Bounding the Treewidth of Outer k-Planar Graphs via Triangulations

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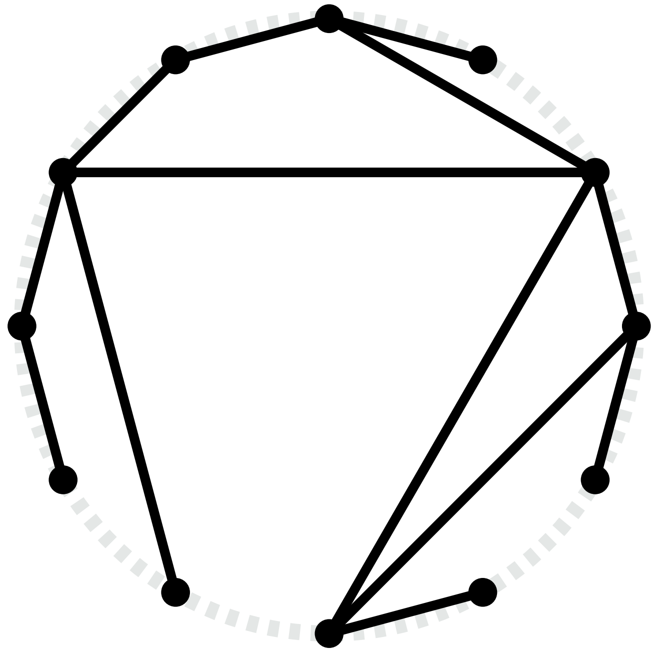
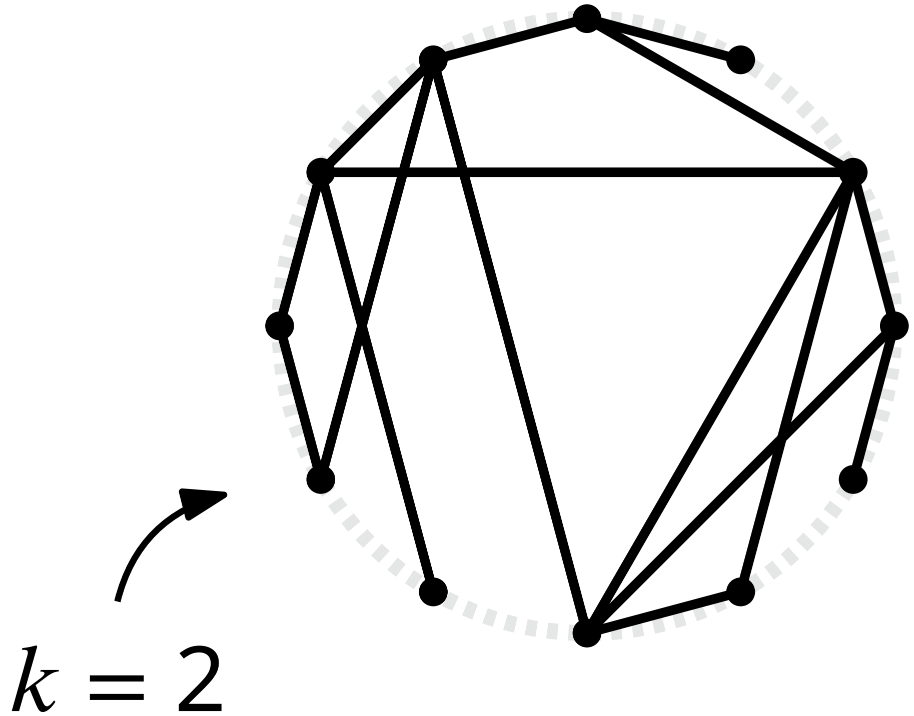
Outer k-Planar Graphs



Outerplanar graph

admits a drawing s.t.

- vertices on a circle
- straight-line
- no crossing



Outer k-planar graph

admits a drawing s.t.

- vertices on a circle
- straight-line
- k-planar

Abstract

Main Contribution

Any outer k-planar graph admits a good triangulation.

Improved Upper Bounds ← gives

- Separation Number

$$2k + 3 \rightarrow k + 2$$

[Chaplick et al., GD 2017] (almost) tight

- Treewidth

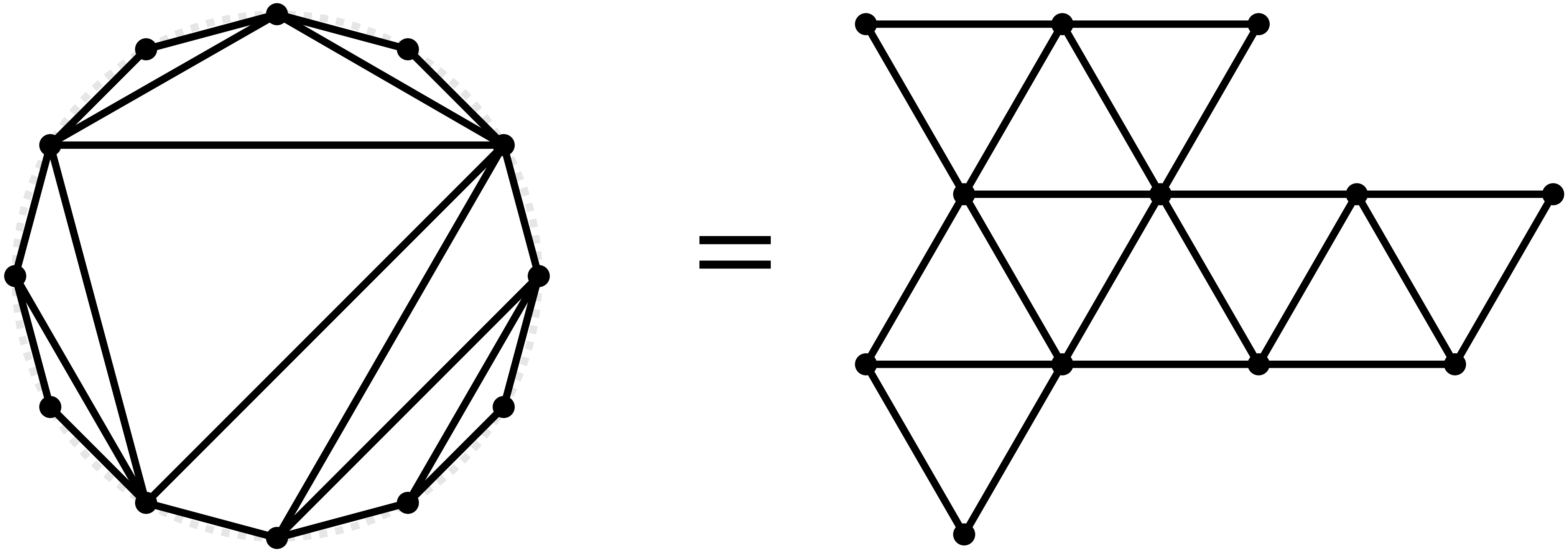
$$3k + 11 \rightarrow 1.5k + 2$$

[Wood and Telle, GD 2006] (lowerbound: $k + 2$)

... and some other results.

Triangulation on Outerplanar Graphs

After adding some edges until it becomes maximal, every inner face will be a triangle.



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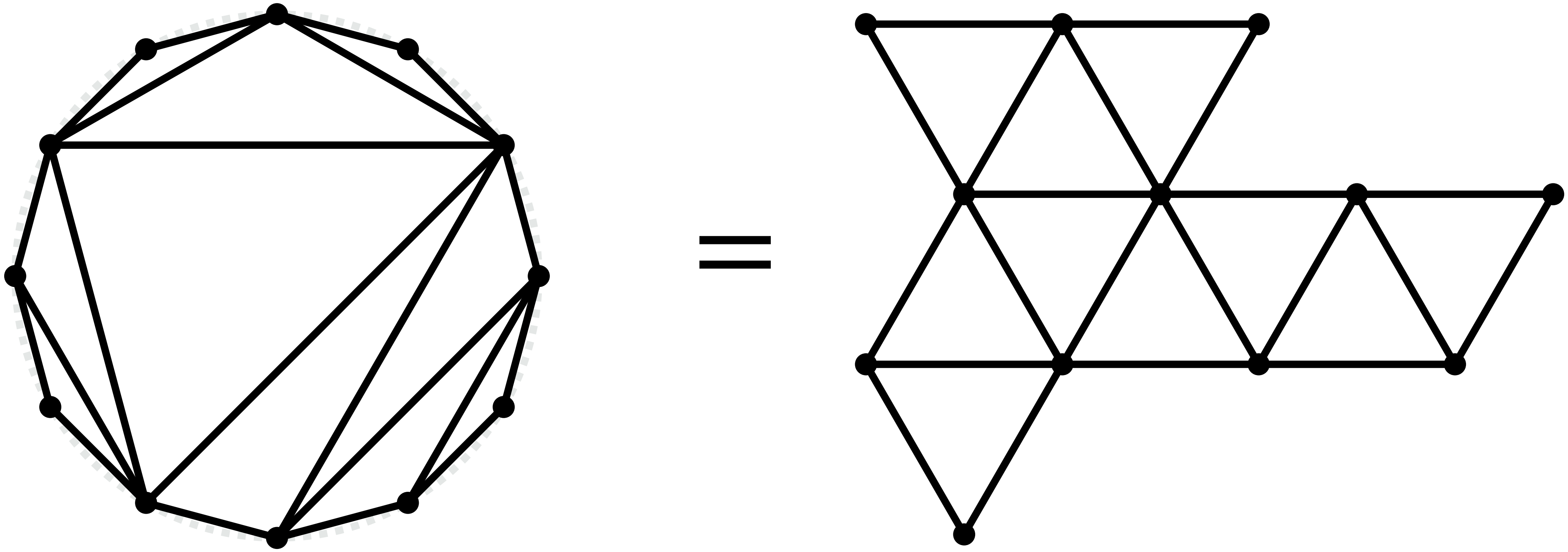
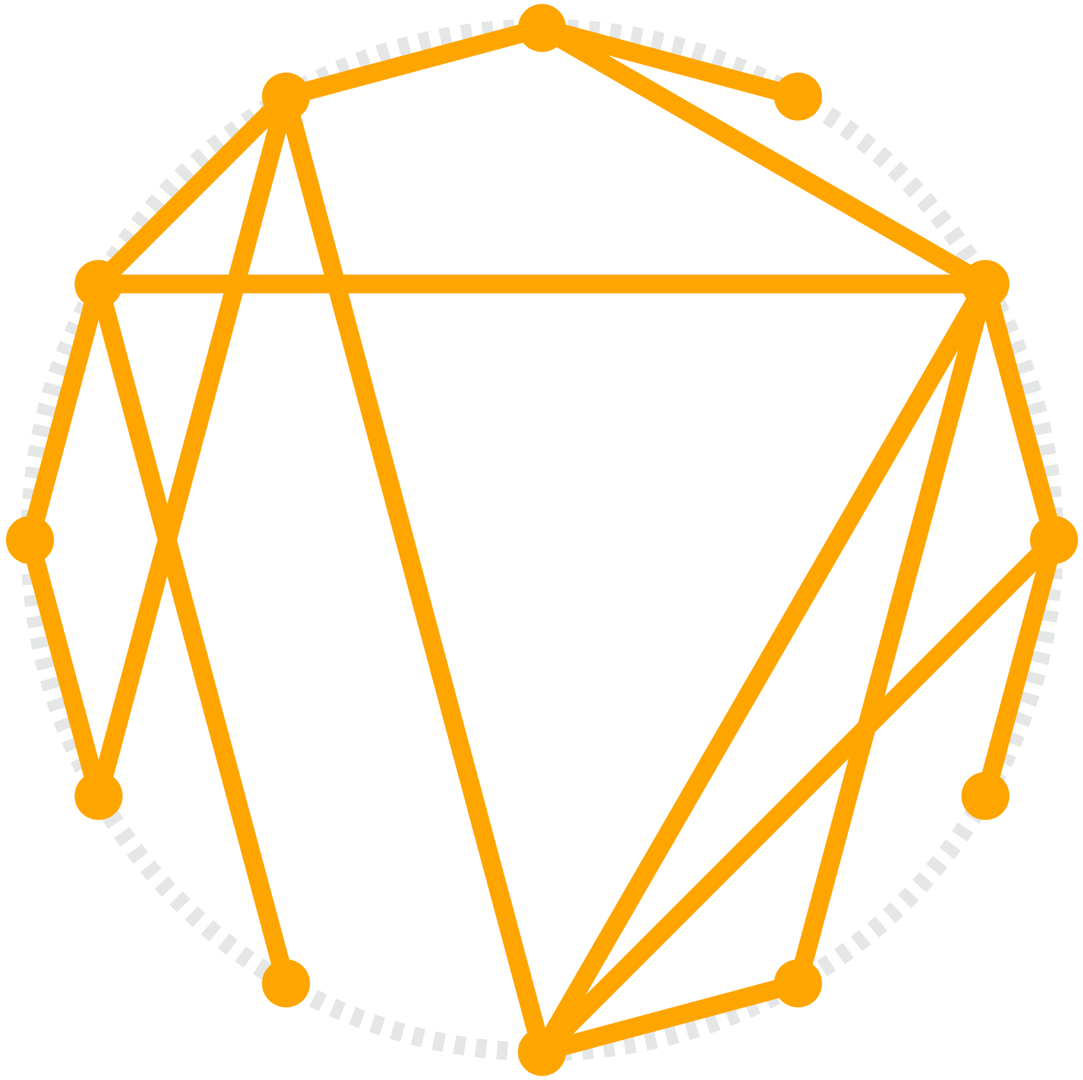


But outer k-planar graphs may have a crossing...?

