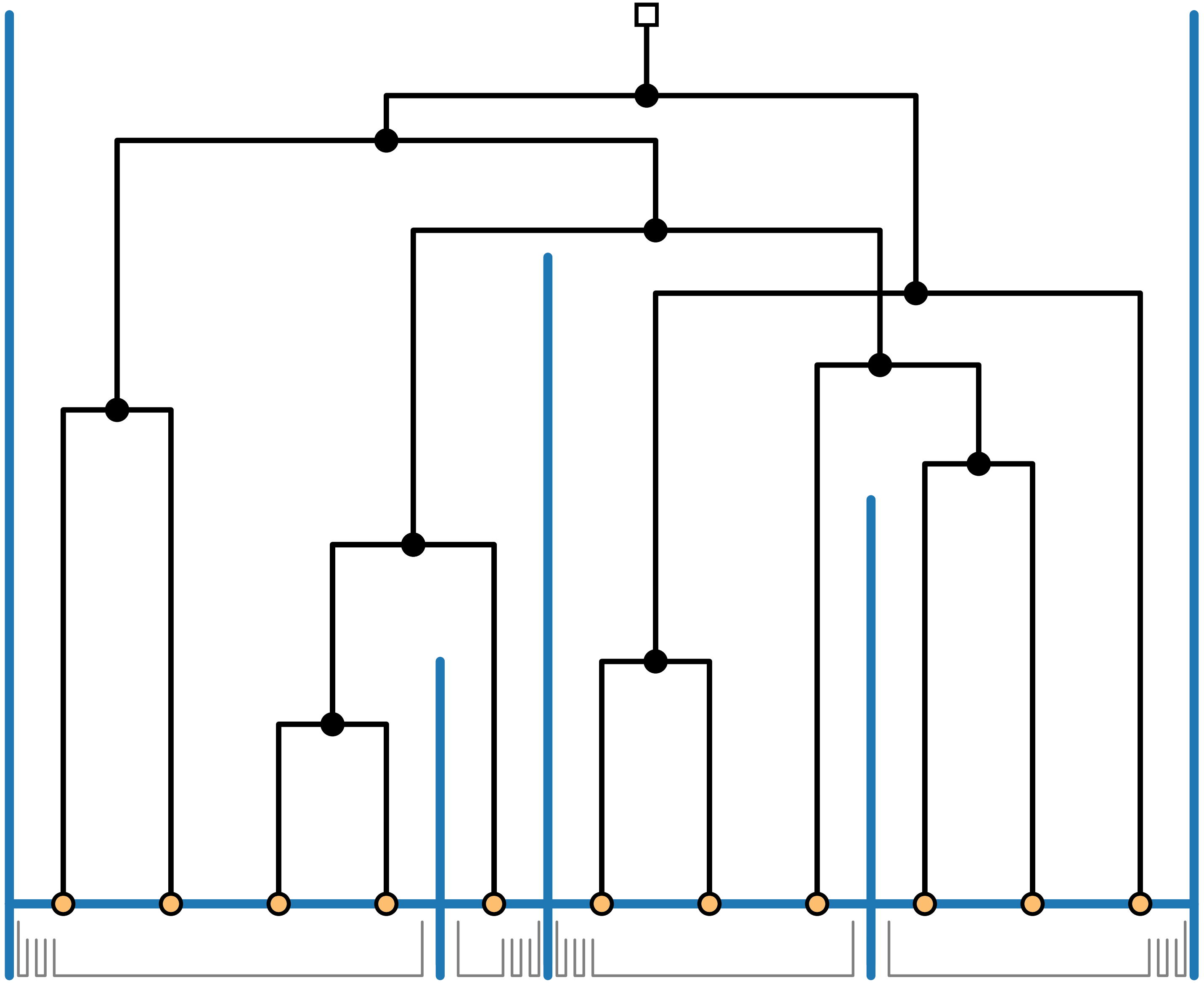
Done! :-)

- runs in linear time



Heuristic for variable species order

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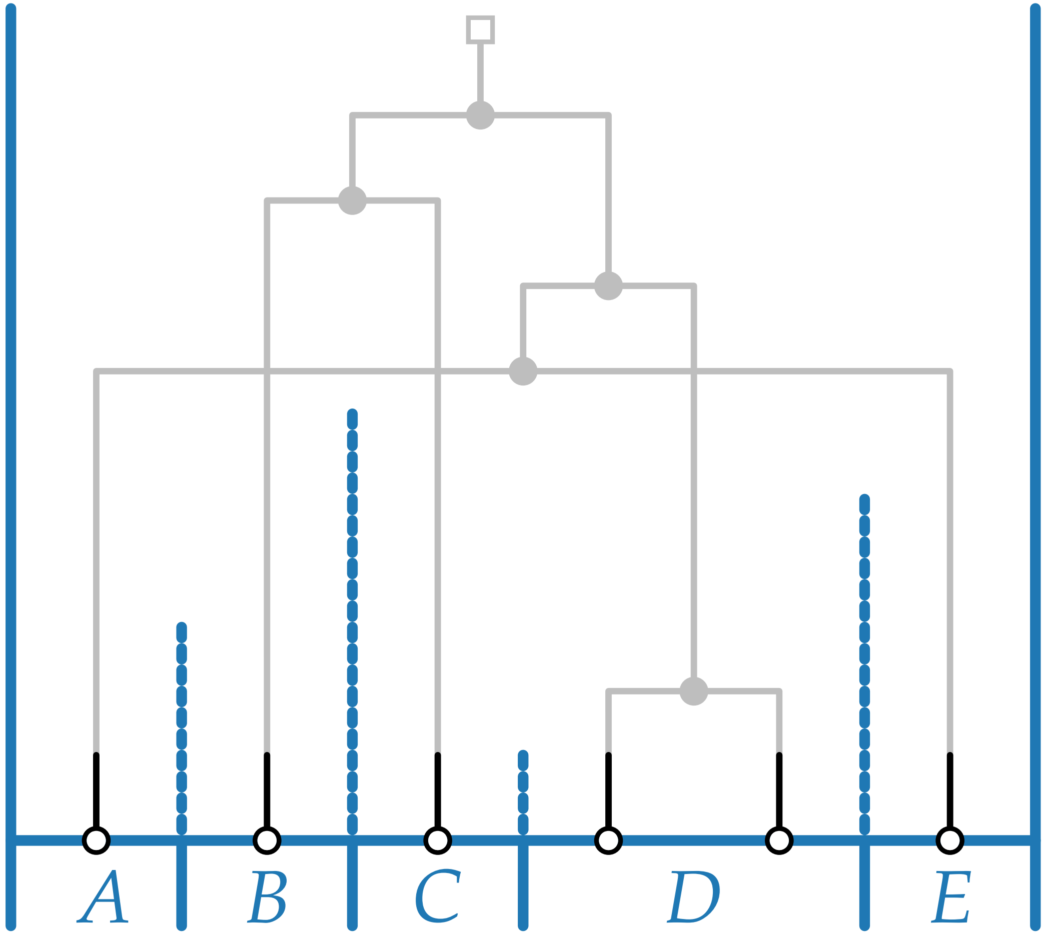

