Morphing Planar Graph Drawings Through 3D

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Morphing. . .

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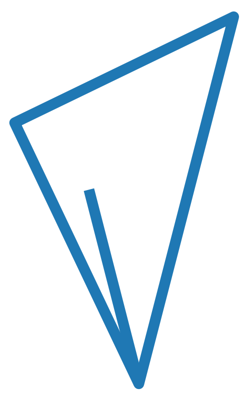
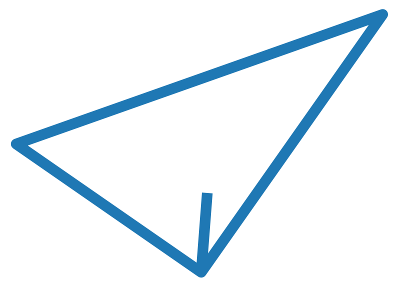


$t = 0$



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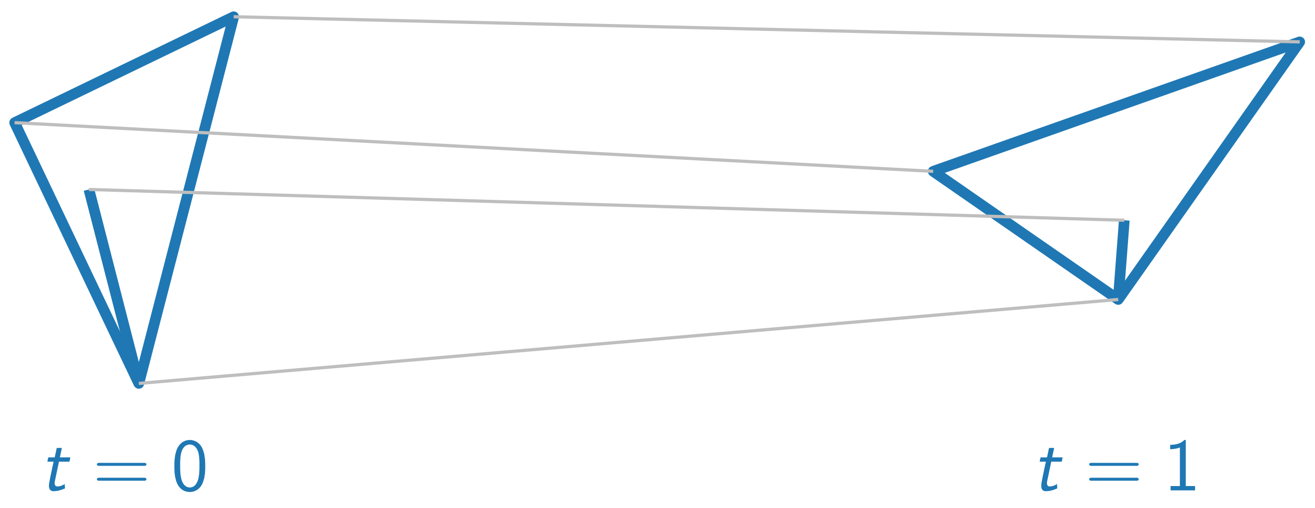

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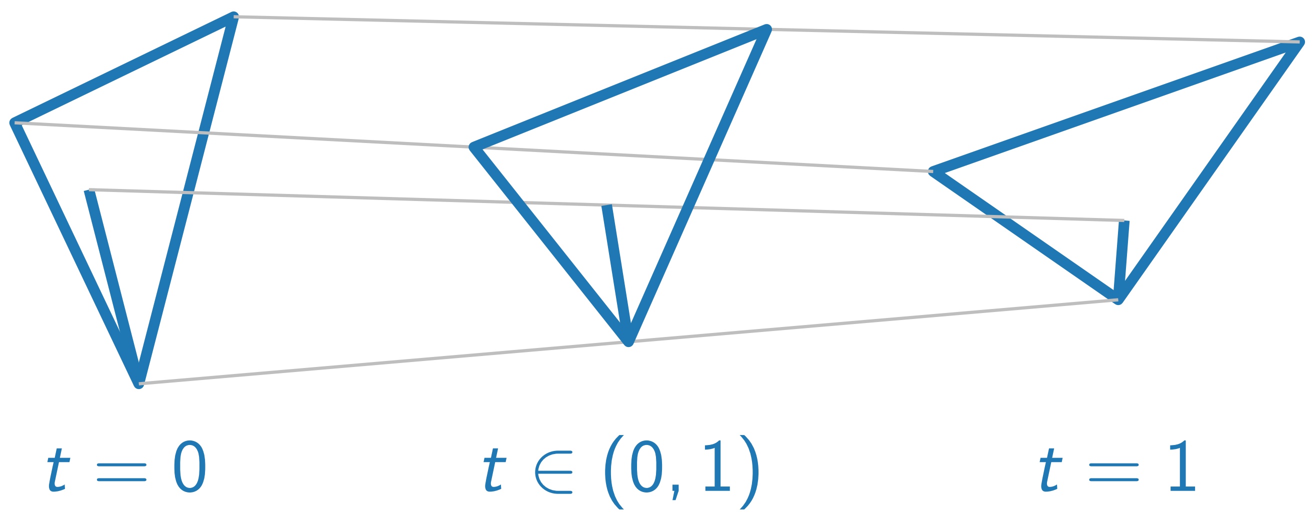

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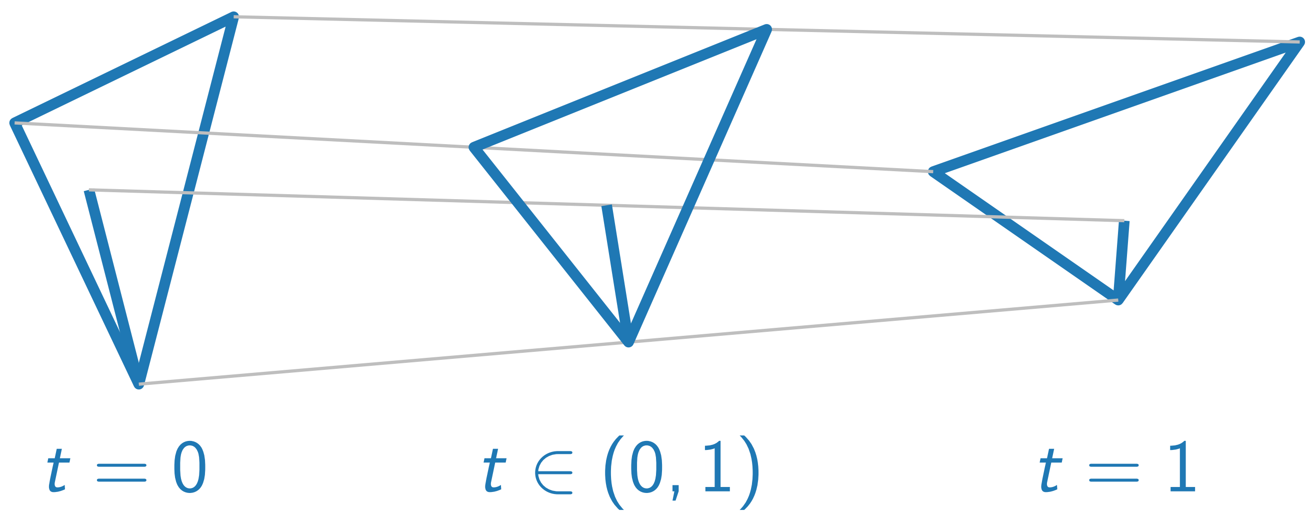

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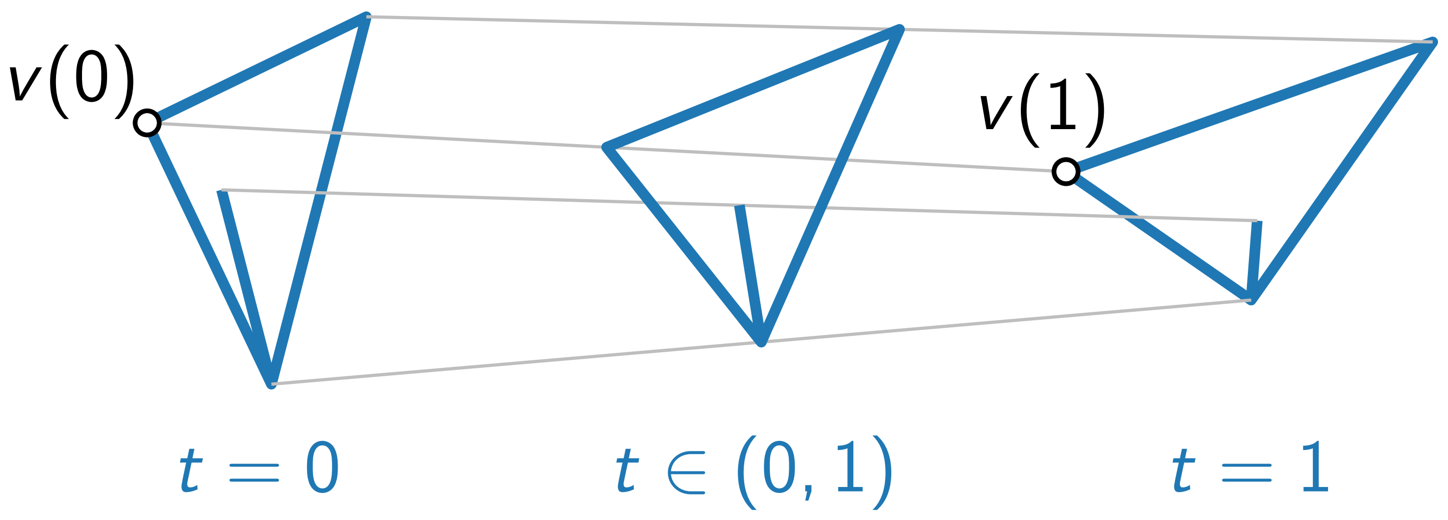

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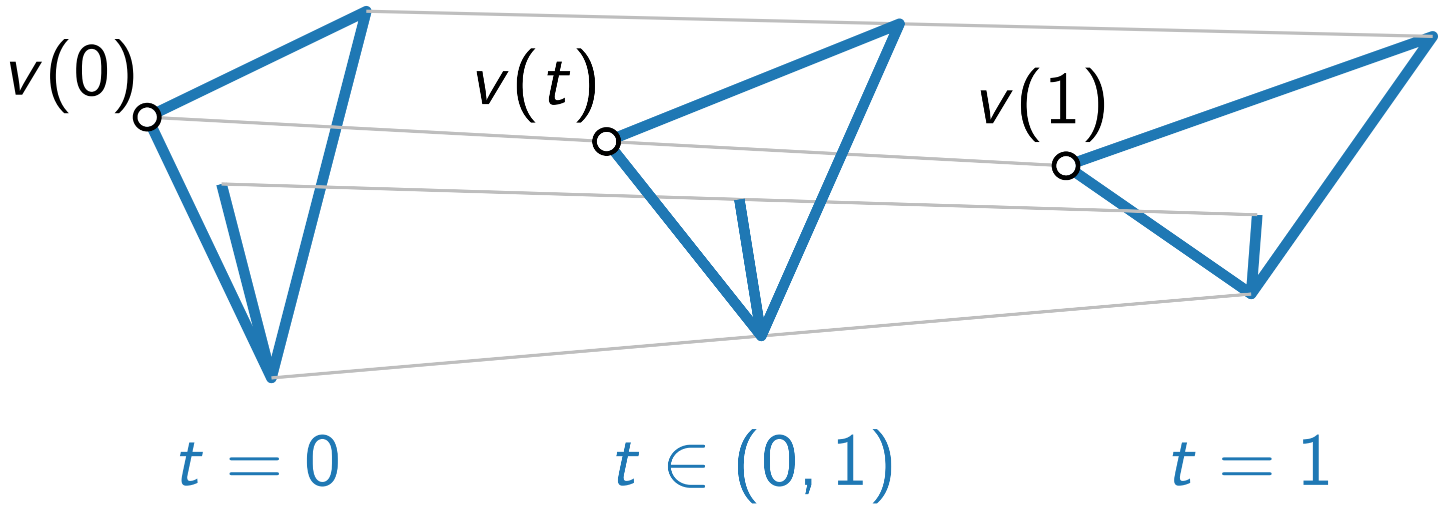

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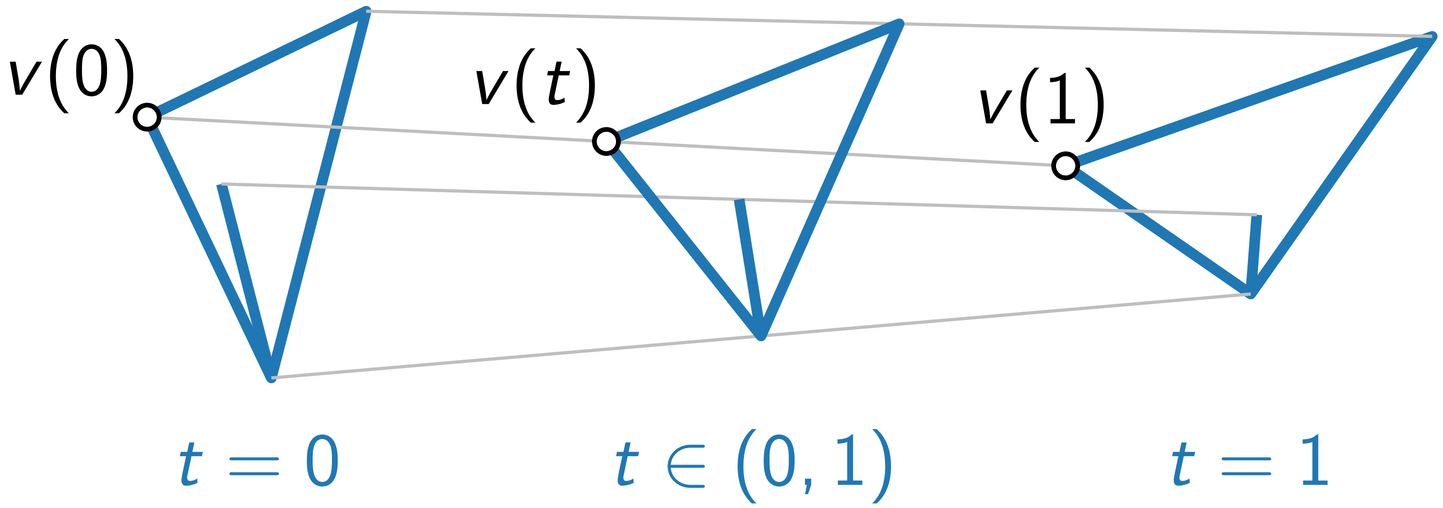

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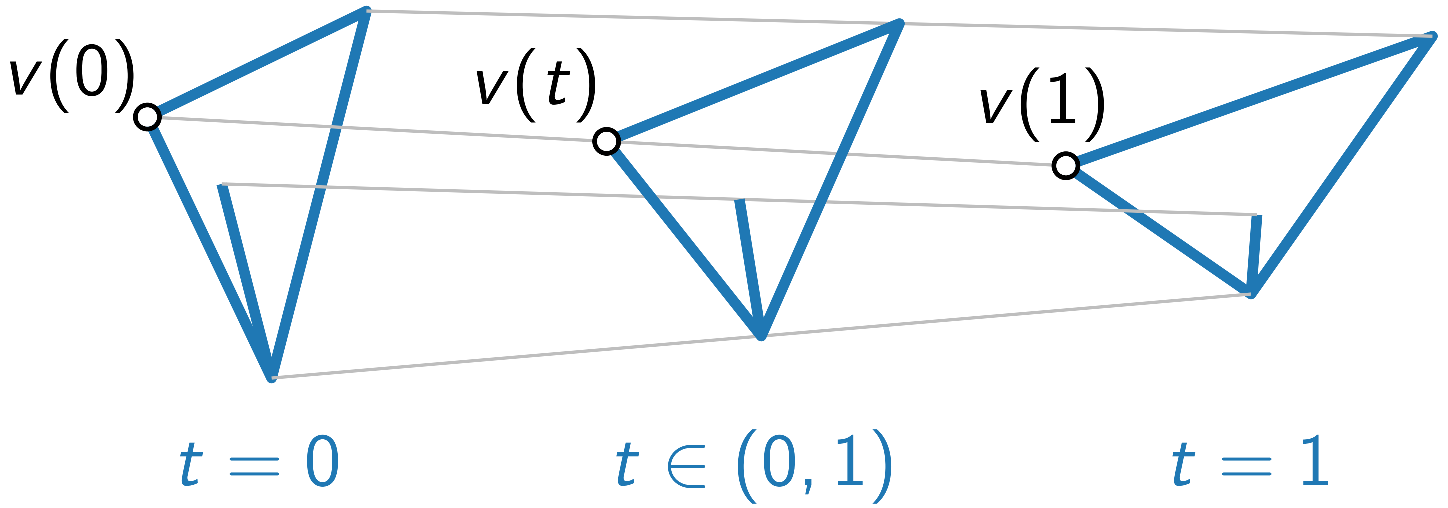

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- straight-line trajectories
- uniform (but possibly different) speed along each trajectory

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Theorem. For any two planar straight-line drawings of the same *triangulation*, there exists a piecewise-linear morph with a finite number of steps. [Cairns '44]

Theorem. For two planar straight-line drawings of the same n-vertex *plane graph*, there exists a k-step morph between the drawings such that $k \in O(n)$. The bound is optimal in the worst case.
[Alamdari, Angelini, Barrera-Cruz, Chan, Da Lozzo, Di Battista, Frati, Haxell, Lubiw, Patrignani, Roselli, Singla, Wilkinson: SIAM J. Comp. '17]

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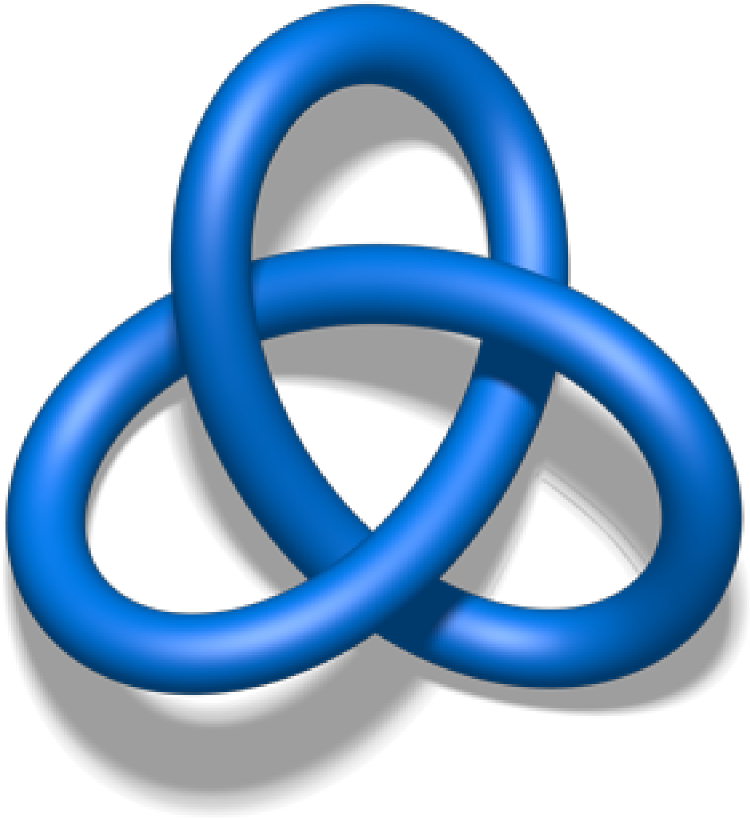

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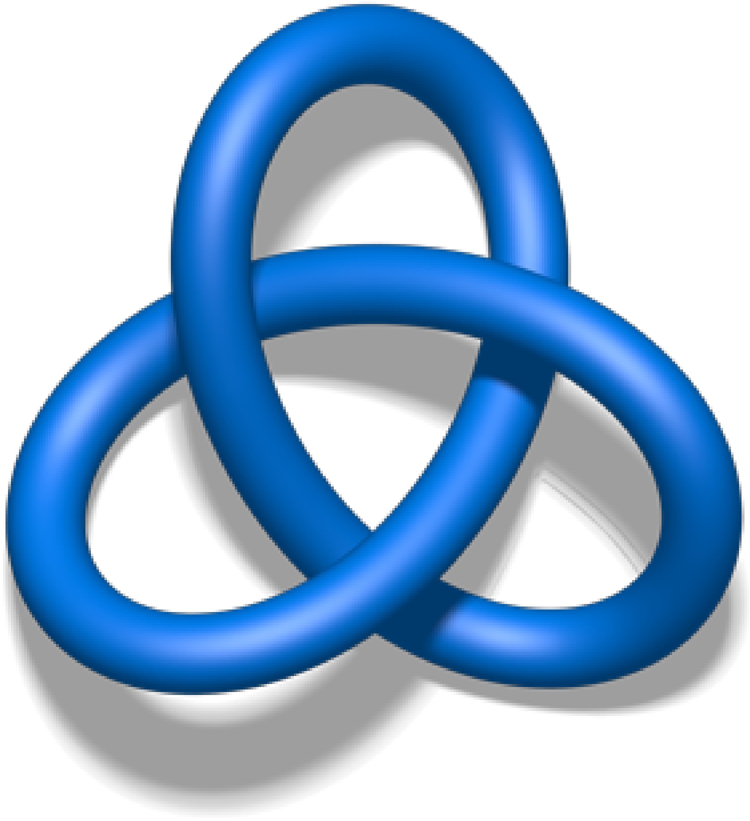

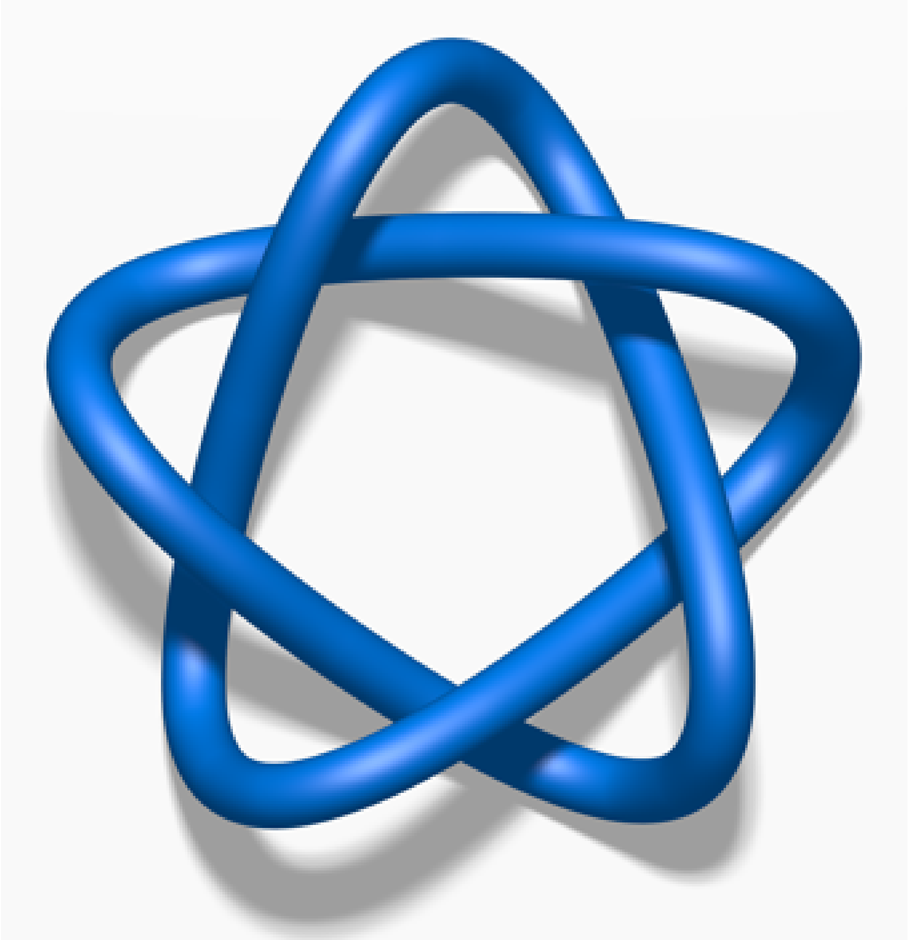

3D – cont'd

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Knot theory!

- Problem known to be in NP and in co-NP (hence most likely not NP-hard).
  [Hass, Lagarias, Pippenger: J. ACM 1999; Kuperberg: Adv. Math. 2014]
- Until solved, not much hope to design algorithms that can morph between crossing-free straight-line drawings of the same graph in *3D*.

2D

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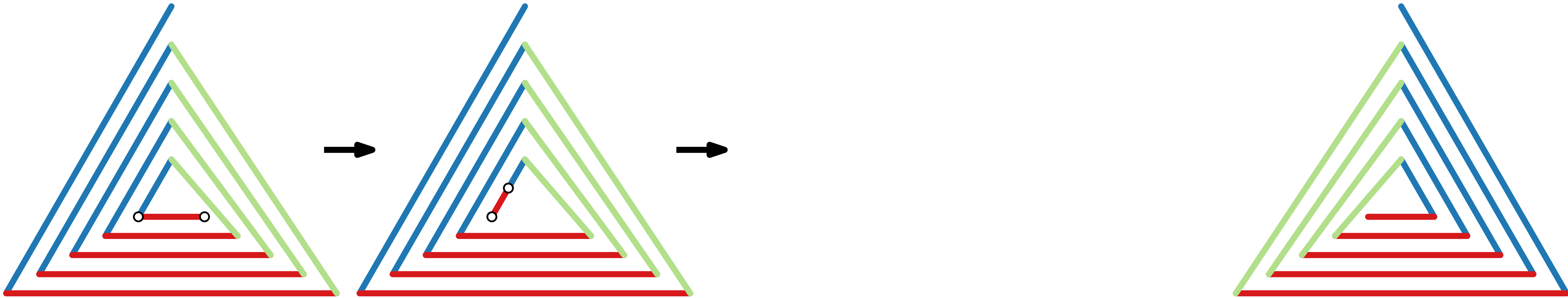

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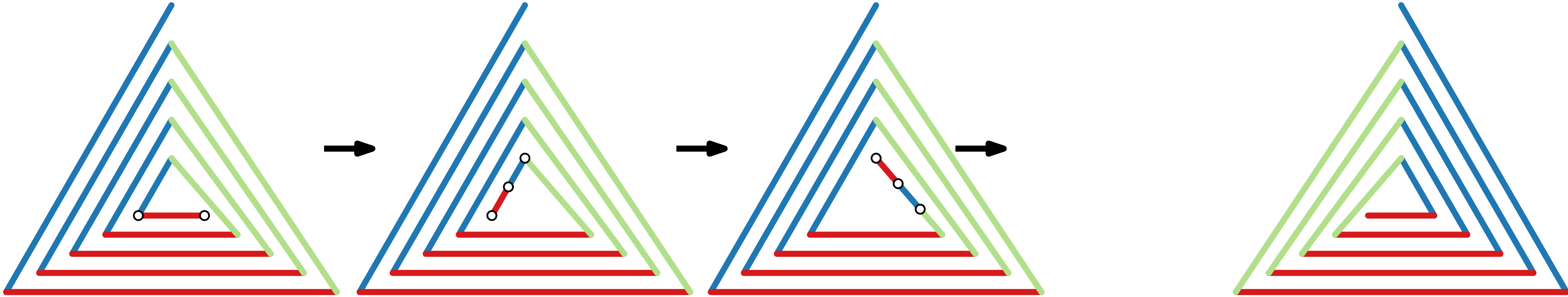

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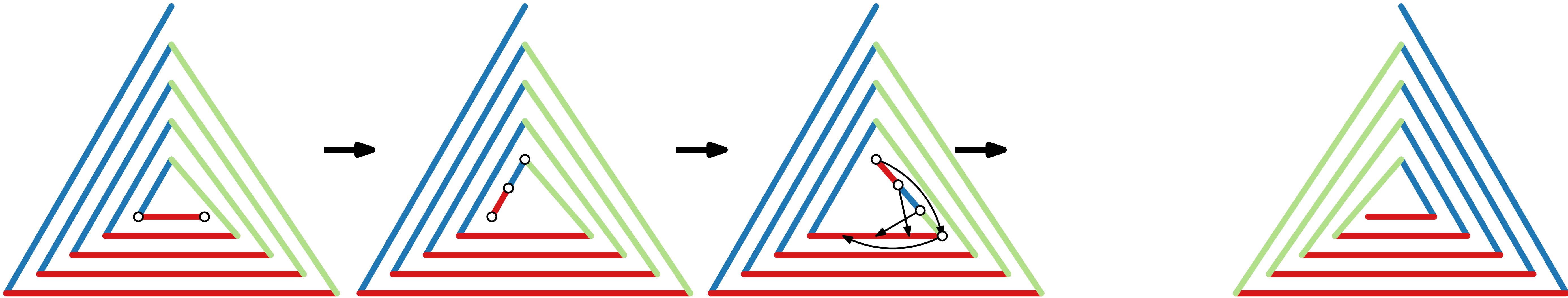