Triangulation on Outer k-Planar Graphs

Lemma 6

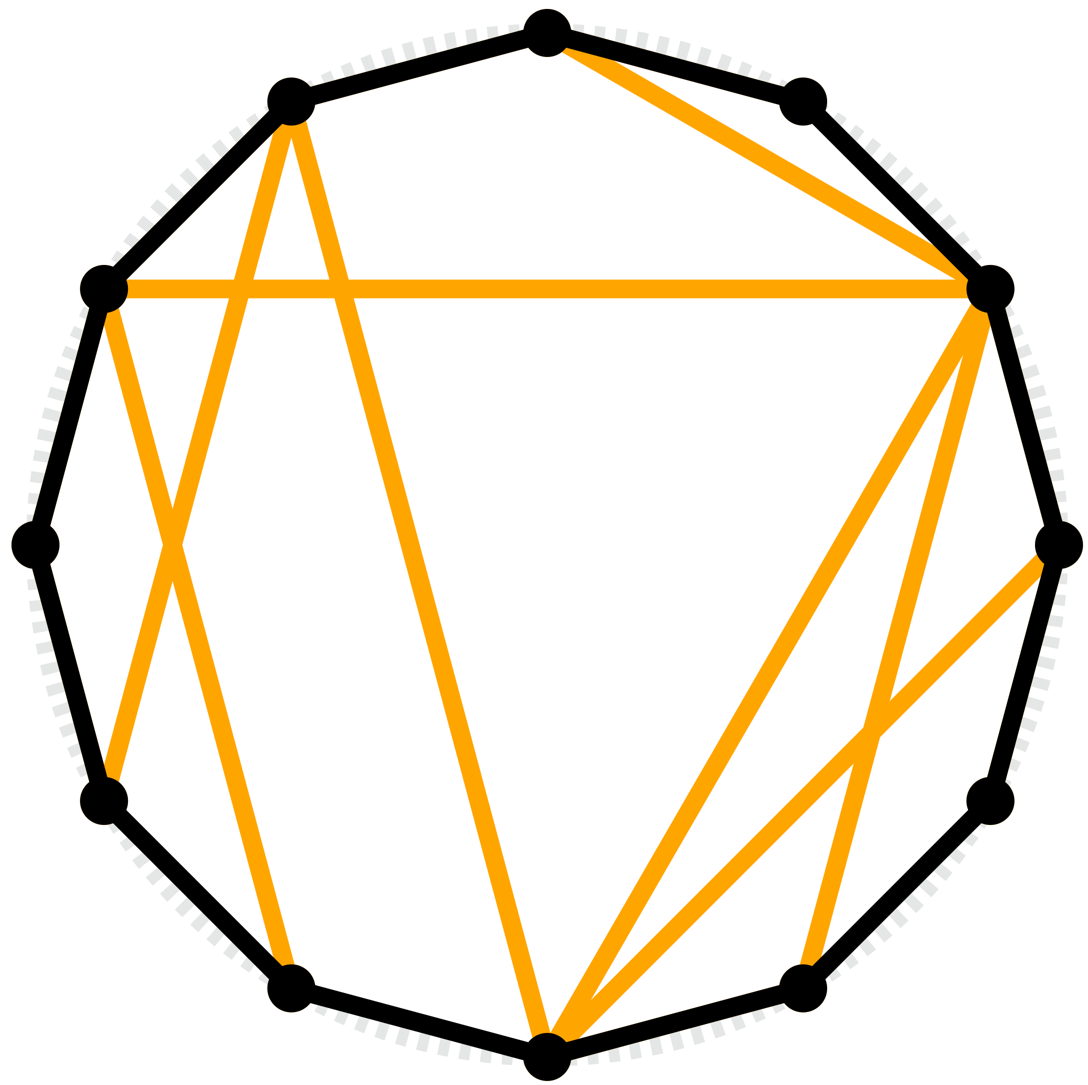
For every $k \geq 0$, given an outer k-planar drawing,



Triangulation on Outer k-Planar Graphs

Lemma 6

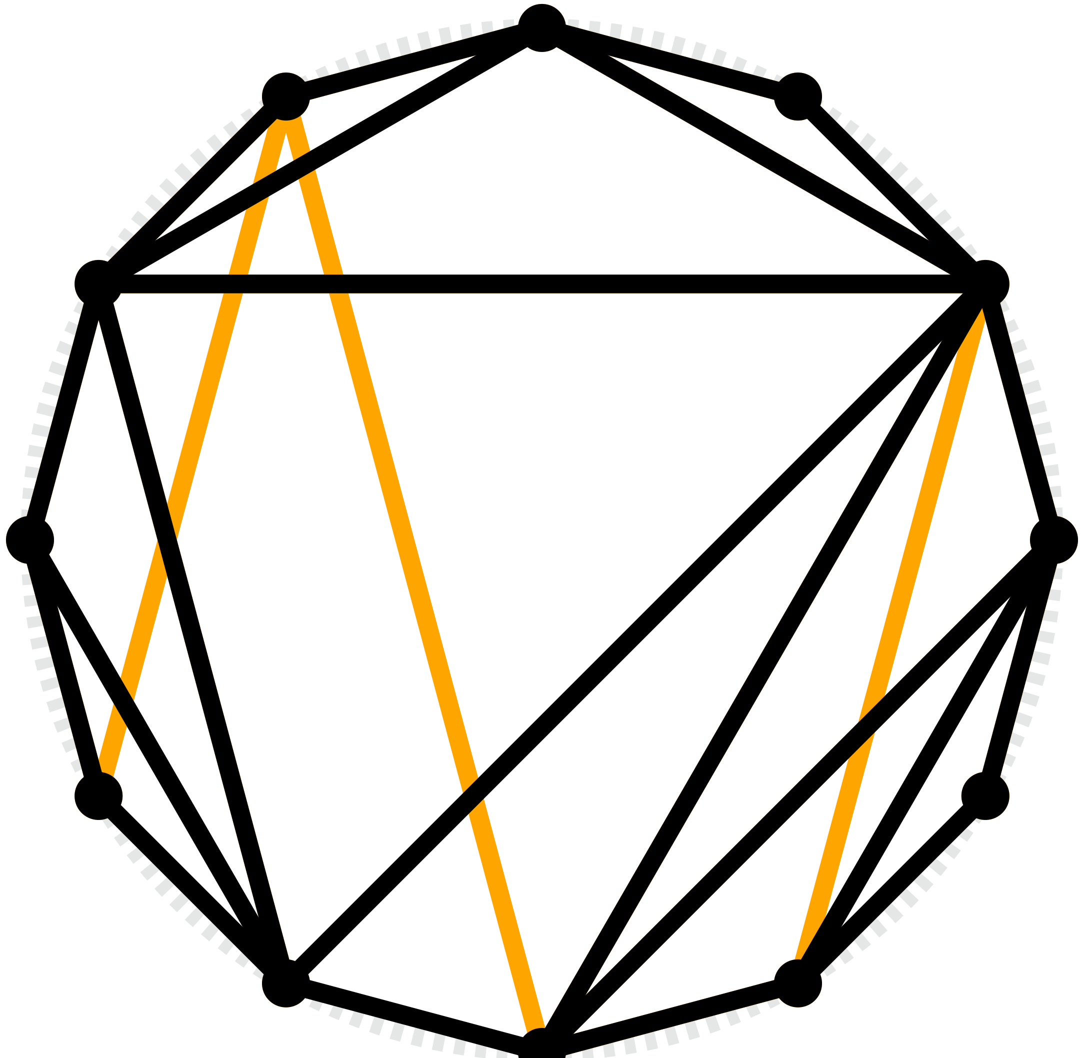
For every $k \geq 0$, given an outer k-planar drawing,
The outer cycle admits a triangulation s.t.



Triangulation on Outer k-Planar Graphs

Lemma 6

For every $k \geq 0$, given an outer k-planar drawing,
The outer cycle admits a triangulation s.t.
each edge crosses the original graph at most k times.

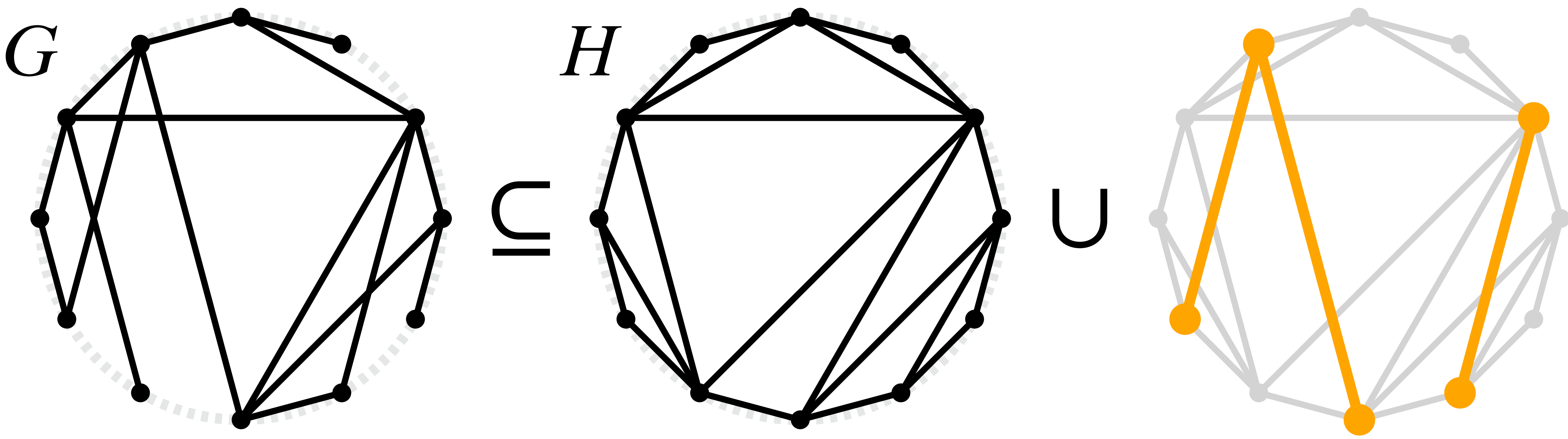


The Advantage of Triangulation

Key Point

The triangulation separates the given graph G into:

- (a subgraph of) maximal outerplane graph H
- crossing edges (sparsely distributed!!)

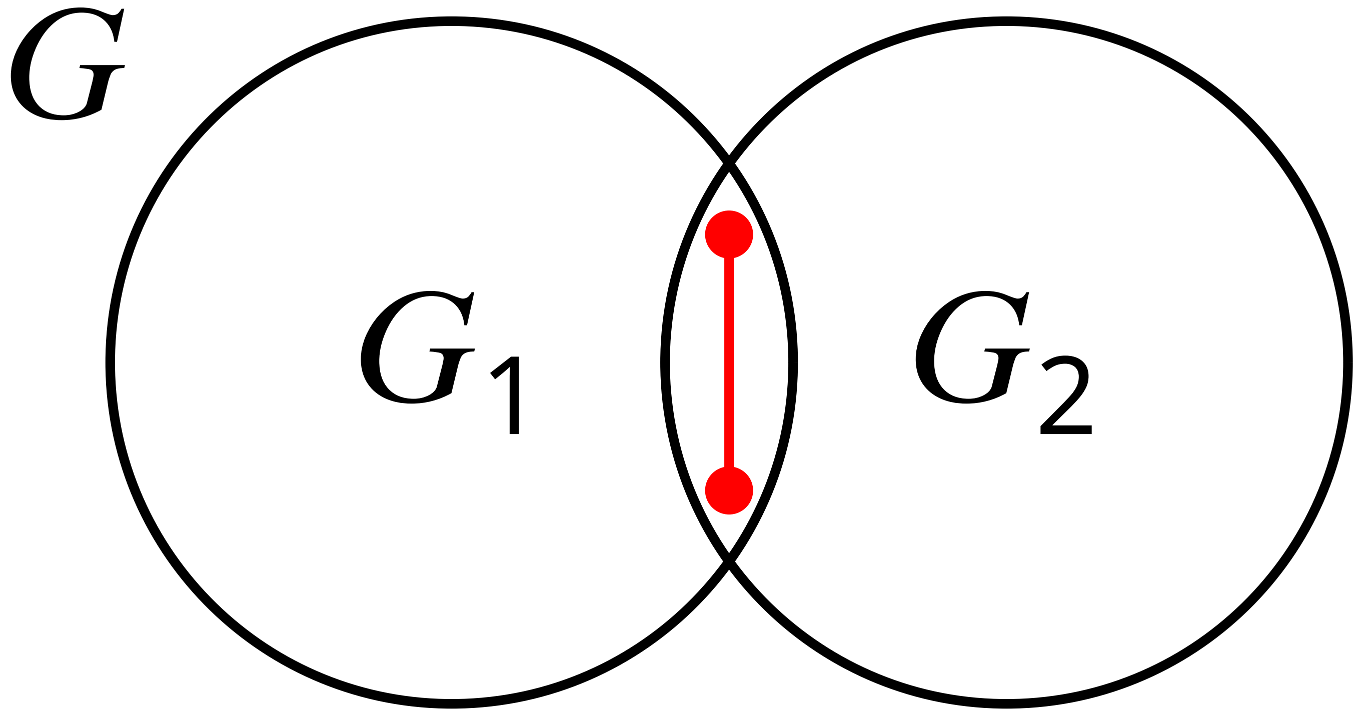


Every edge of H has at most k crossing edges.
Every triangle of H has at most $3k$ crossing edges.
$\rightarrow$ We can re-use properties of outerplanar graphs.

Upper Bound on Separation Number

On outerplanar graphs...

Every maximal outerplanar graph
is known to have an edge that is a balanced separator.



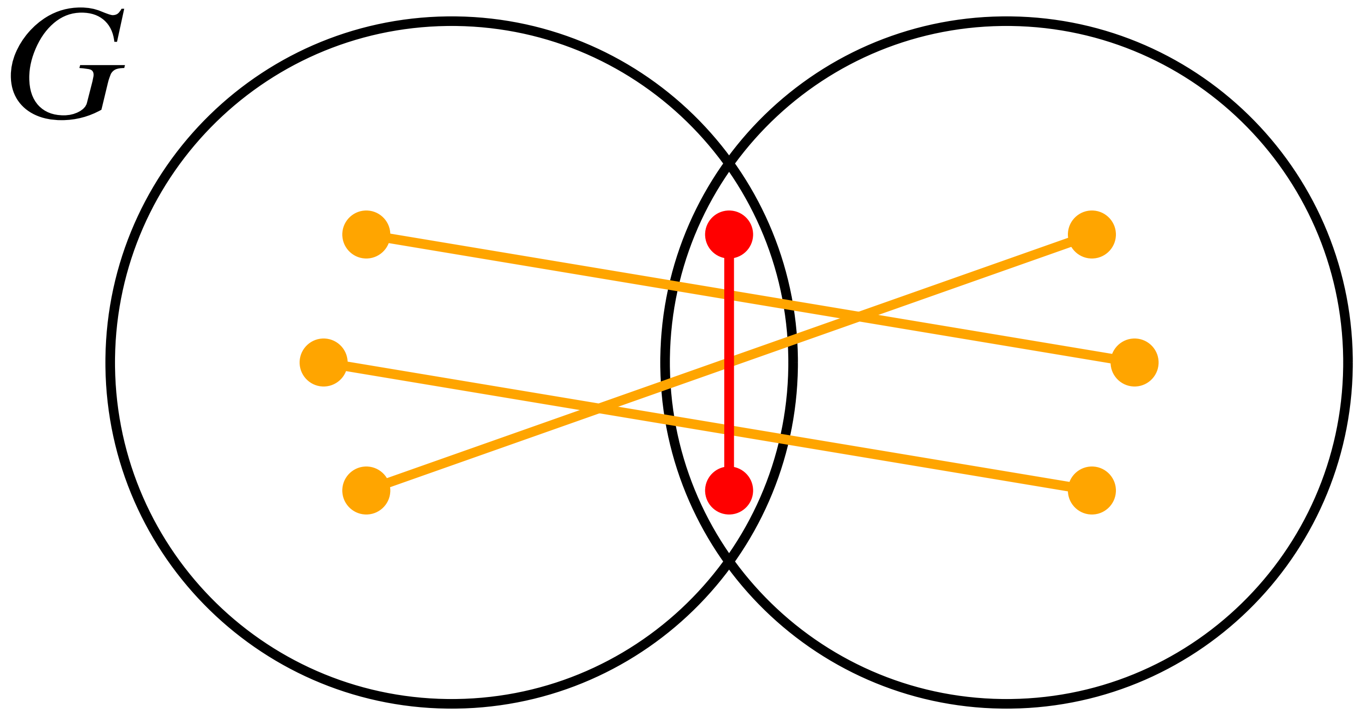
$|G_1|, |G_2| \leq \frac{2}{3}|G| + 2$

Note: A balanced separator of G is a vertex set
whose removal yields components of size at most $\frac{2}{3}|G|$.

Upper Bound on Separation Number

On outer k-planar graphs...

The maximal outerplane graph H also has
a balanced separator edge with $\leq k$ crossing edges.