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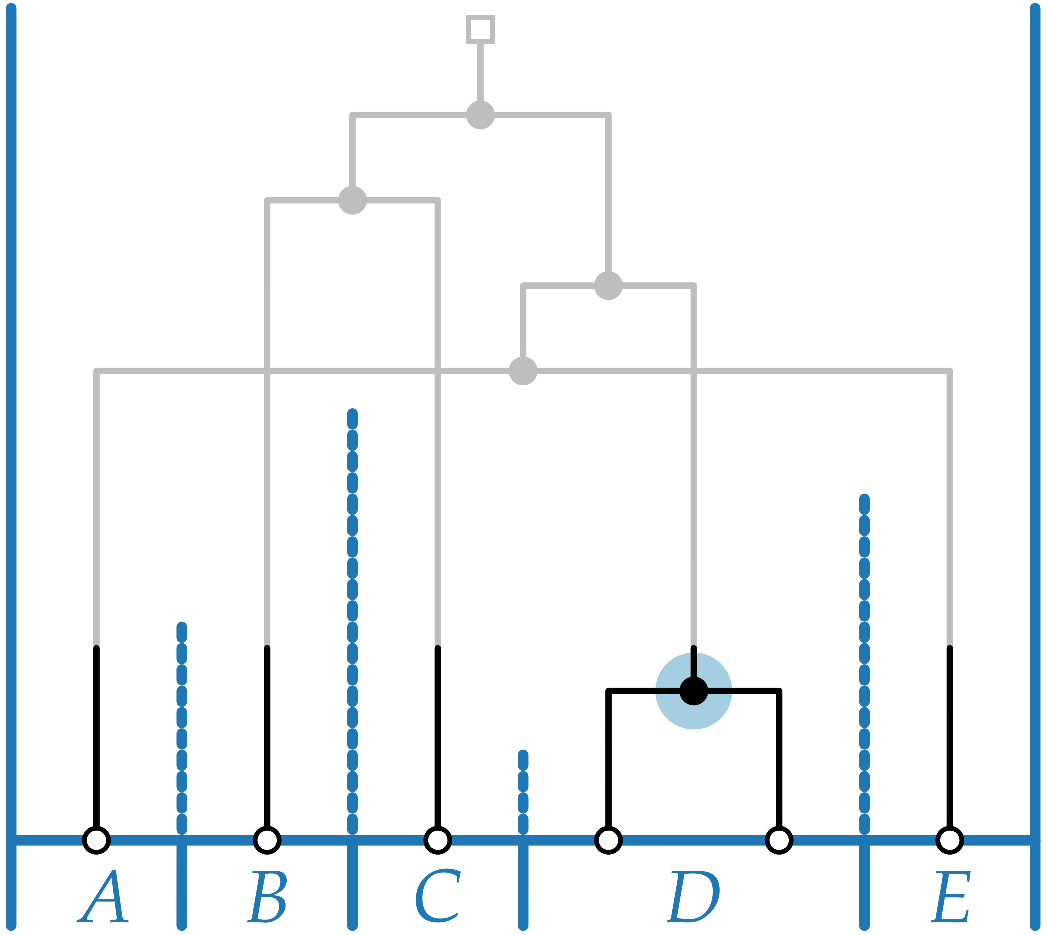

Heuristic for variable species order

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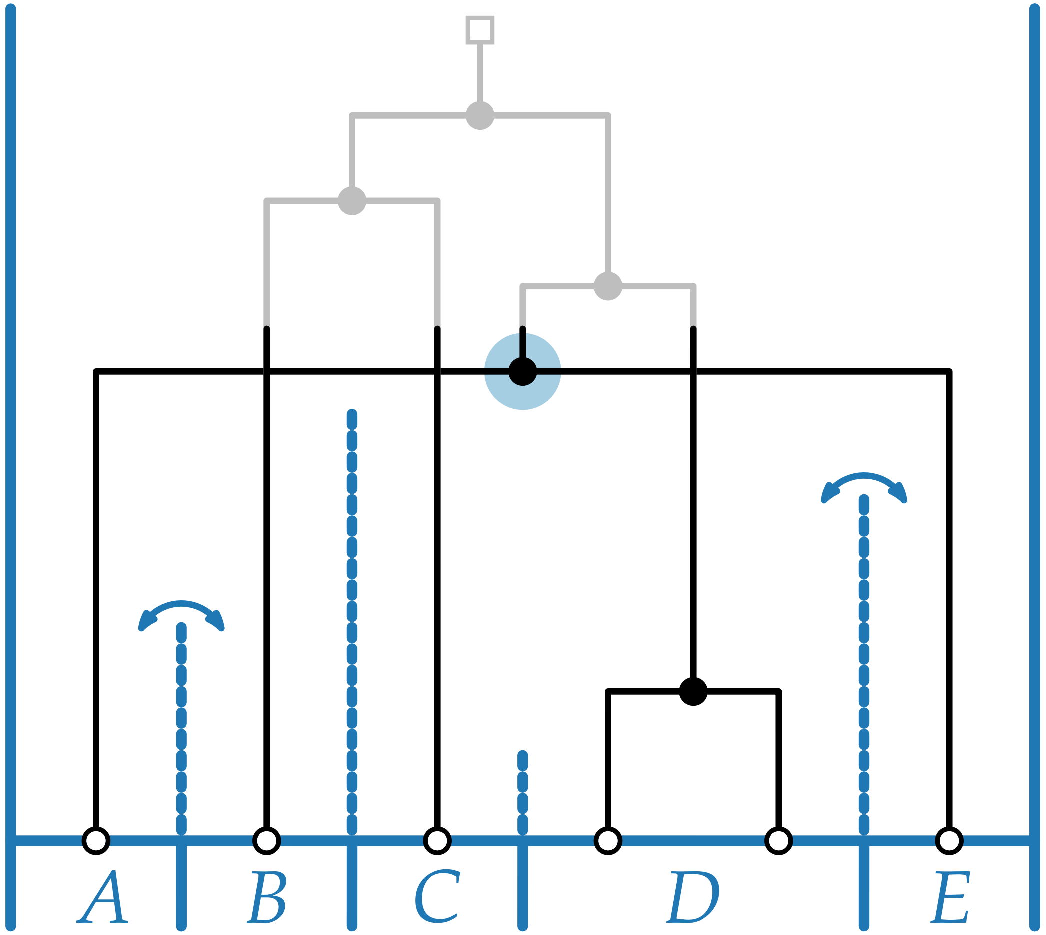
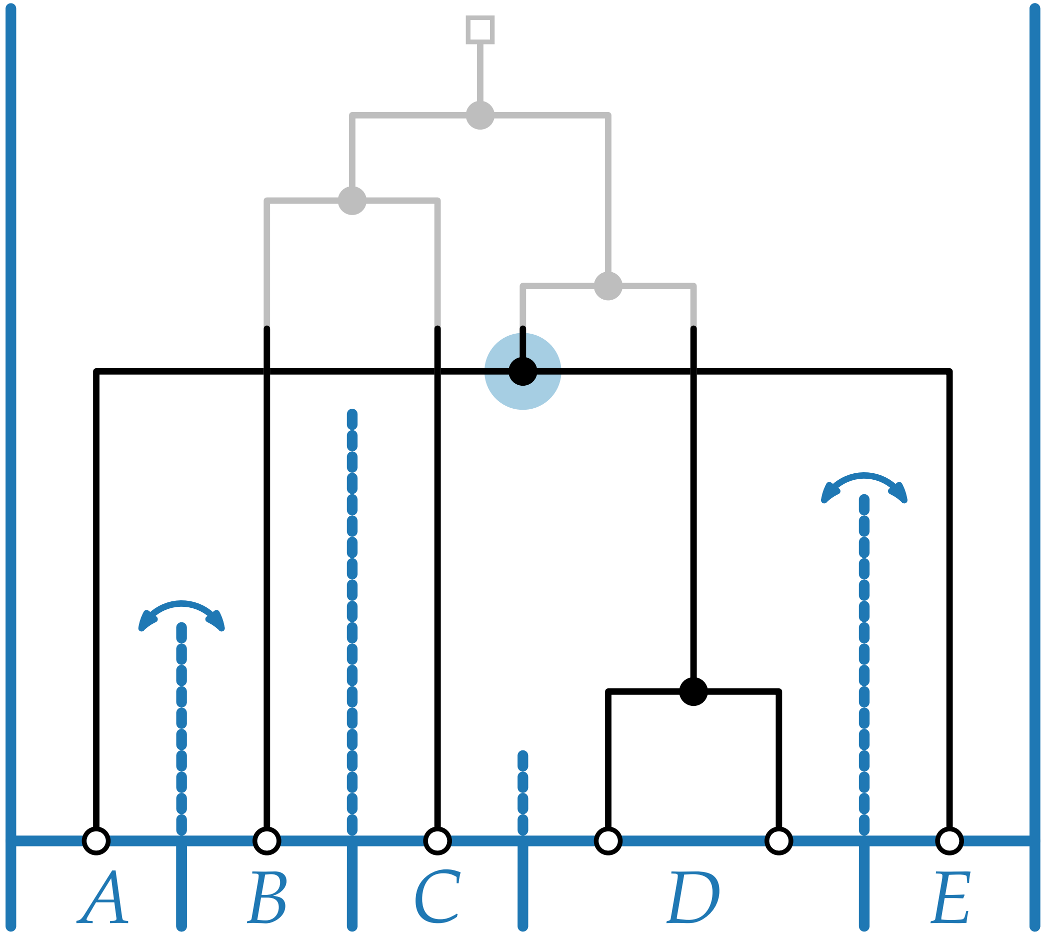