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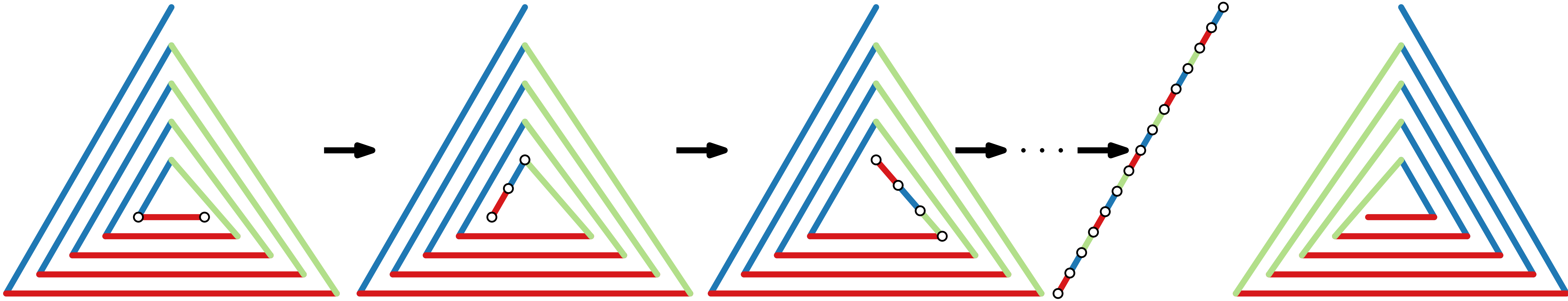

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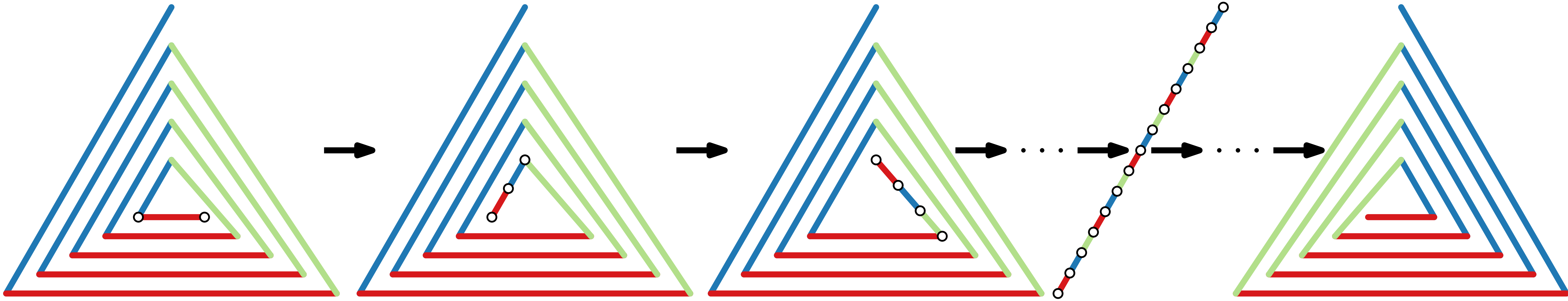

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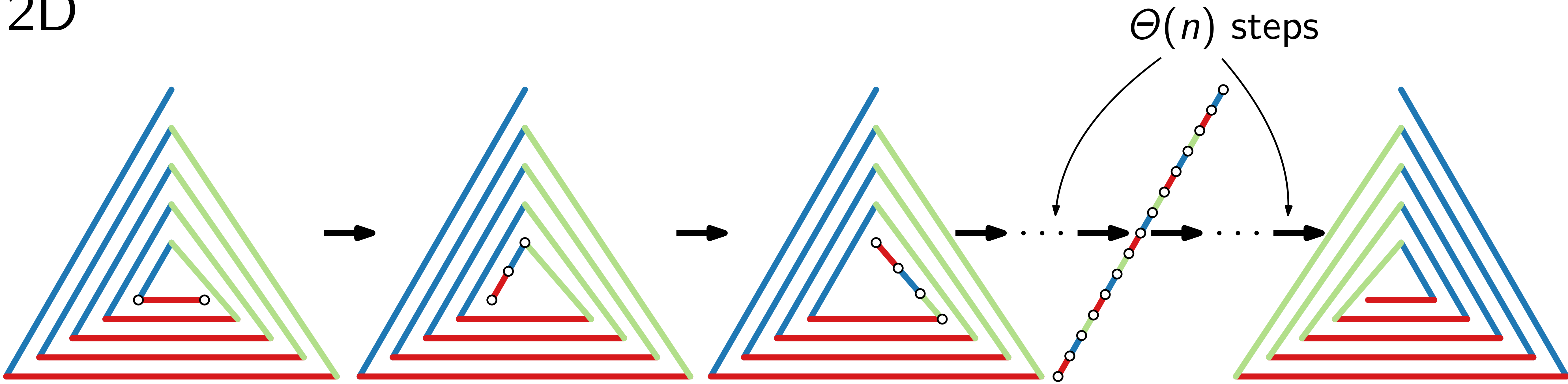

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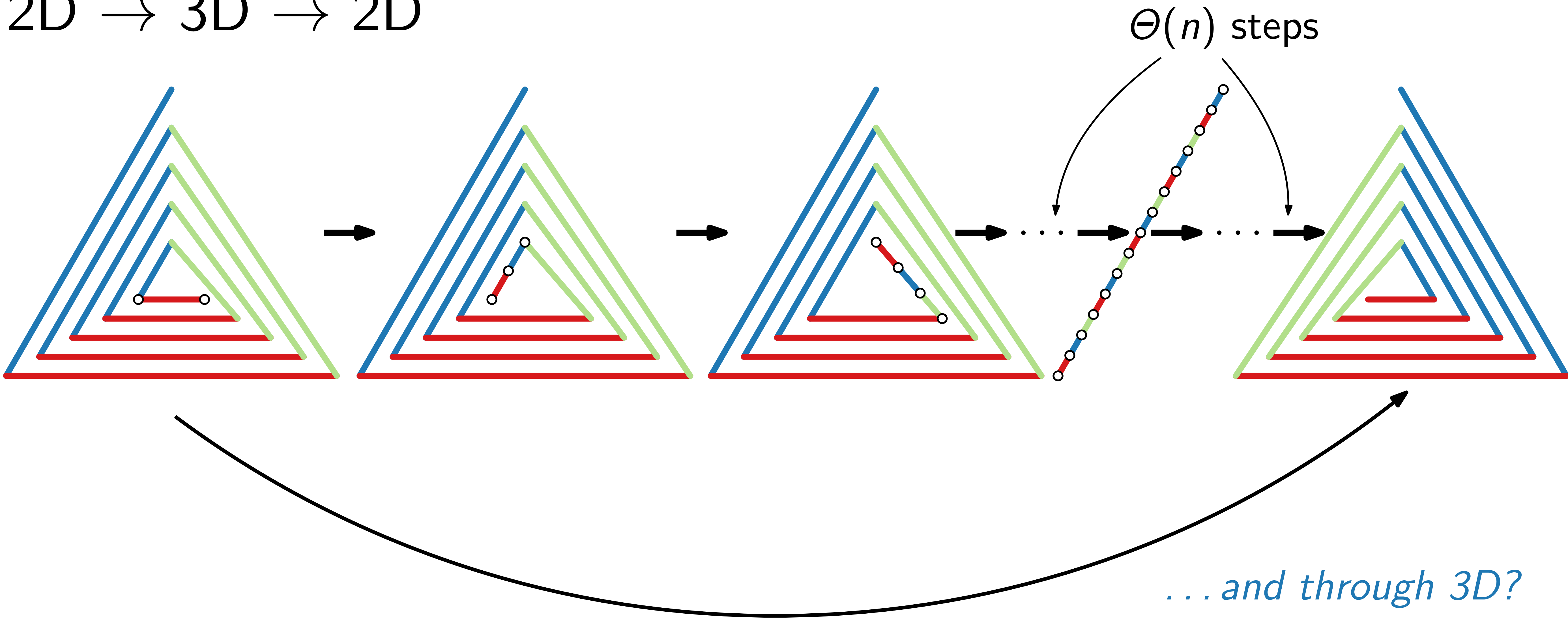

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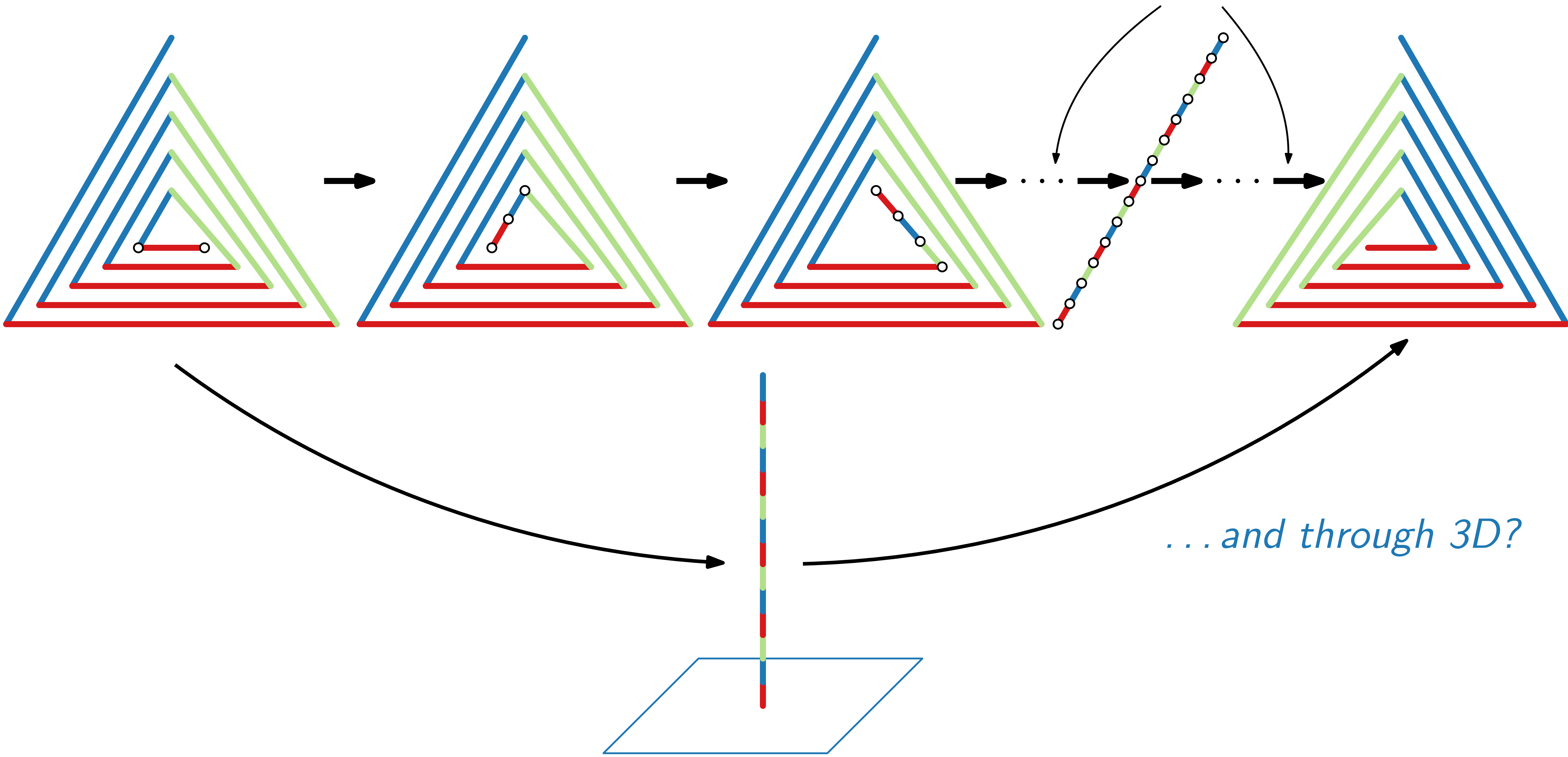

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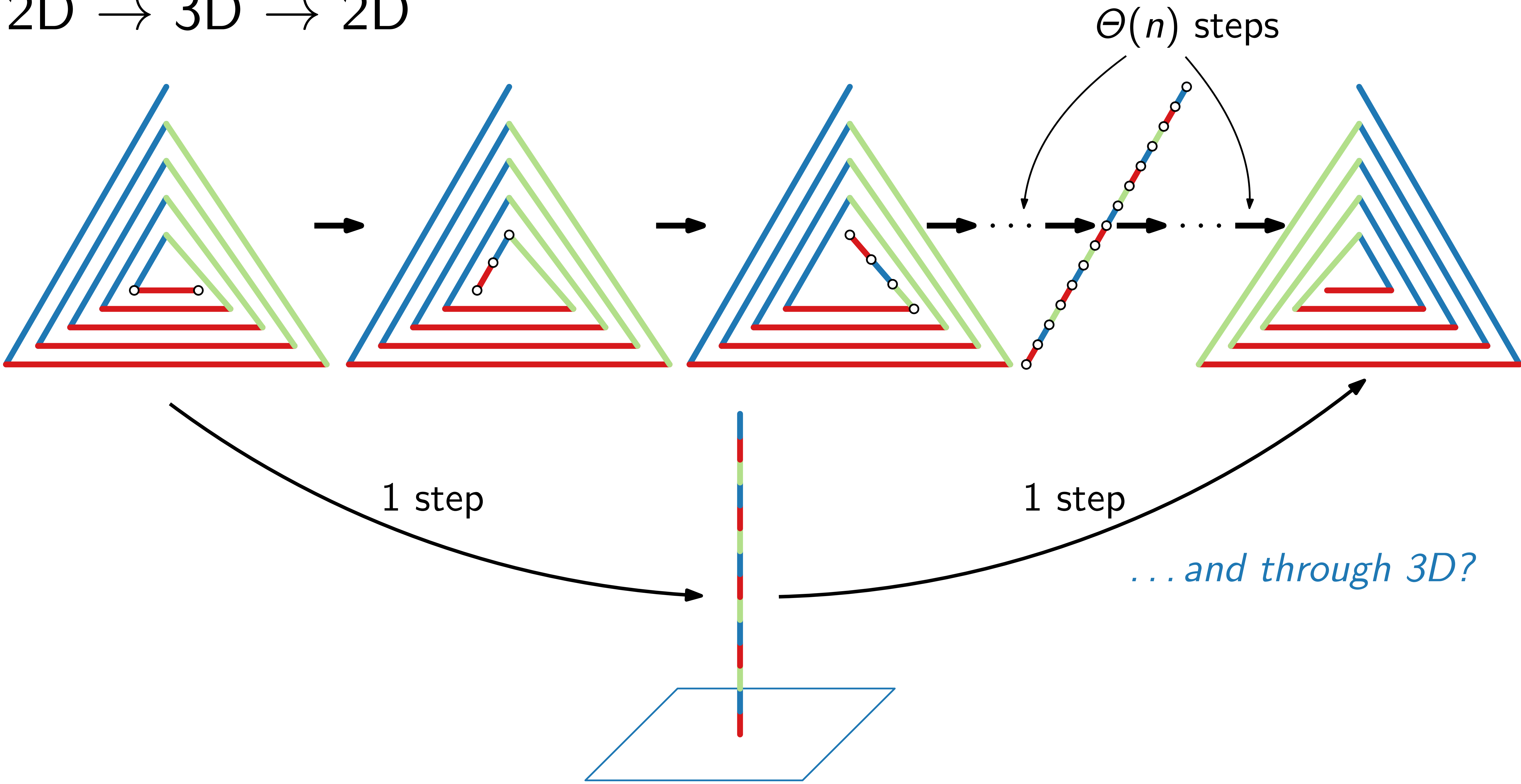

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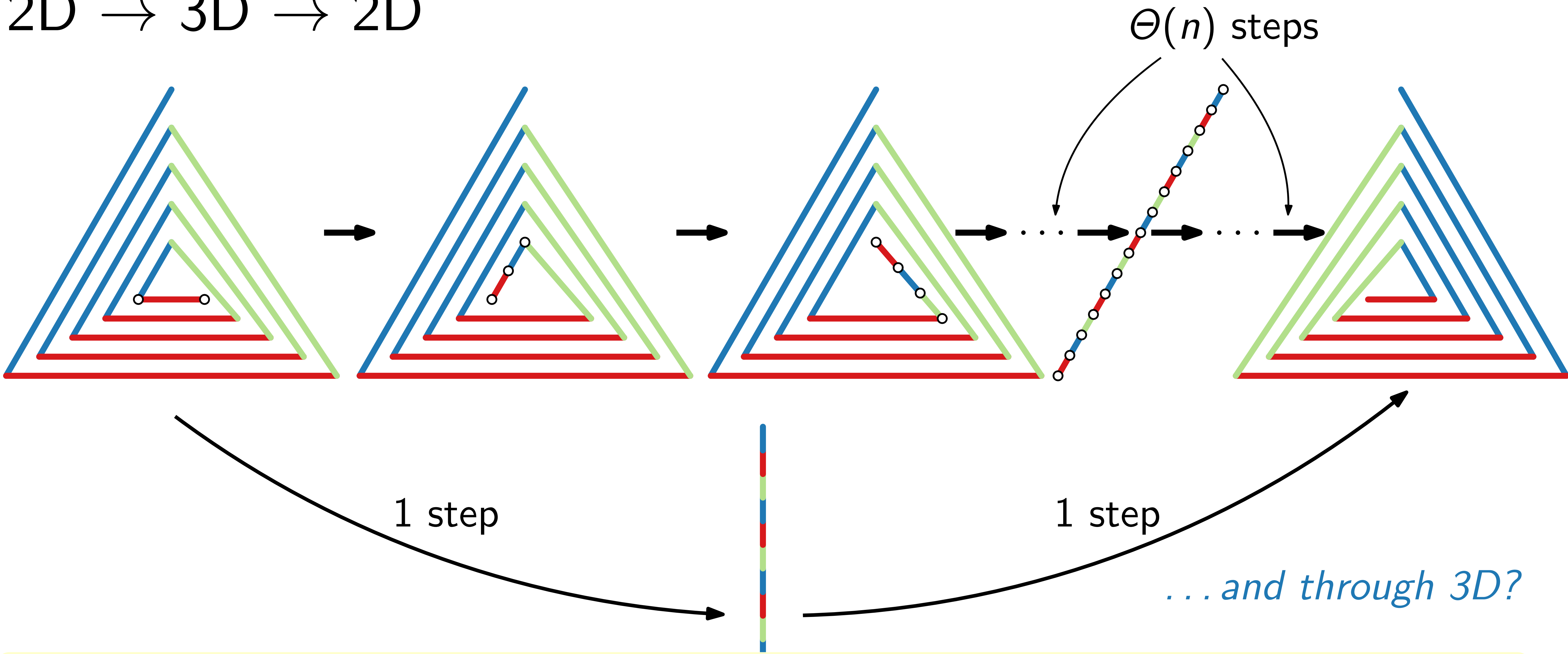

Between any two 2D straight-line crossing-free drawings of the same n-vertex *tree*, there is a k-step morph through 3D s.t. $k \in O(\log n)$. (For 3D input, $k \in O(n)$.)
[Arseneva, Bose, Cano, D'Angelo, Dujmovic, Frati, Langerman, Tappini: JGAA 2019]

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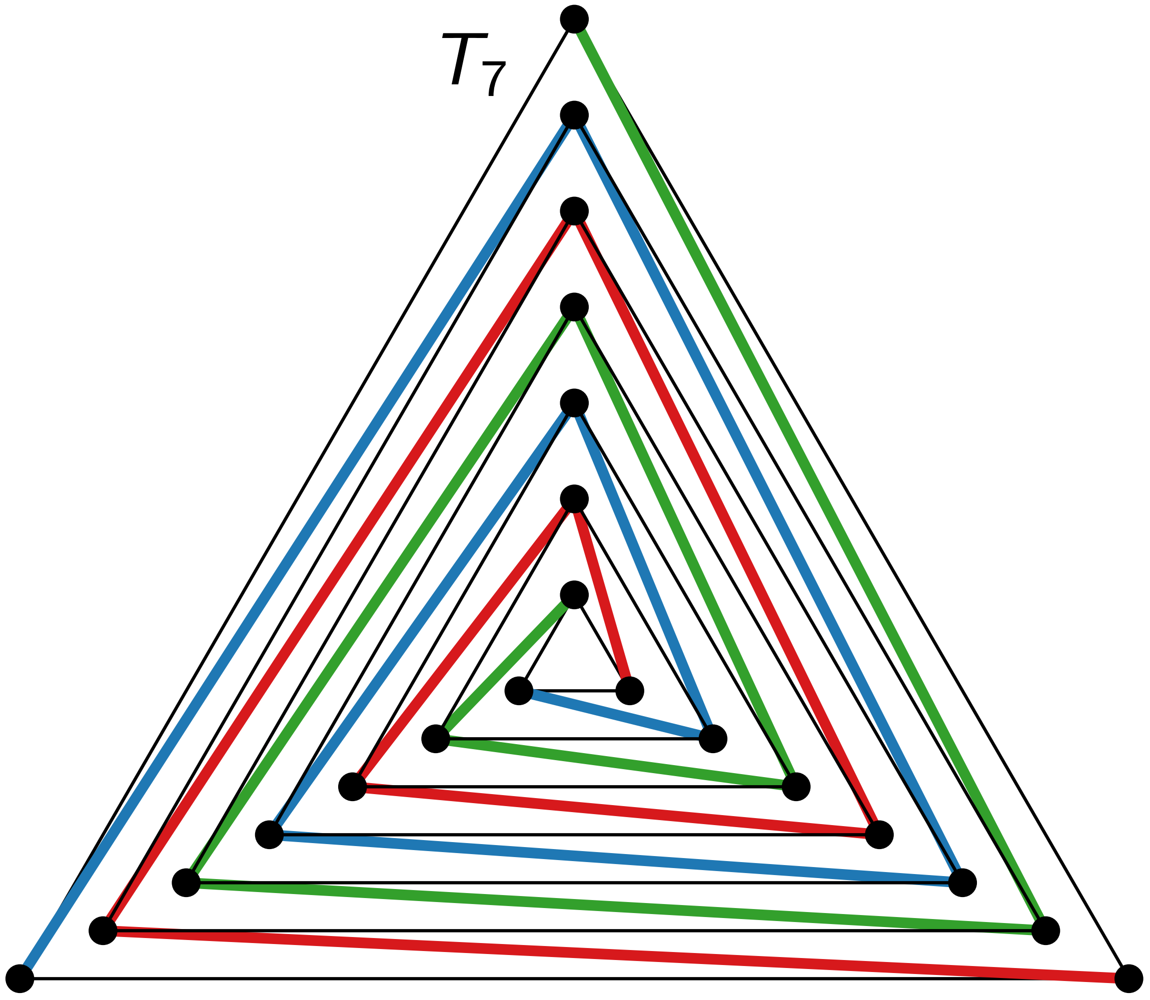
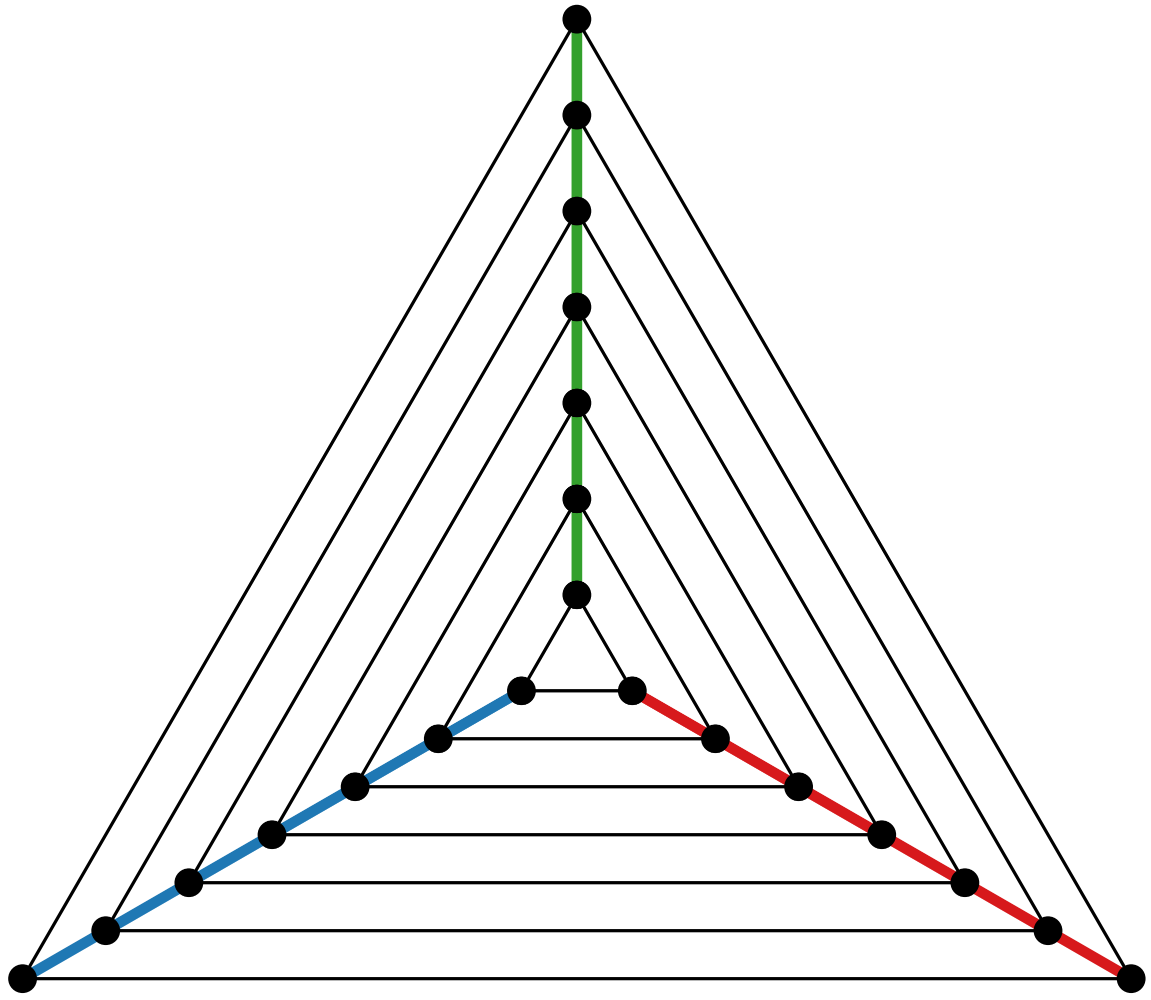
Show that T_n needs $\Omega(n)$ steps to untangle, even with the help of 3D!