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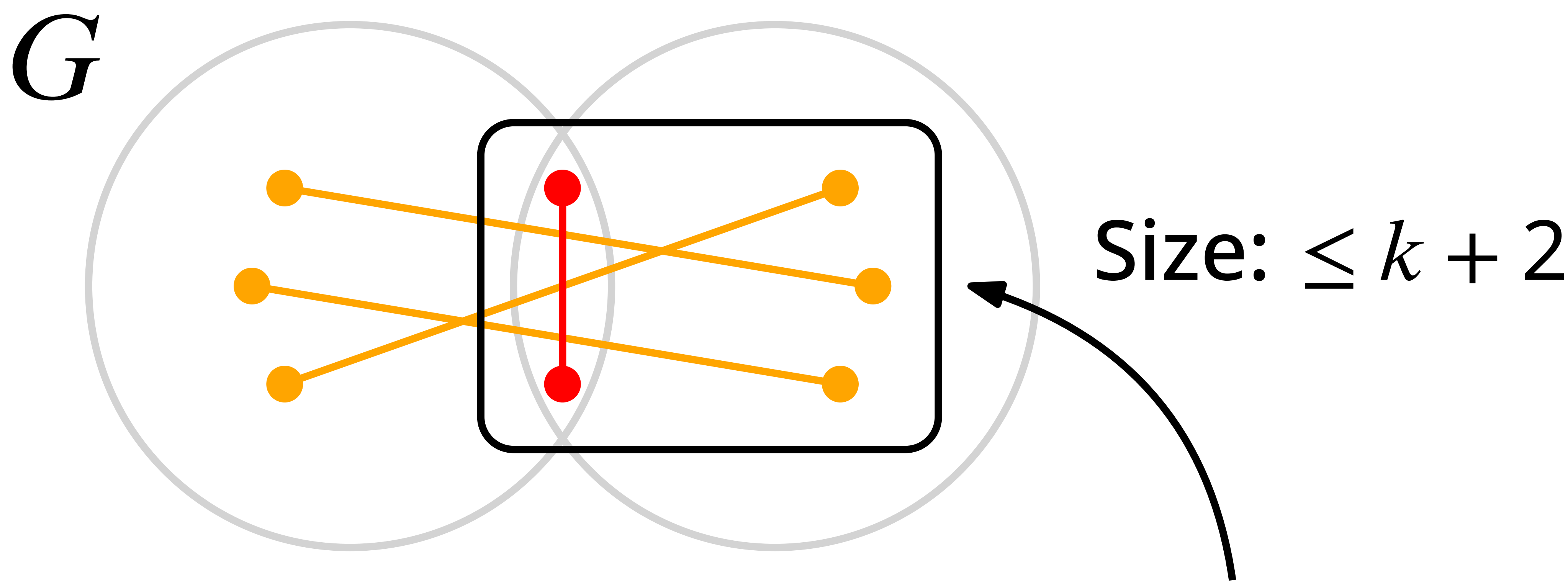
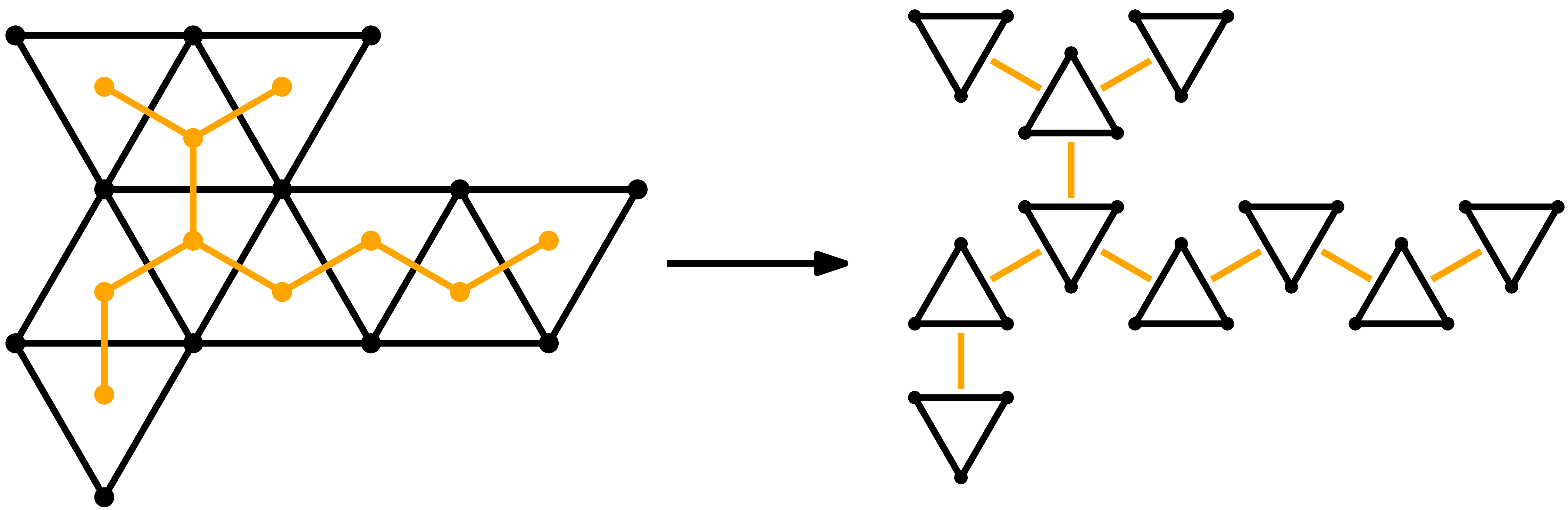


The edge and k endpoints form a balanced separator.
$\rightarrow$ separation number at most $k + 2$.

Upper Bound on Treewidth

On outerplanar graphs...

The weak dual of a maximal outerplane graph is a tree.

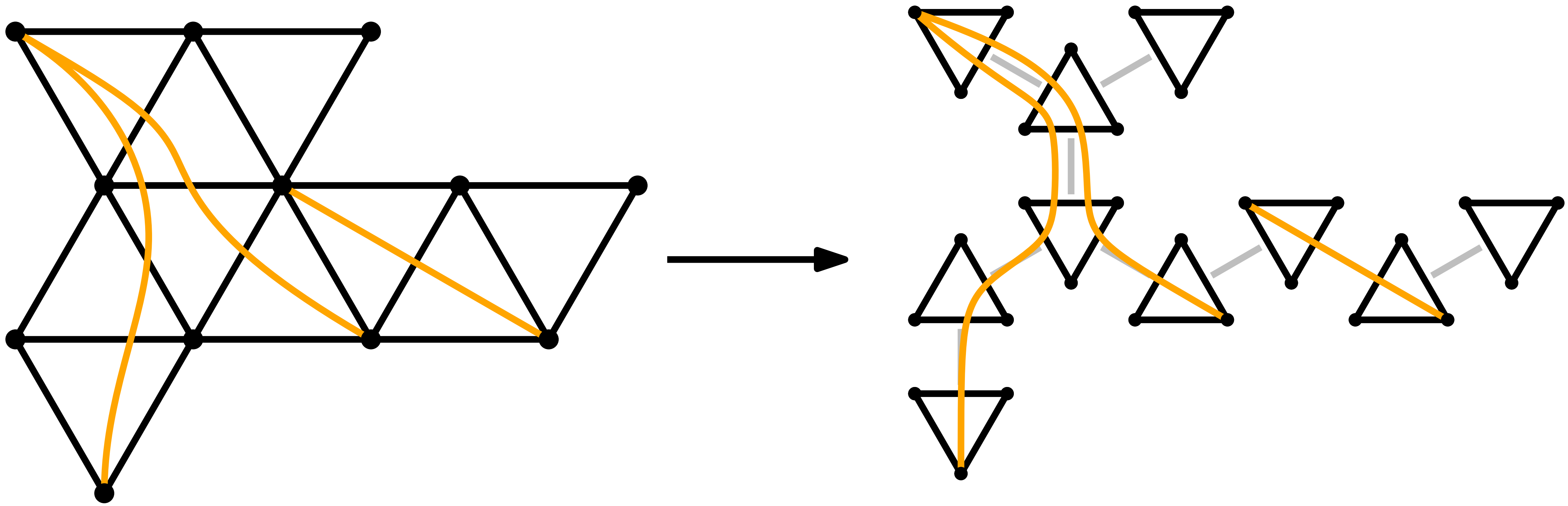


Replacing each vertex with the corresponding triangle yields a tree decomposition of width 2.

Upper Bound on Treewidth

On outer k-planar graphs...

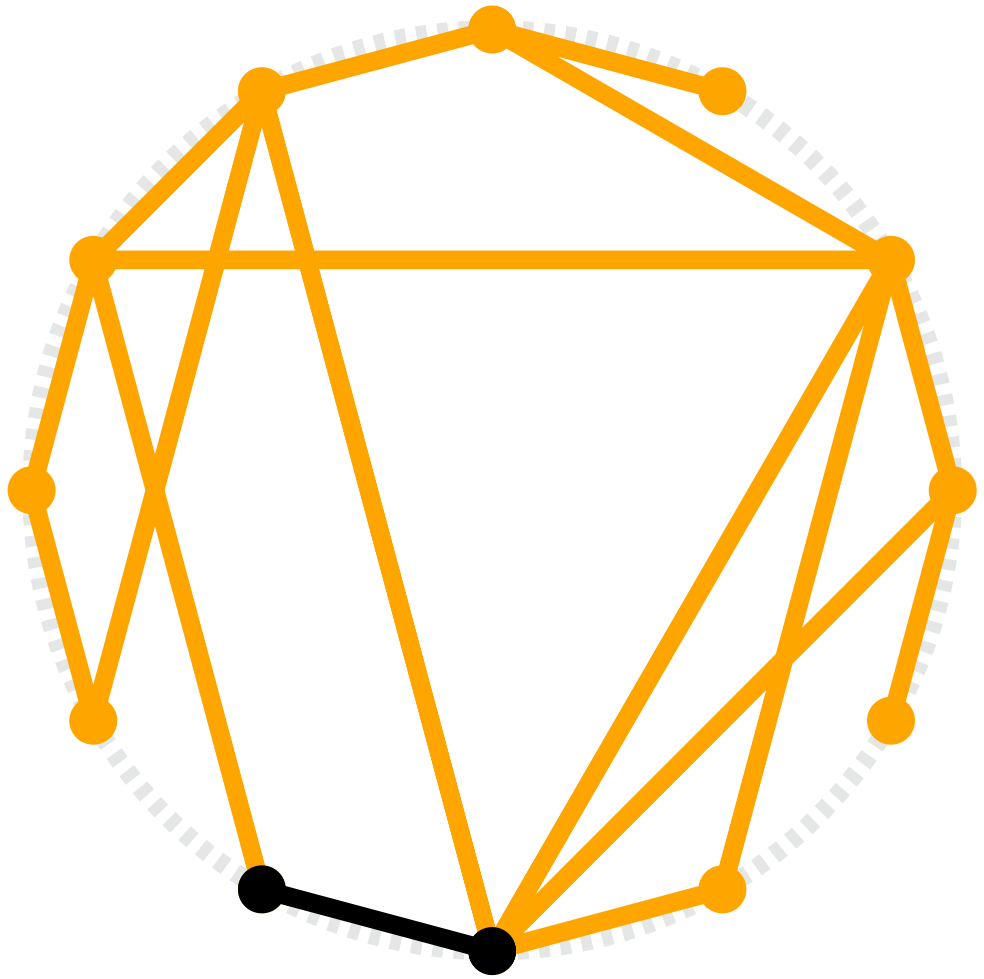
We use a tree decomposition of the graph H.
Then we only need to deal with the crossing edges.



Each triangle (= bag) has at most $3k$ crossing edges.
$\rightarrow$ treewidth $3k + 2$ (naïve), can be improved to $1.5k + 2$.

Construction of the Triangulation

A greedy algorithm yields the triangulation.



1. Take any edge from the outer cycle.
2. There always exists a suitable vertex w.
3. Create a triangle with the vertex, and recurse.

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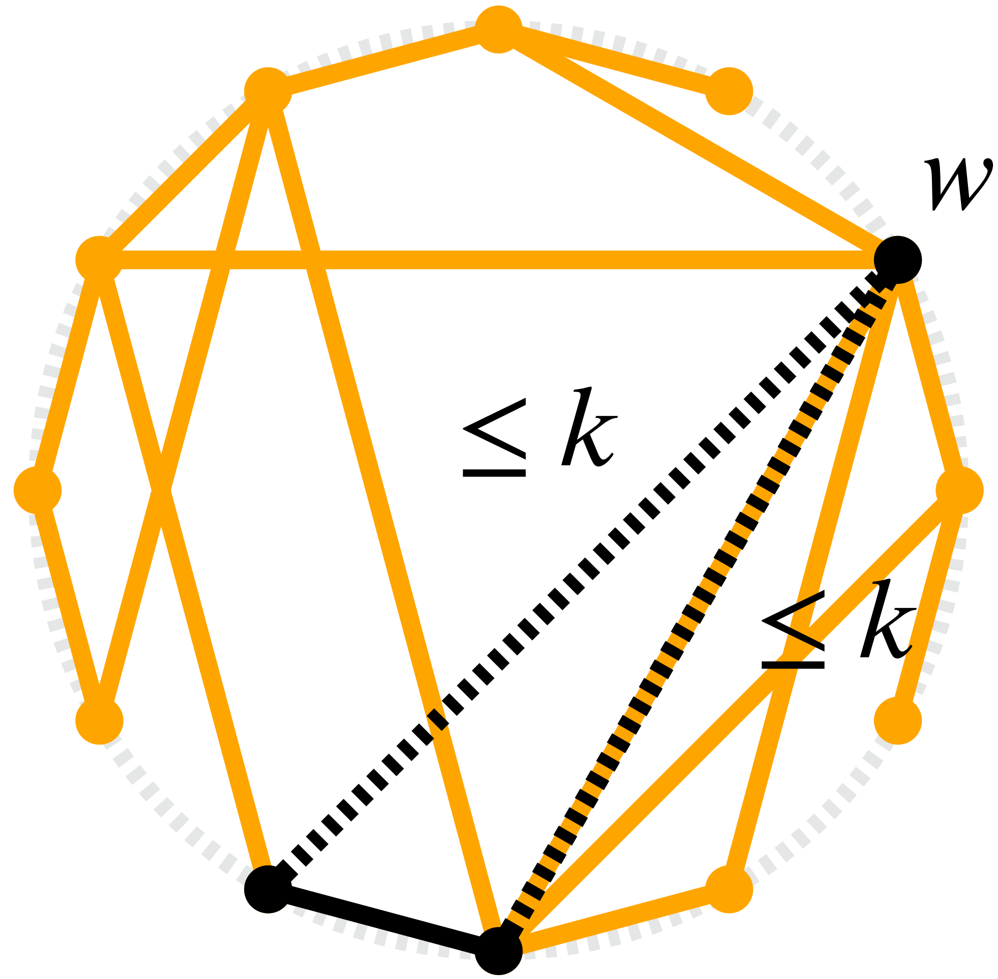