Heuristic for variable species order

- bottom-up
- rotate species together, fix rotations



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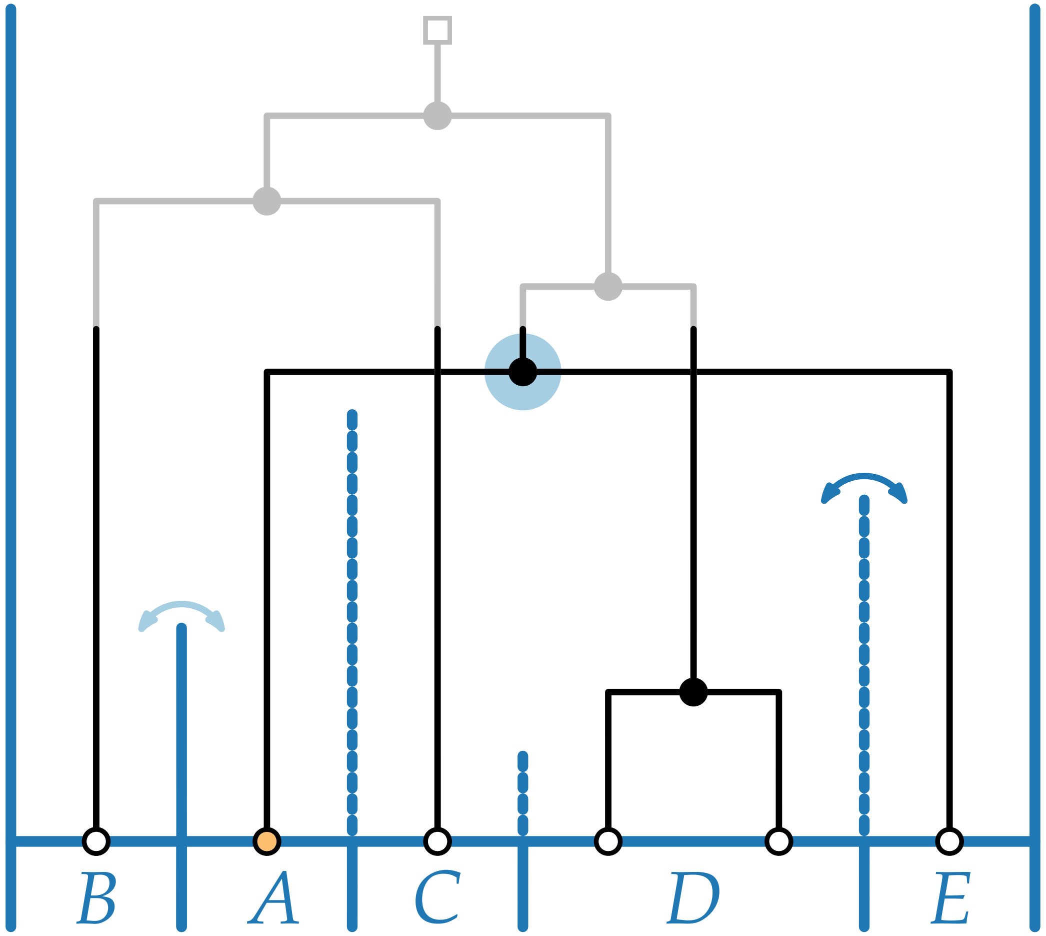

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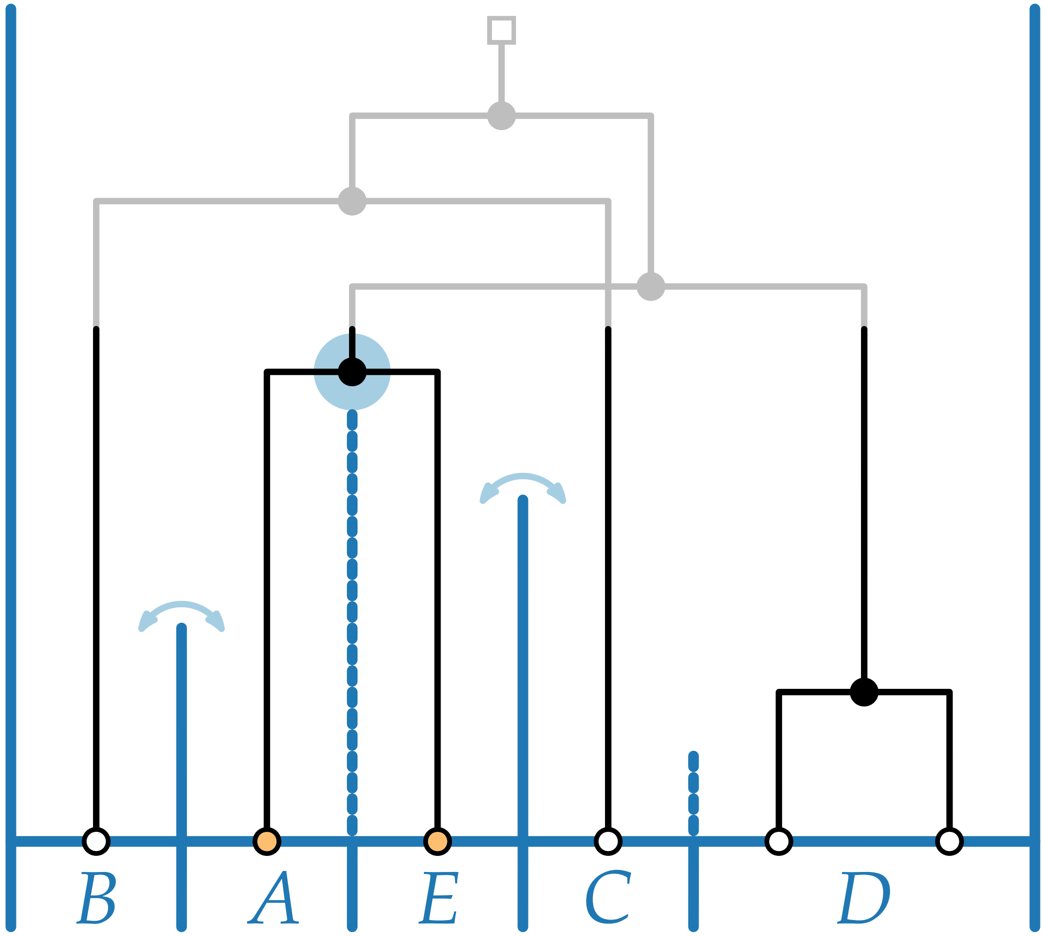