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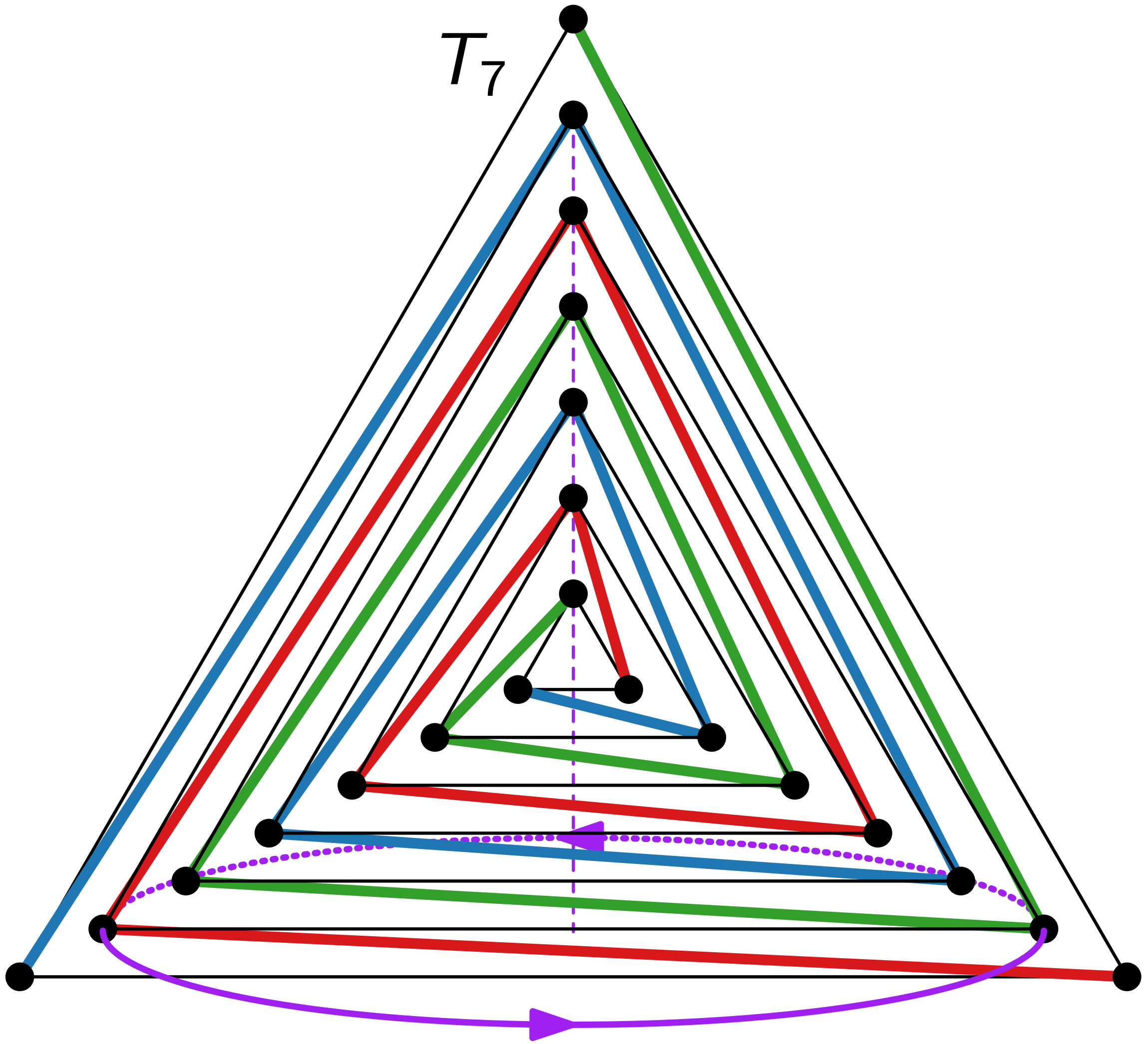

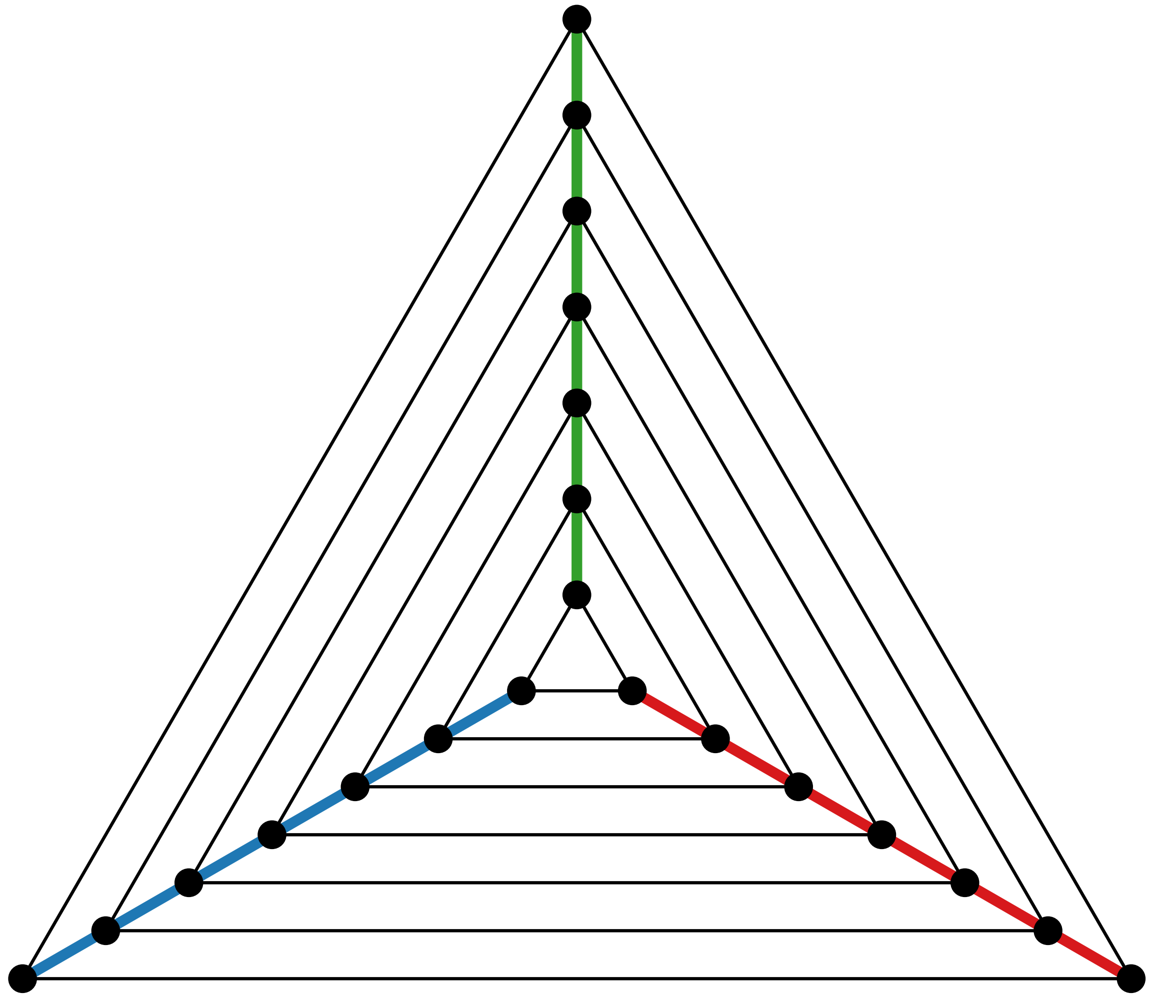

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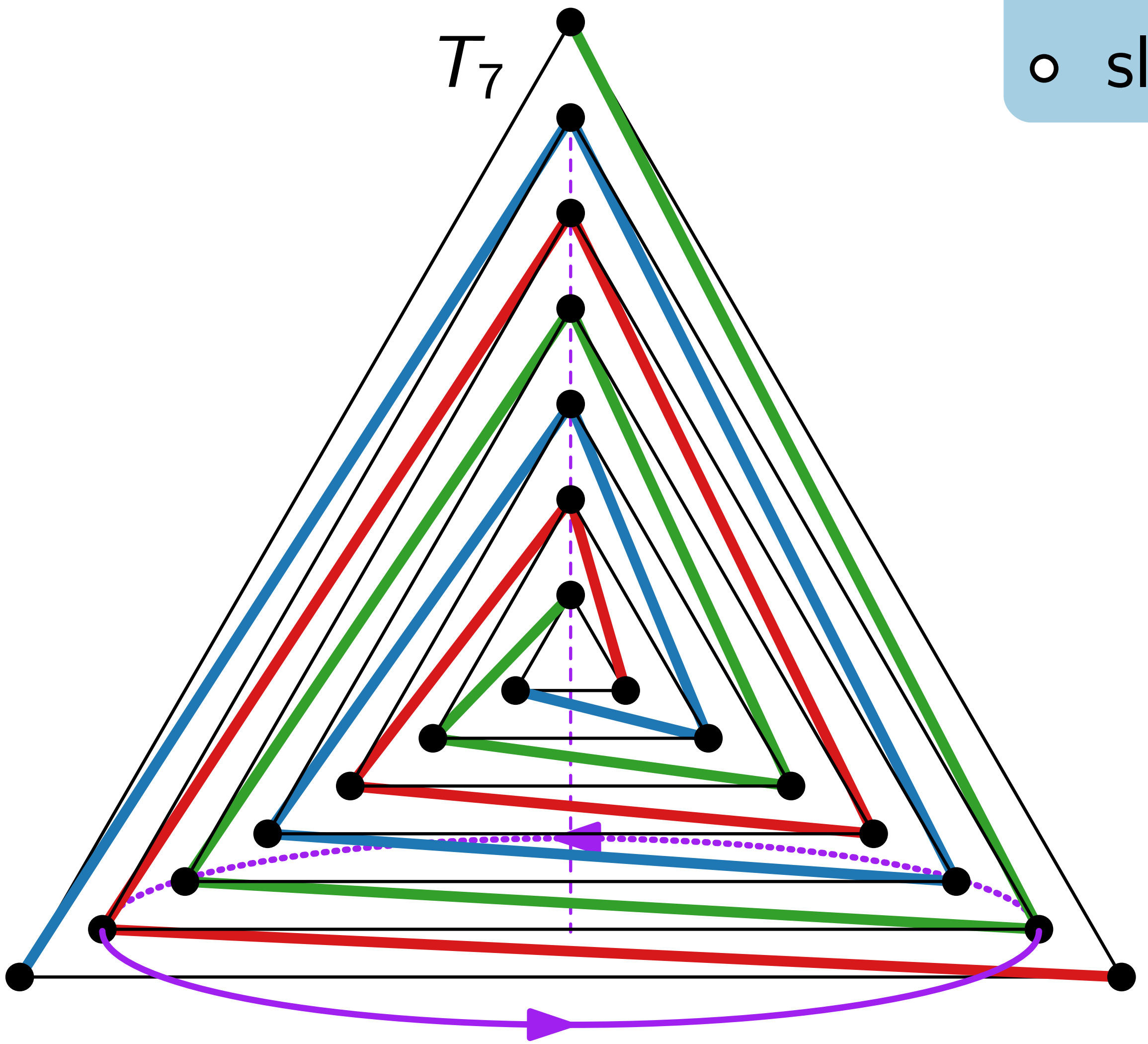
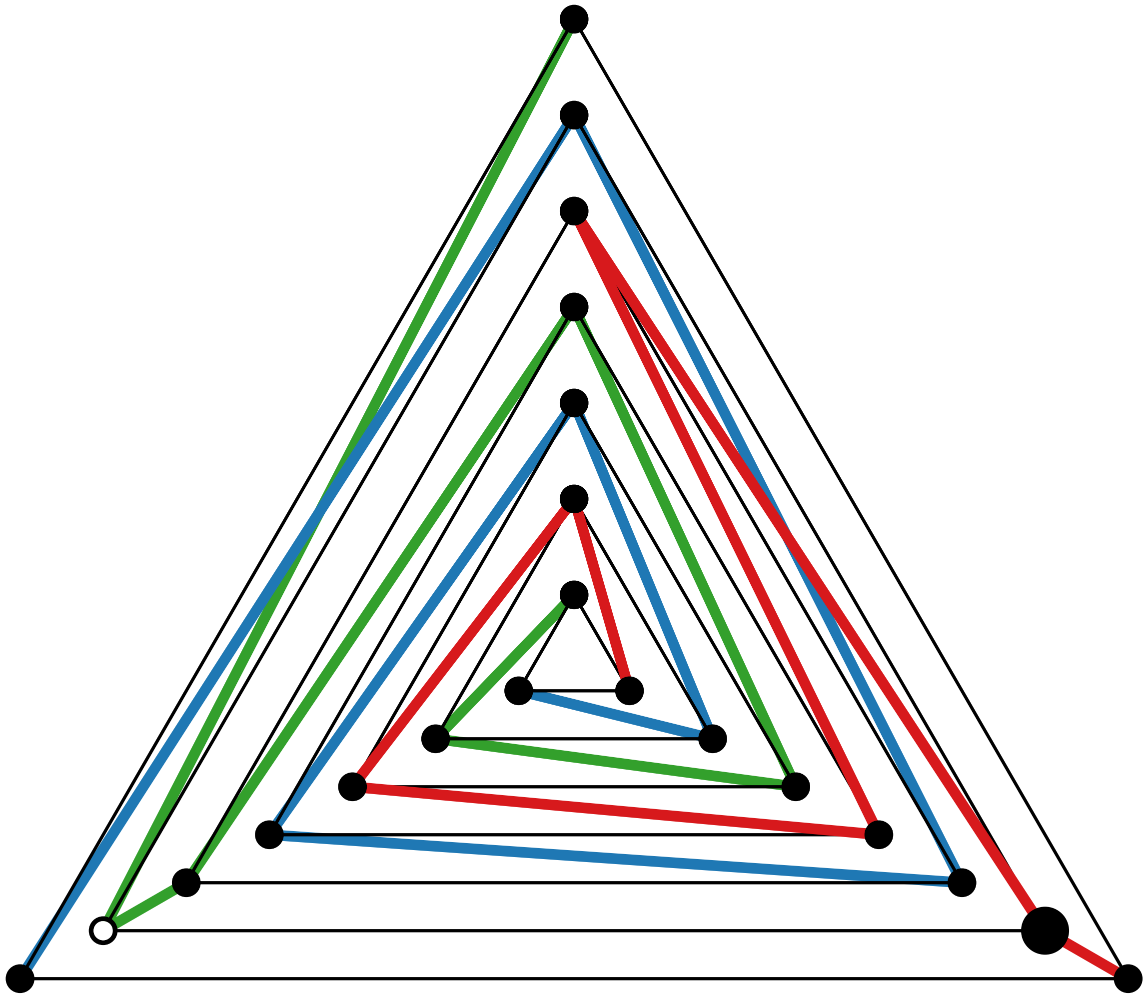
- slighly above
- slighly below





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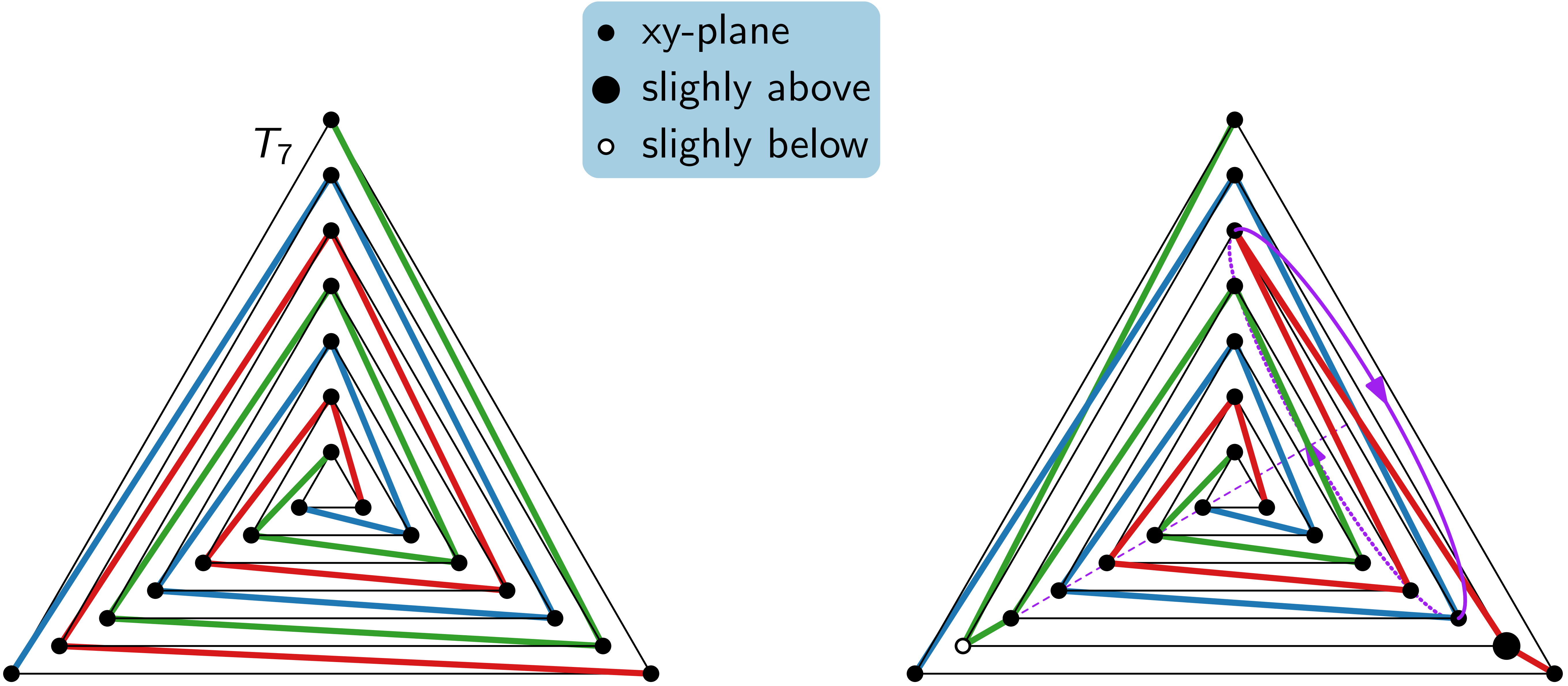

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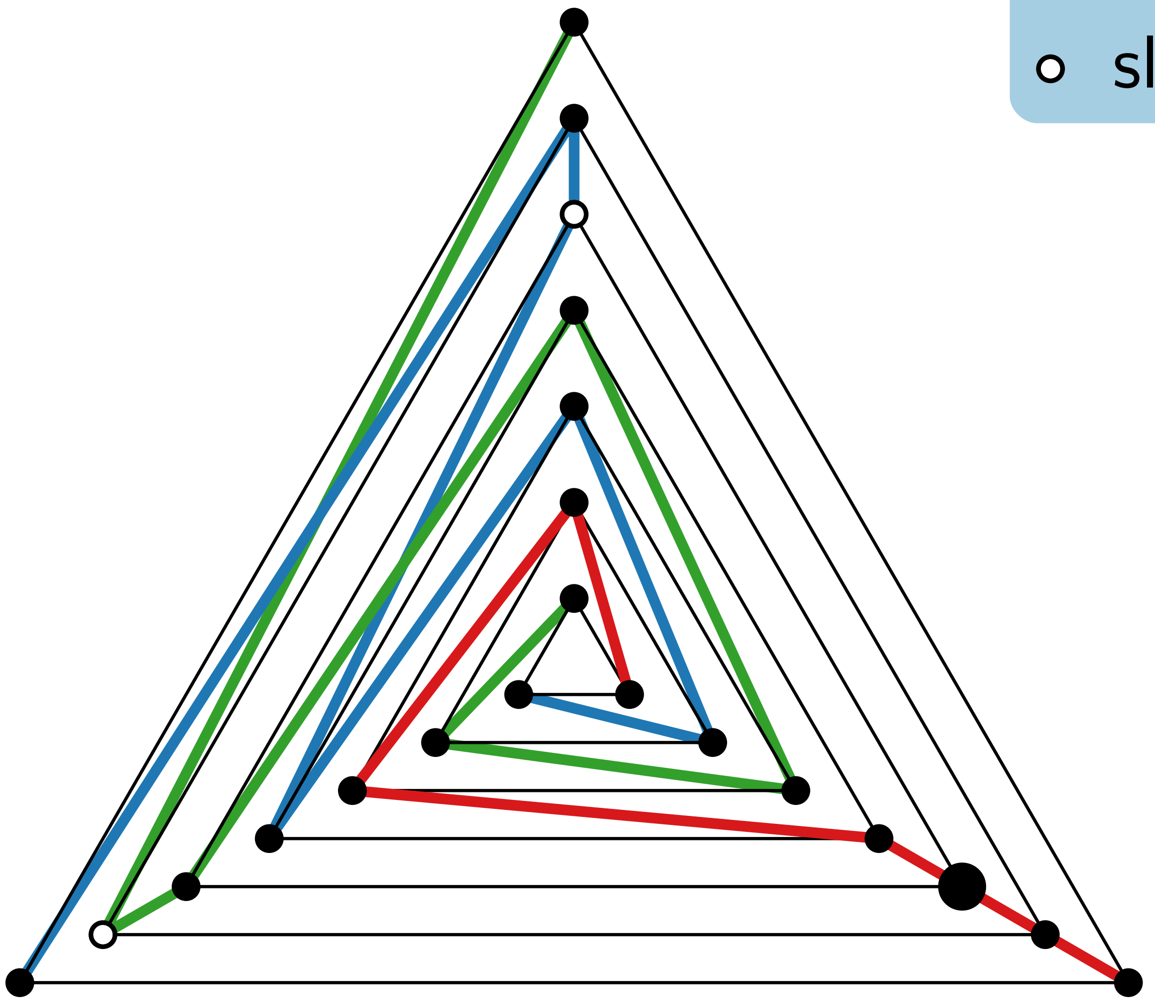

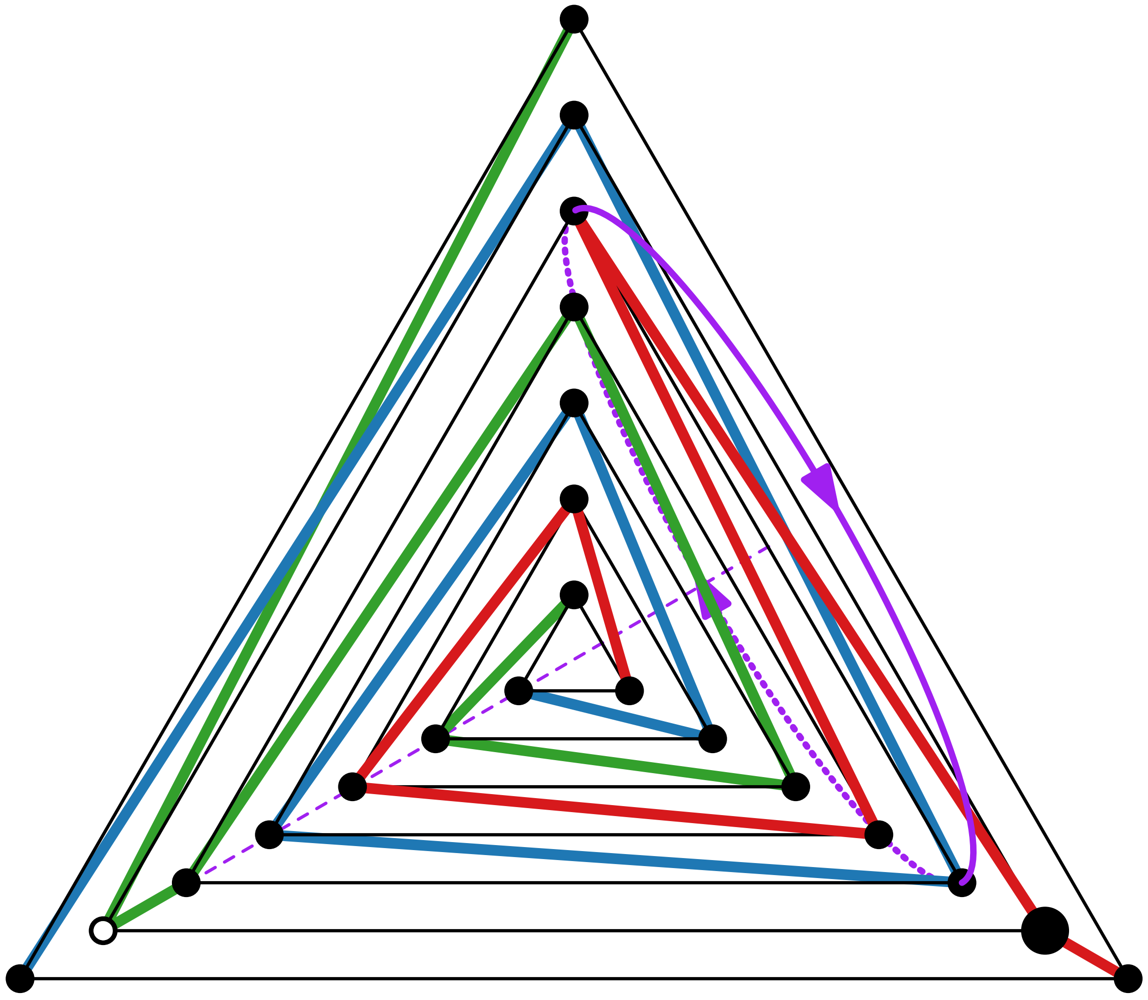

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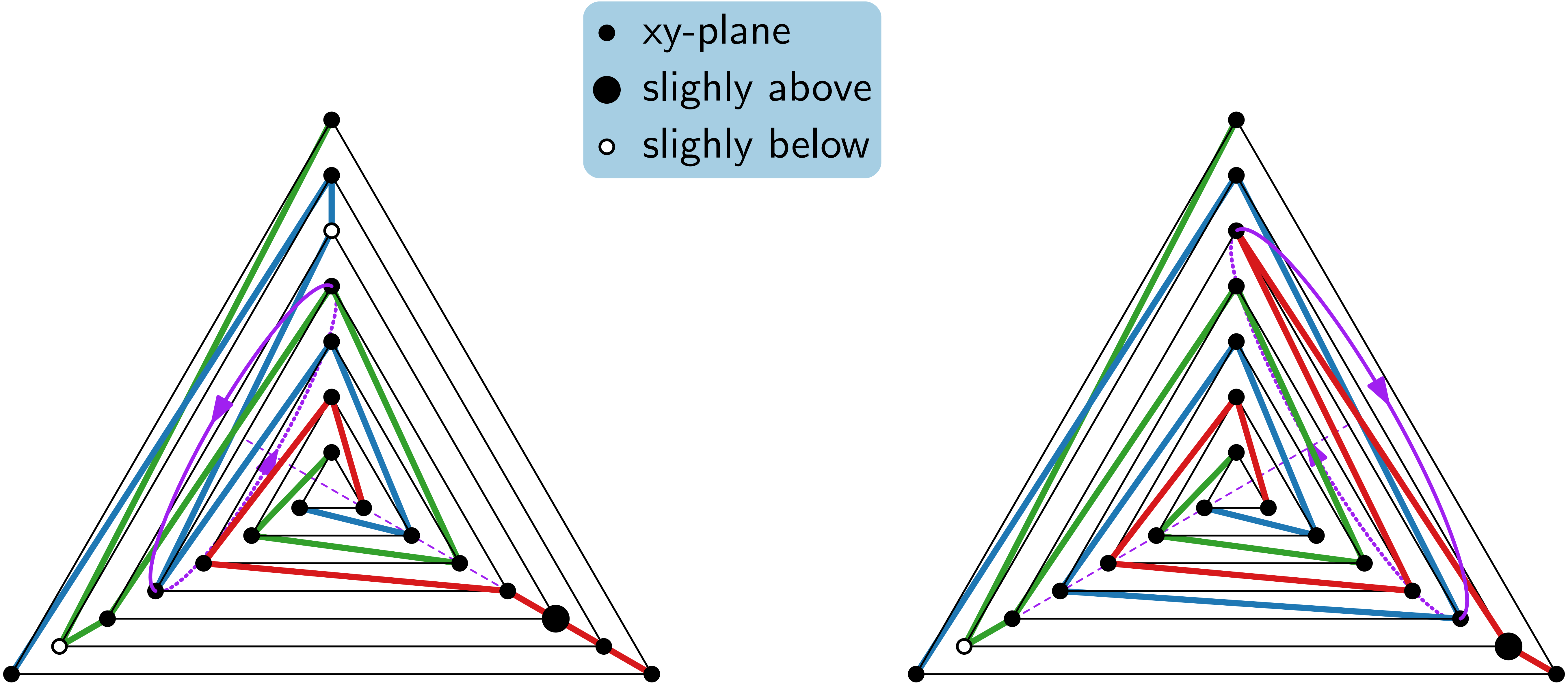

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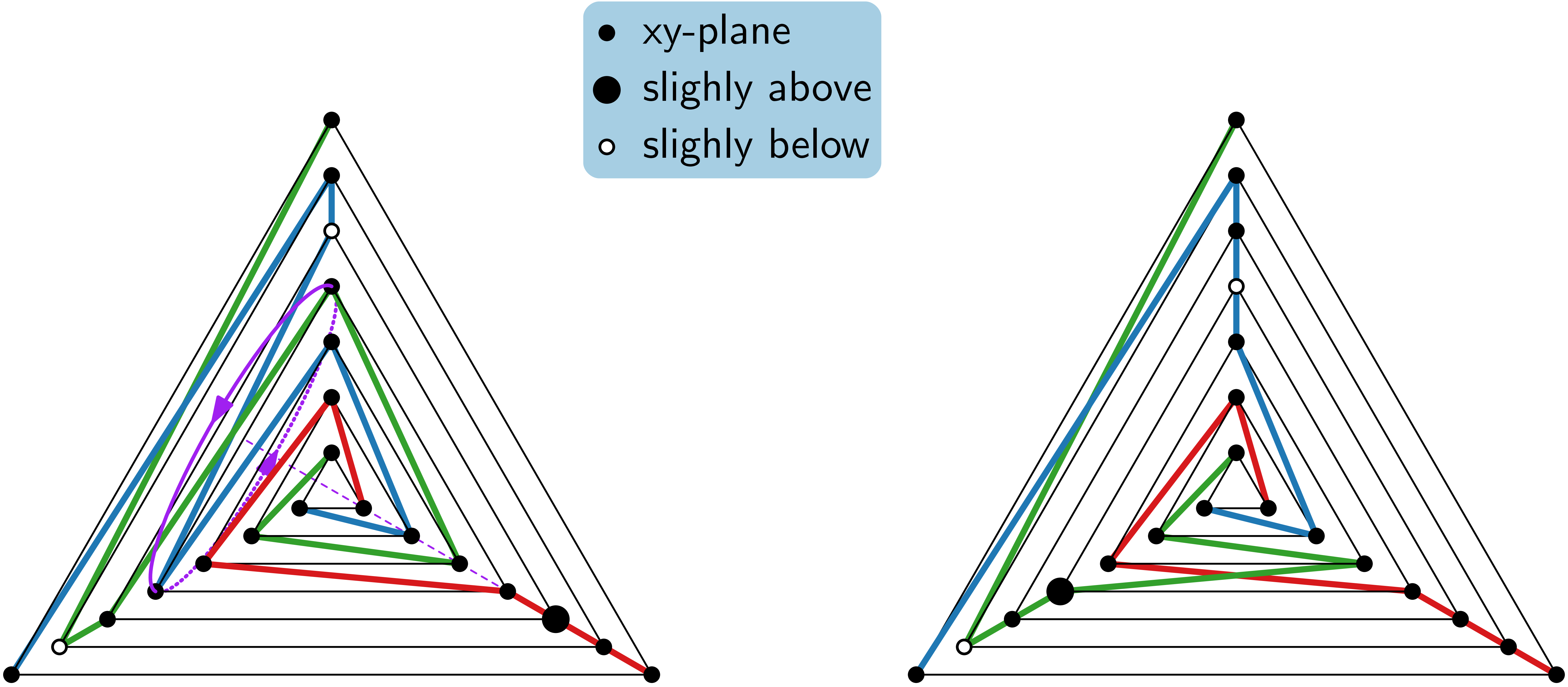