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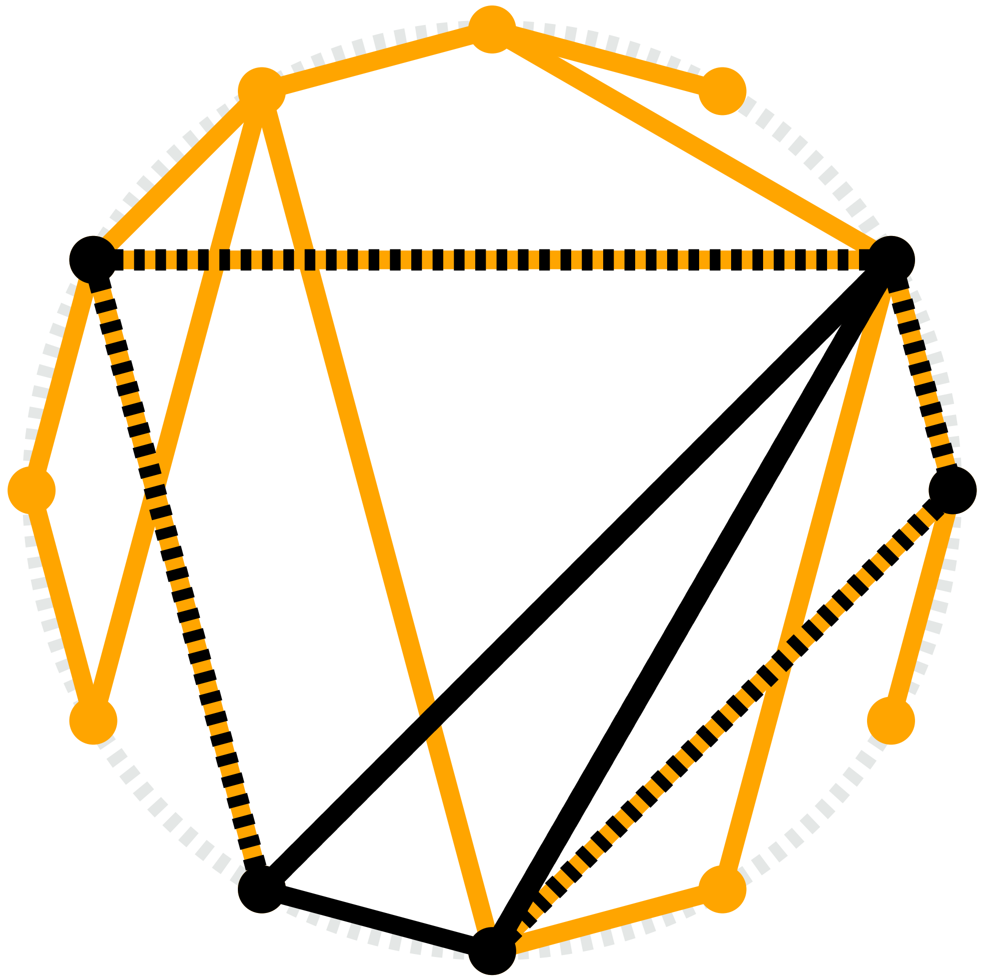

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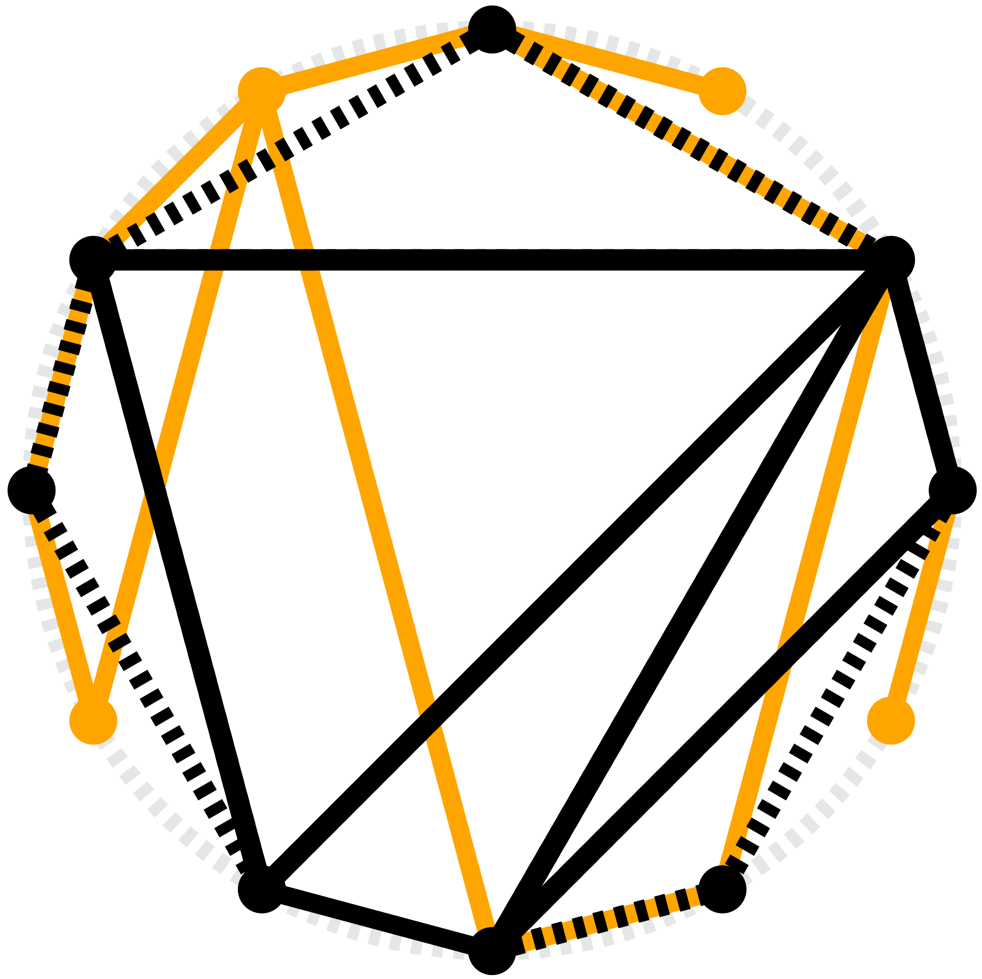

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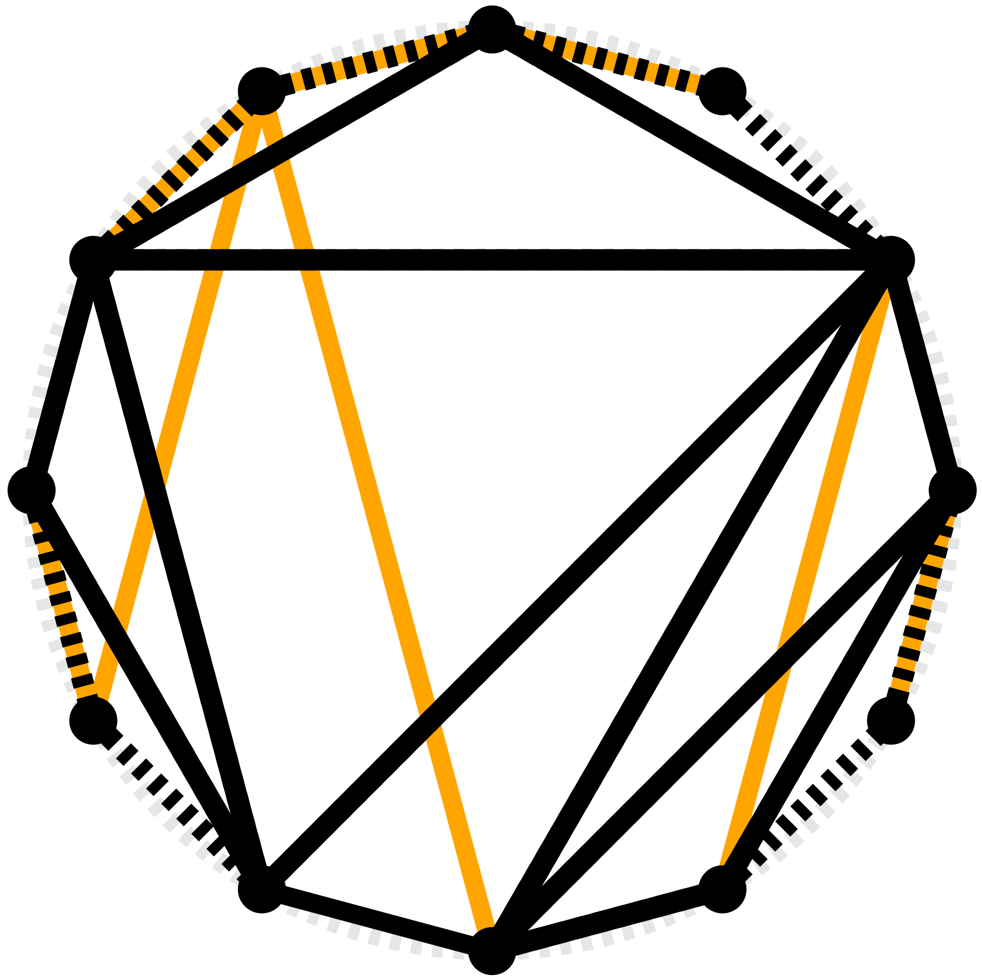

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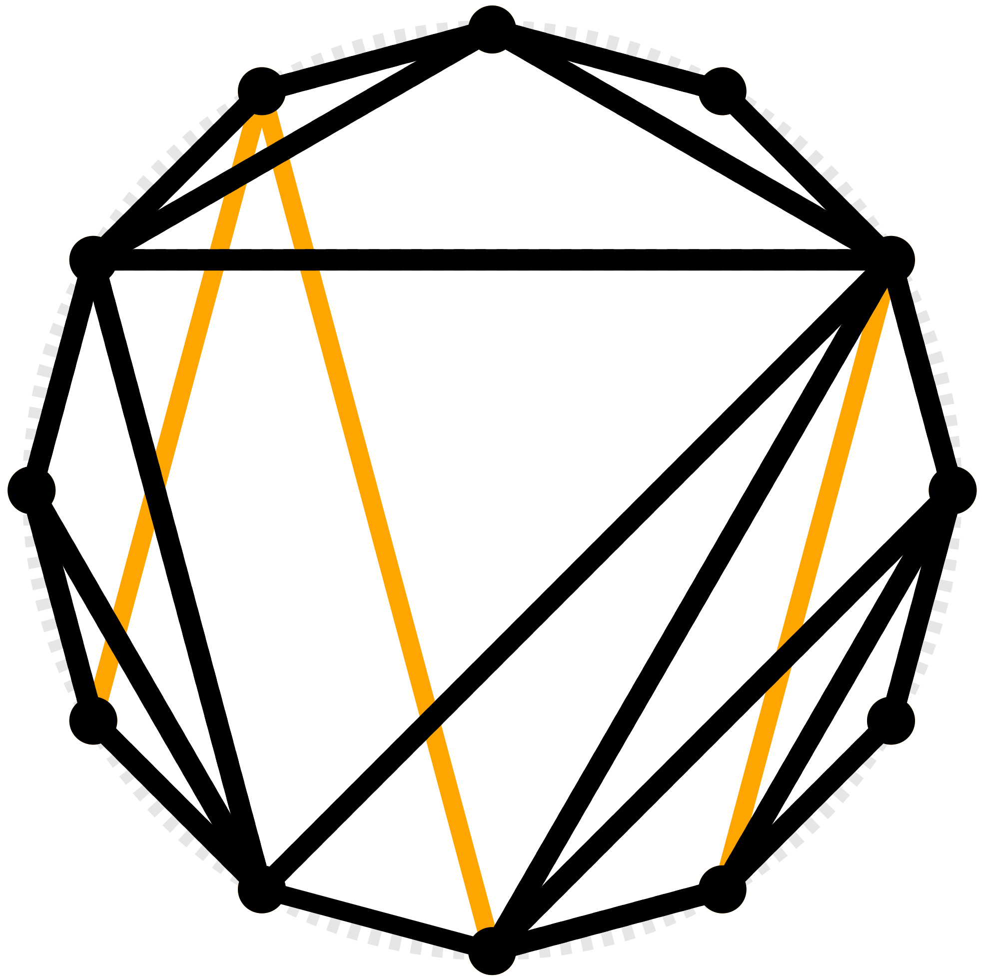

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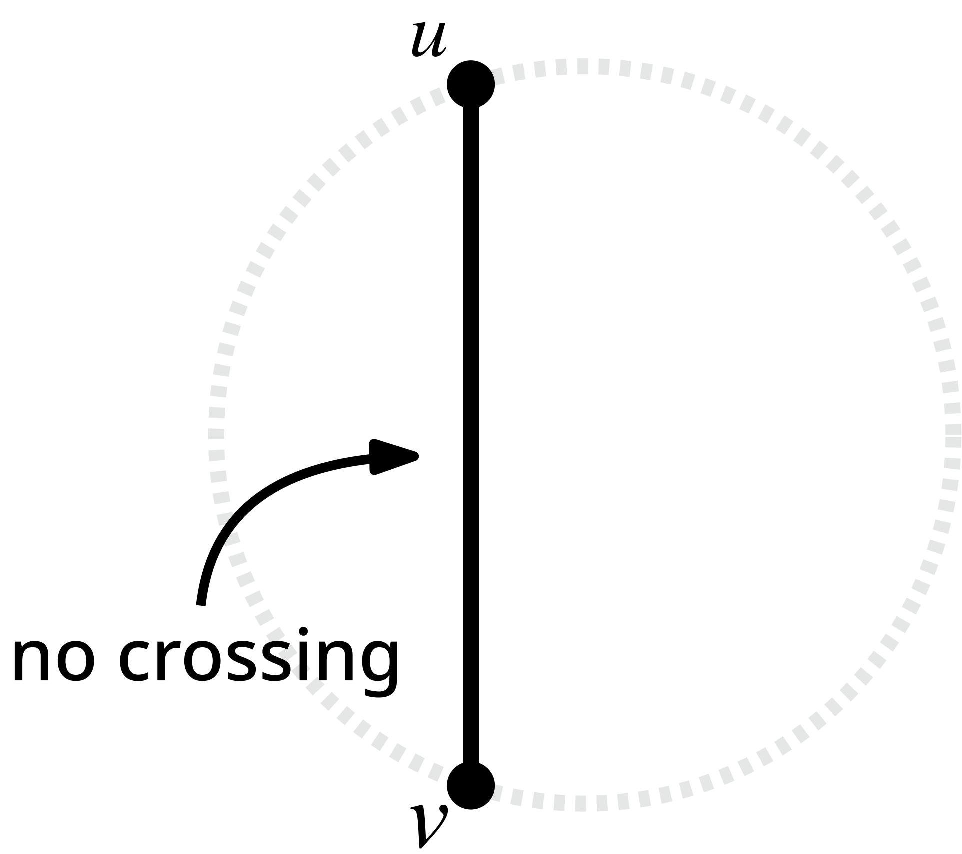
Correctness of the Greedy Algorithm