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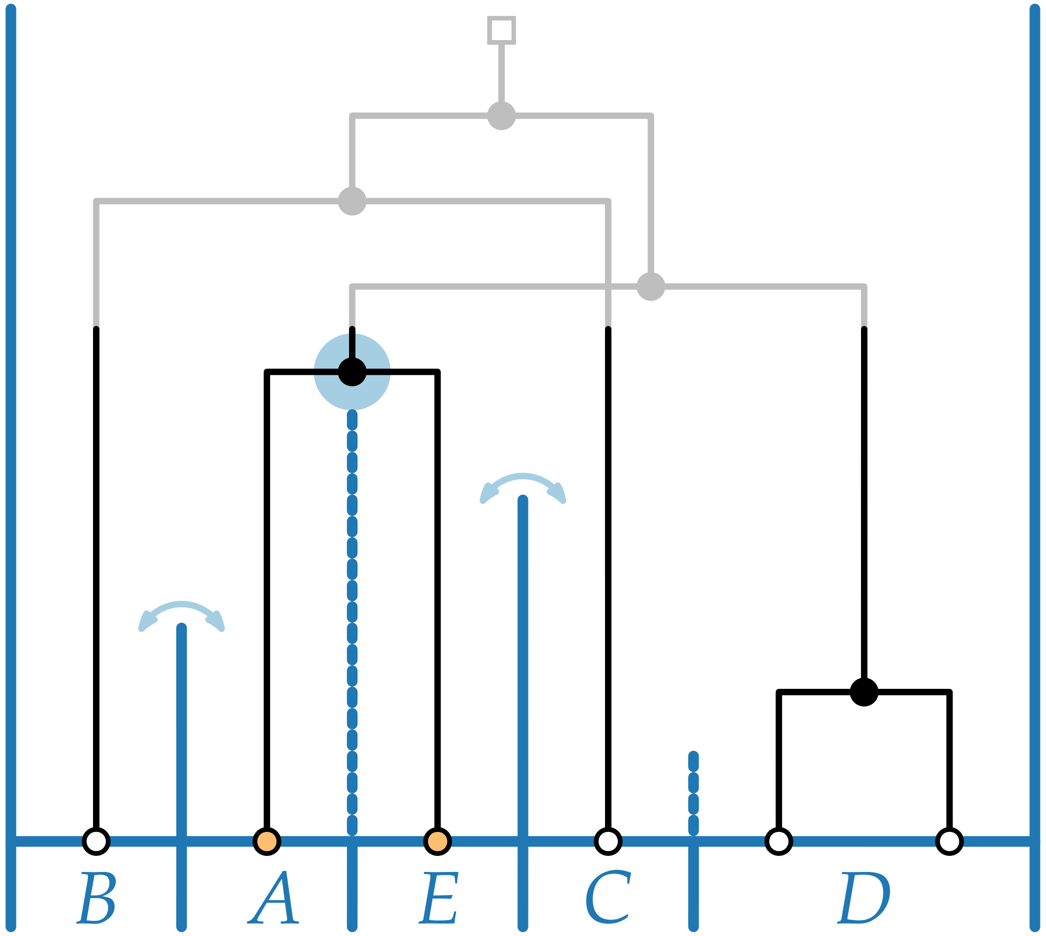

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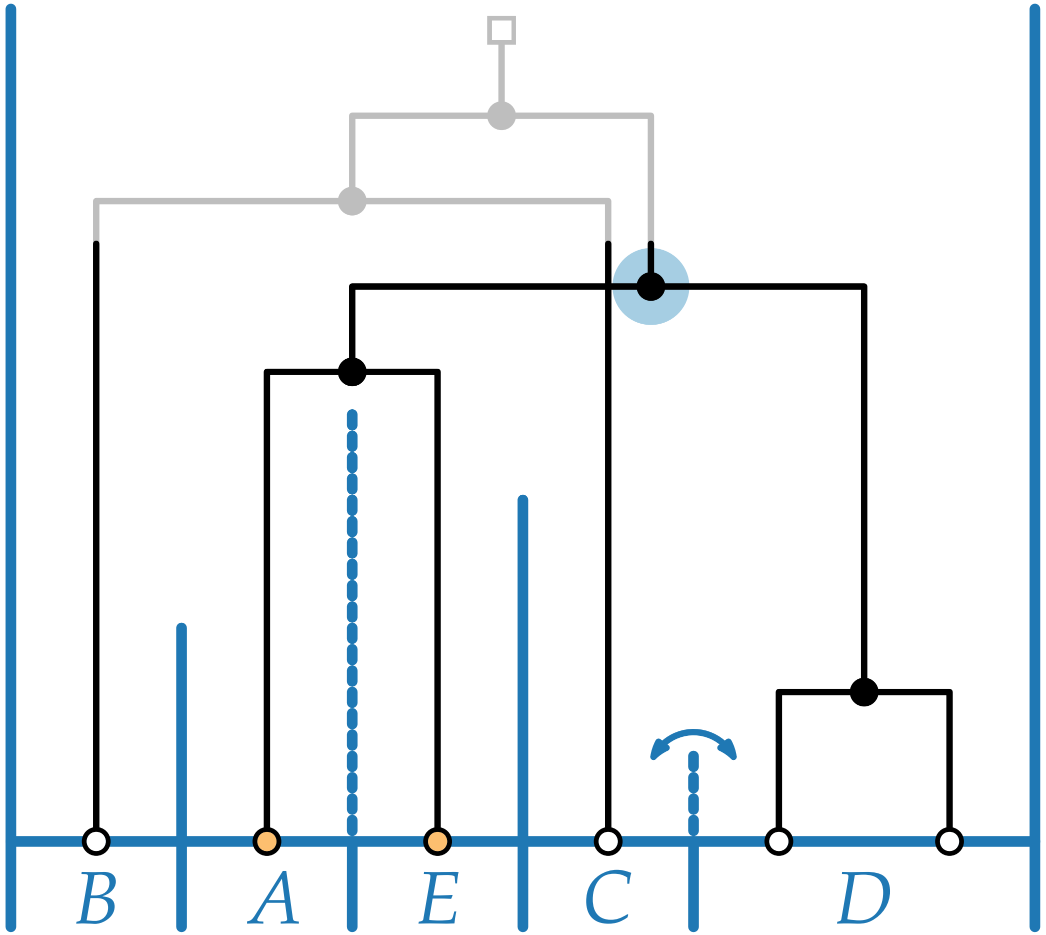