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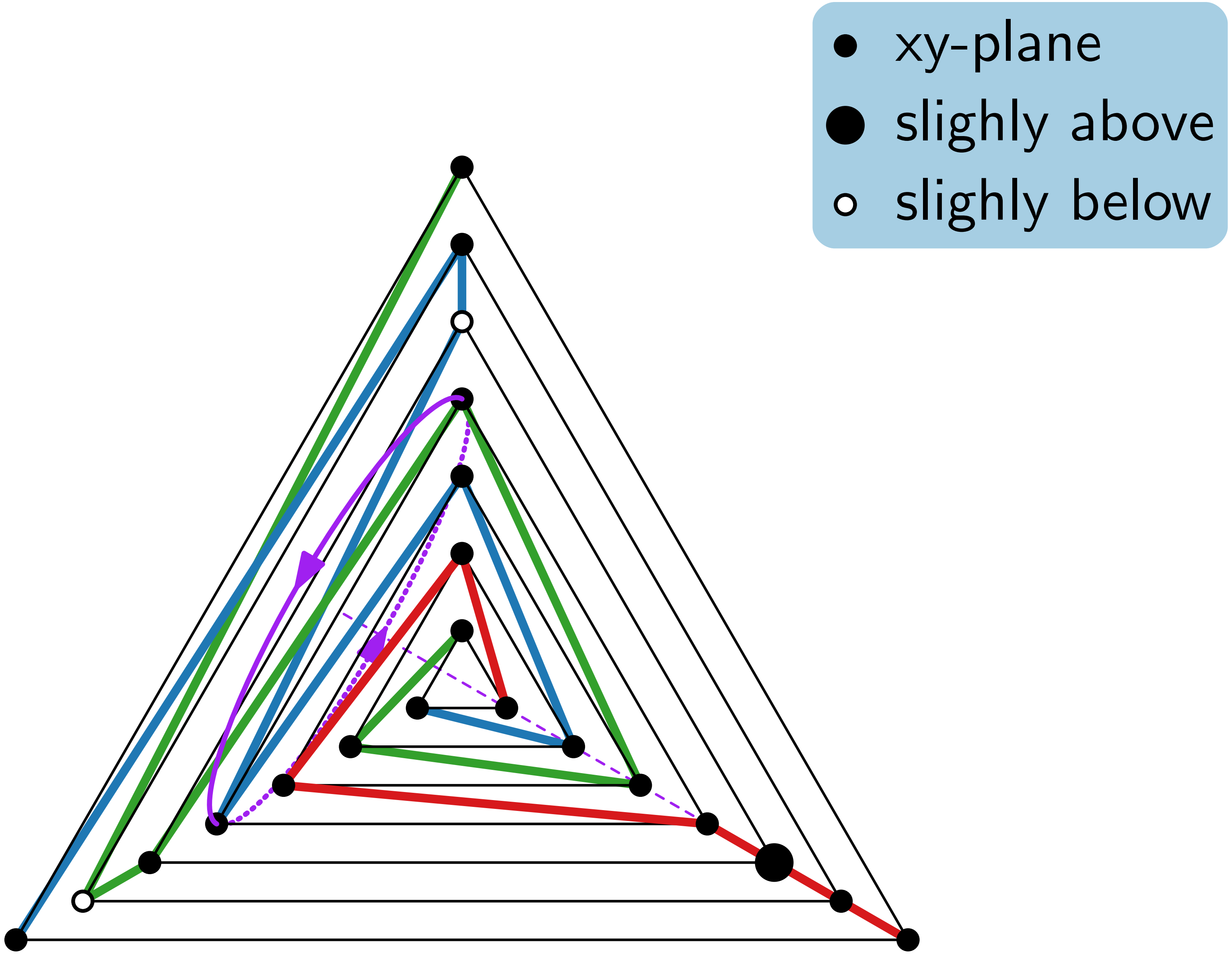

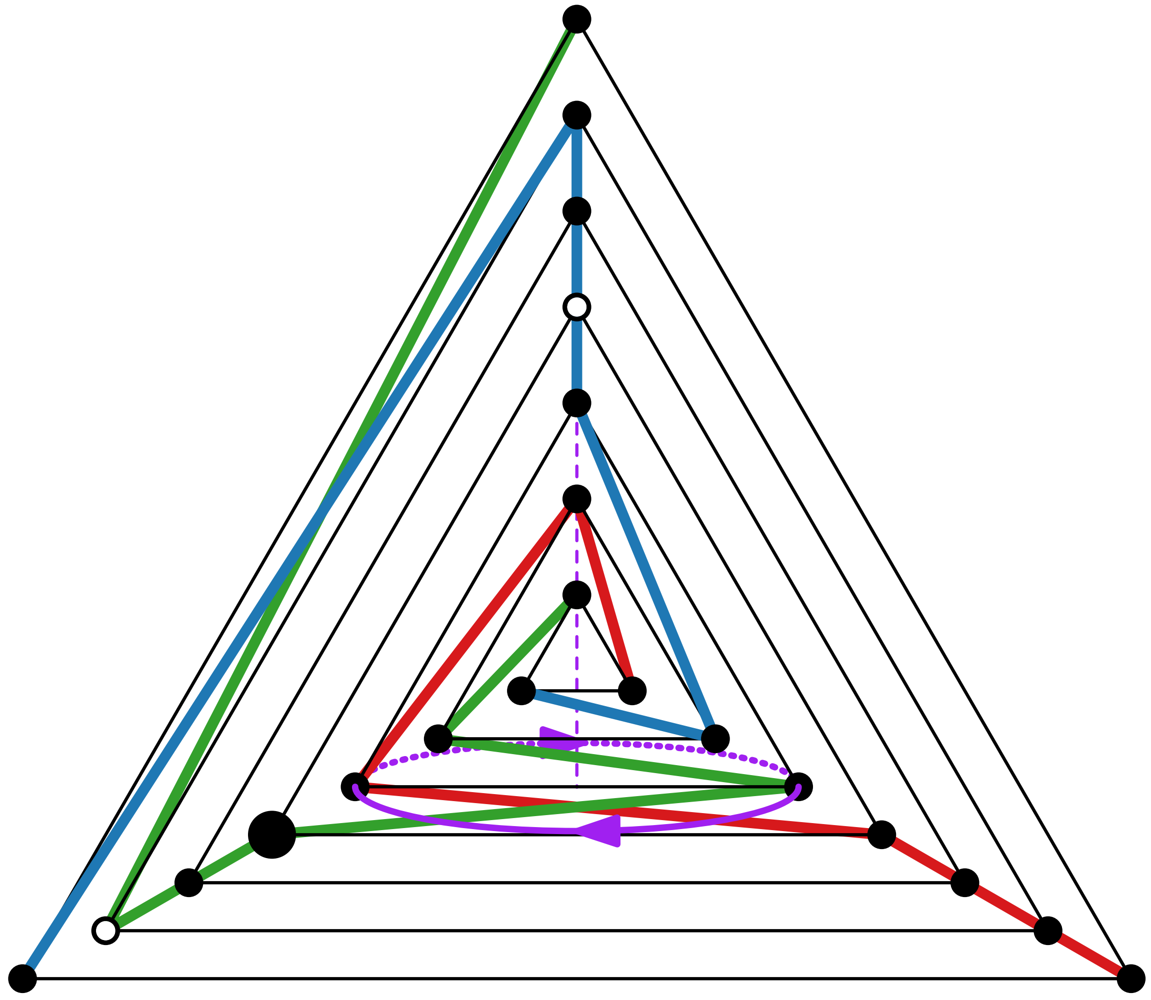

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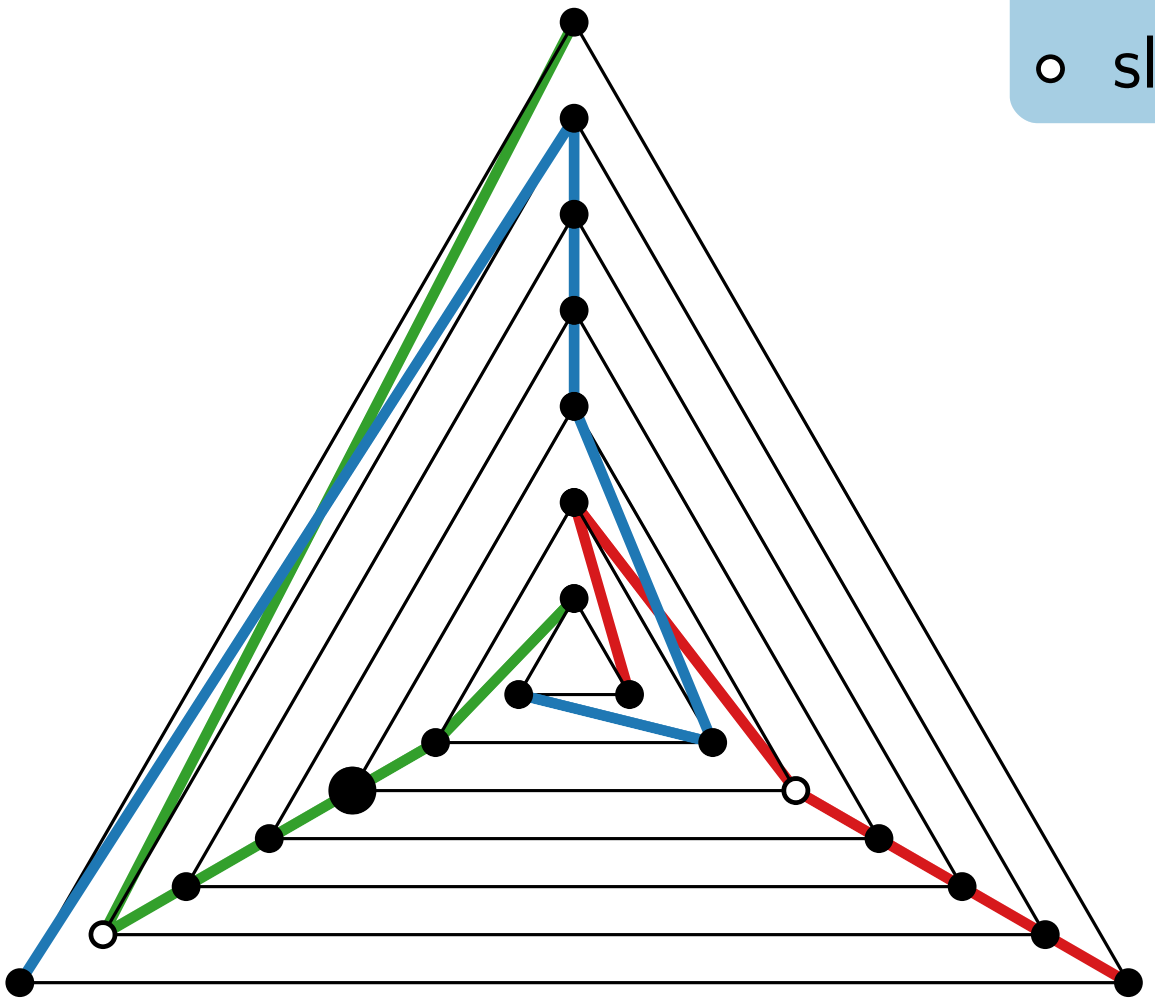

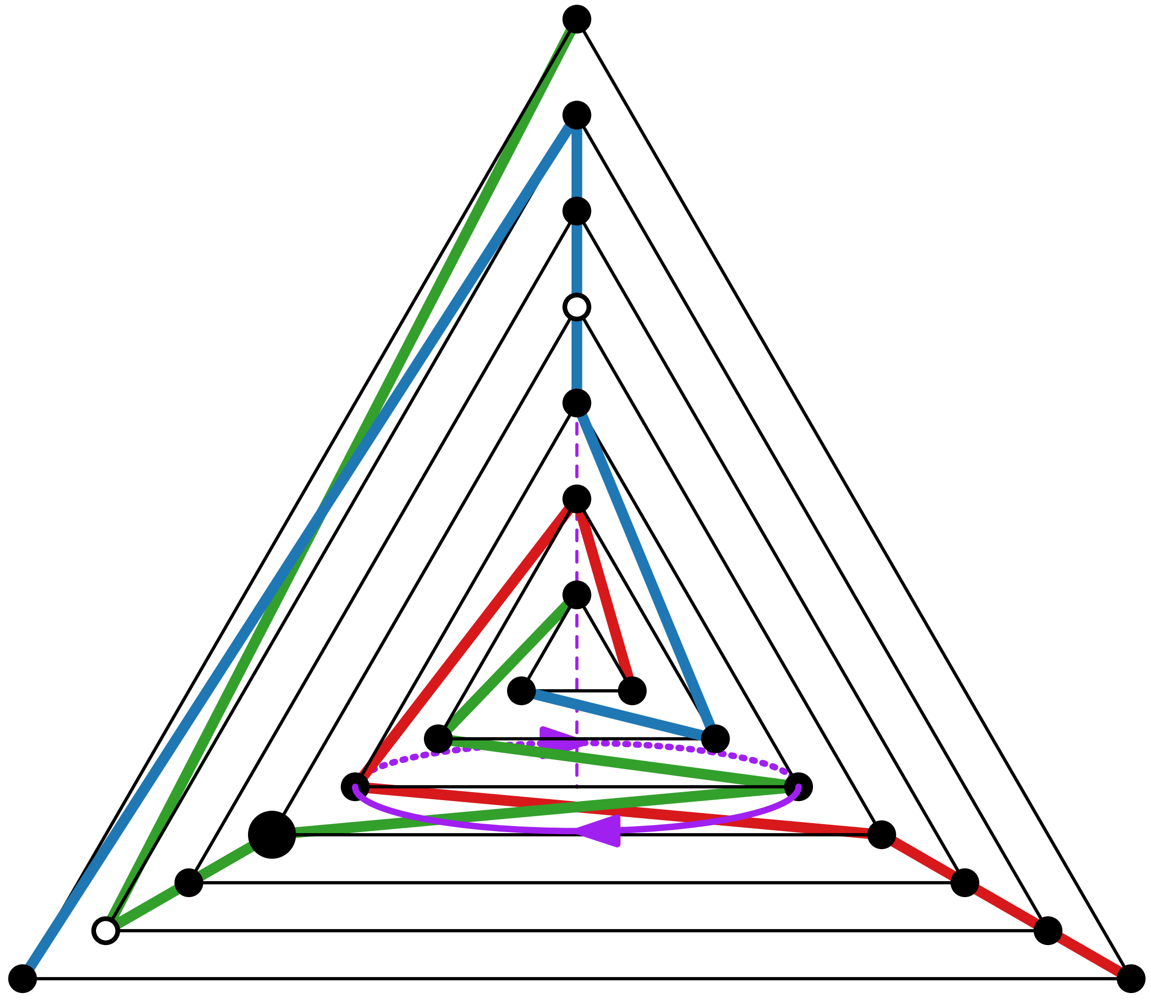

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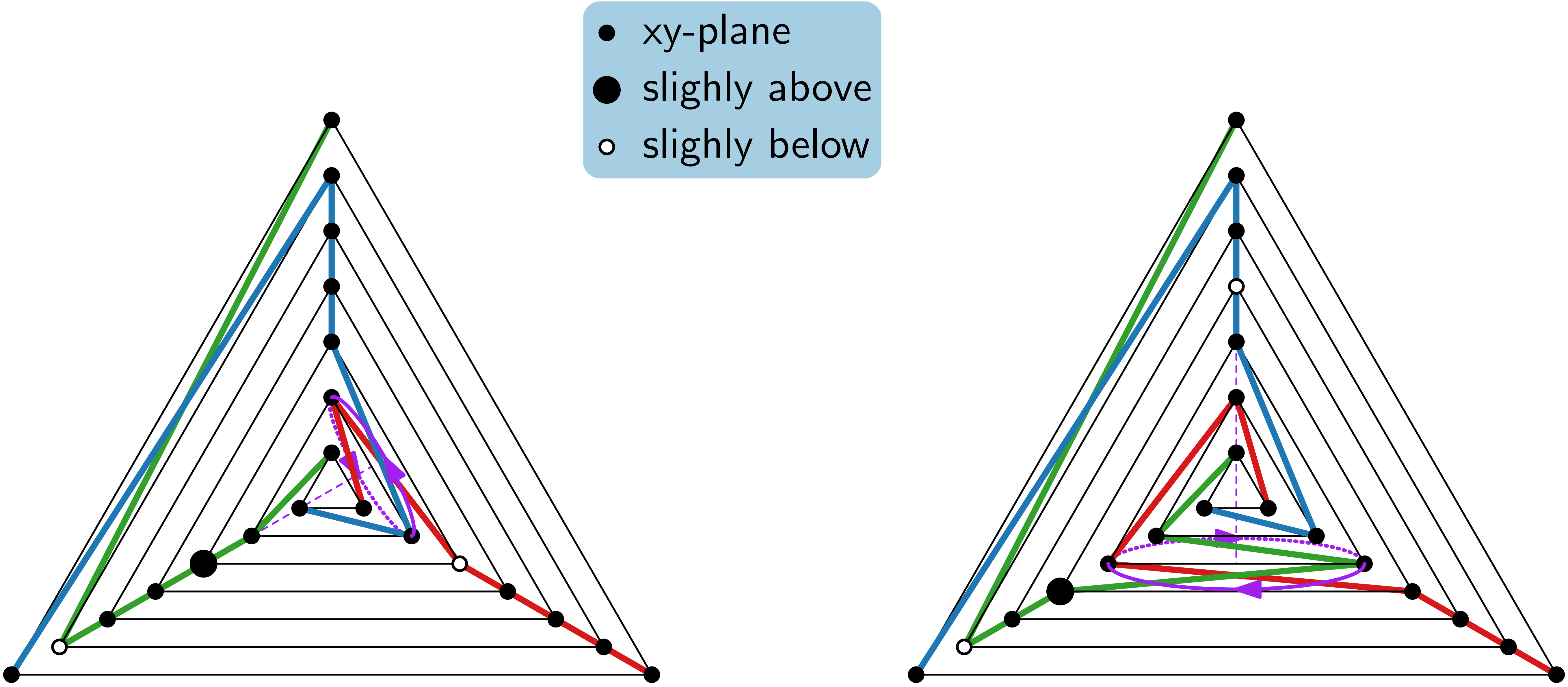

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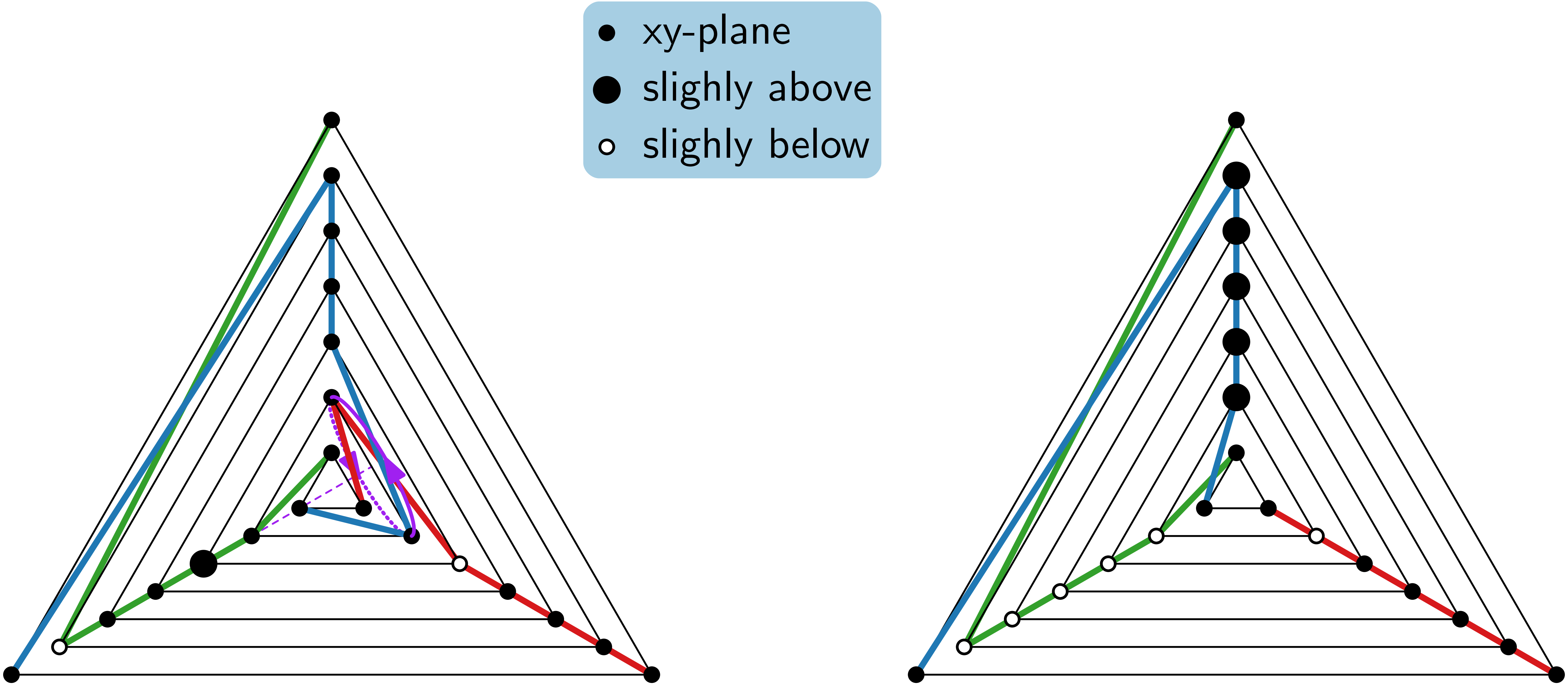

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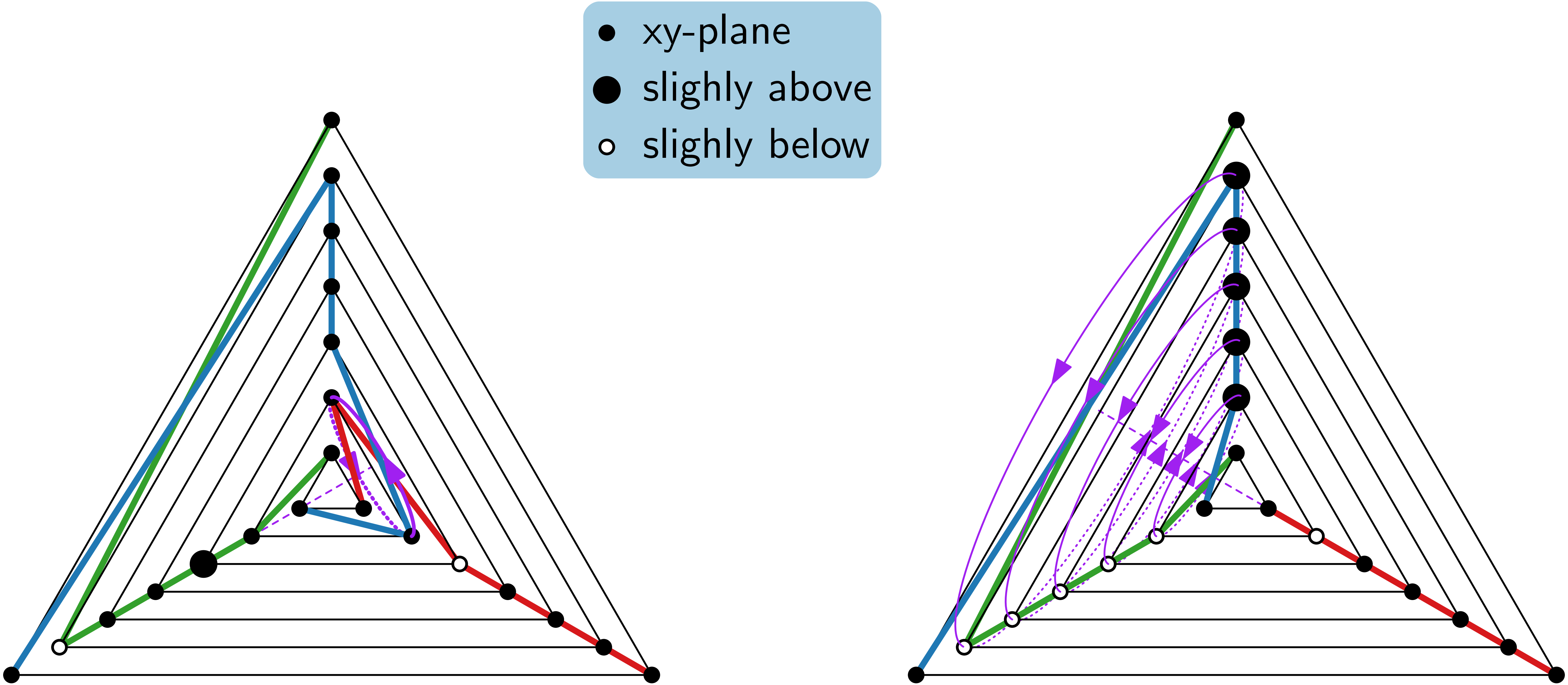

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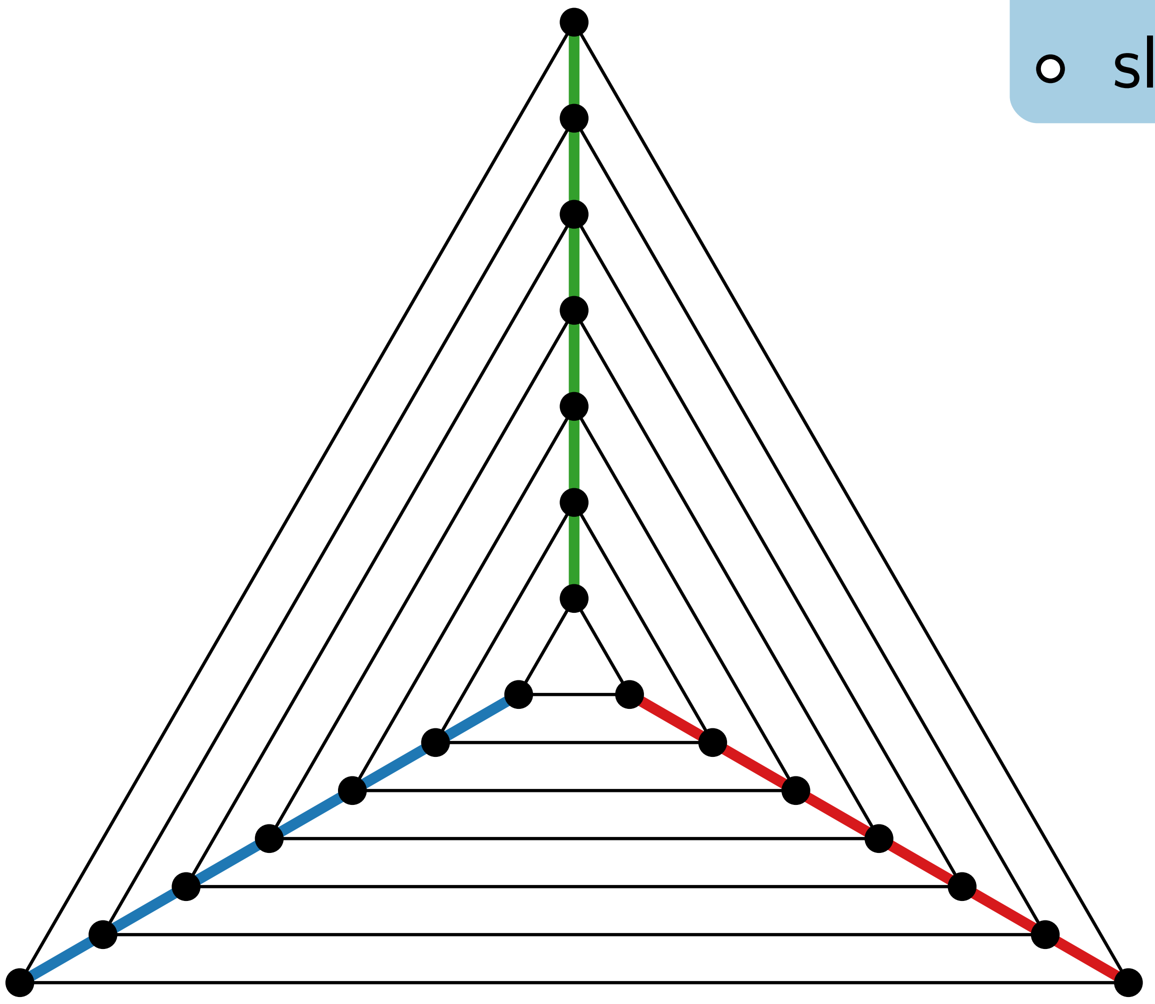