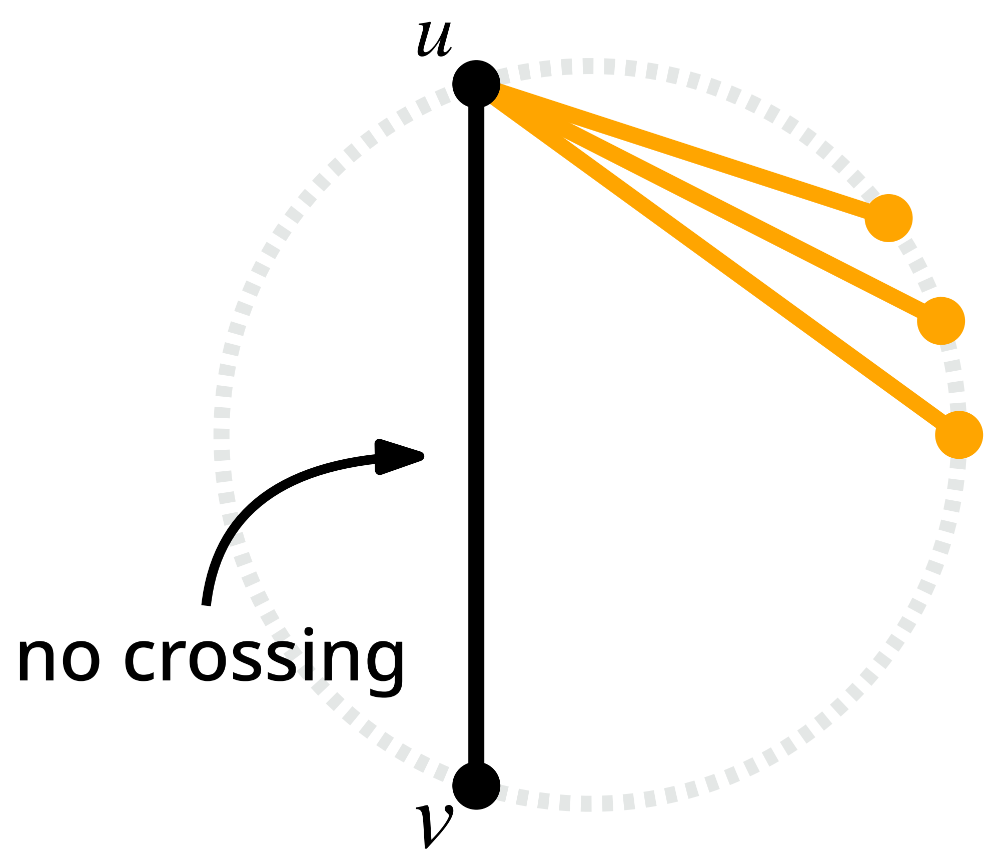
(in Lemma 6)

If $\{u, v\}$ crosses the original graph at most k times, there always exists a suitable vertex w.

Proved by case analysis. If $\{u, v\}$ has no crossing,



(i) u has no neighbor



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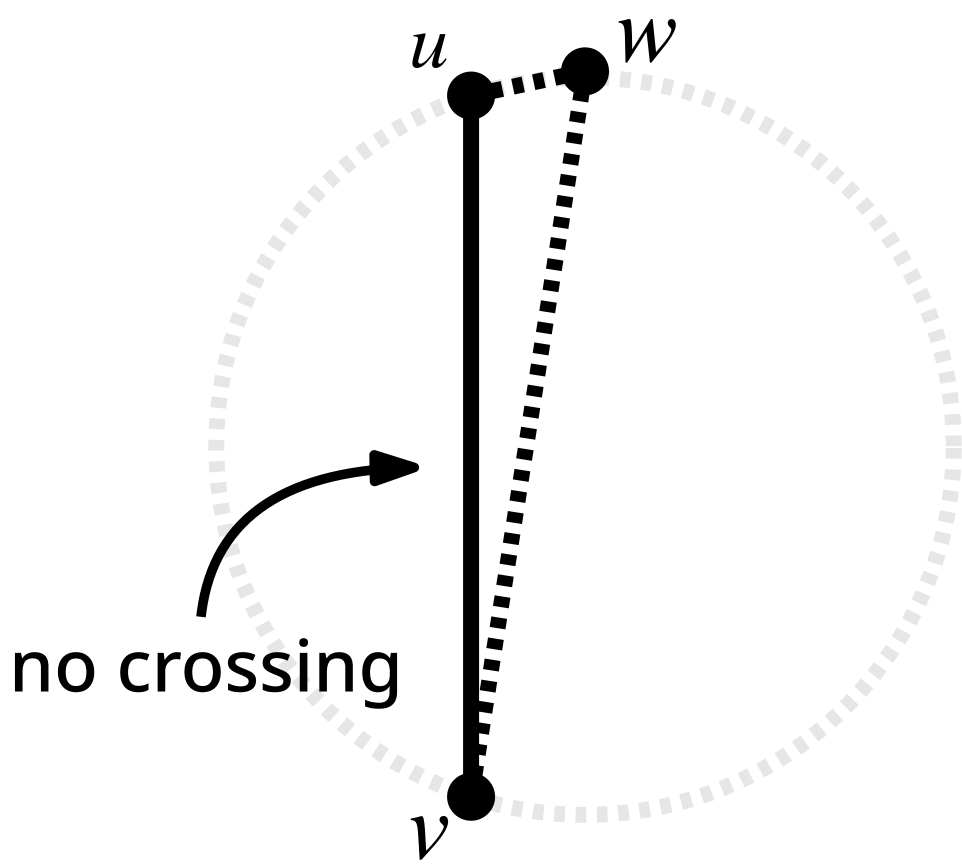

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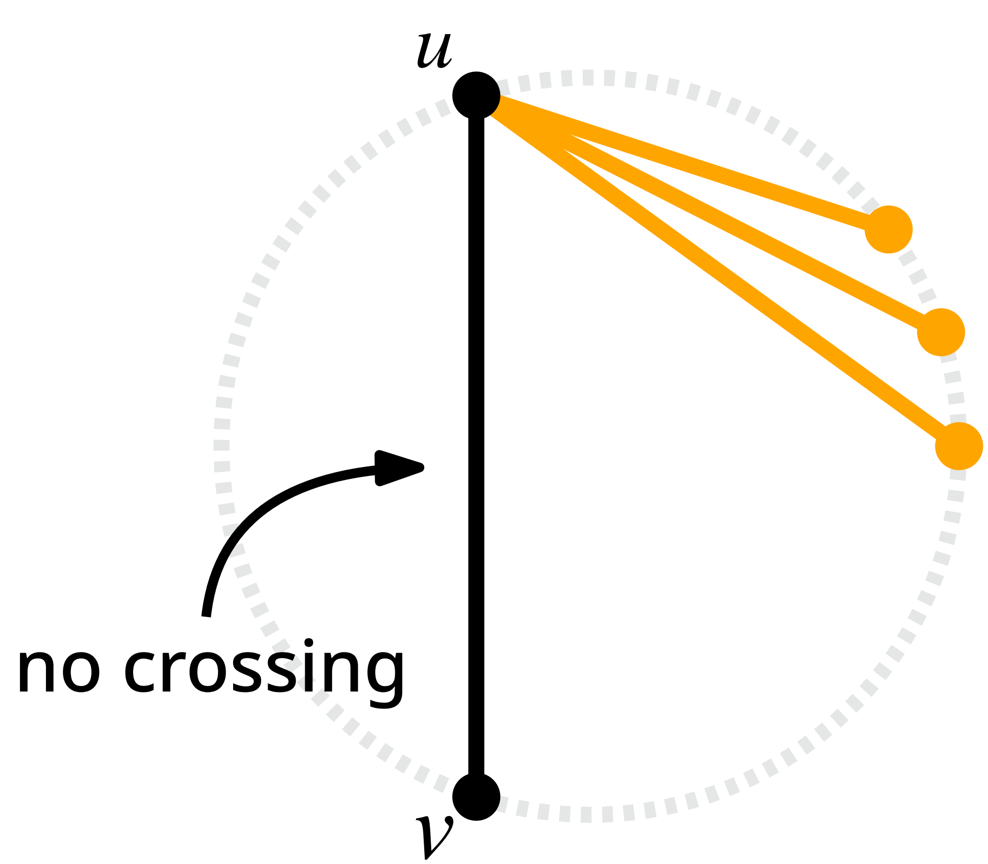

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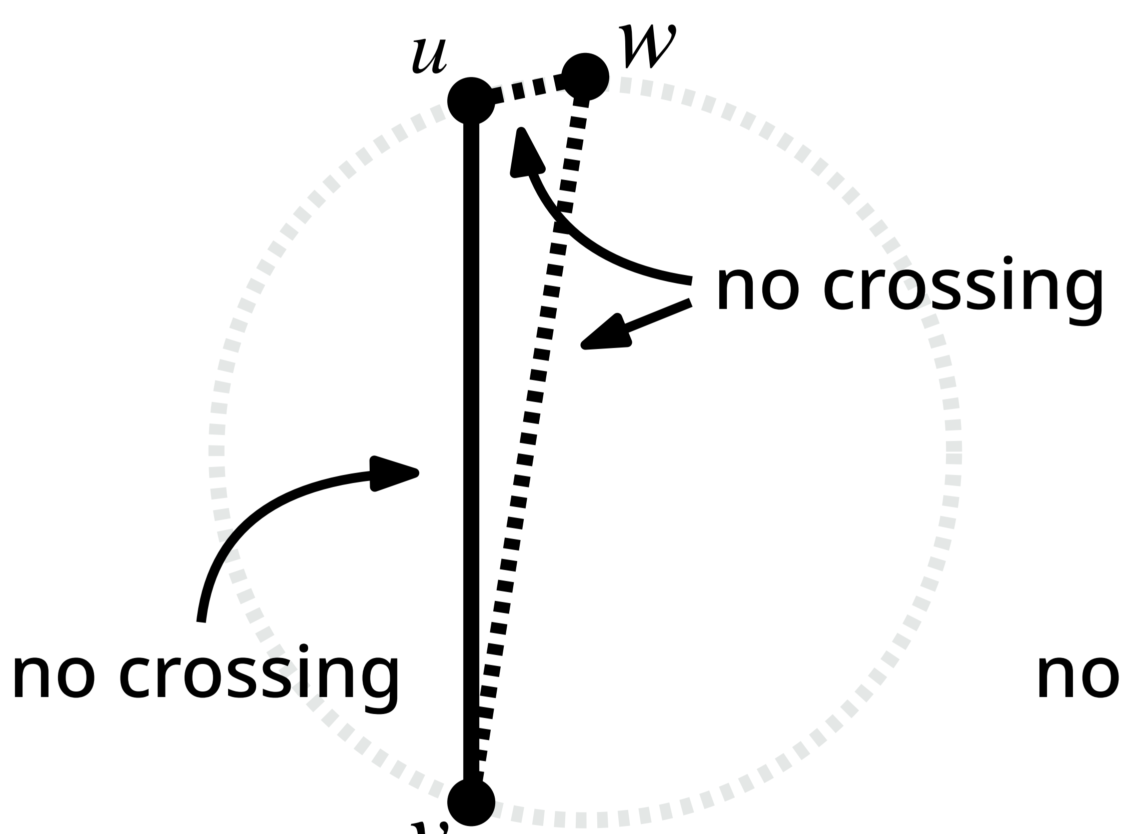

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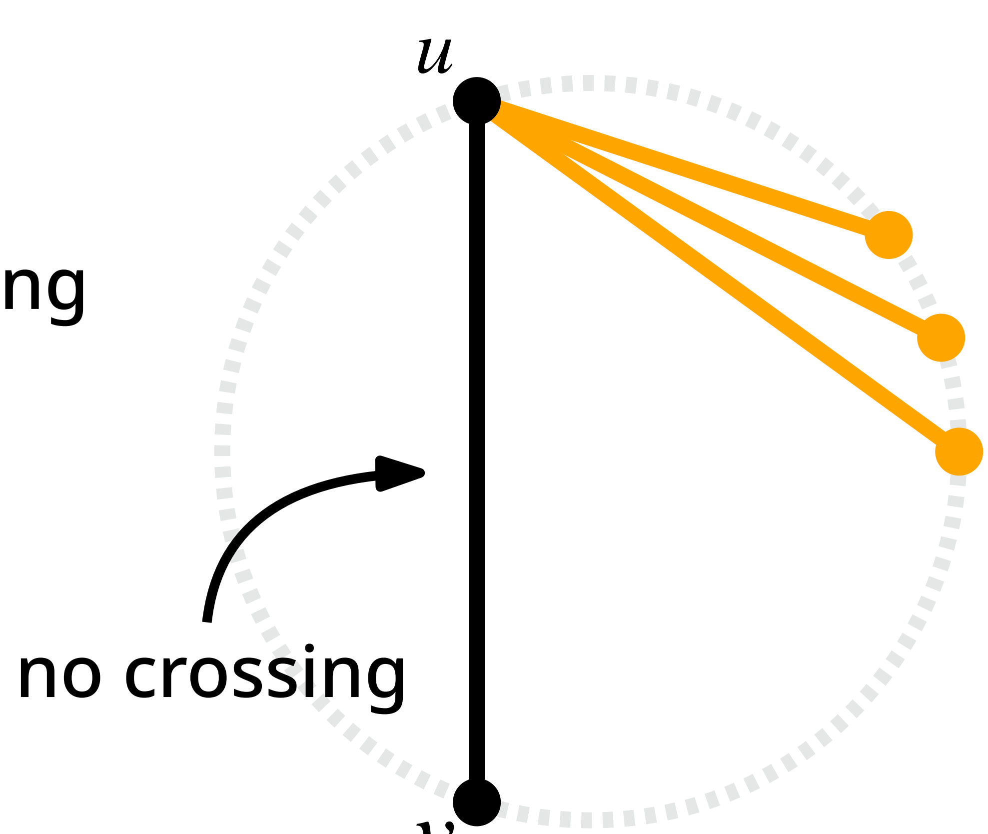

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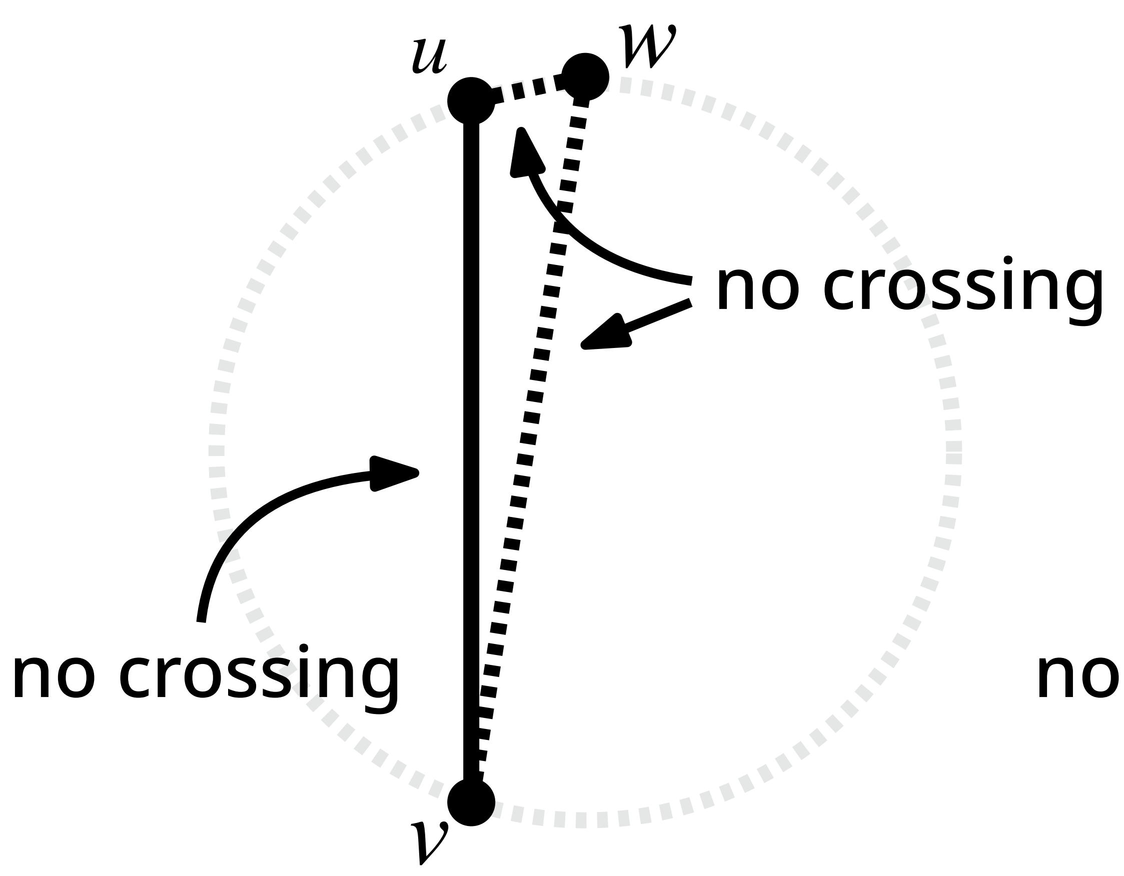