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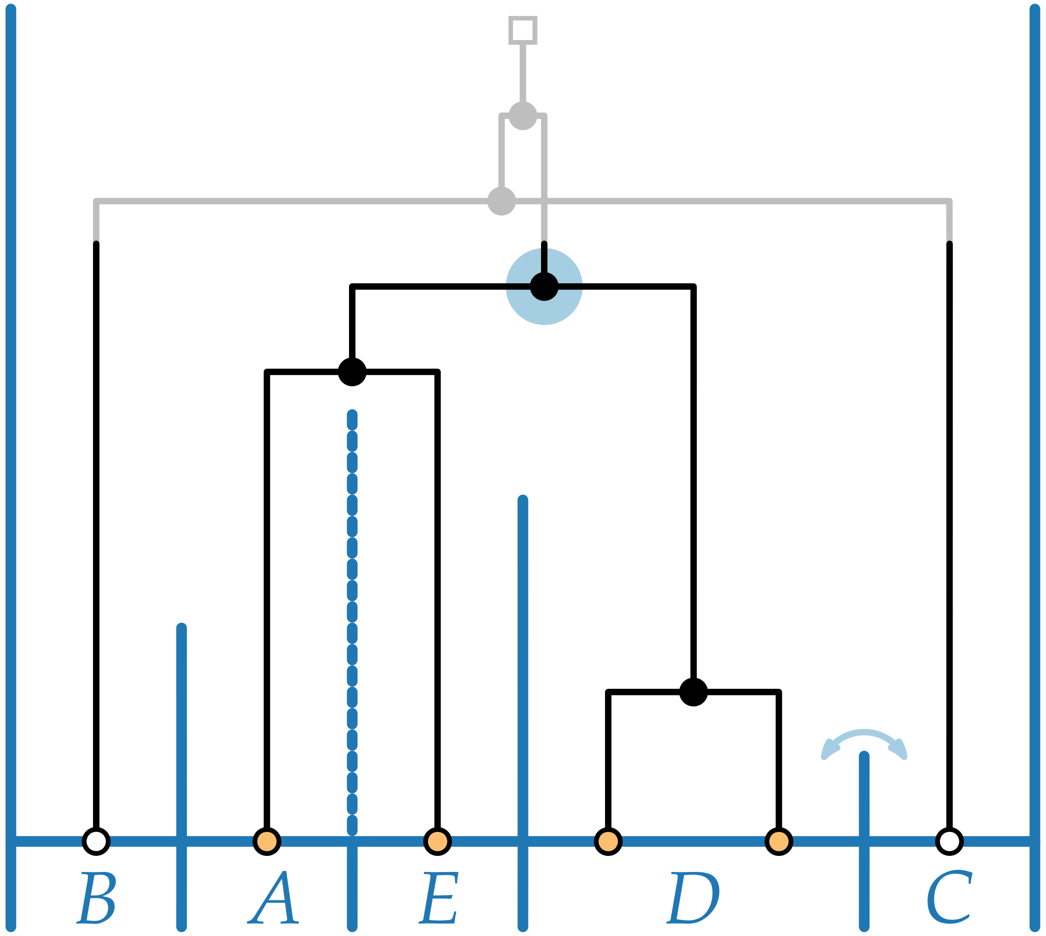

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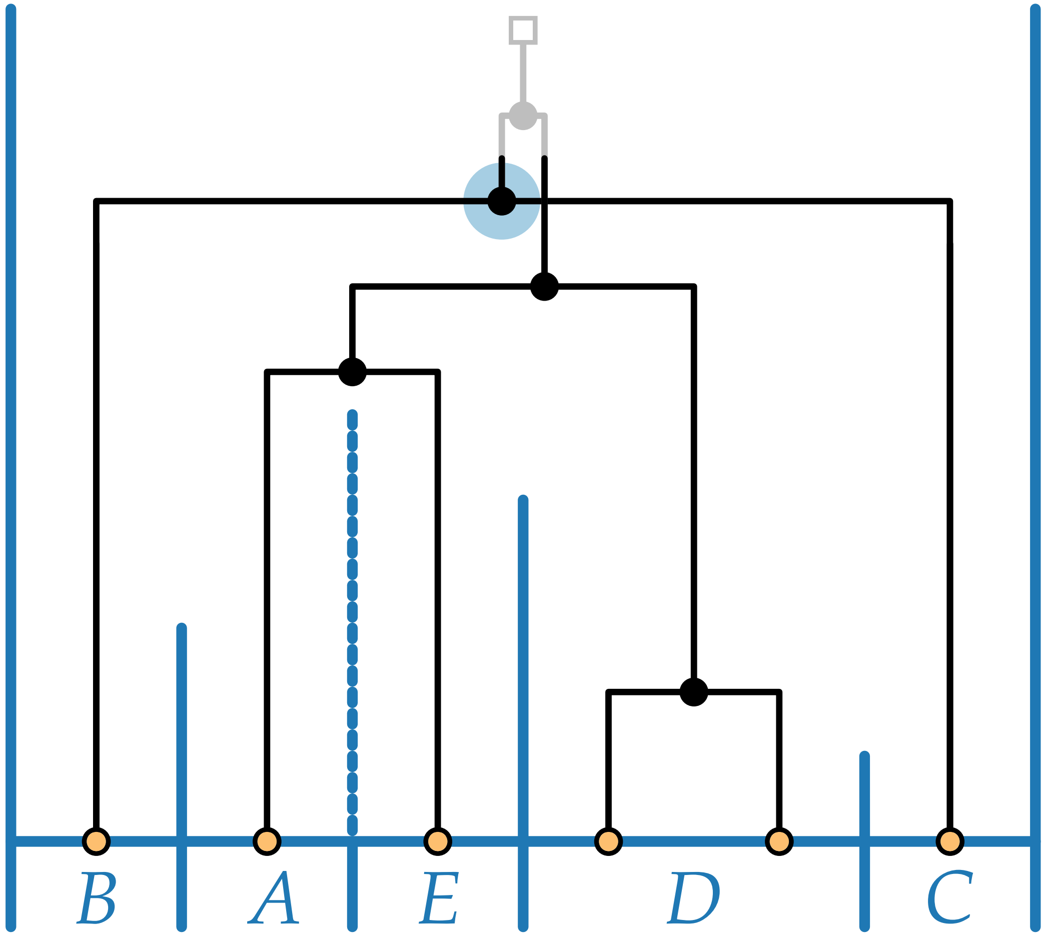

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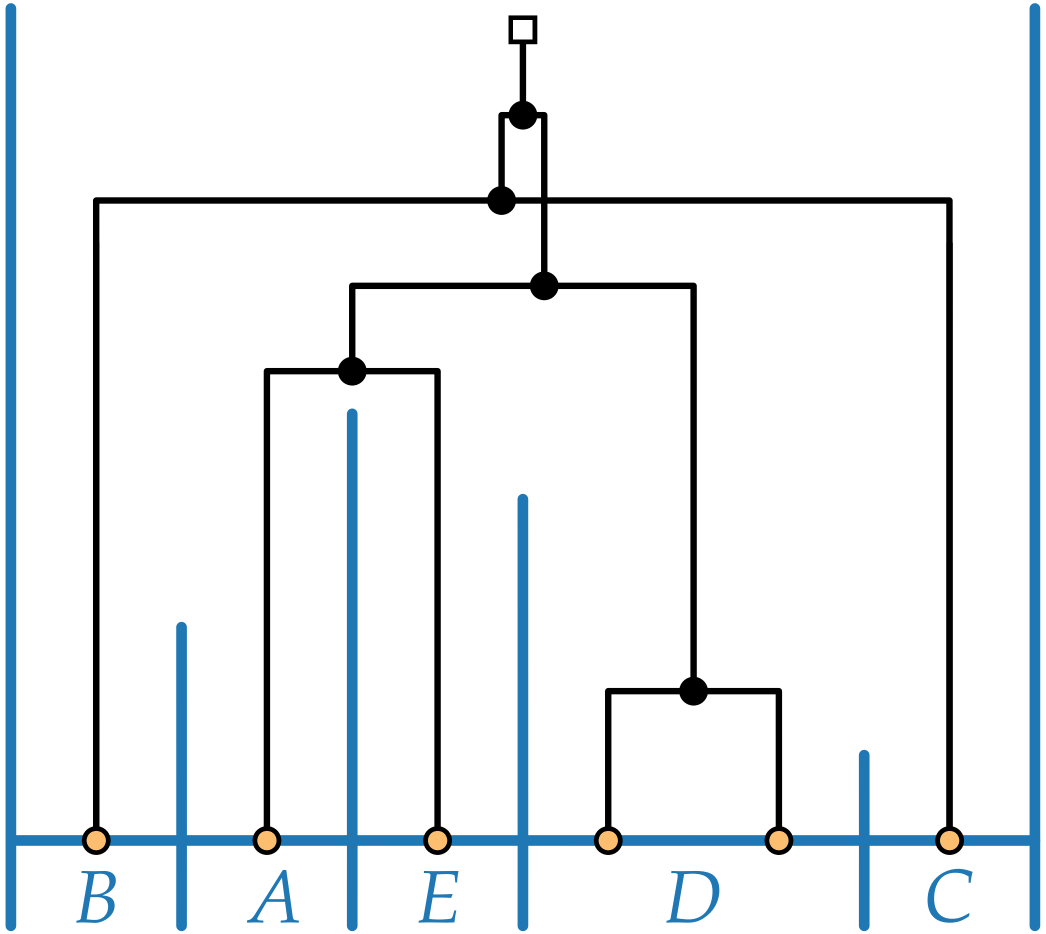