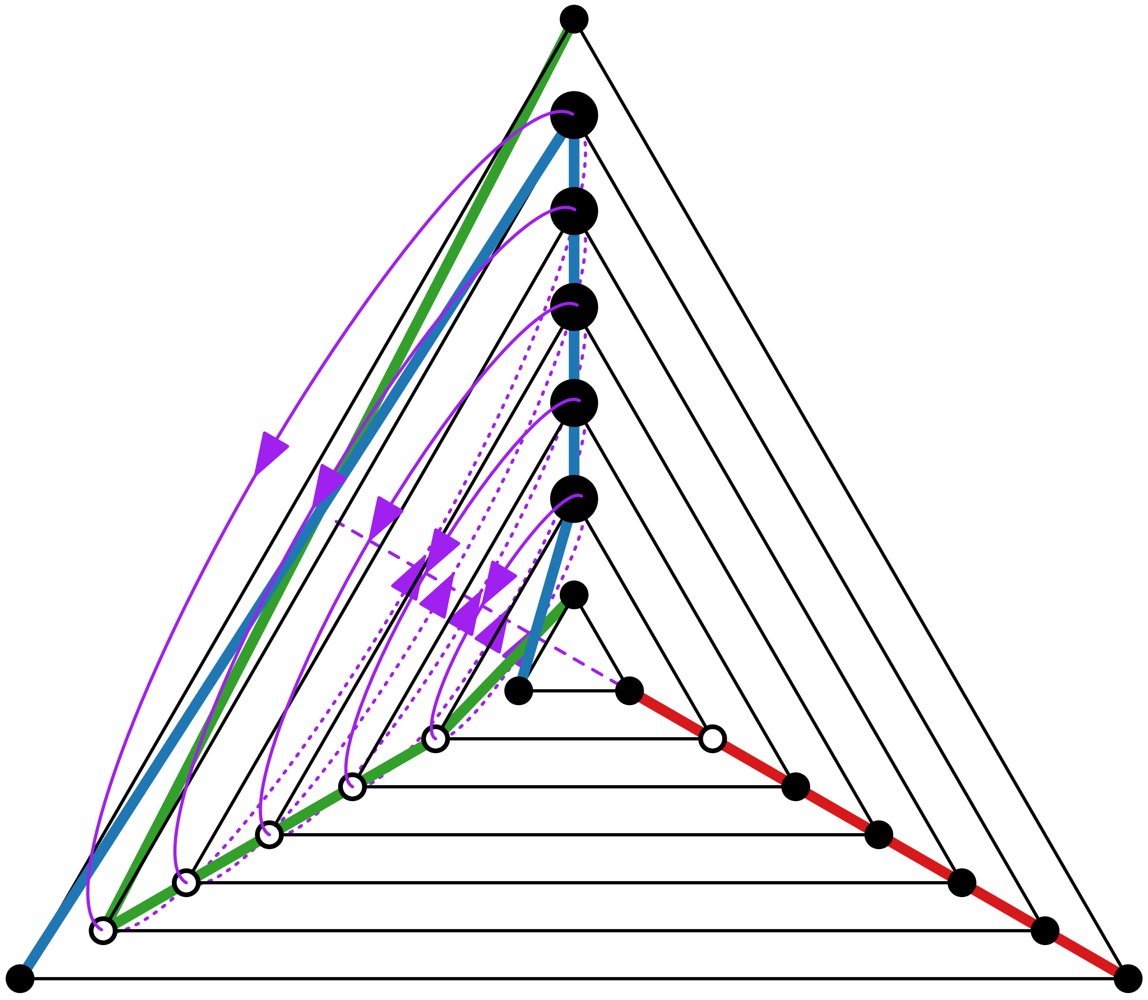

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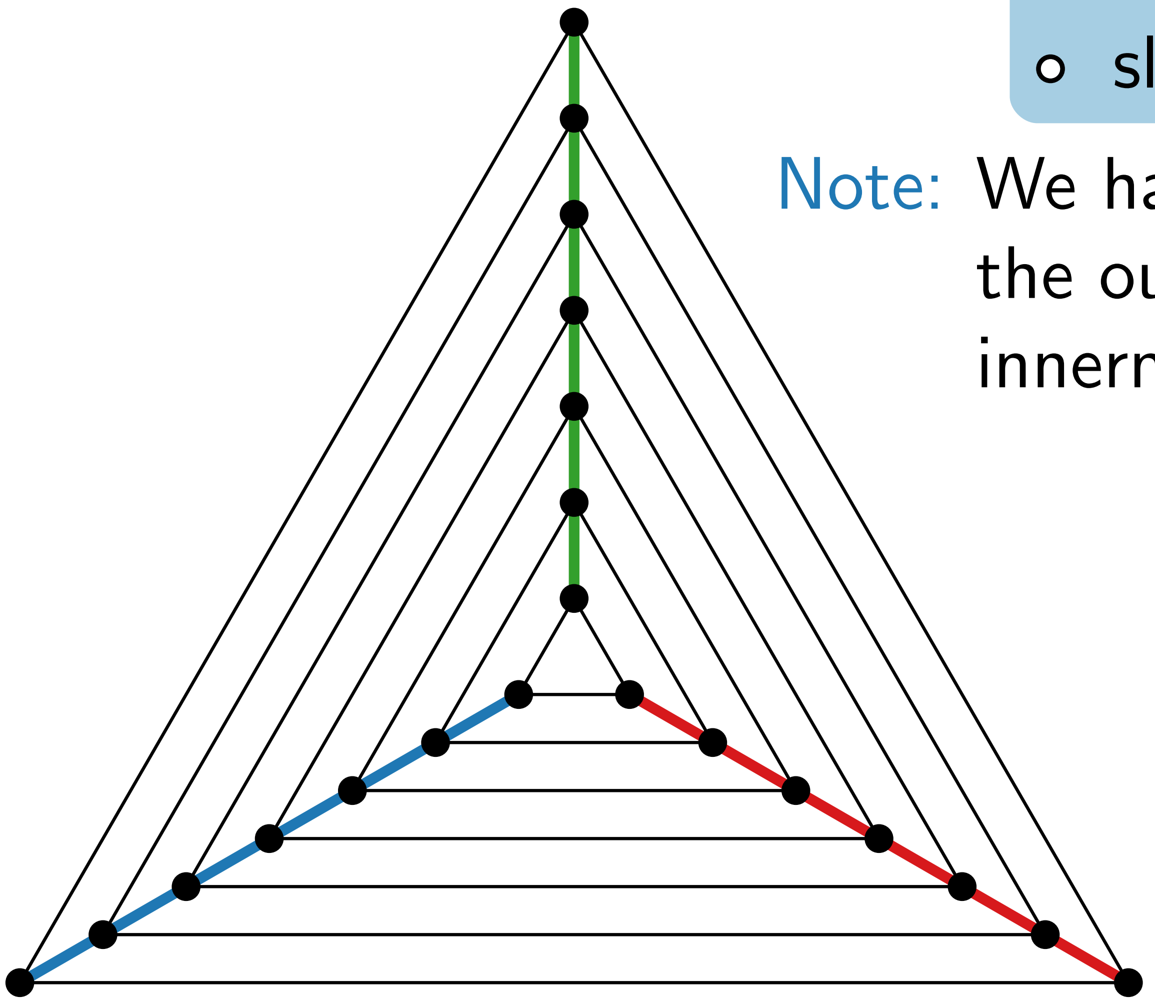
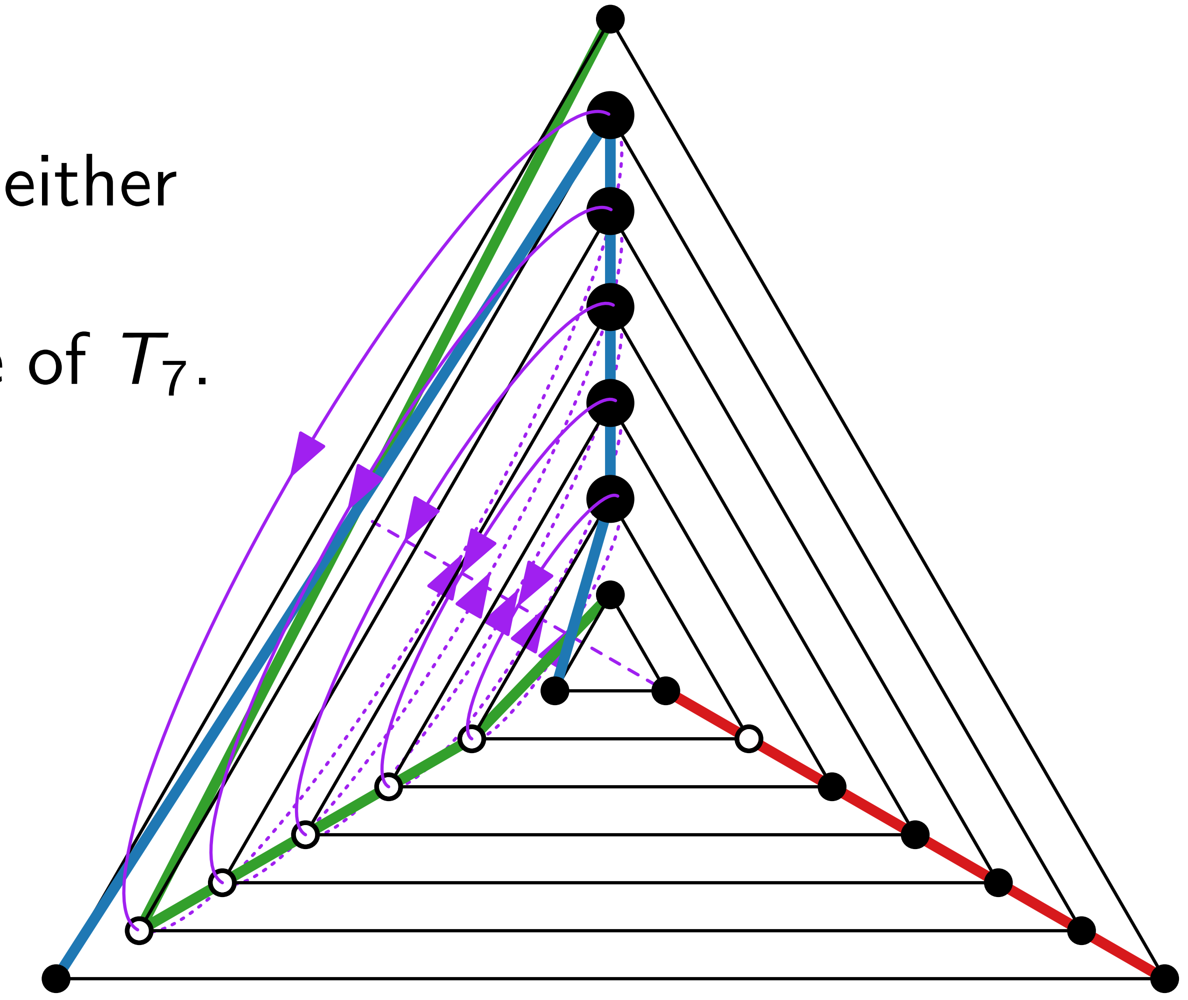
Note: We have moved neither the outer- nor the innermost triangle of T_7.





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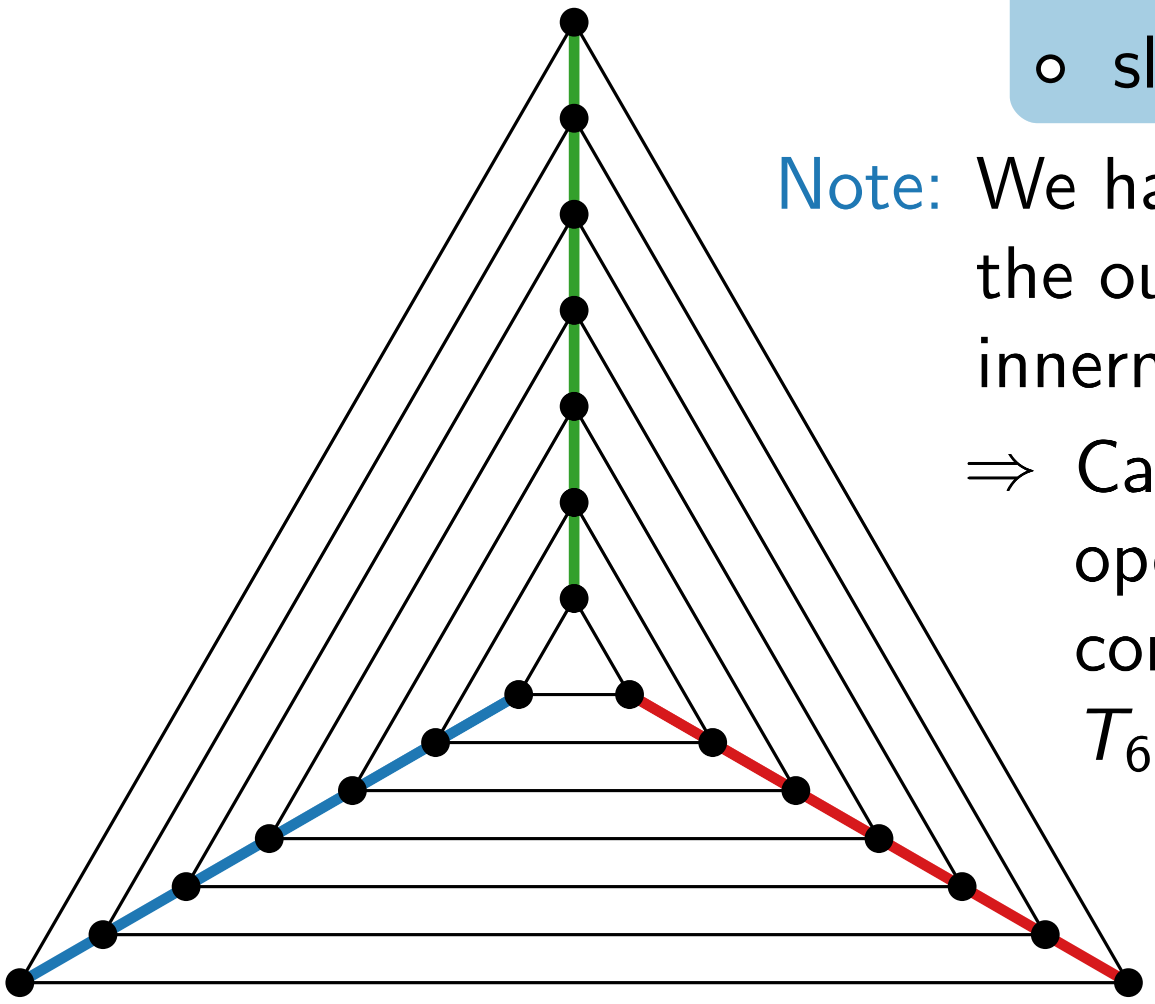
Show that T_n needs $\Omega(n)$ steps to untangle, even with the help of 3D!



- ● xy-plane
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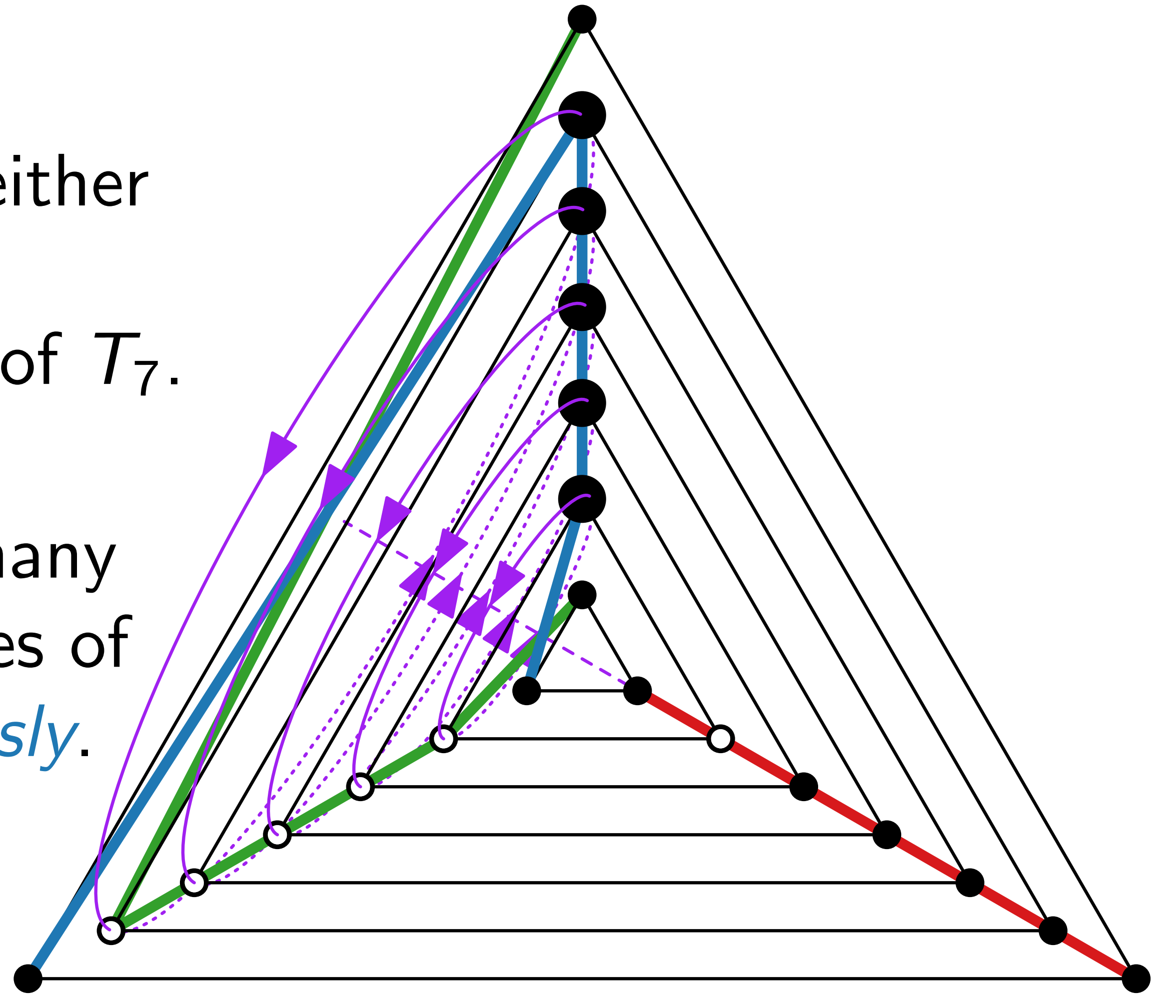
Note: We have moved neither the outer- nor the innermost triangle of T_7.

$\Rightarrow$ Can do same operations to many concentric copies of T_6 *simultaneously*.



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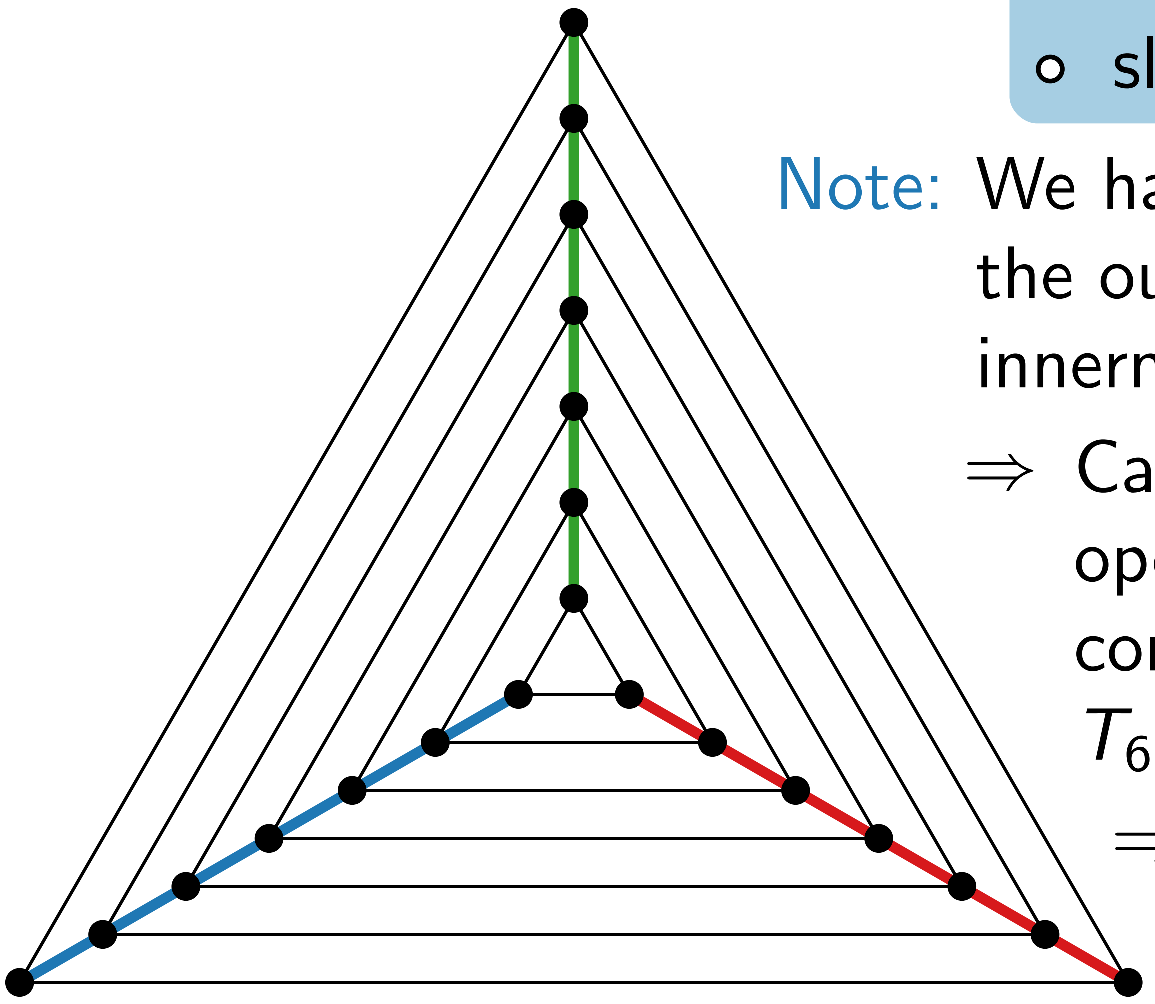
Try our video!



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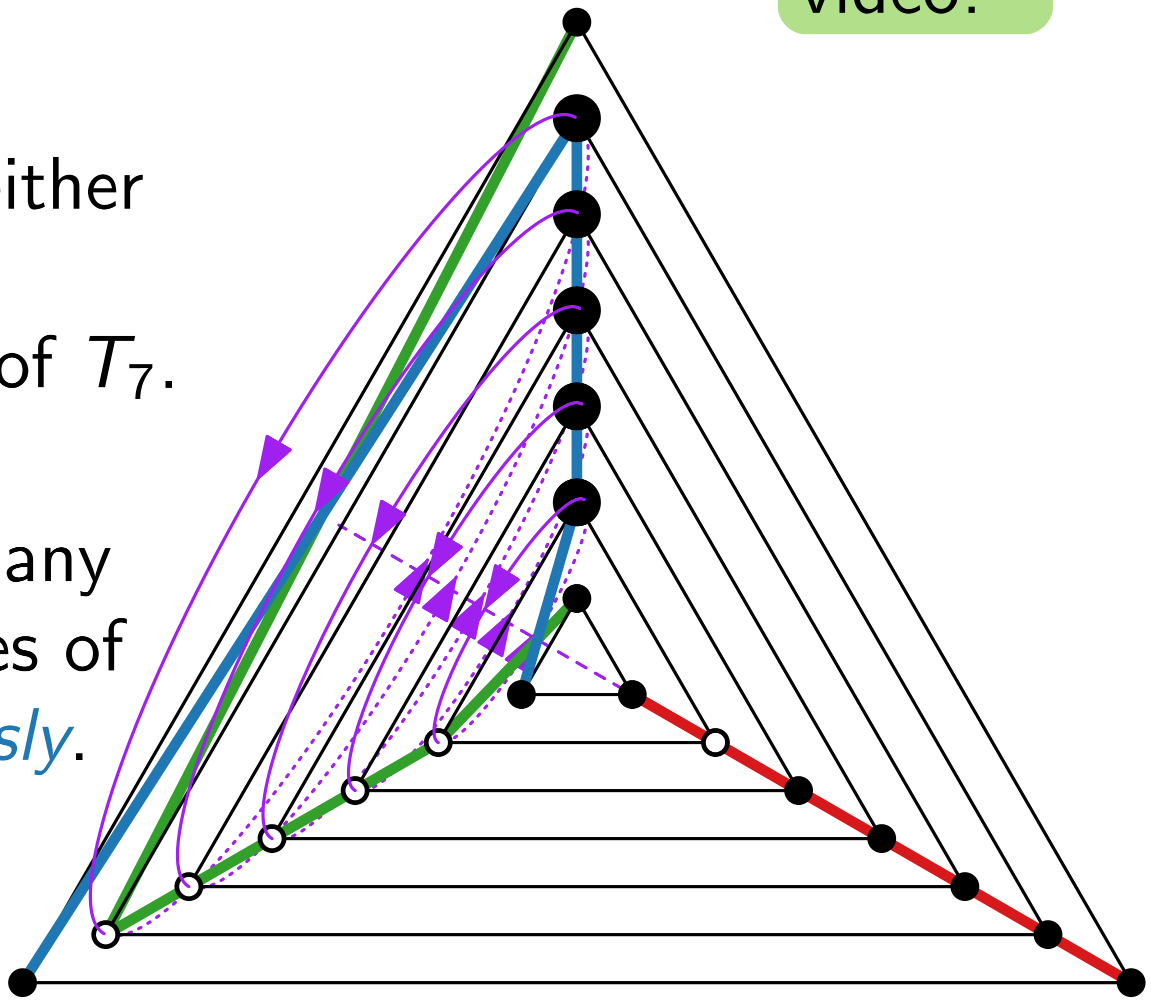
⇒ Need only $O(1)$ steps for any T_n.



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A *split component* of G with respect to a split pair $\{u, v\}$ is the edge uv or a maximal subgraph G_{uv} of G such that $\{u, v\}$ is not a split pair of G_{uv}.