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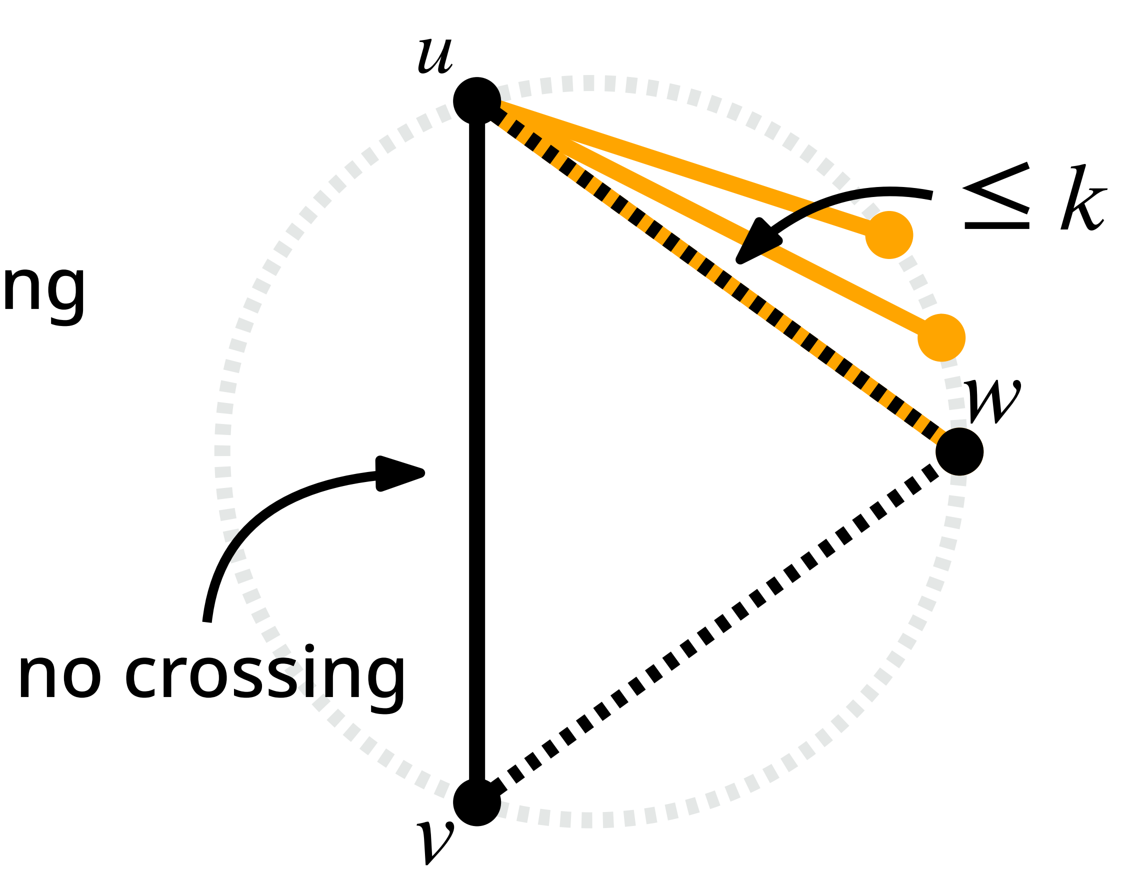

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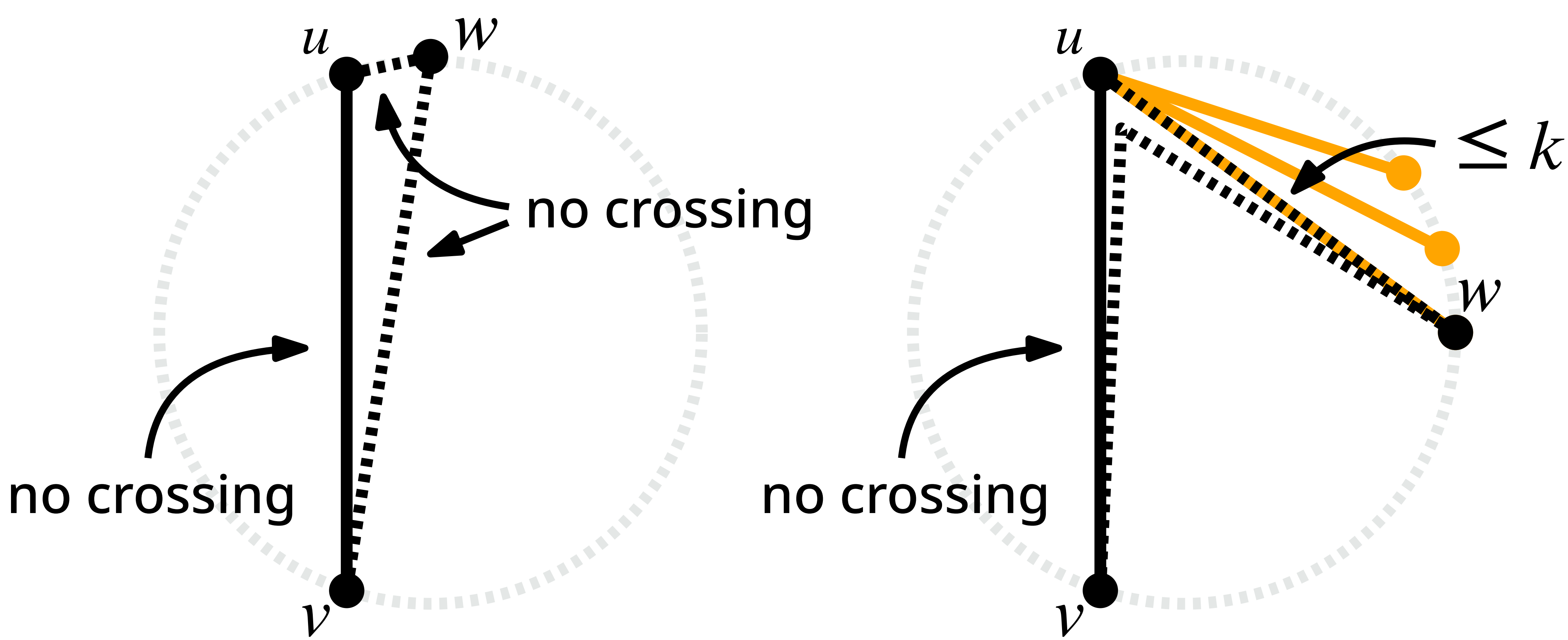

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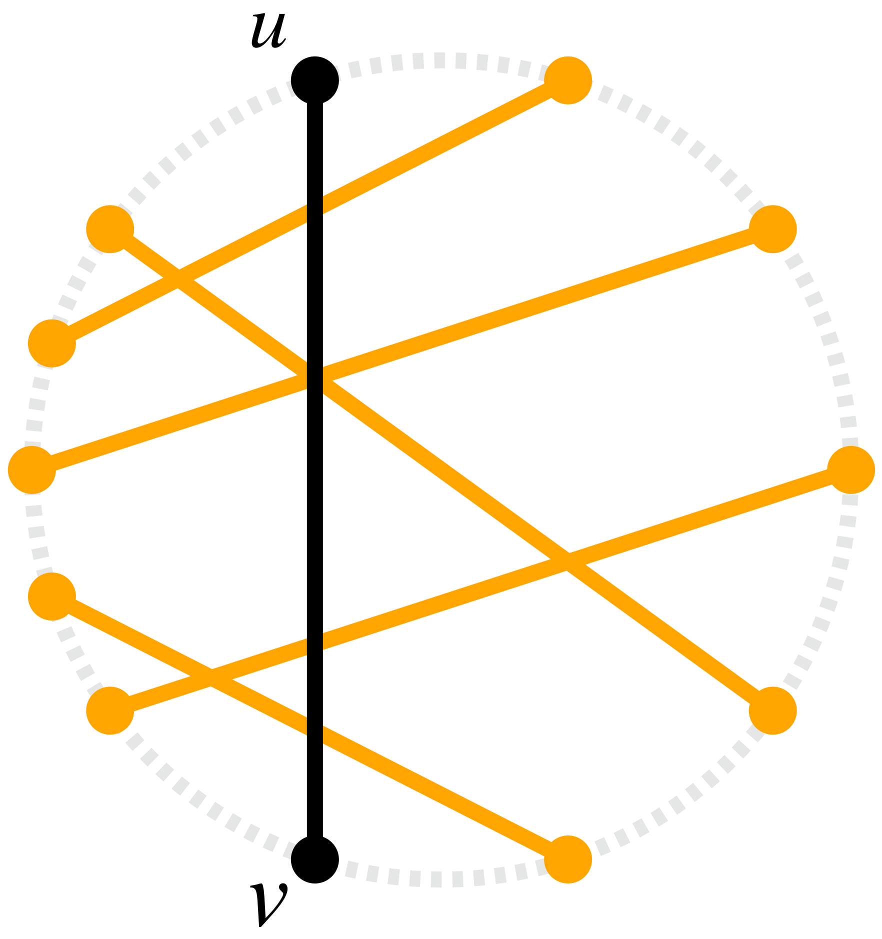

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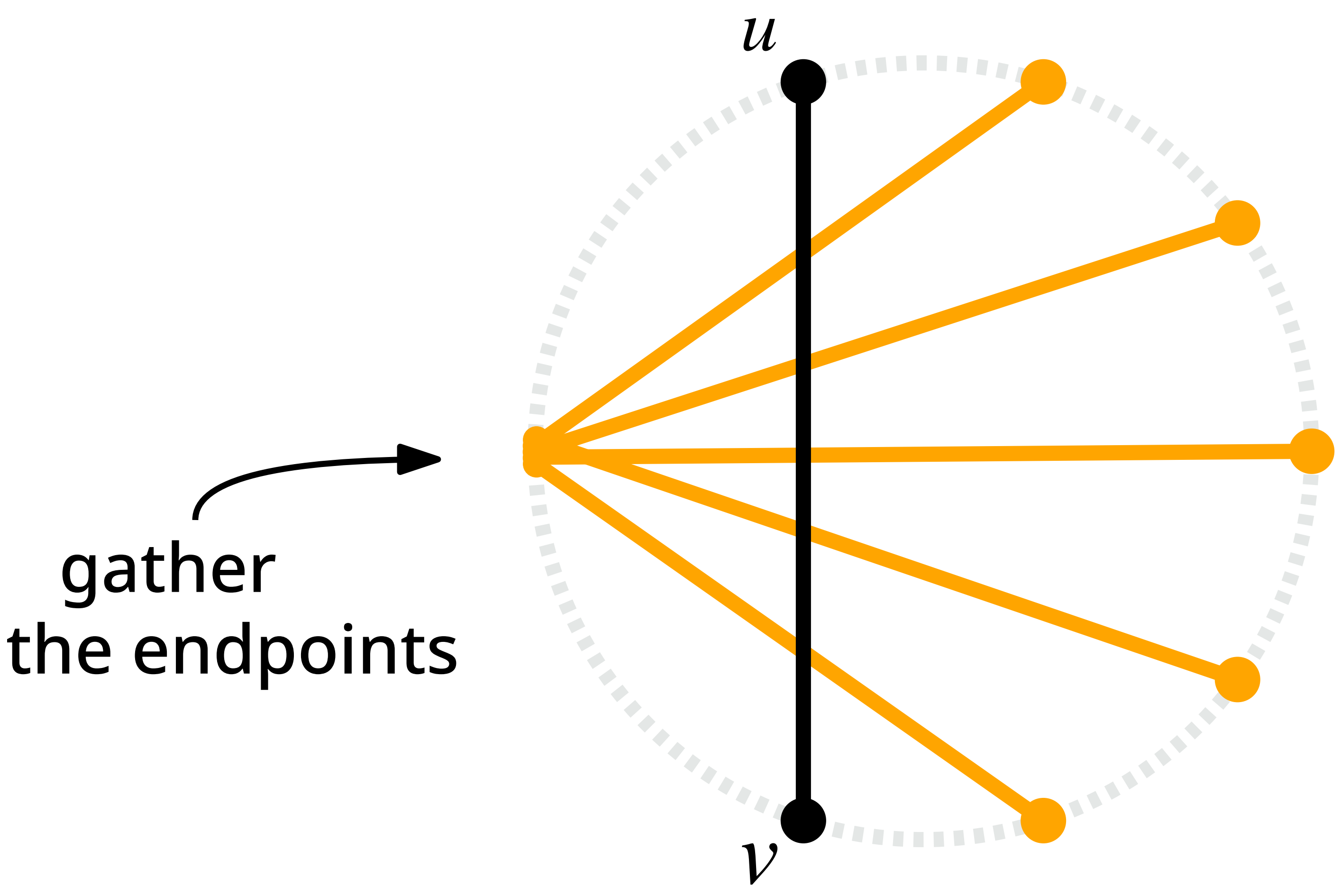