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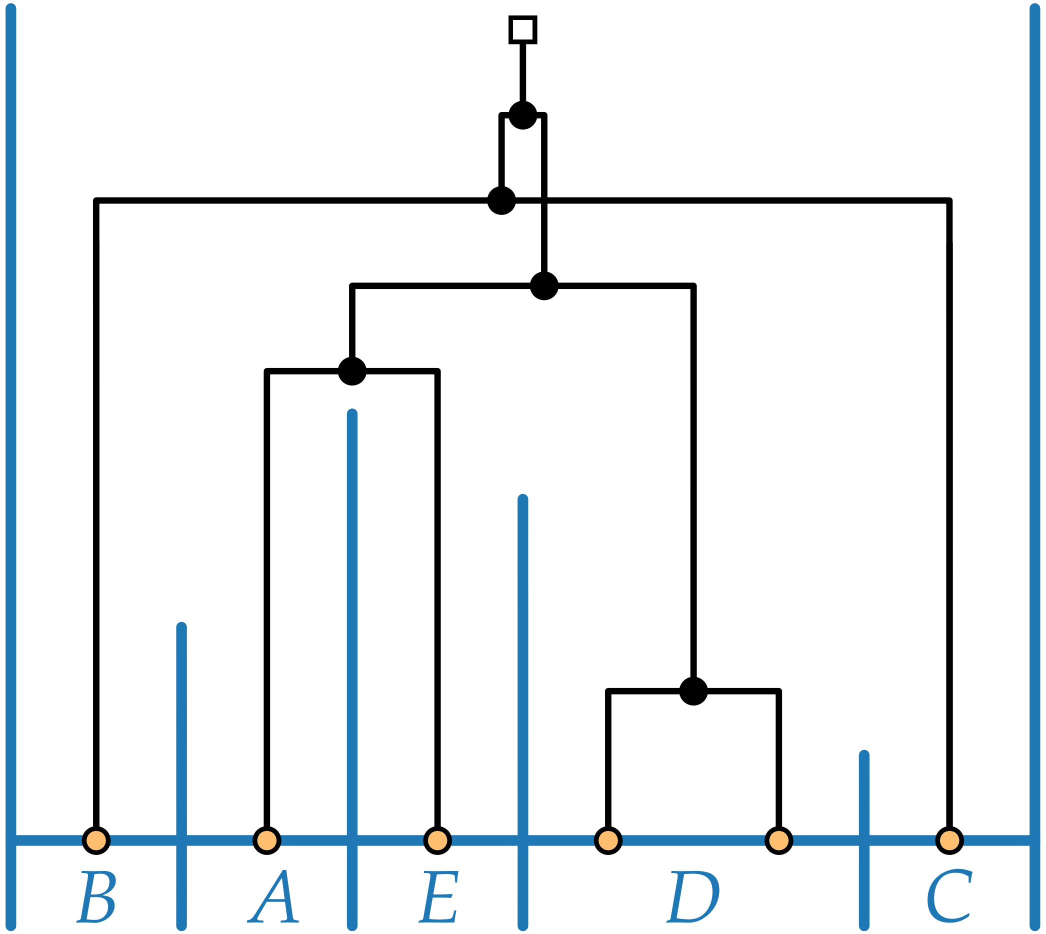

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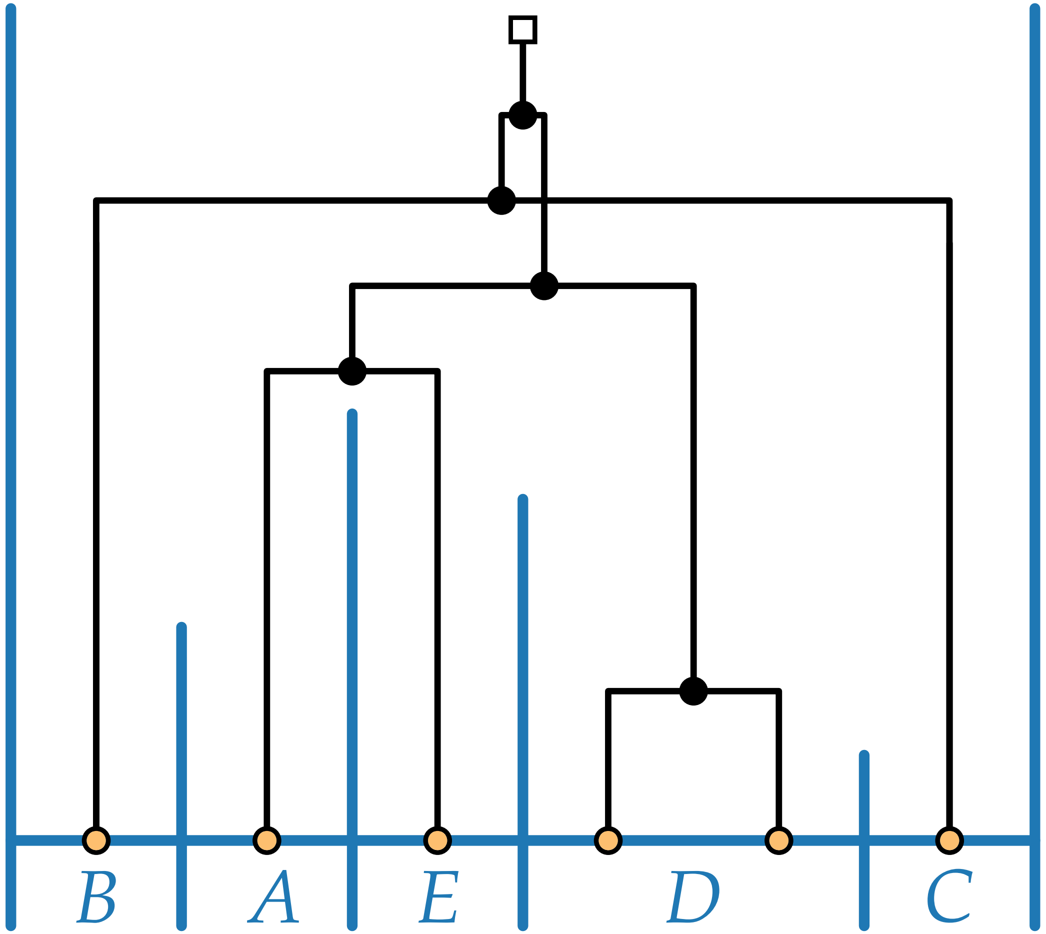

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- We need $O(n^2 + m)$ variables and $O(n^2 + n \cdot m)$ constraints,
  where $n = |V(T)|$ and $m = |V(S)|$

Table of contents

- We have defined drawing style for MSC trees.
- Planarity testing is easy.
- Crossing minimisation is NP-hard (for fixed & variable species order).
- We present two simple heuristics and an integer linear program (ILP).
- We do an experimental evaluation of the heuristics.