Our Operations

Given a biconnected planar graph with two embeddings $\mathcal{E}_1$ and $\mathcal{E}_2$, then there is a sequence of $O(n)$ flip and skip operations that transforms $\mathcal{E}_1$ into $\mathcal{E}_2$. The sequence can be computed in linear time. [Angelini, Cortese, Di Battista, Patrignani: TCS'13]

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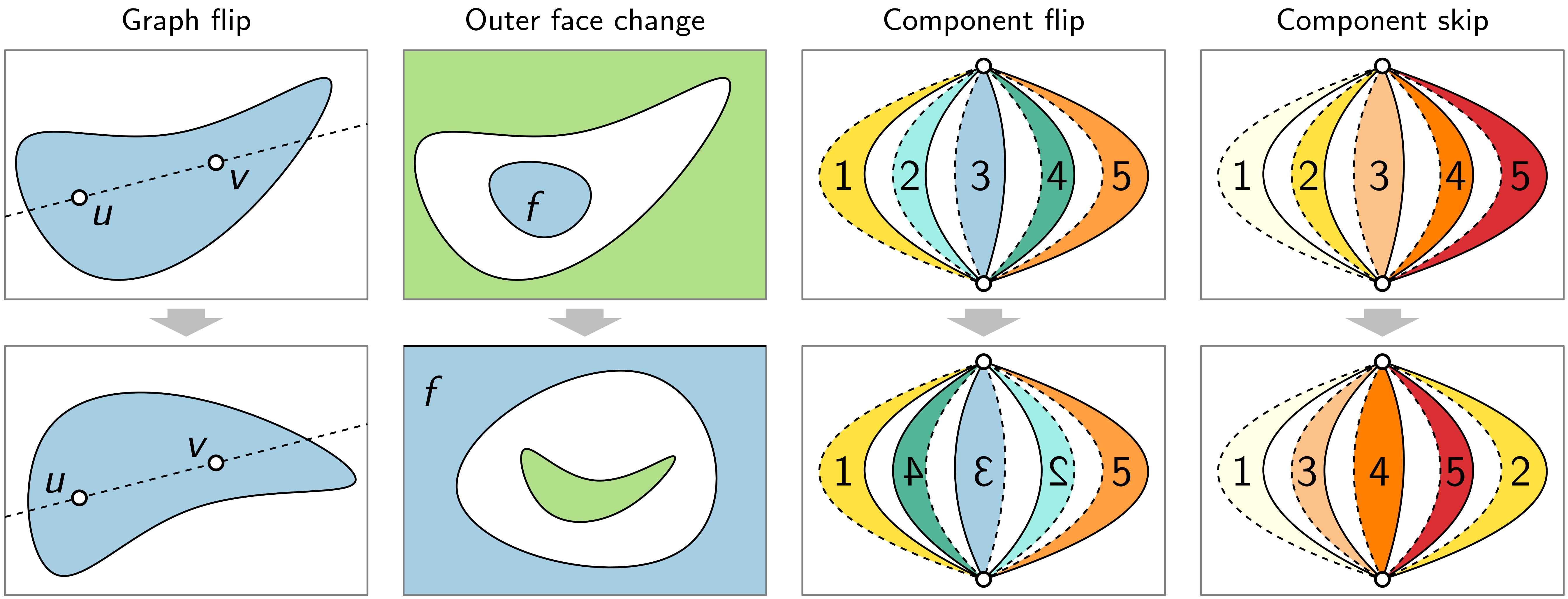

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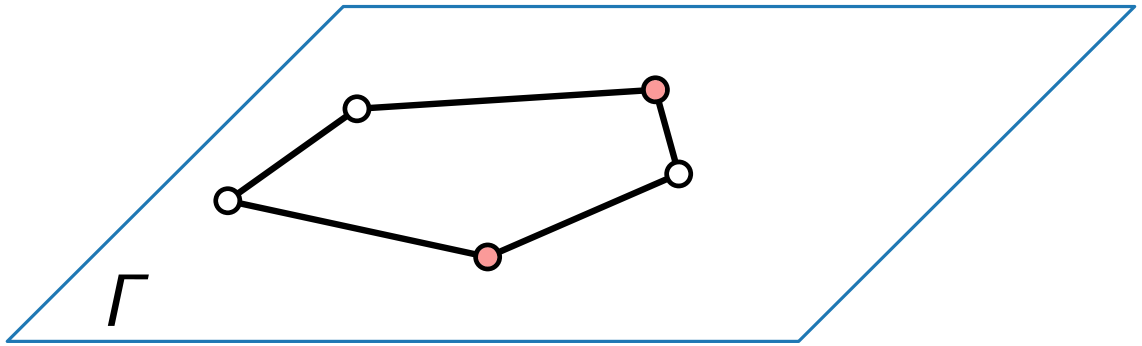

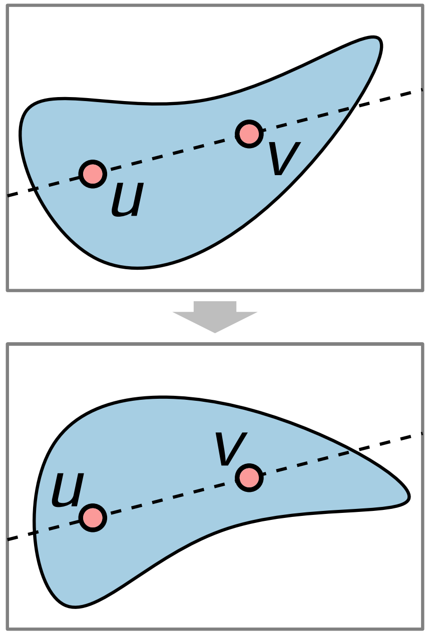

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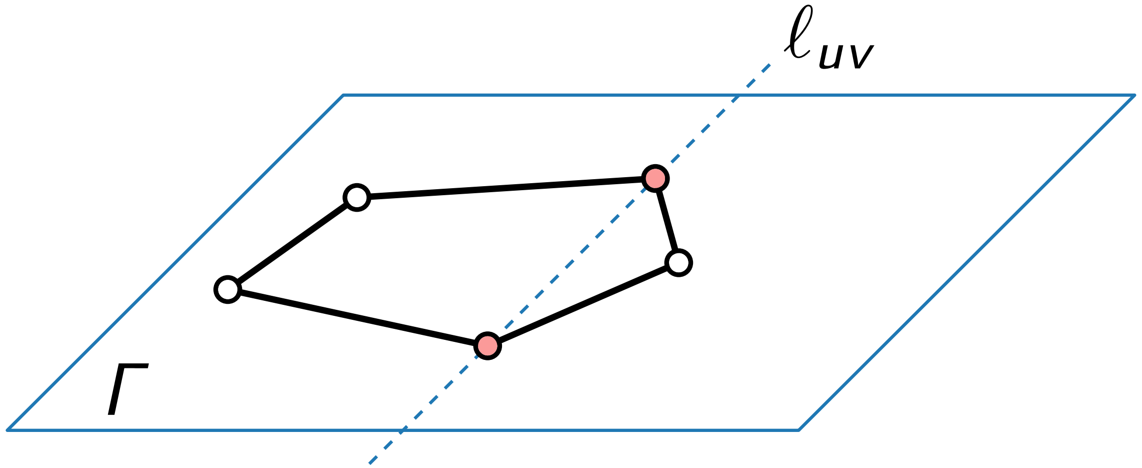

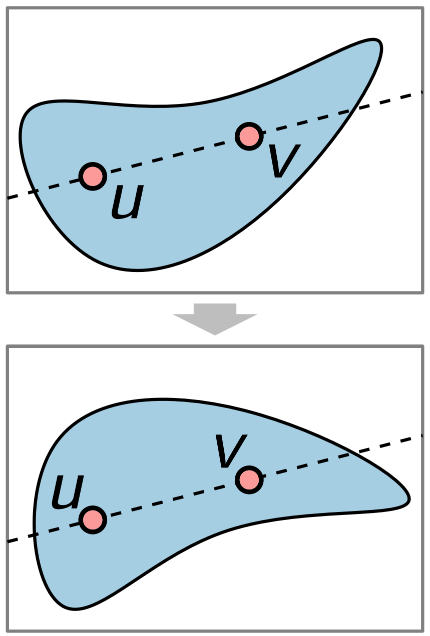

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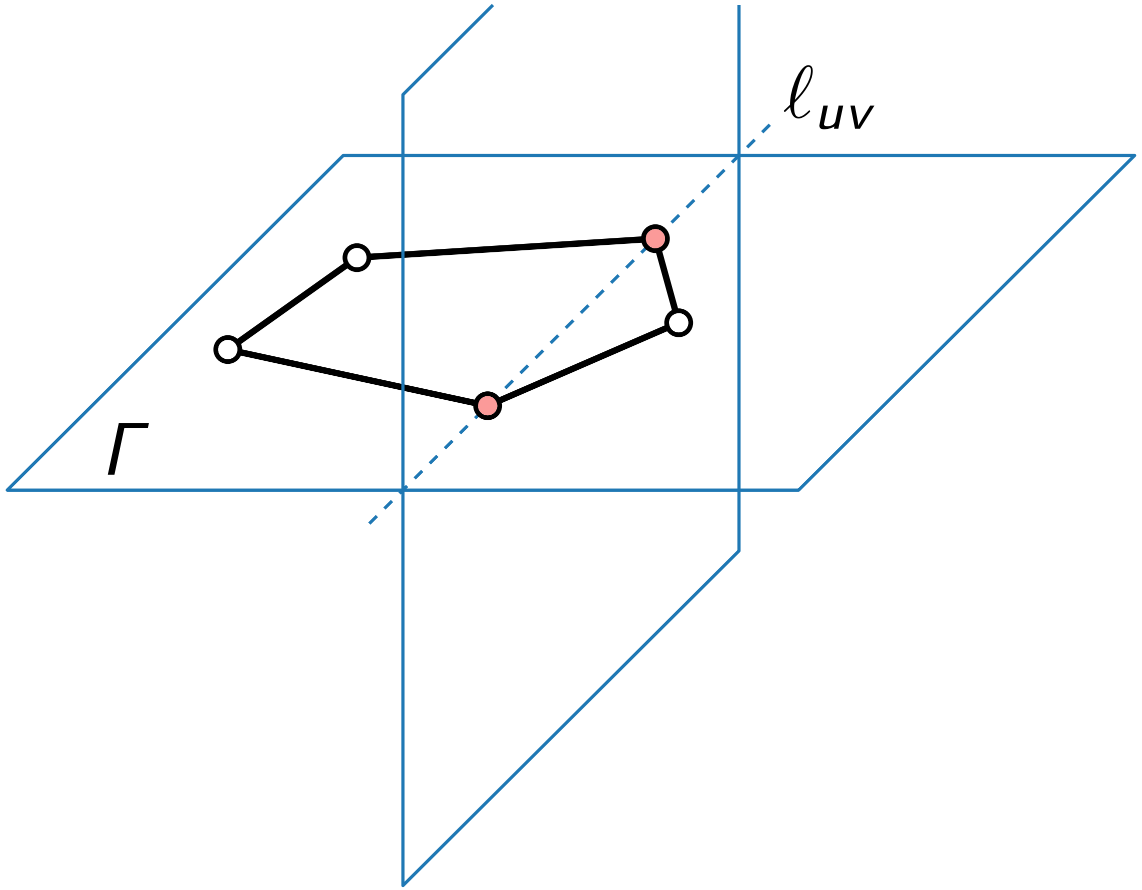

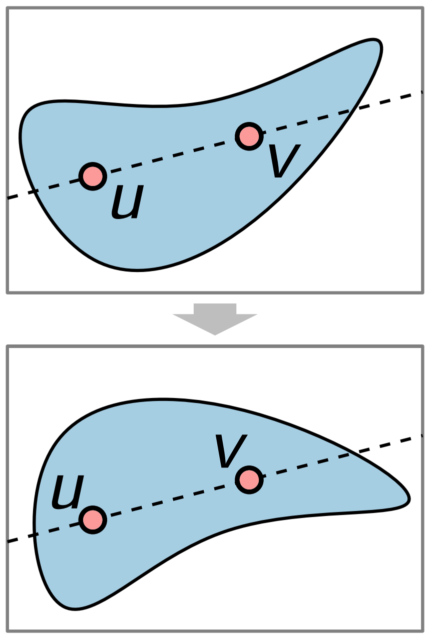

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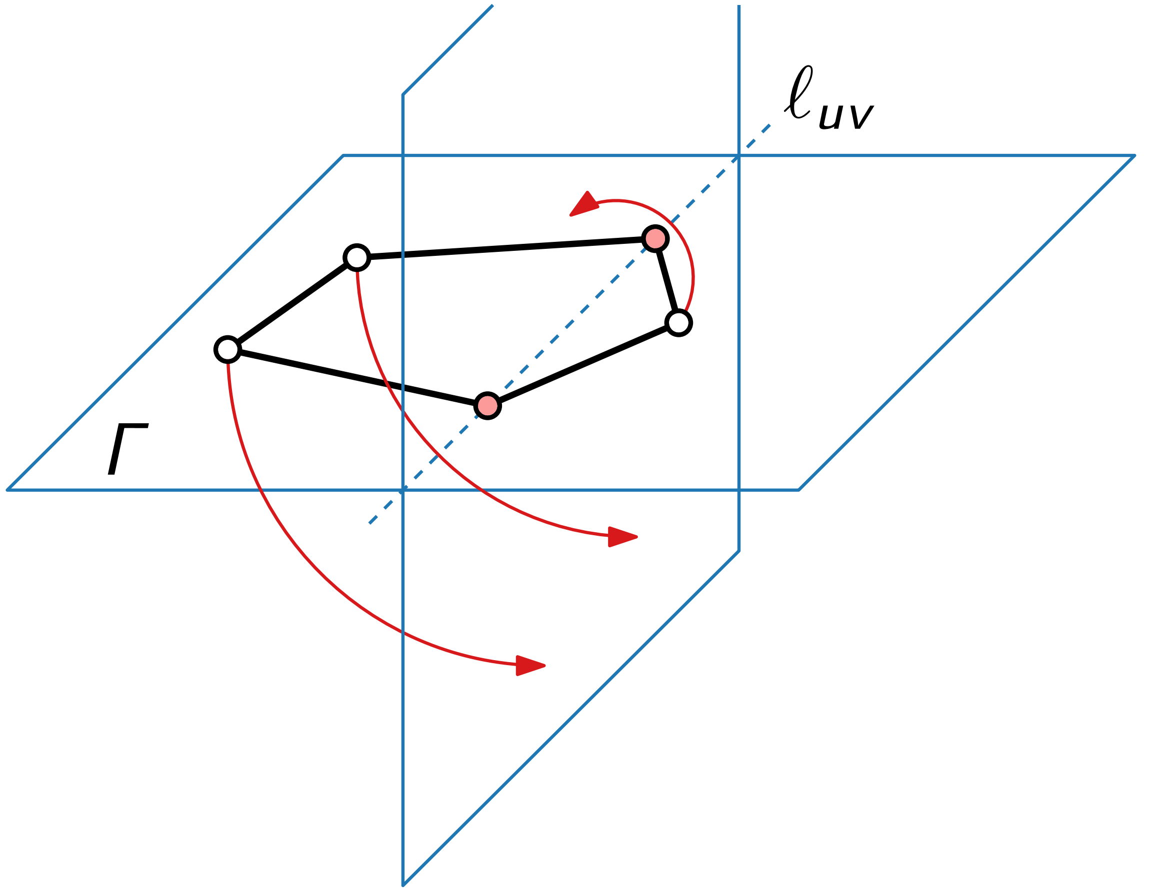

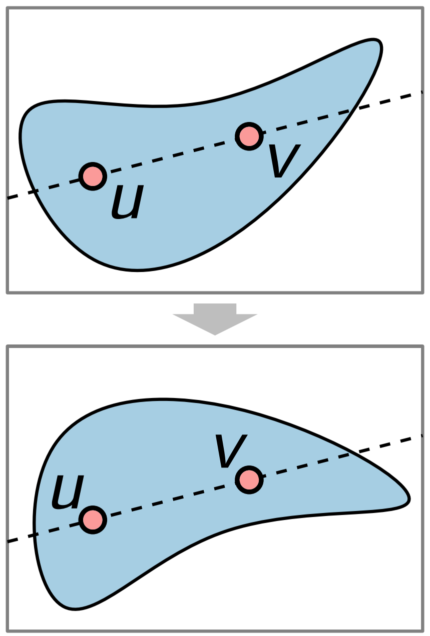

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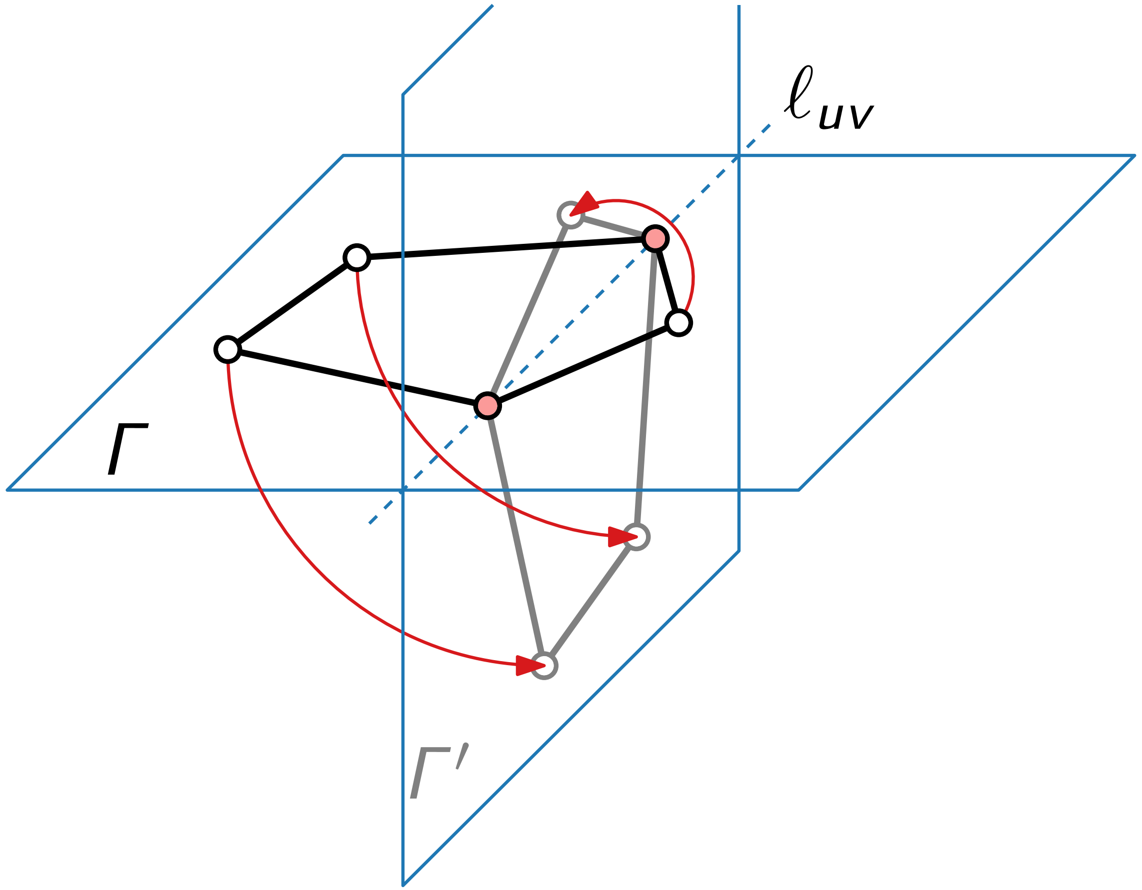

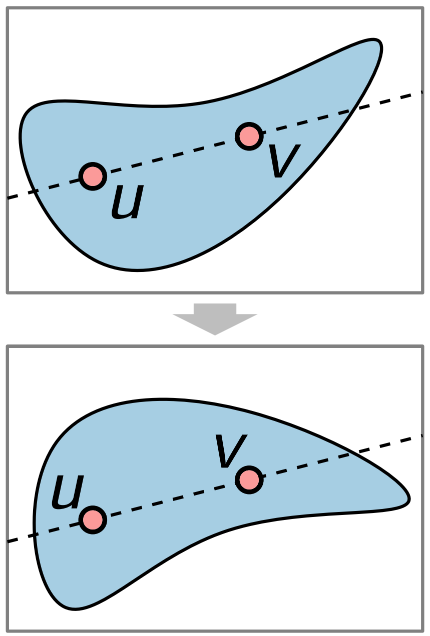

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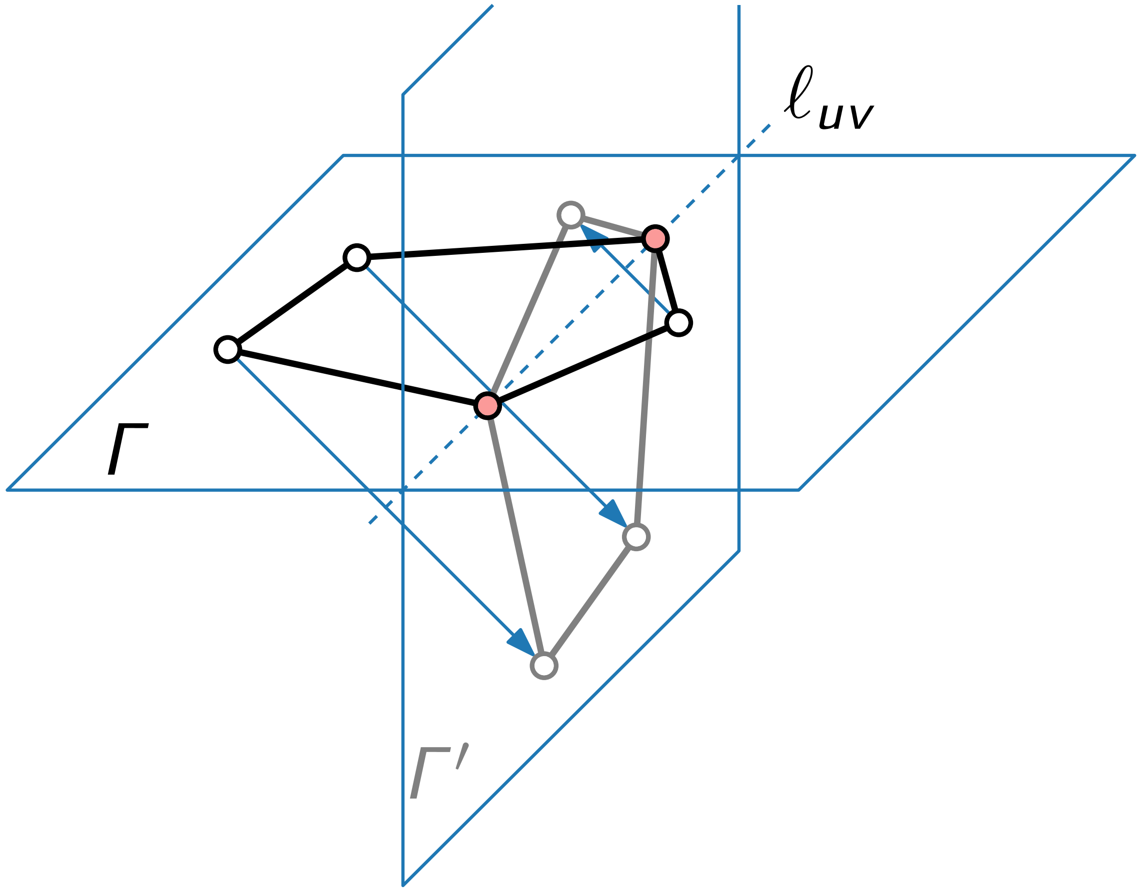

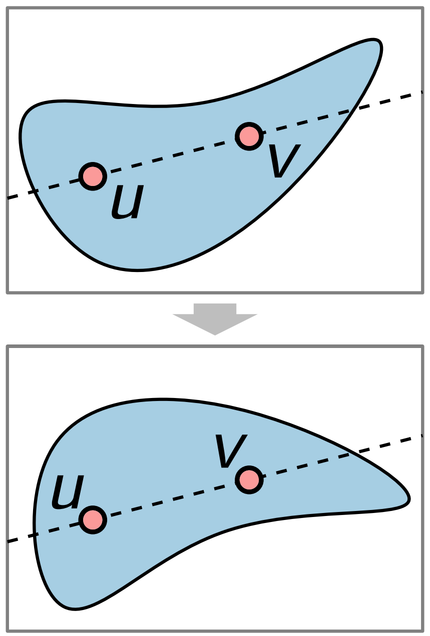

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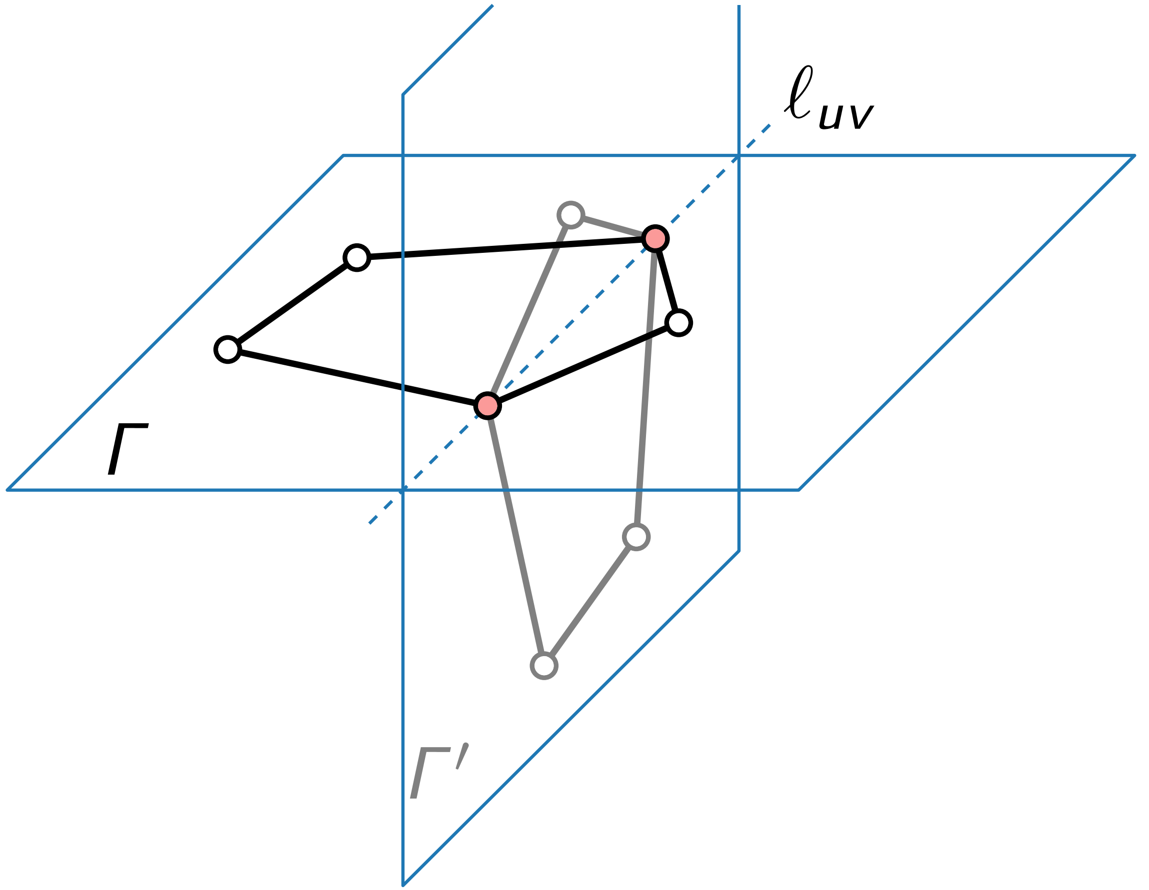

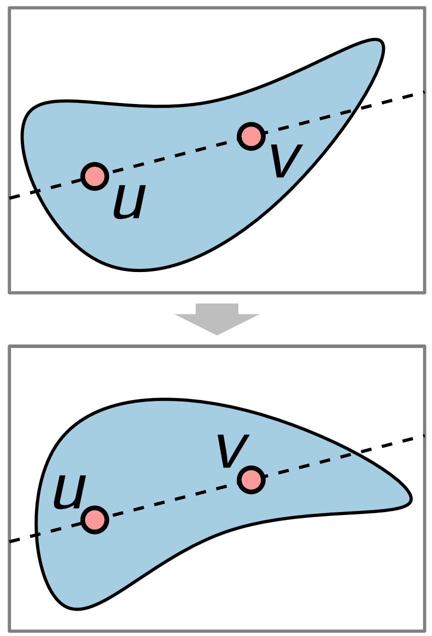

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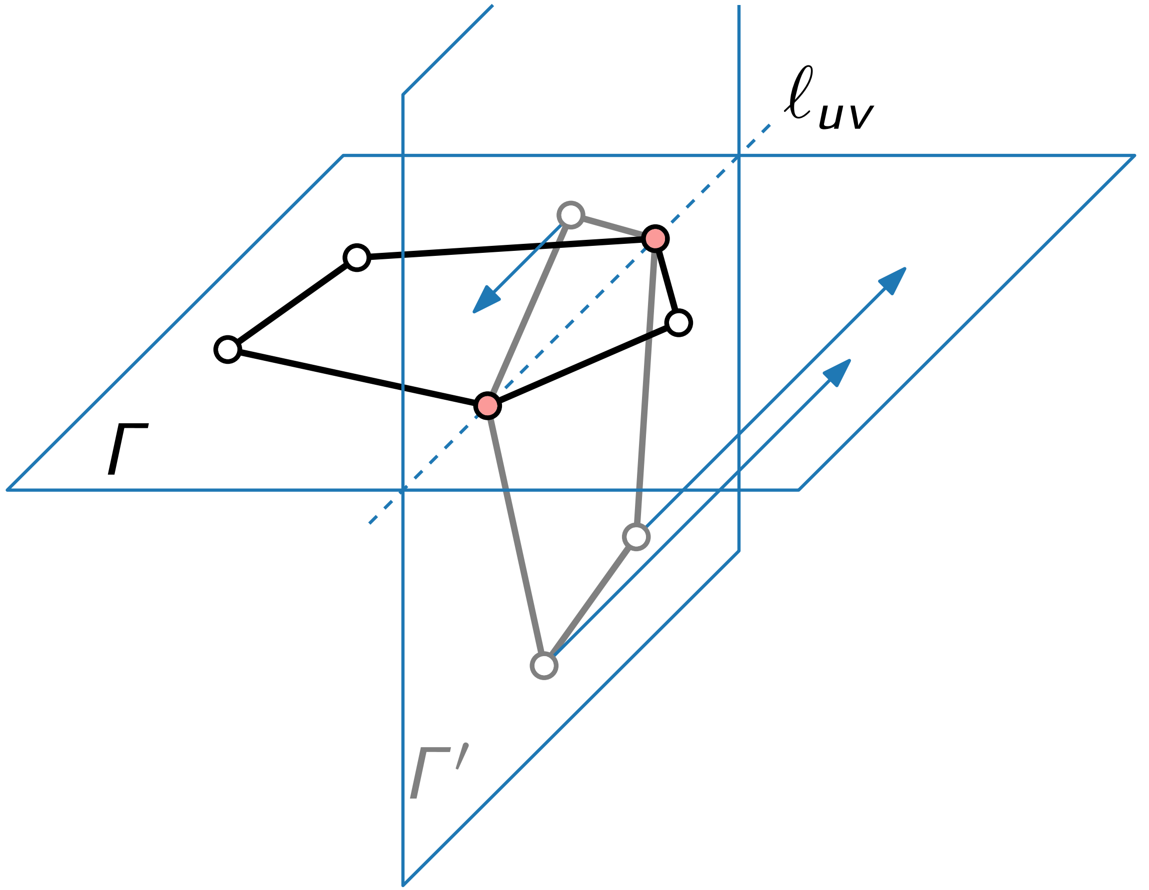