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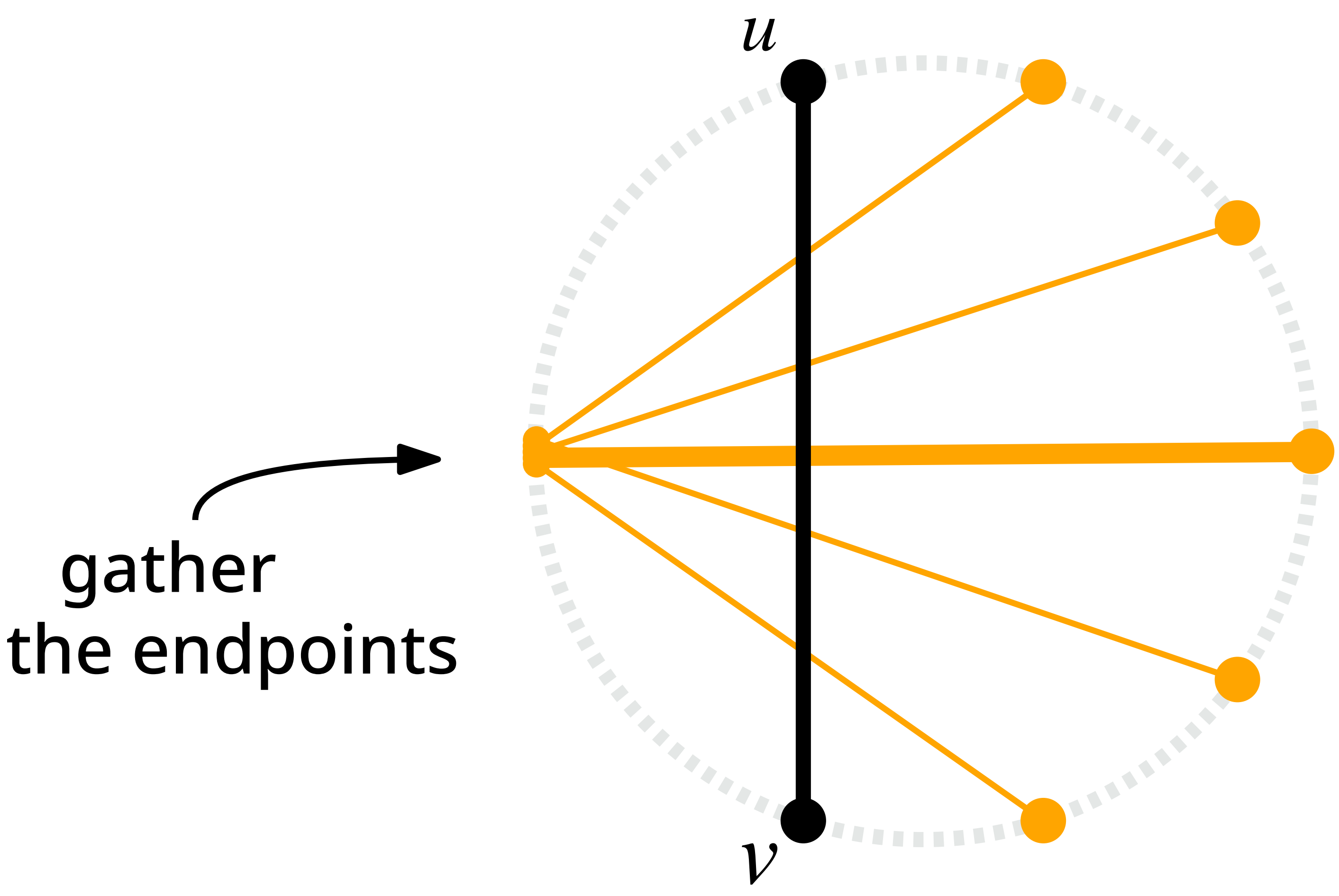

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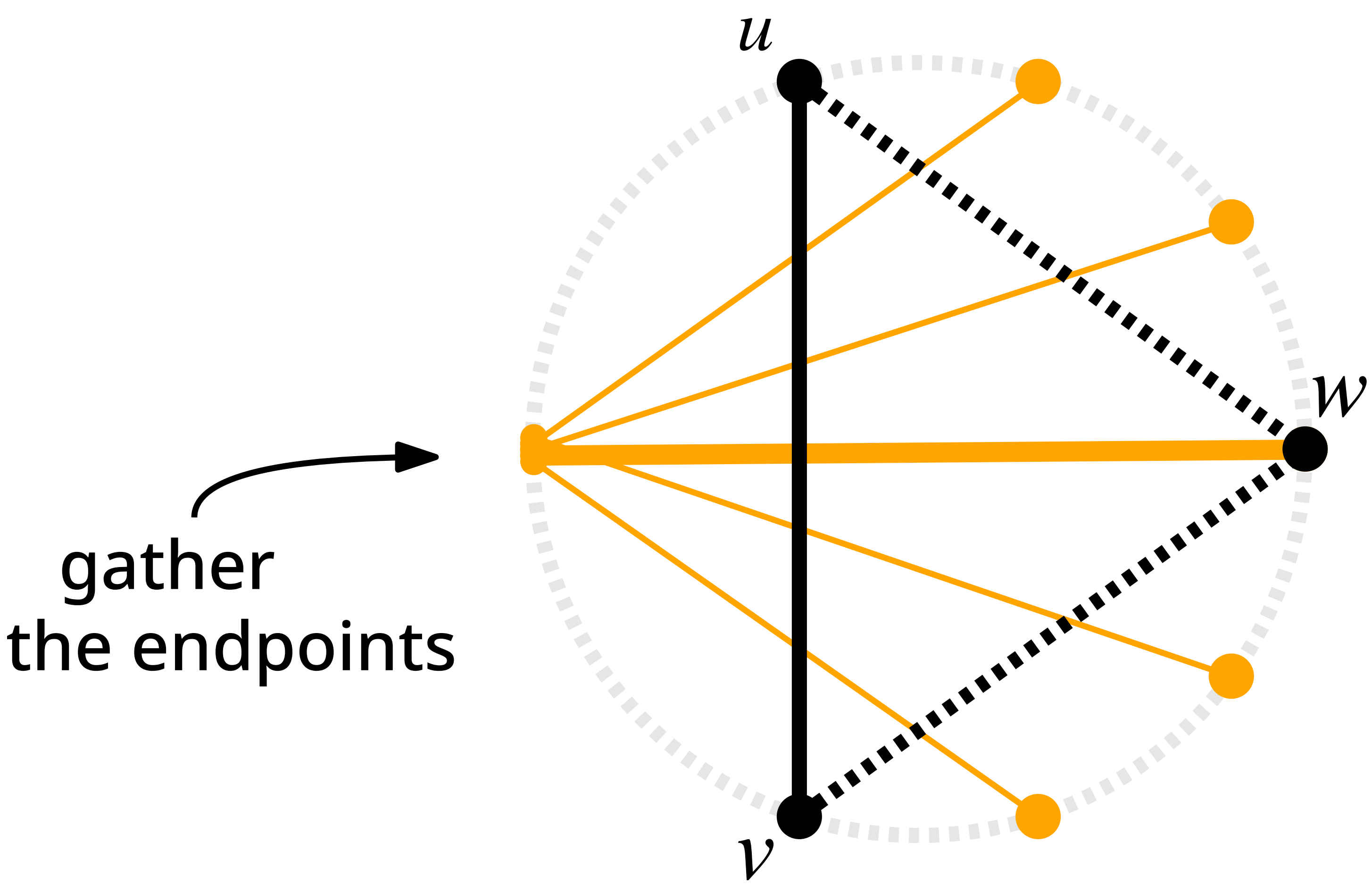

The middle edge has (iii) no crossing

Correctness of the Greedy Algorithm

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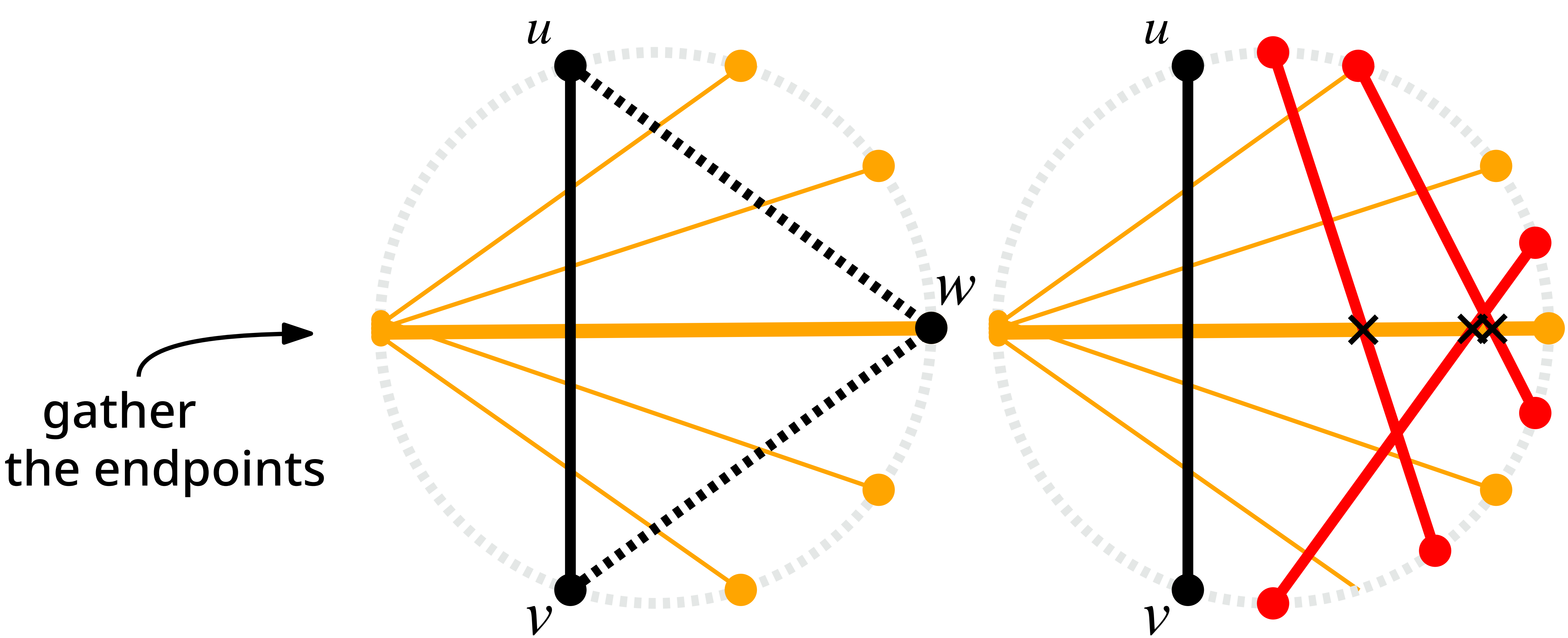

The middle edge has (iii) no crossing (iv) some crossings

Correctness of the Greedy Algorithm

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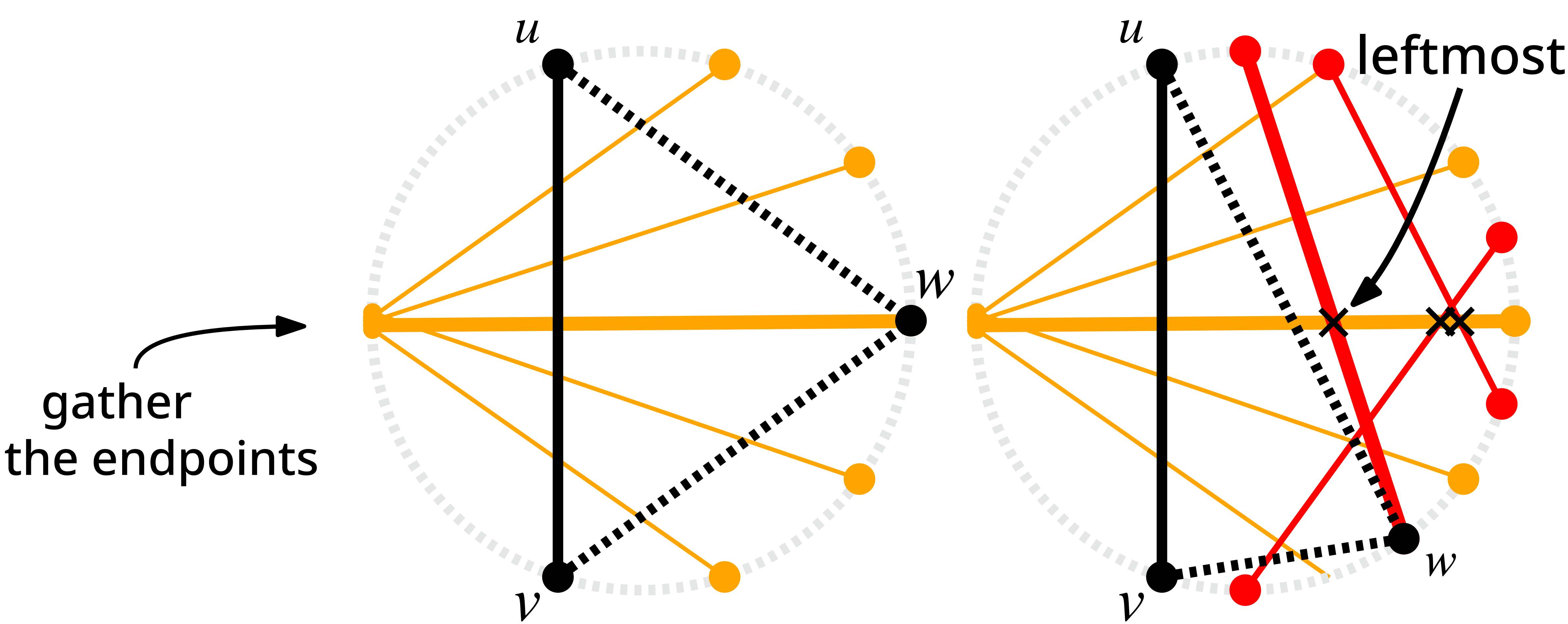
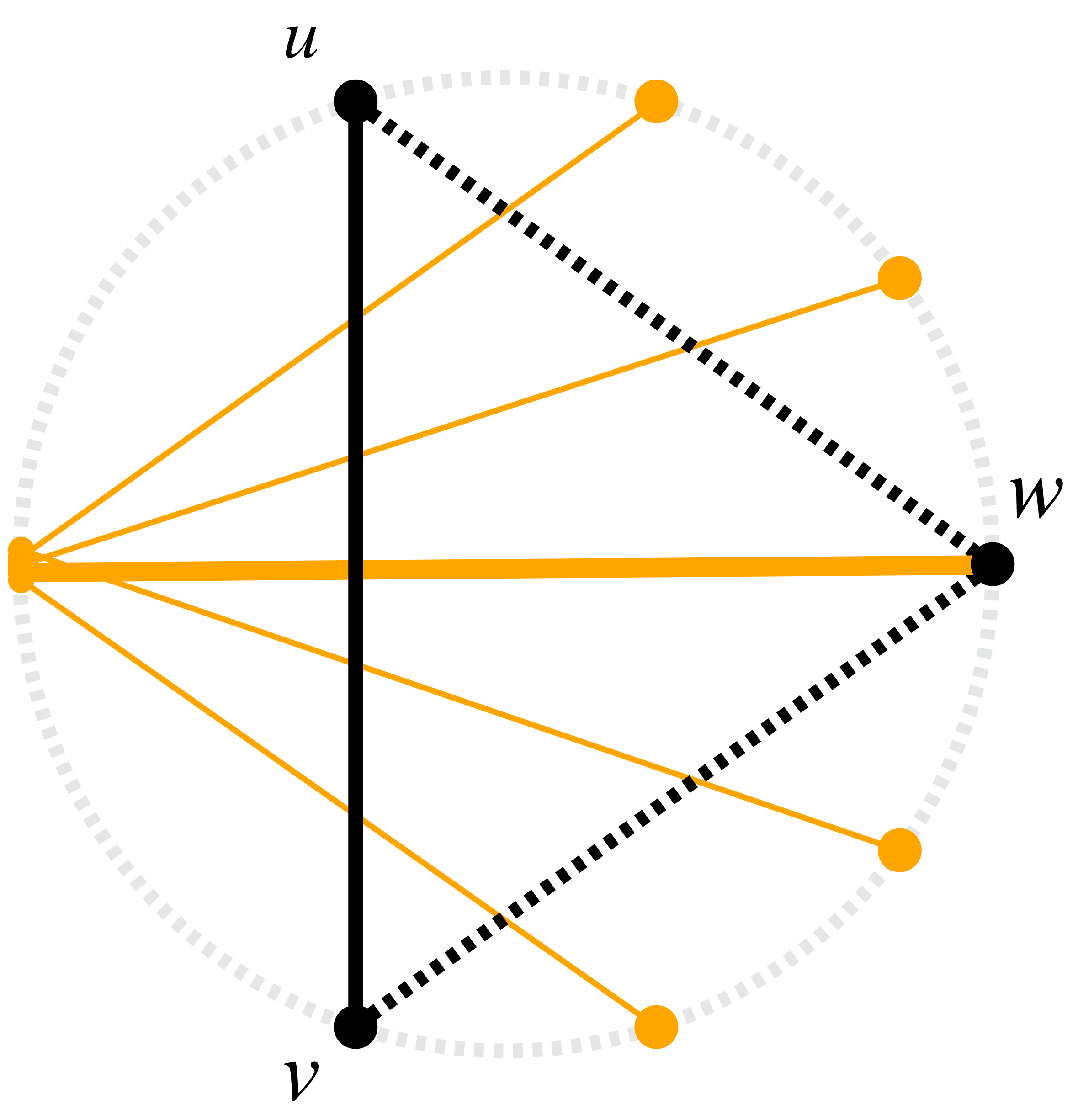


The middle edge has (iii) no crossing (iv) some crossings

Correctness of the Greedy Algorithm

(iii) The middle edge has no crossing.