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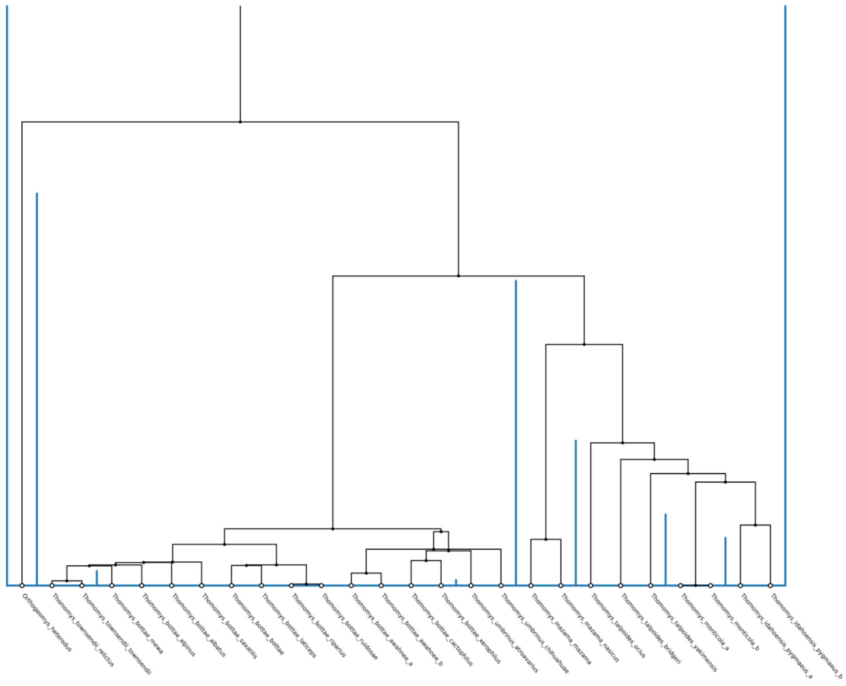
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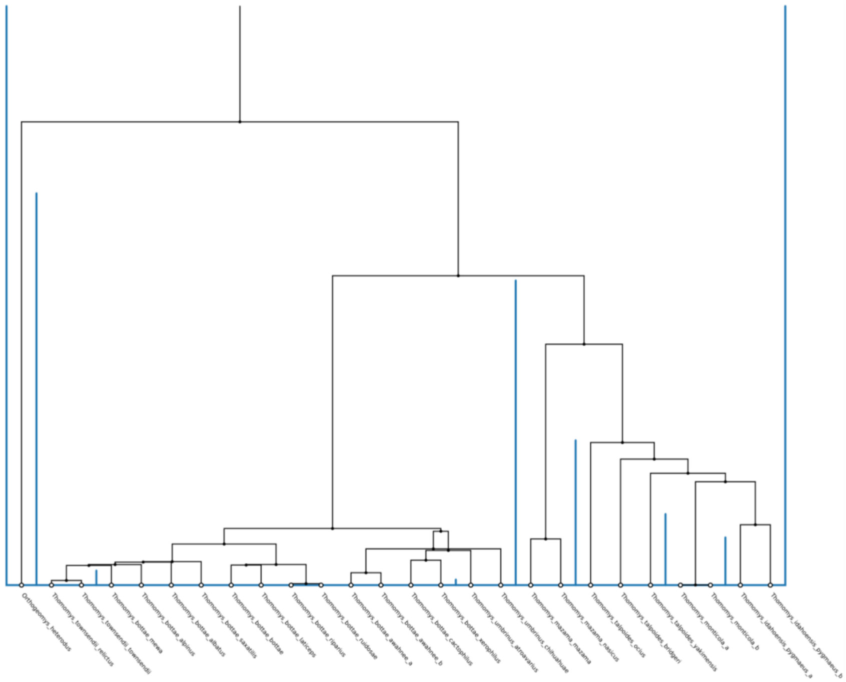
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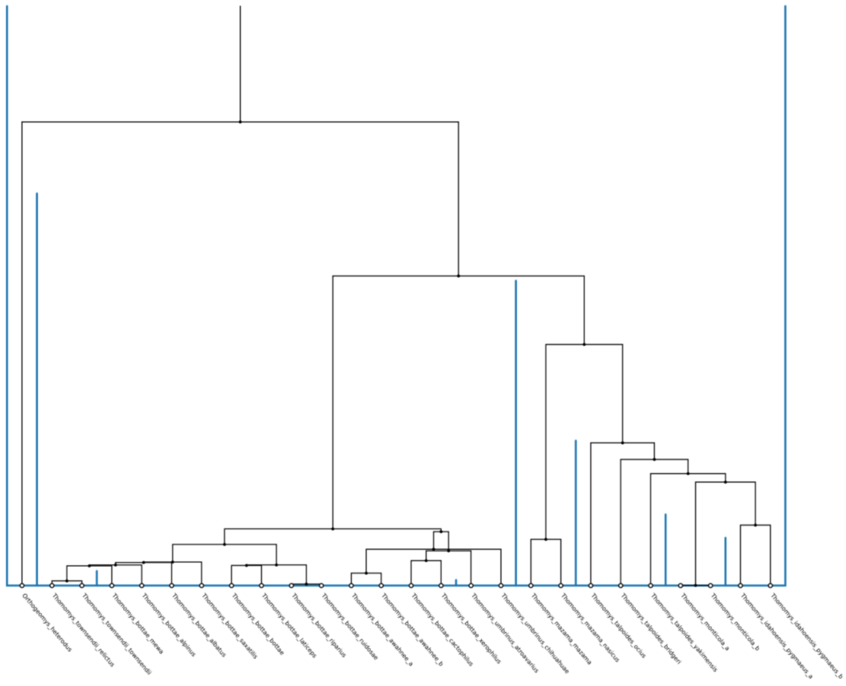
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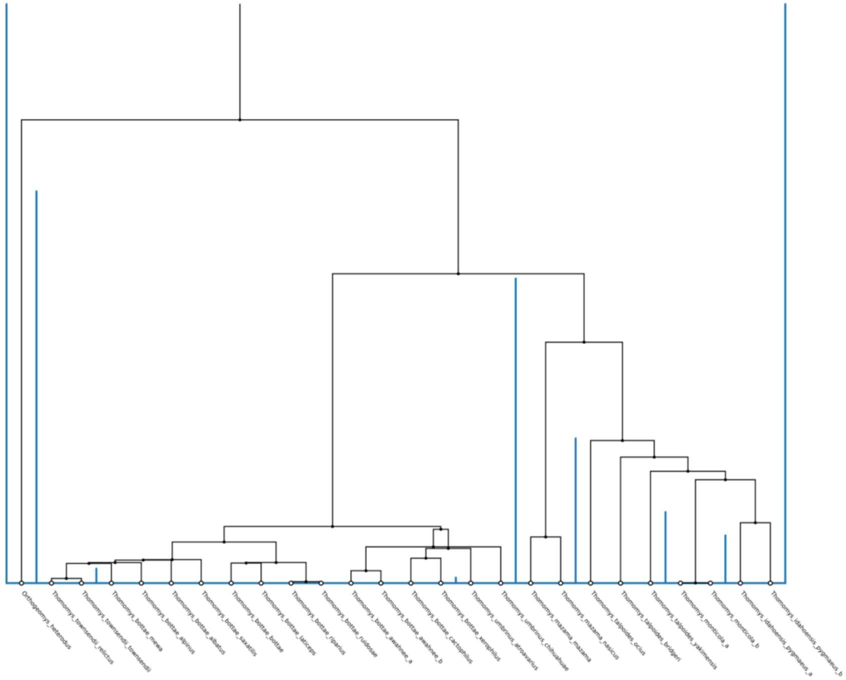
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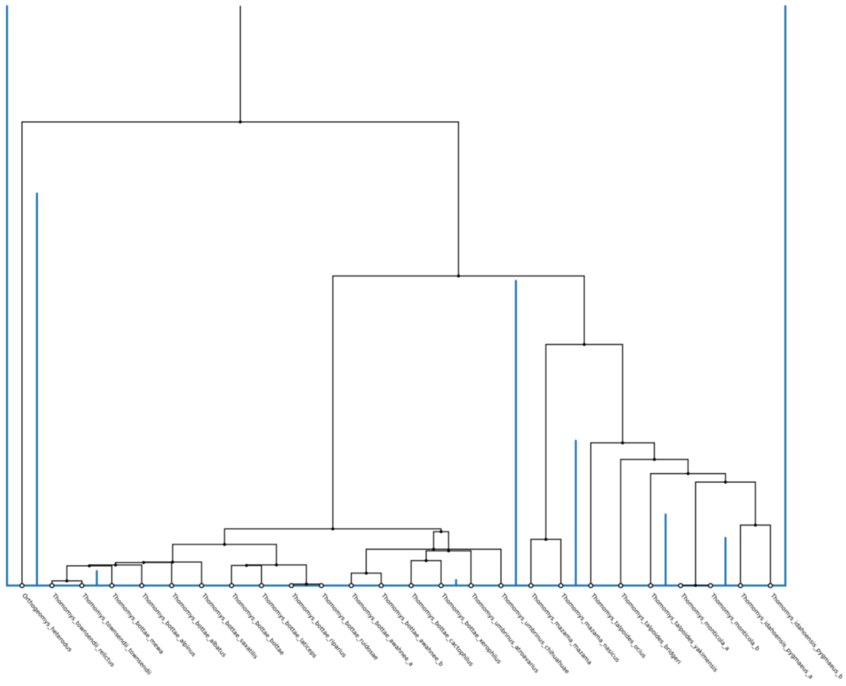