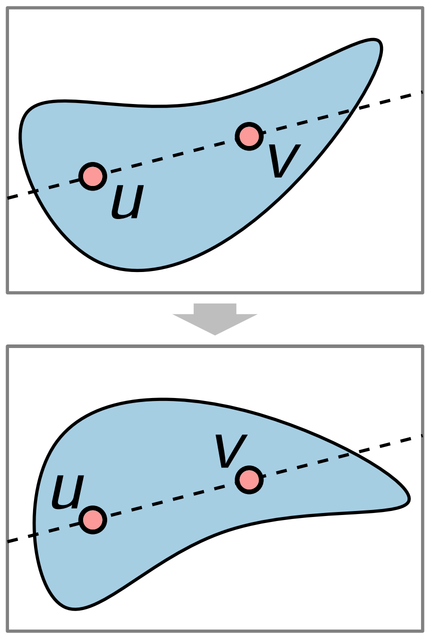

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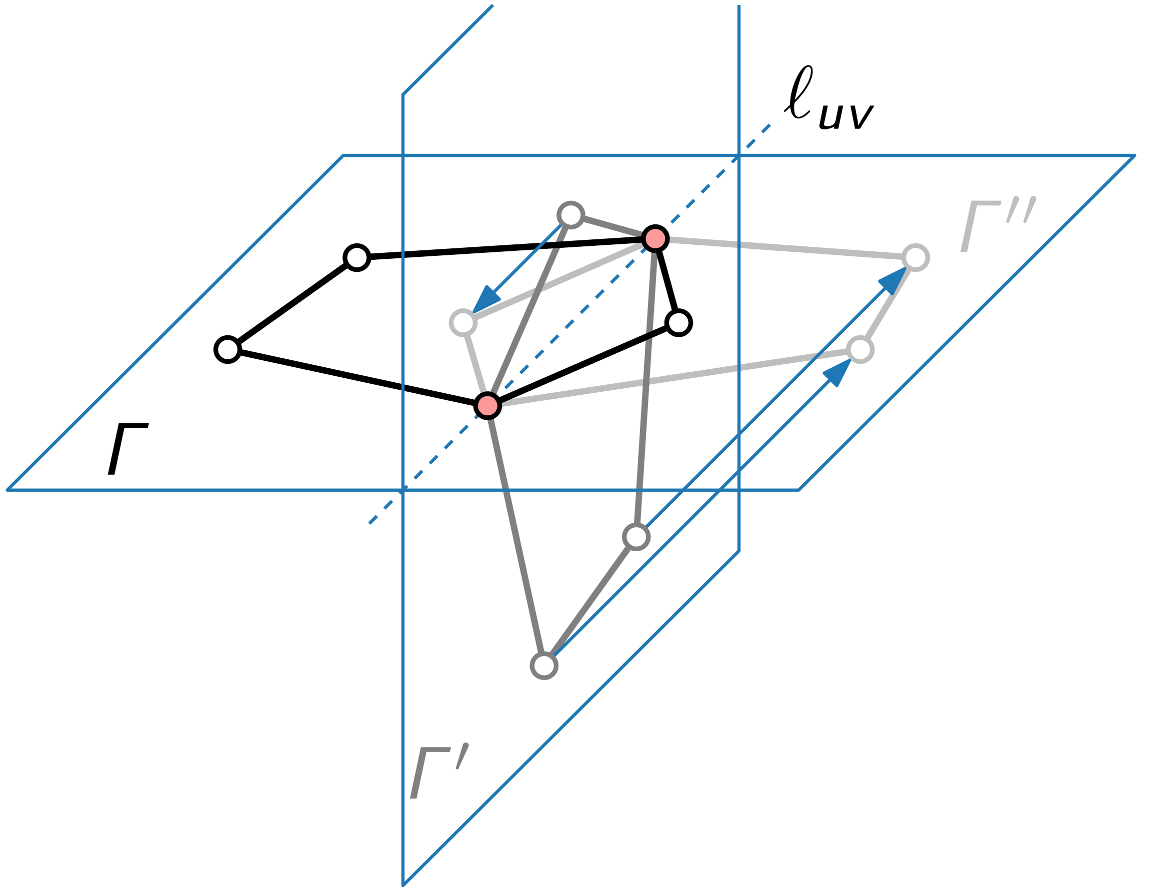

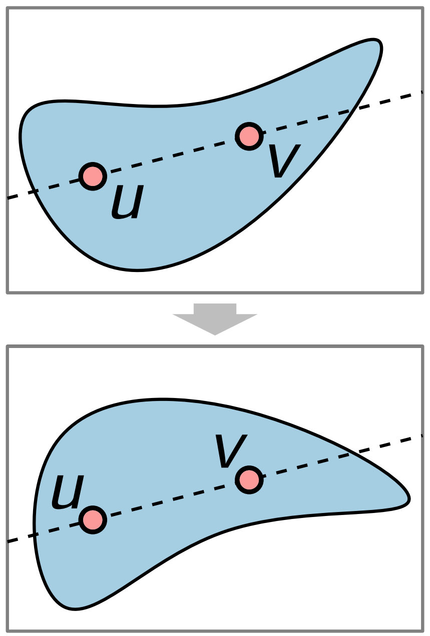

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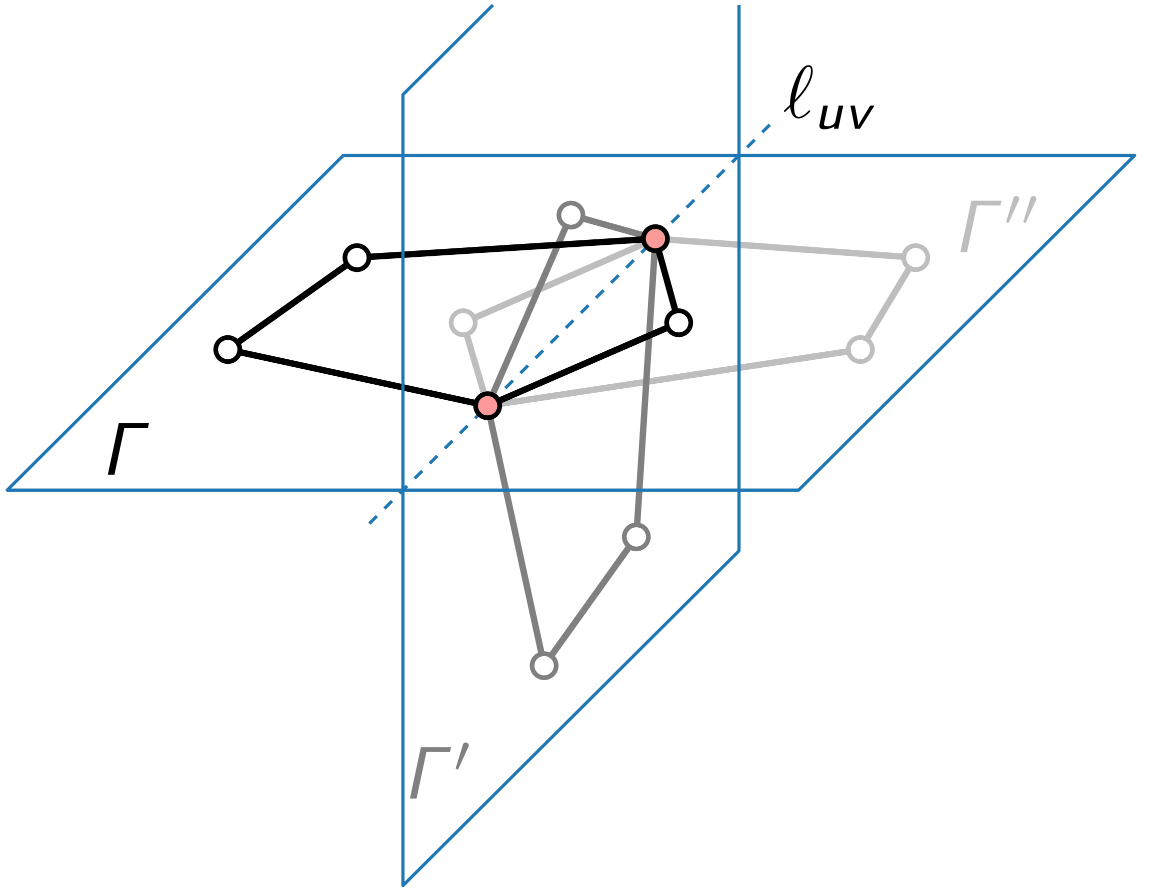

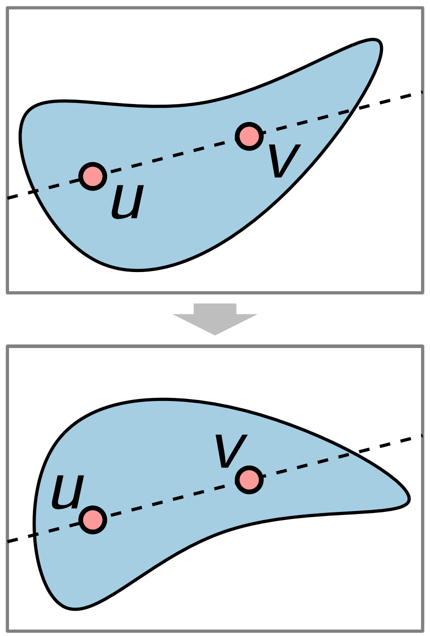

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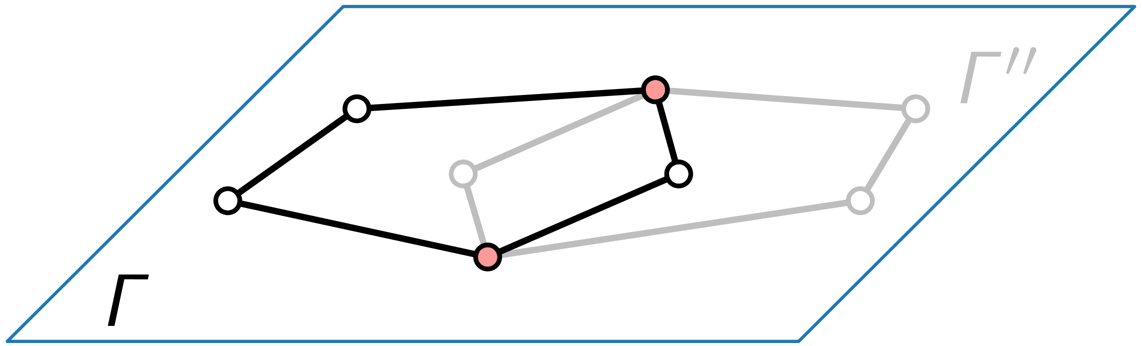

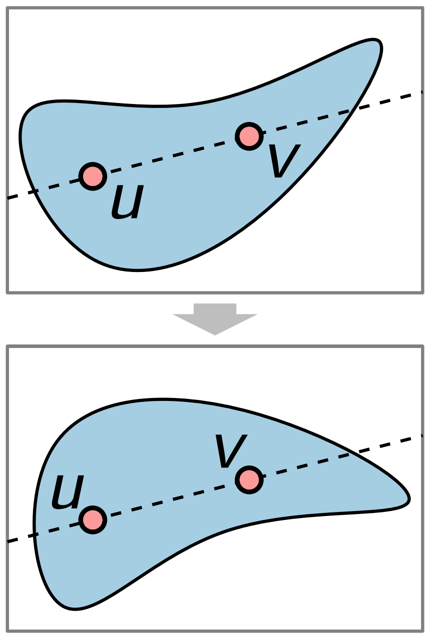

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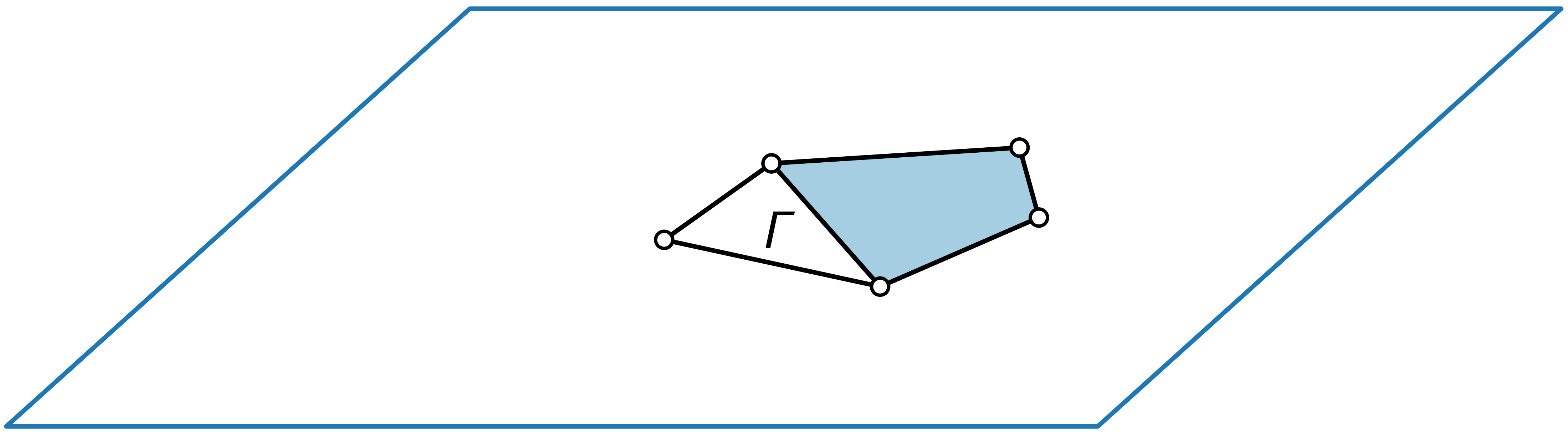

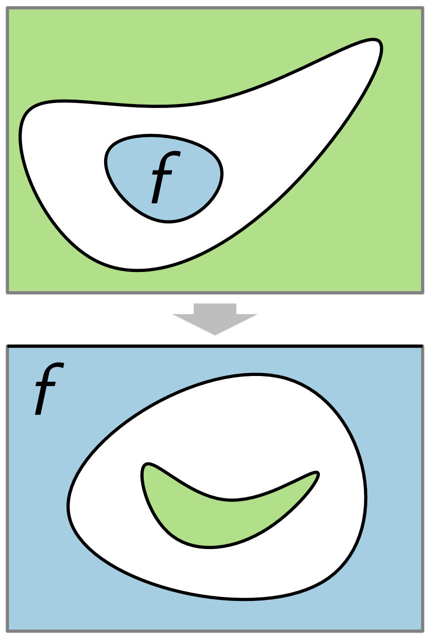

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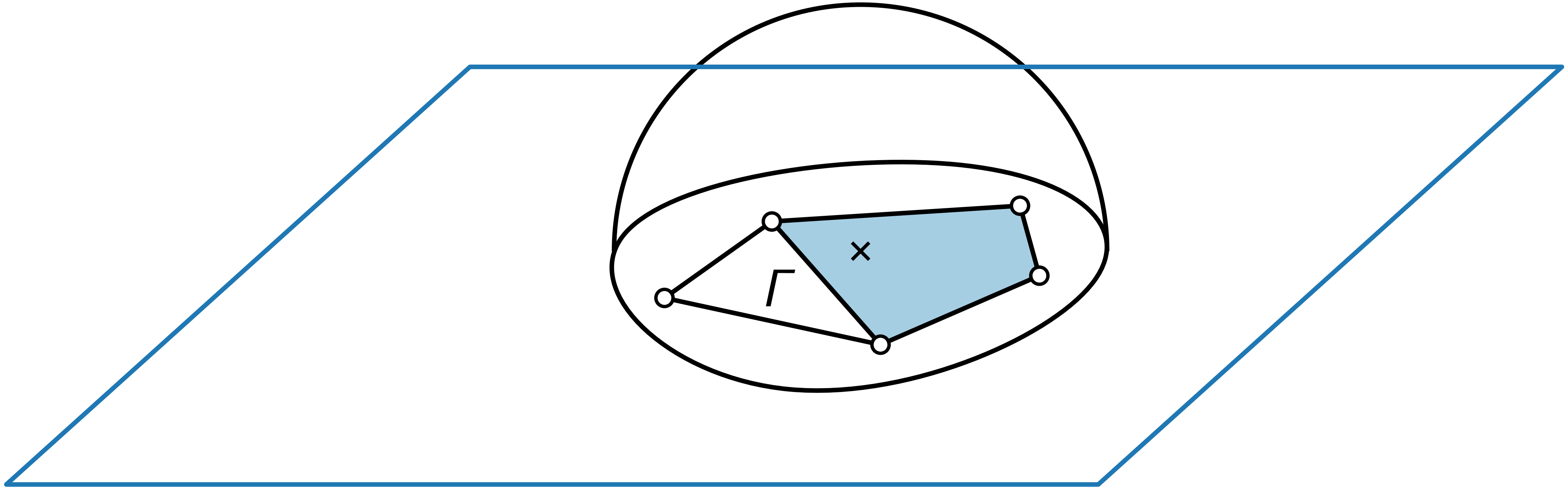

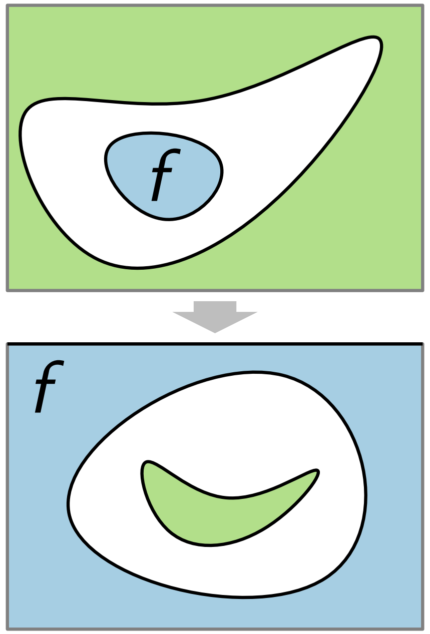

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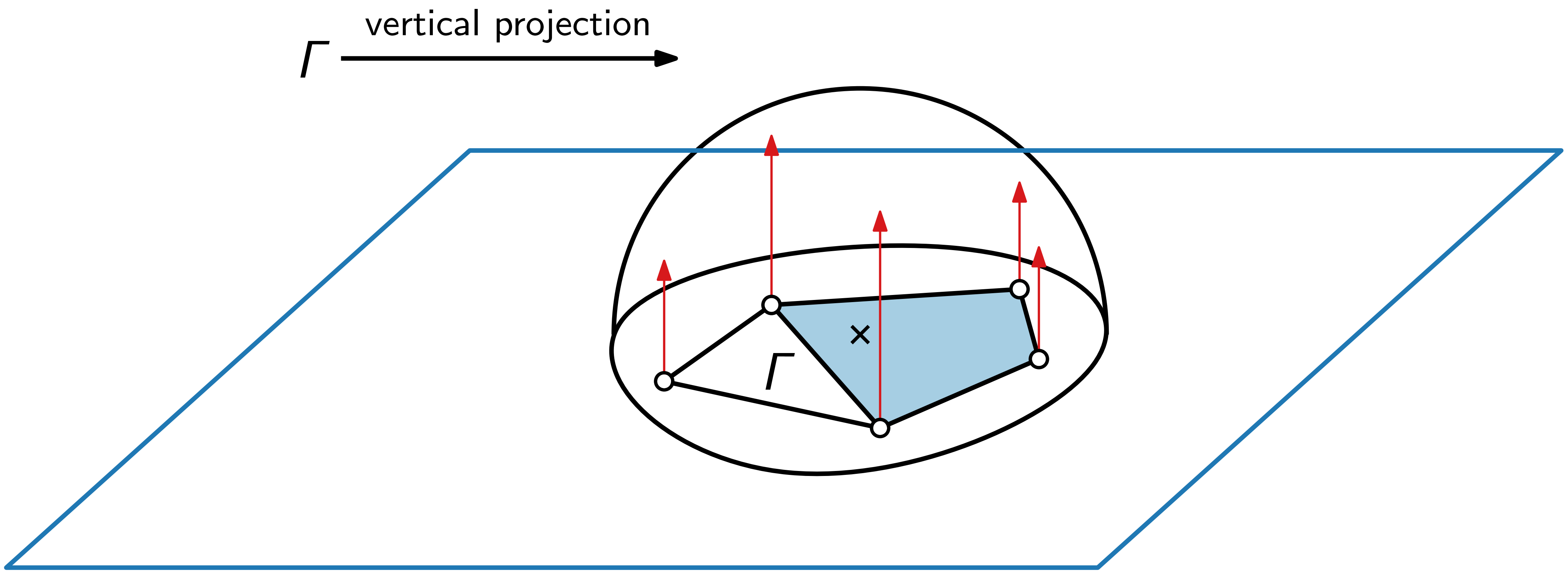

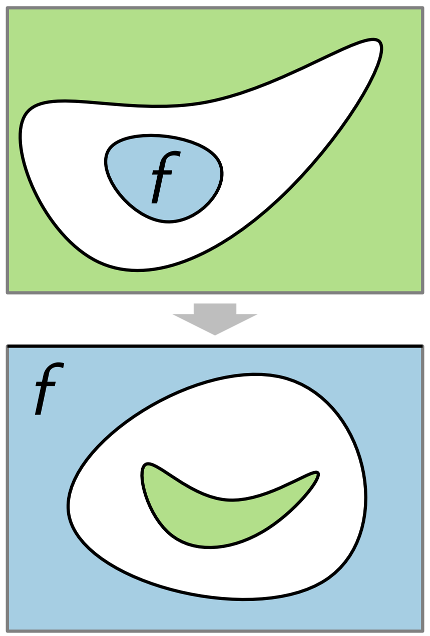

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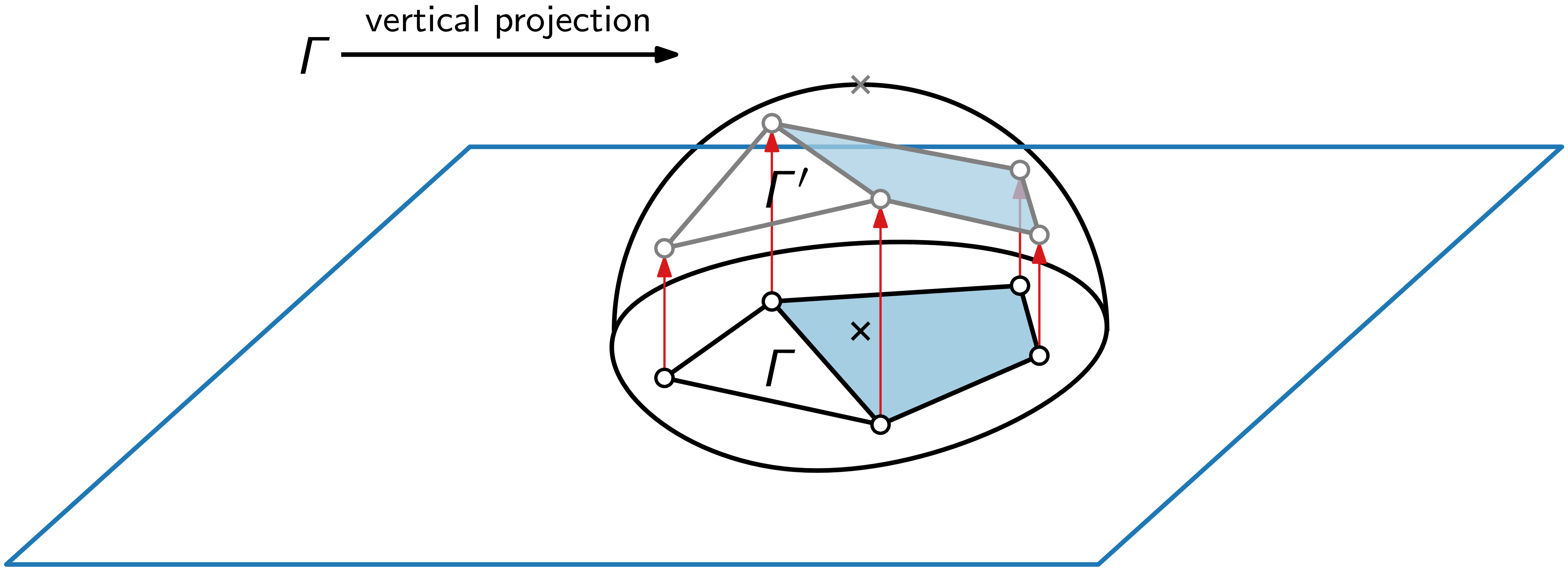

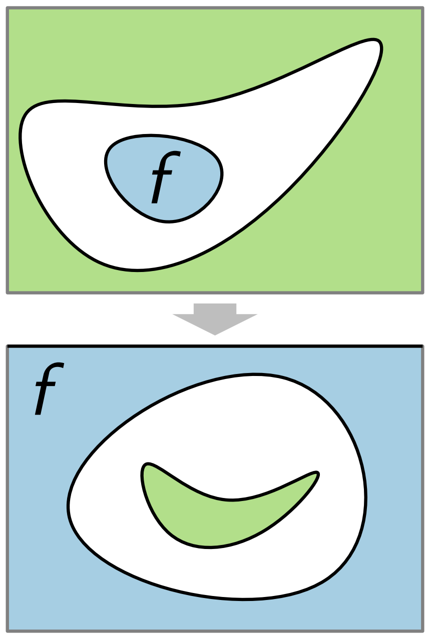

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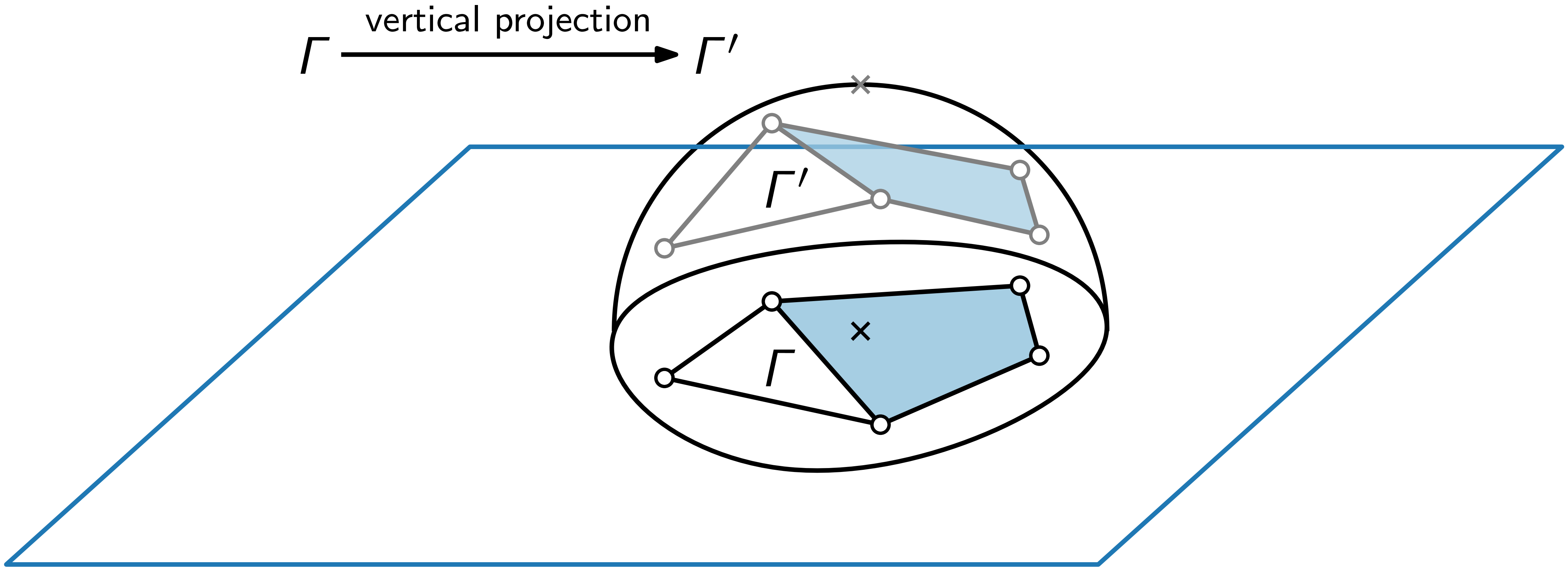