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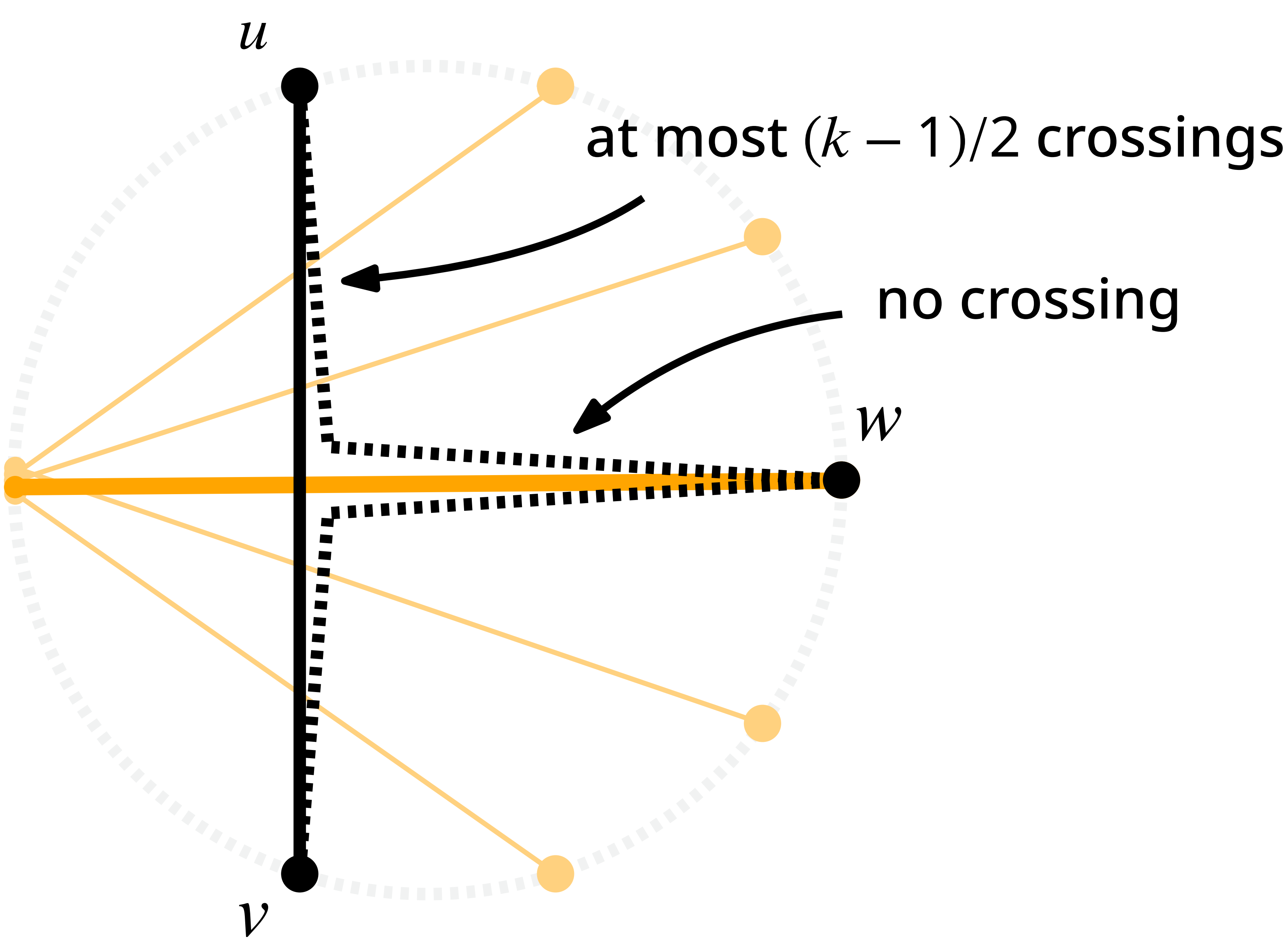
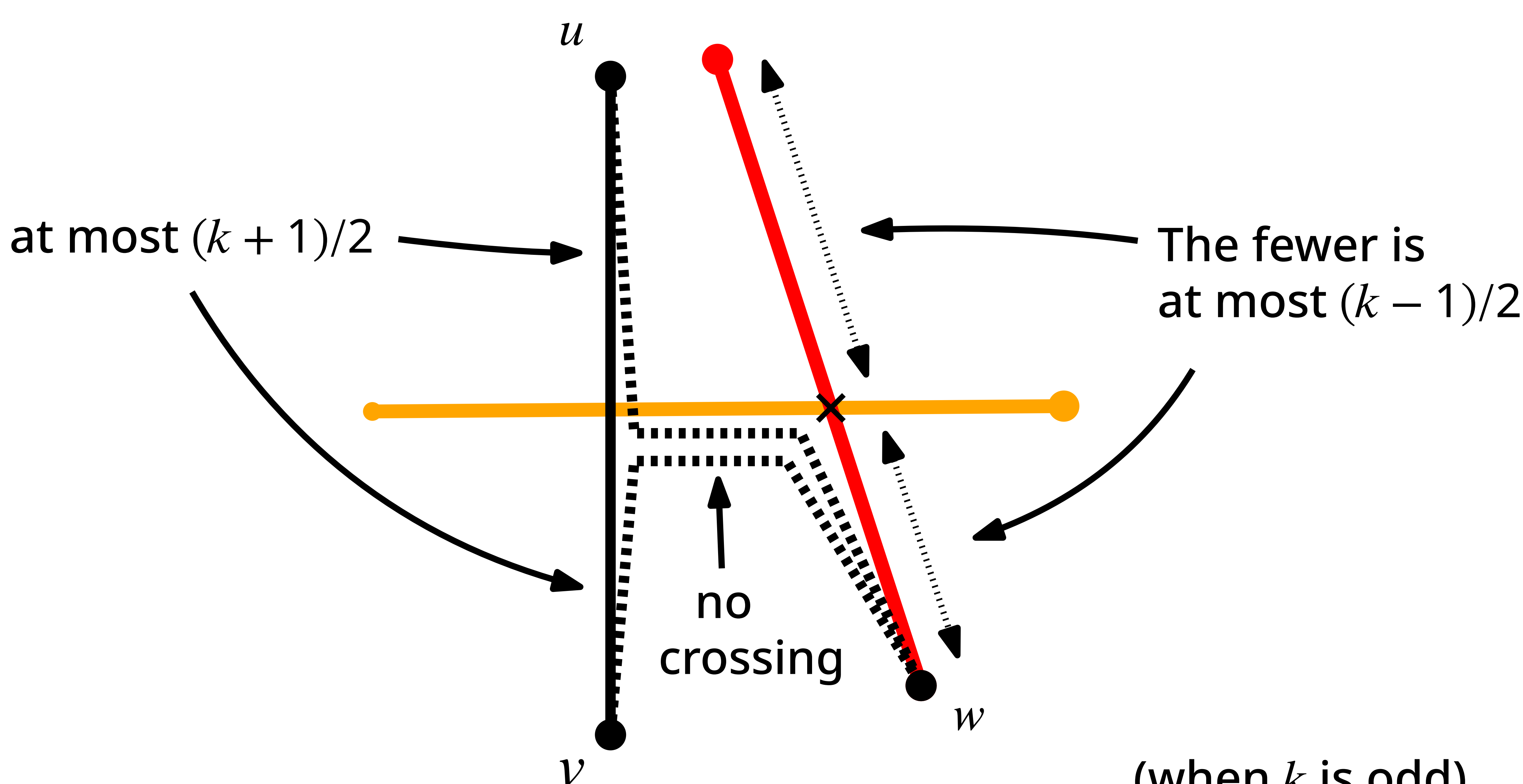


Correctness of the Greedy Algorithm

(iv) The middle edge has some crossings.

Choose the endpoint of fewer side as w.



(when k is odd)

Summary & Other Results

Main Results

- Improved upper bounds via a good triangulation.

	Upper	Lower
k	$k+2$	$k+2$

separation number
(Orange results are ours.)

	Upper	Lower
0	2	2 (K_3)
1	3	3 (K_4)
2	4	4 (K_5)
k	$1.5k+2$	$k+2$

treewidth

Other Results

- Lower bounds with stacked prisms for even k.
- Treewidth 4 for $k=2$ by a specialized triangulation.
- Similar results on outer min-k-planar graphs.

Future Work & Open Problems

Other application of the triangulation?

Fill the gap between bounds of the treewidth.

Poly-time recognition of outer k-planar graphs.

Linear-time algorithm known for $k = 1$ [Auer et al., '16].

For fixed k, $O(2^{\text{poly}(\log(n))})$-time [Chaplick et al., '17].

Triangulation of k-planar graphs?

Our triangulation can be stated as this:

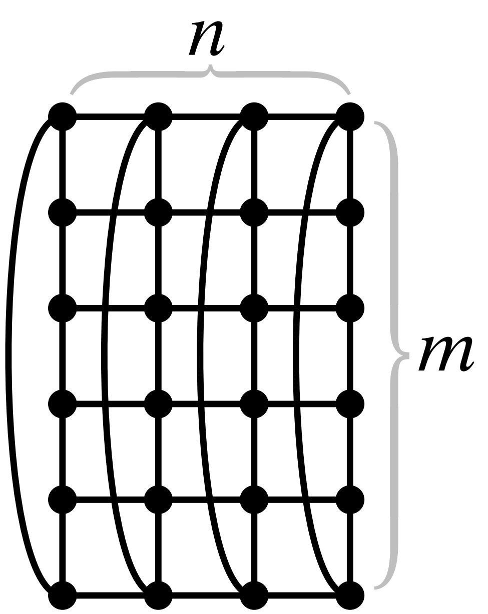
> On every outer k-planar drawing, we can draw
> a maximal outerplane graph with the same vertices
> s.t. each edge crosses the drawing at most k times.

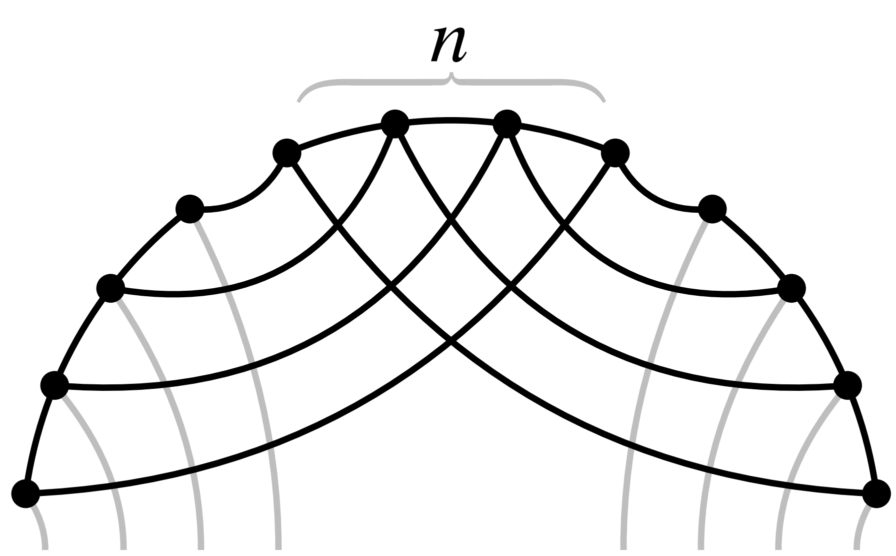
Does this also hold for k-planar graphs?

Appendix: Lower Bounds

Stacked Prisms

are obtained by connecting the top & bottom of grids.





The $m \times n$ grid satisfies: (for large enough m)

- Outer $(2n-2)$-planar graph
- Treewidth $2n$
- Separation number $2n$

$\rightarrow$ lower bounds $k+2$ on both for every even $k > 0$.

Appendix: Treewidth 4 for $k = 2$

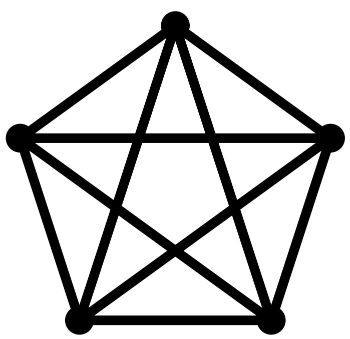
Lemma 8

Every outer 2-planar graph admits a triangulation s.t. each triangle has at most 4 crossing edges.

$+$

Lemma 11

If the outer cycle of G admits a triangulation s.t. each triangle has at most c crossings, then tw$(G) \leq (c + 5)/2$.



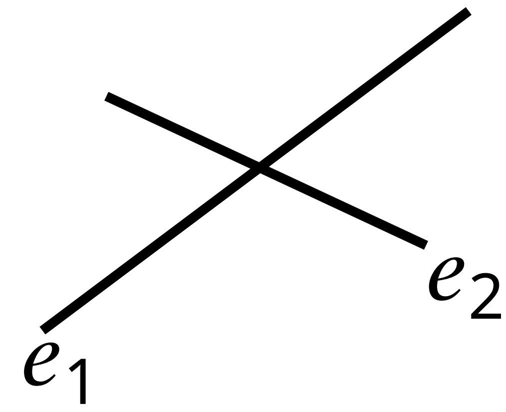
Theorem 12

Every outer 2-planar graphs have treewidth at most 4, which is tight because of K_5.

Appendix: Results on Outer Min-k-Planar

Min-k-Planar

If edges e_1 and e_2 cross, then either e_1 or e_2 have at most k crossings.



Lemma 9

Every outer min-k-planar graph admits a triangulation s.t. each edge has at most $2k - 1$ crossing edges.

↓

Theorem 14 (Treewidth)

Every outer min-k-planar graph has tw at most $3k + 1$.

Theorem 16 (Separation Number)

Every outer min-k-planar graph has sn at most $2k + 1$.