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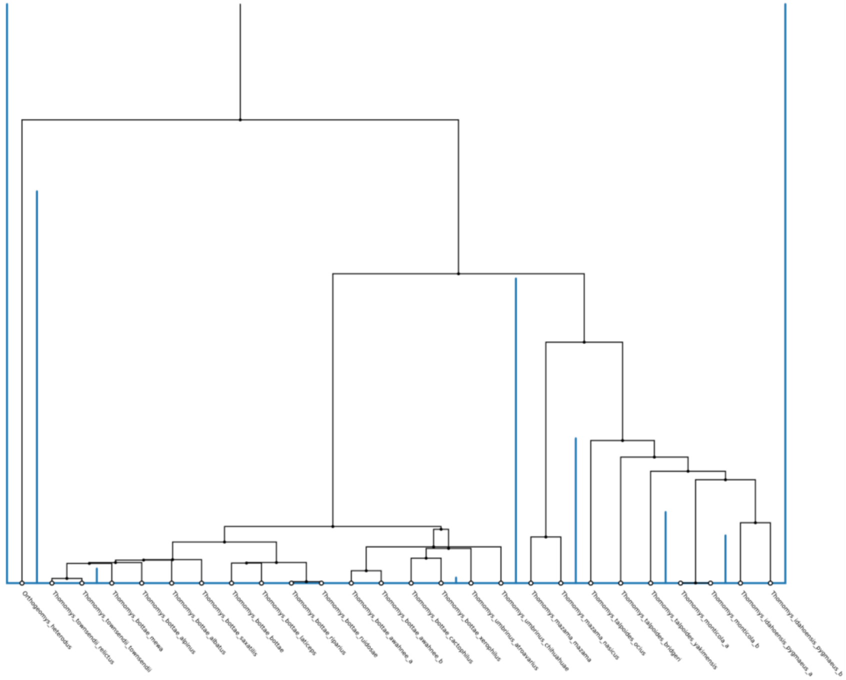

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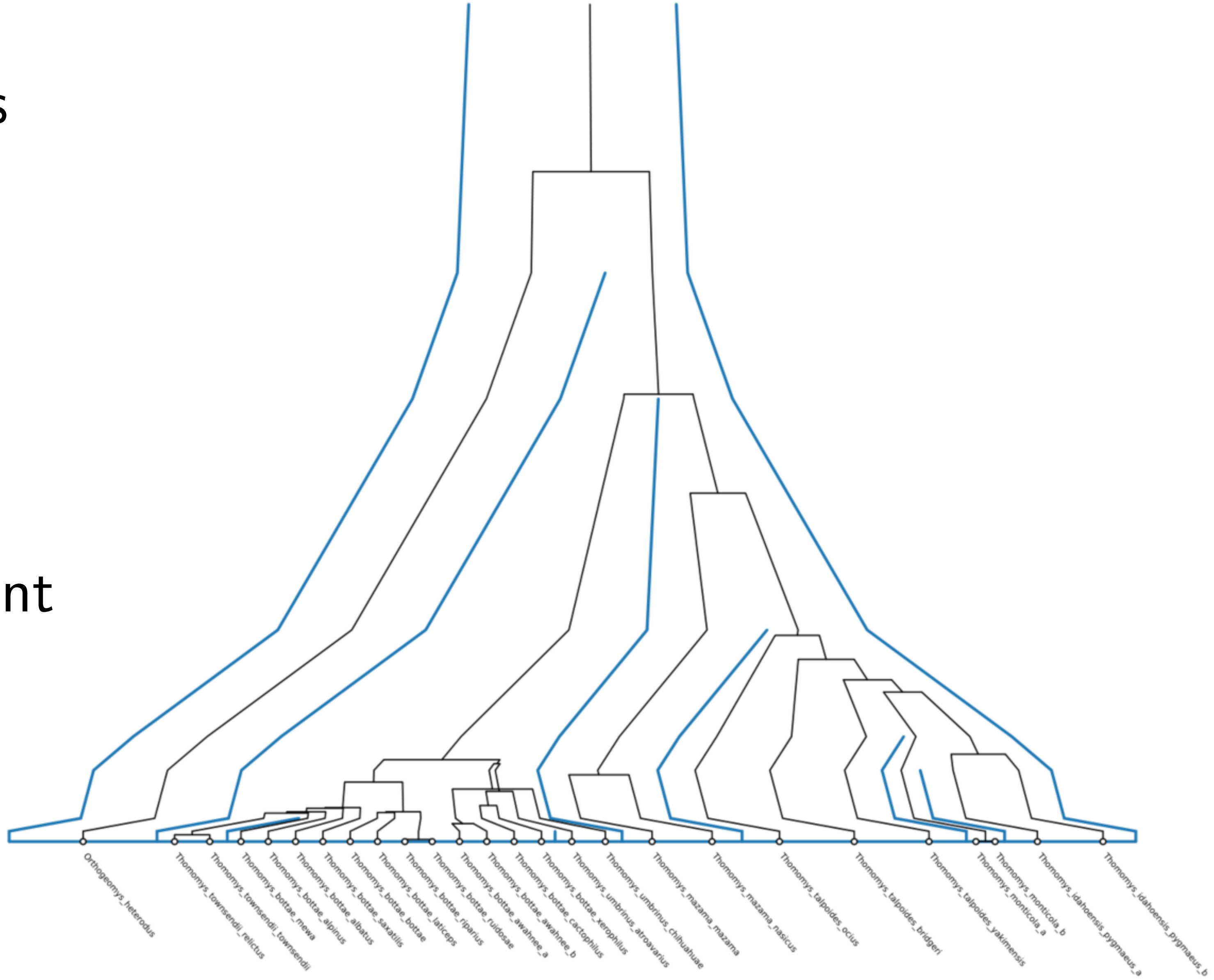
Recommendation.
Try heuristics on random start embedding while improving; ILP only on small instances.

Summary

- Defined drawing style for multispecies coalescent trees.
- Planarity testing is easy.
- Crossing minimisation is NP-hard.
- Described simple heuristics and ILP.
- Recommend using heuristic on different start embeddings in practice.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