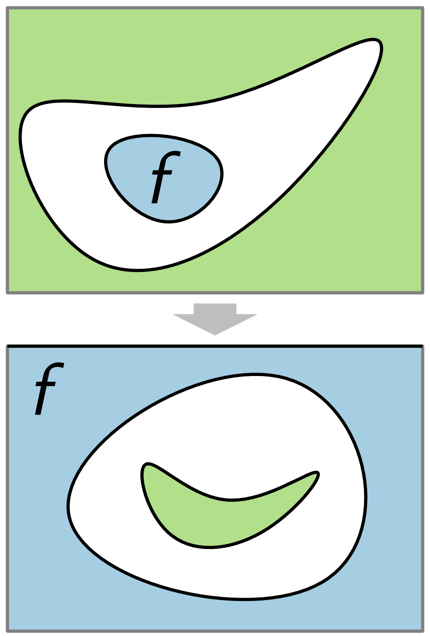

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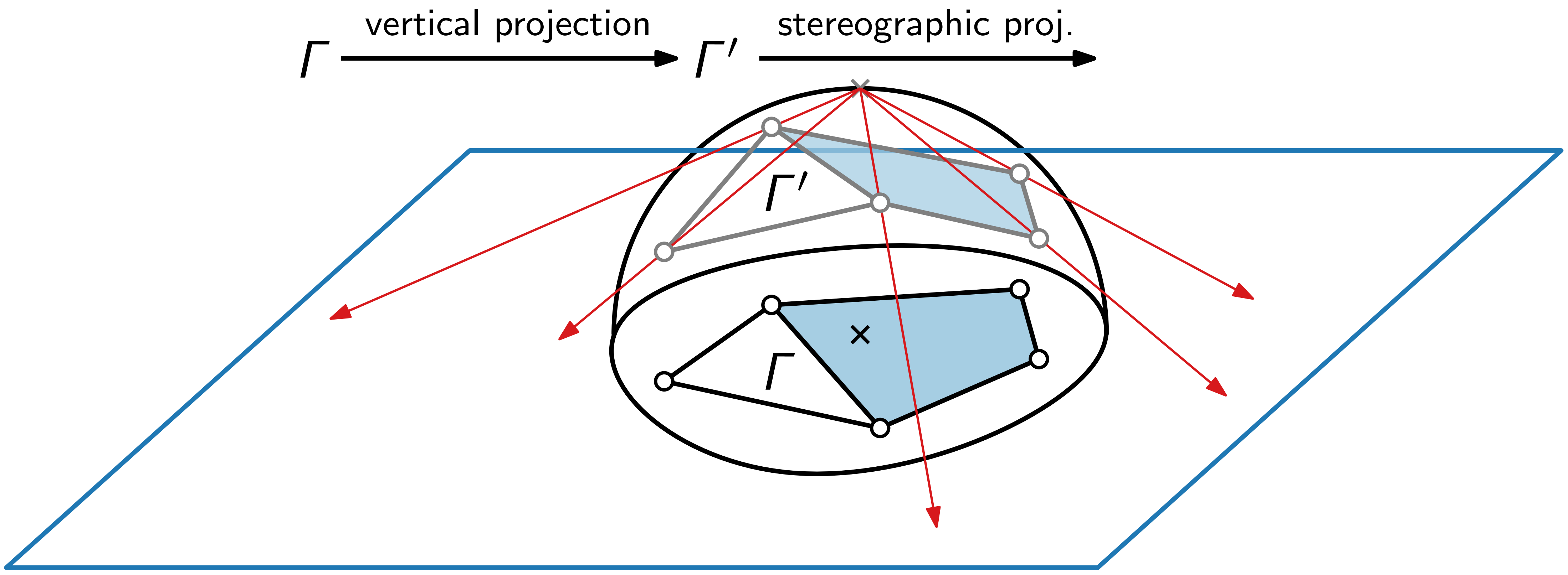

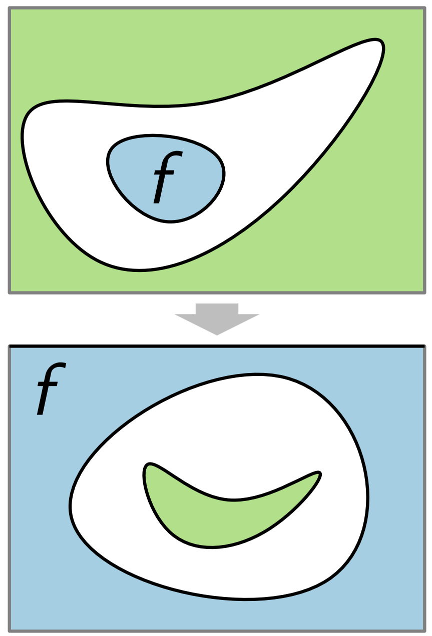

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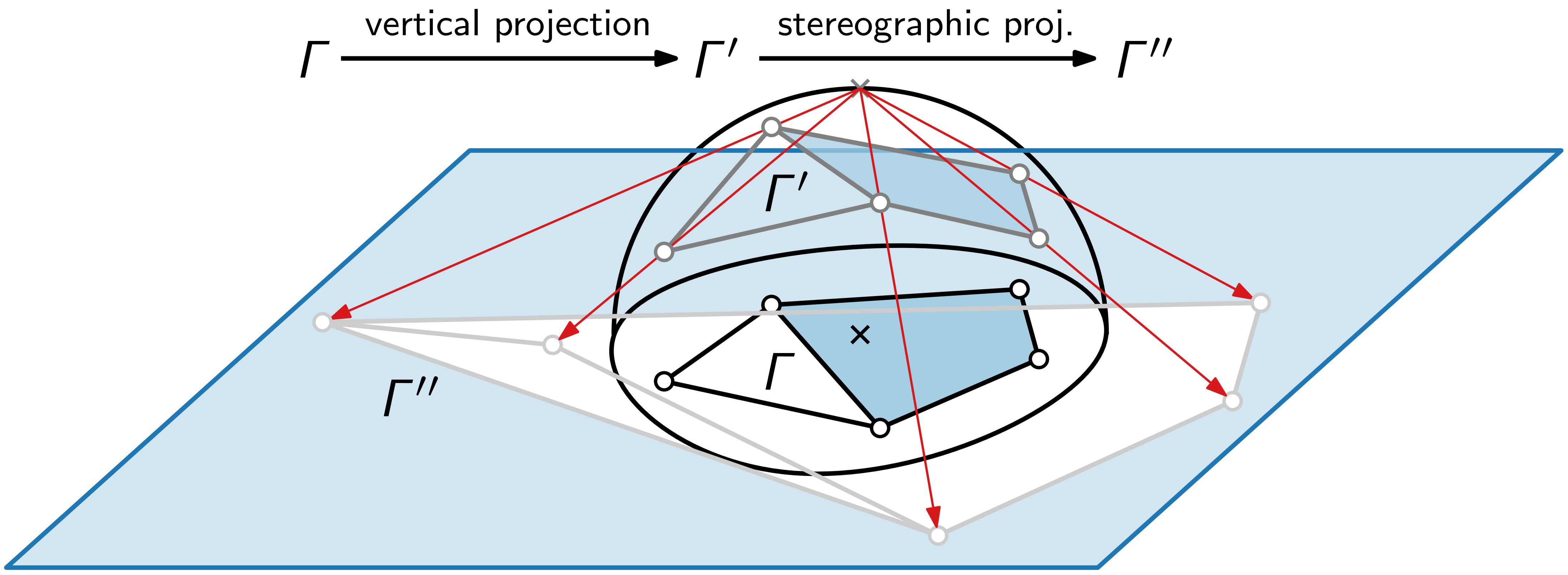

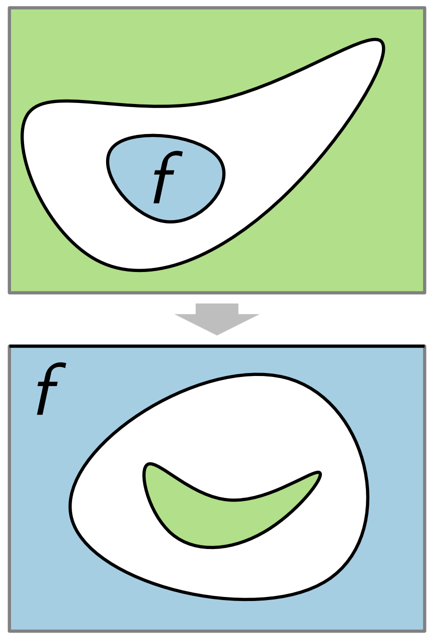

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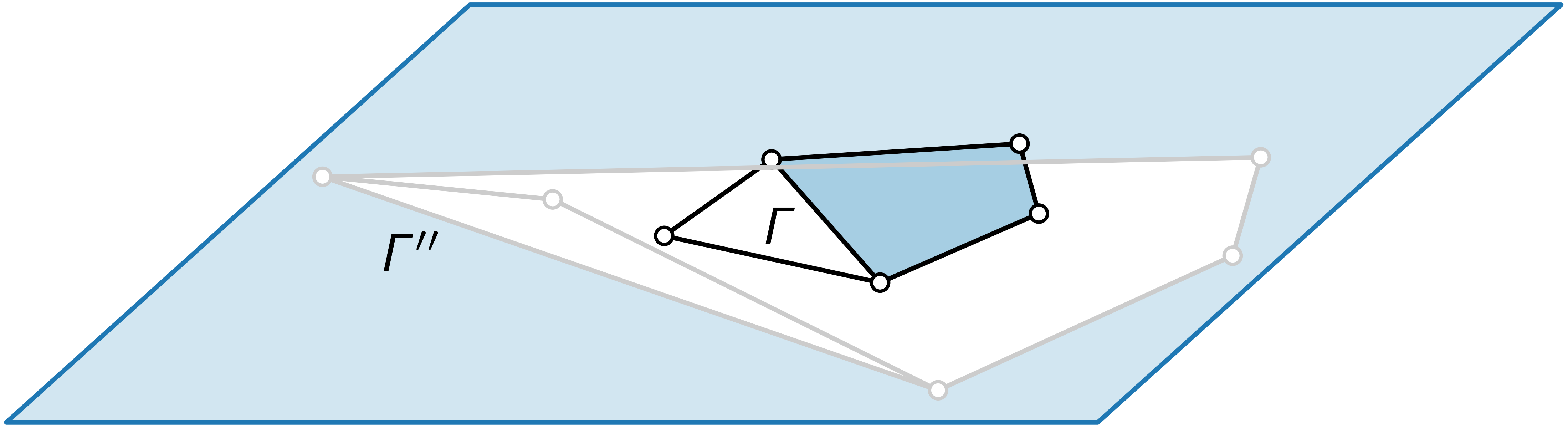
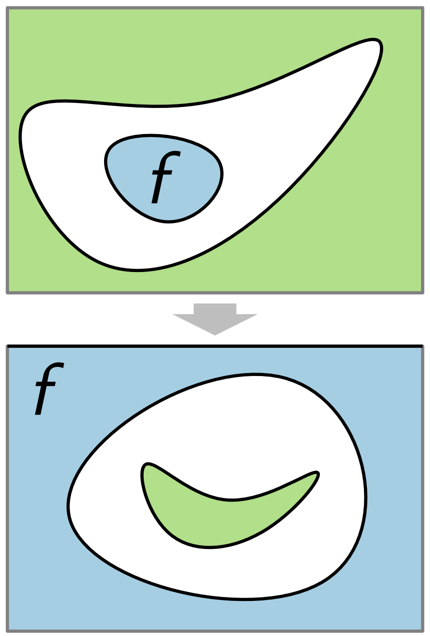
$$\Gamma \xrightarrow{\text{vertical projection}} \Gamma' \xrightarrow{\text{stereographic proj.}} \Gamma''$$





Operation 2: Changing the Outer Face

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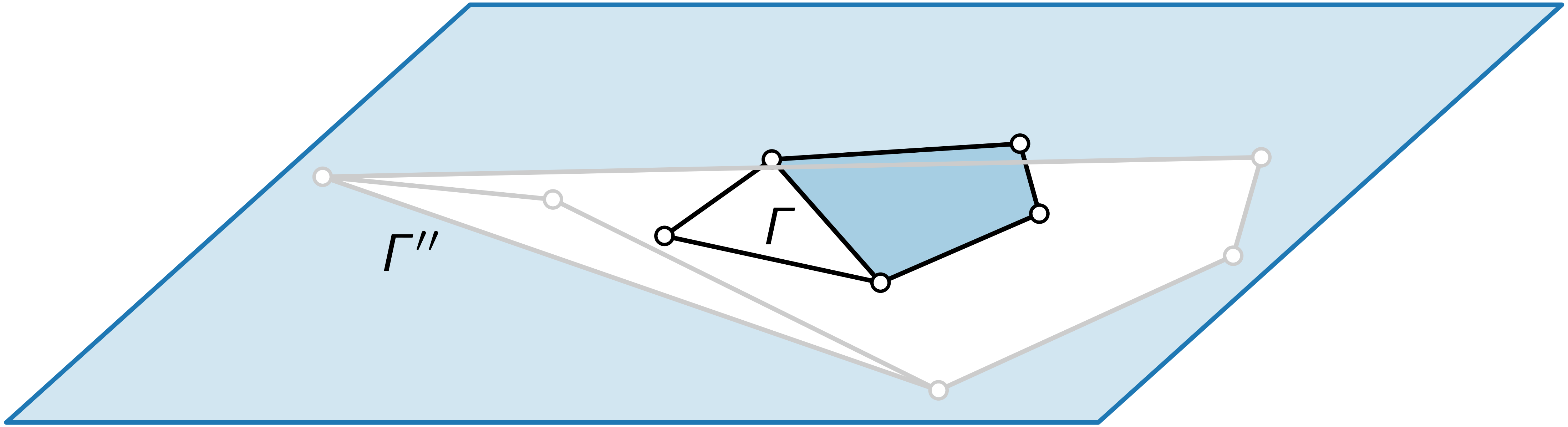
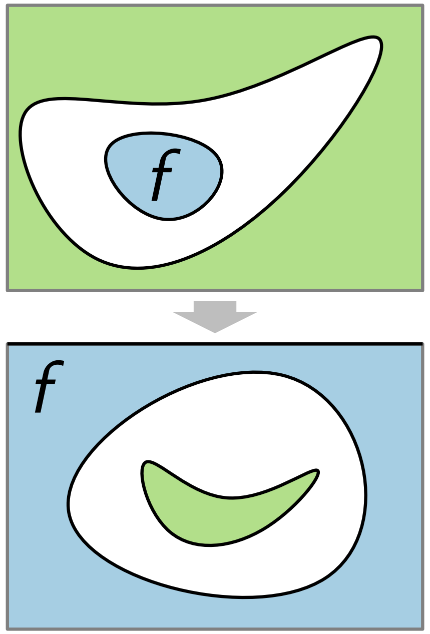
$$\Gamma \xrightarrow{\text{vertical projection}} \Gamma' \xrightarrow{\text{stereographic proj.}} \Gamma'' \longrightarrow \Gamma'''$$





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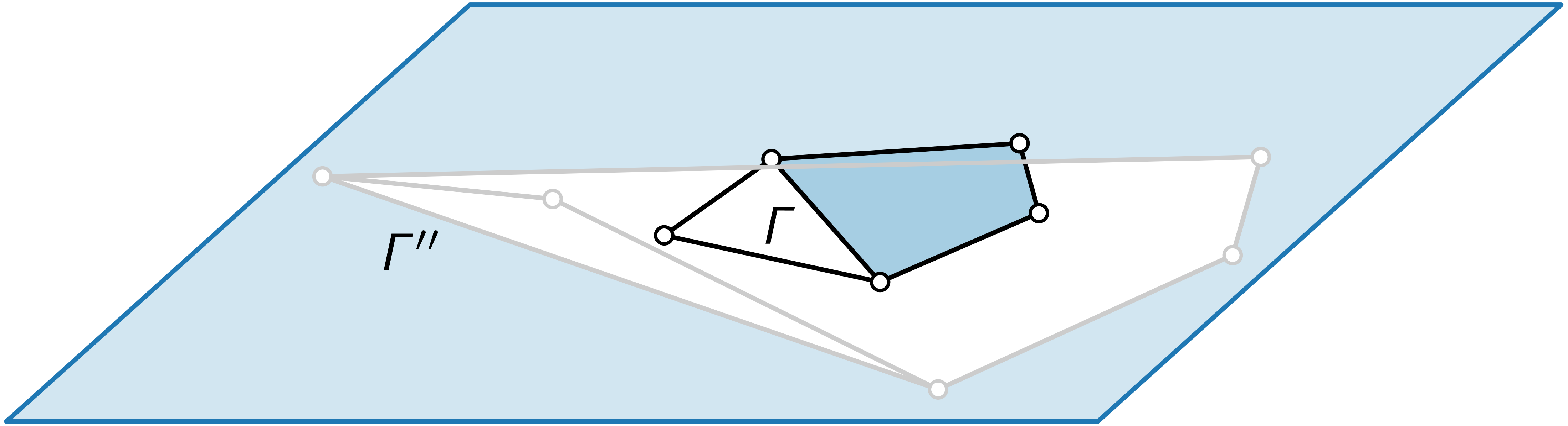

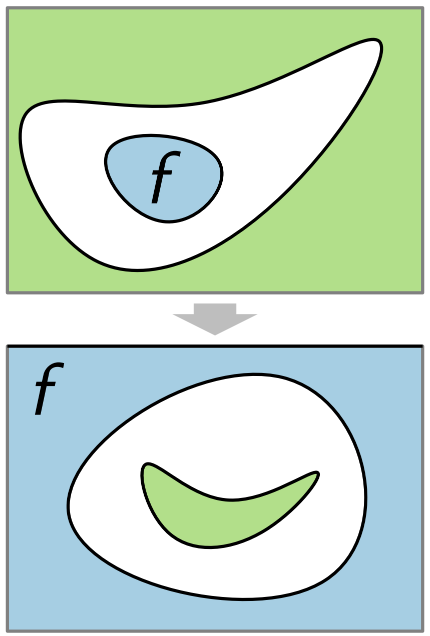

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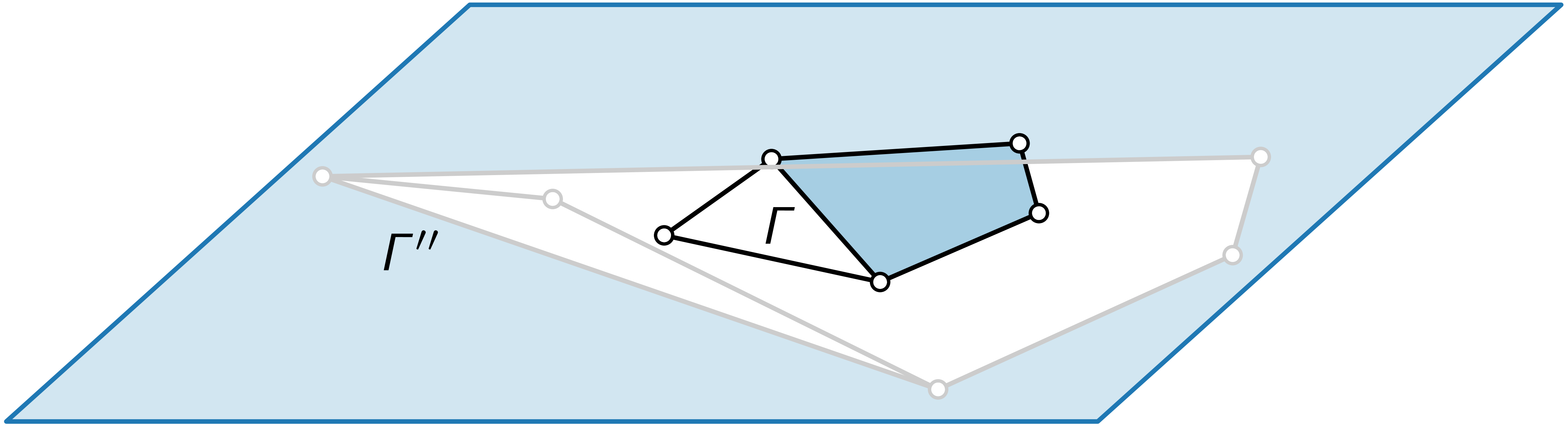
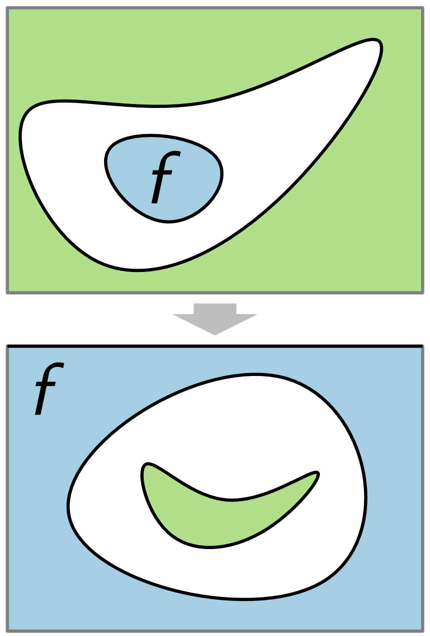
$$\Gamma \xrightarrow[\text{1 step}]{\text{vertical projection}} \Gamma' \xrightarrow[\text{1 step}]{\text{stereographic proj.}} \Gamma'' \xrightarrow[\text{2 steps}]{\text{flip}} \Gamma'''$$



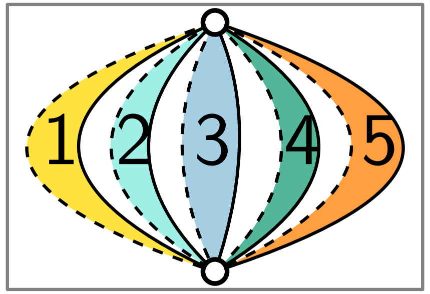


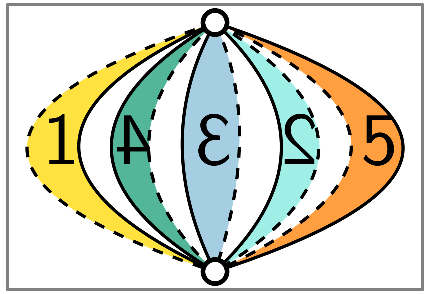
Operation 3: Flipping Split Components

Lemma 3.

- Let $\{u, v\}$ be a split pair. Let Γ be a planar straight-line drawing of G where u and v are incident to the outer face.
- Let $G_1, \ldots, G_k$ be the split components w.r.t. $\{u, v\}$ in cw order around u s.t. G_1 and G_k are incident to the outer face.
- Let $1 \leq i \leq j \leq k$. If $uv \in E(G)$, then uv is one of $G_i, \ldots, G_j$.

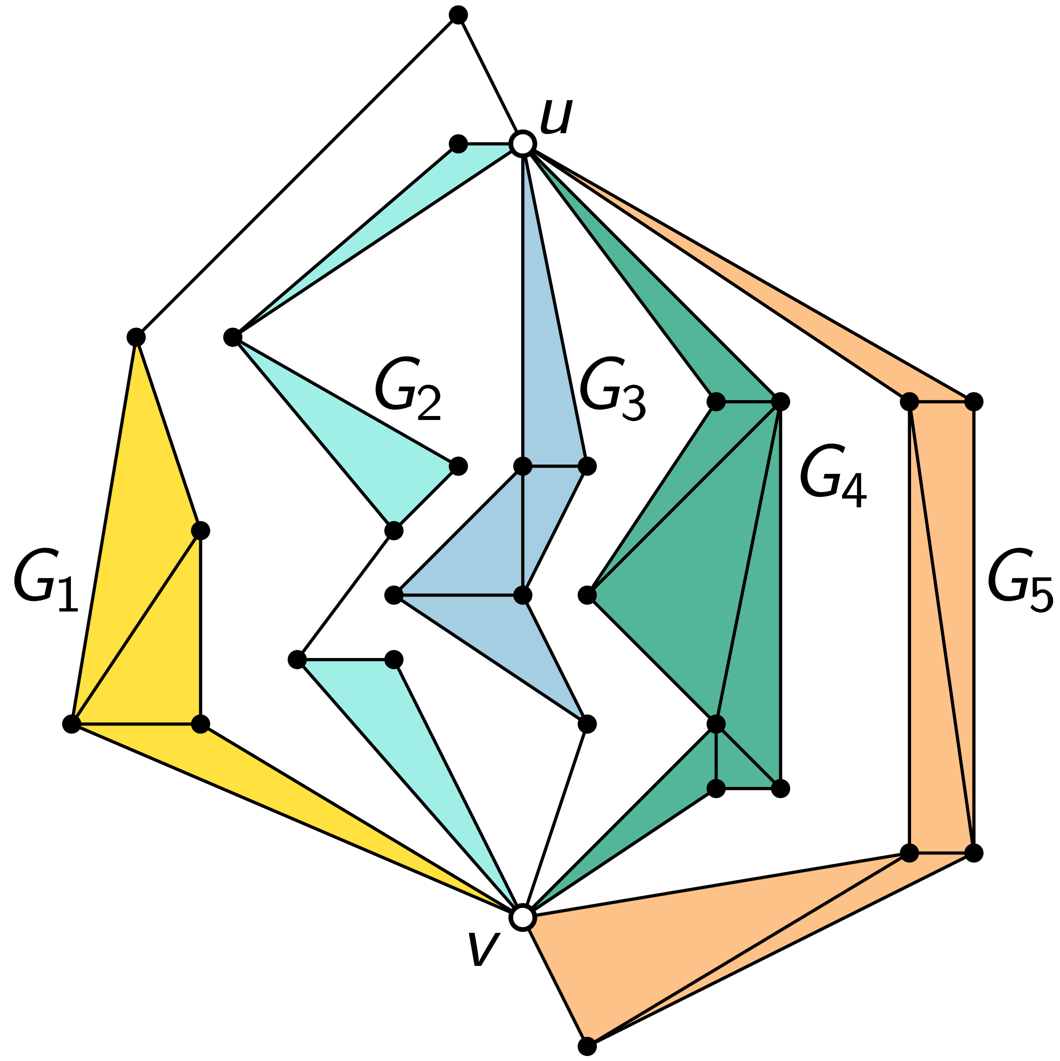
There exists an $O(n)$-step 3D crossing-free morph from Γ to a planar straight-line drawing Γ' of G in which exactly $G_i, \ldots, G_j$ are flipped. In Γ', the cw order around u is G_1, ..., G_{i-1}, G_j, G_{j-1}, ..., G_i, G_{j+1}, ..., G_k.





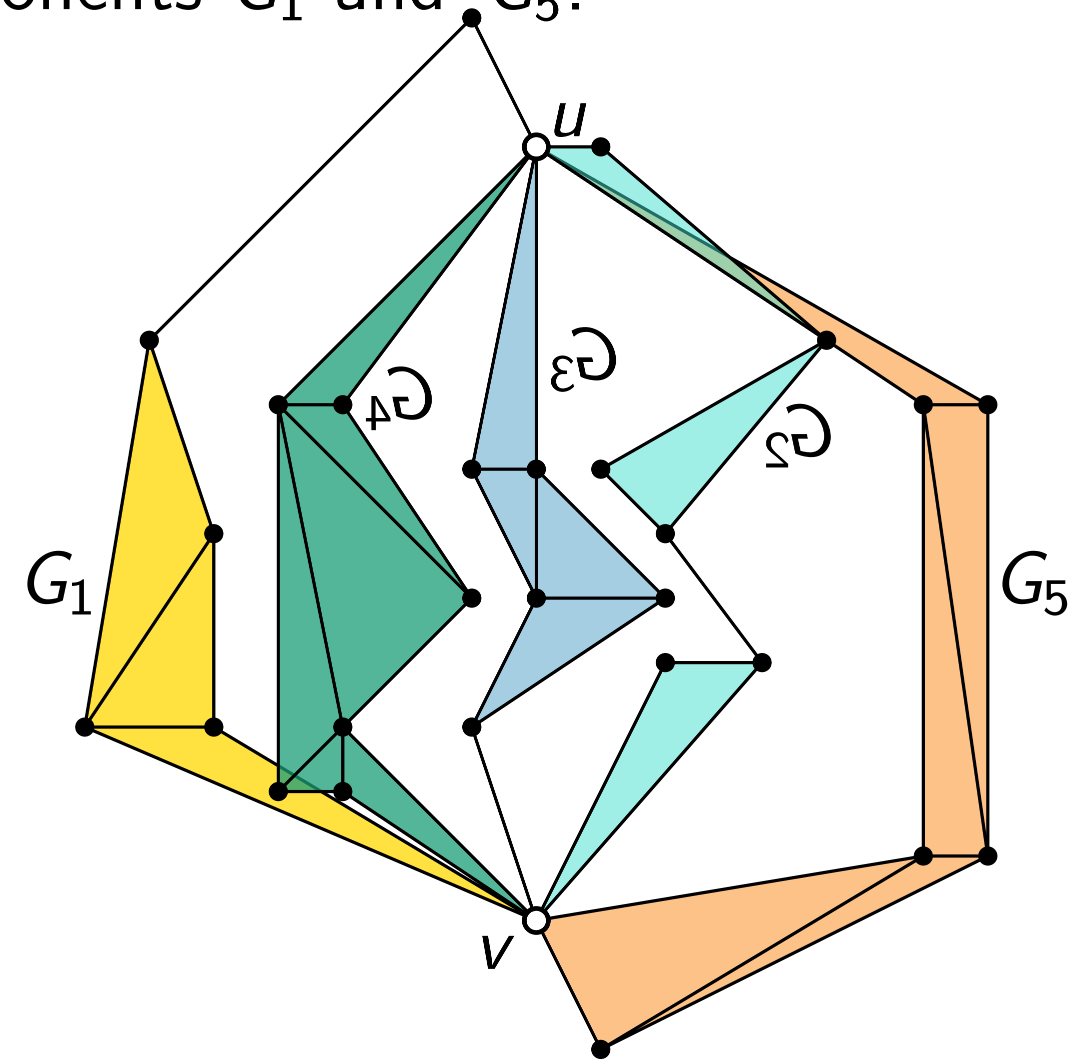
Algorithm for Operation 3

- We want to flip split components G_2, G_3, and G_4.



Algorithm for Operation 3

- We want to flip split components G_2, G_3, and G_4.
- But they may intersect split components G_1 and G_5.



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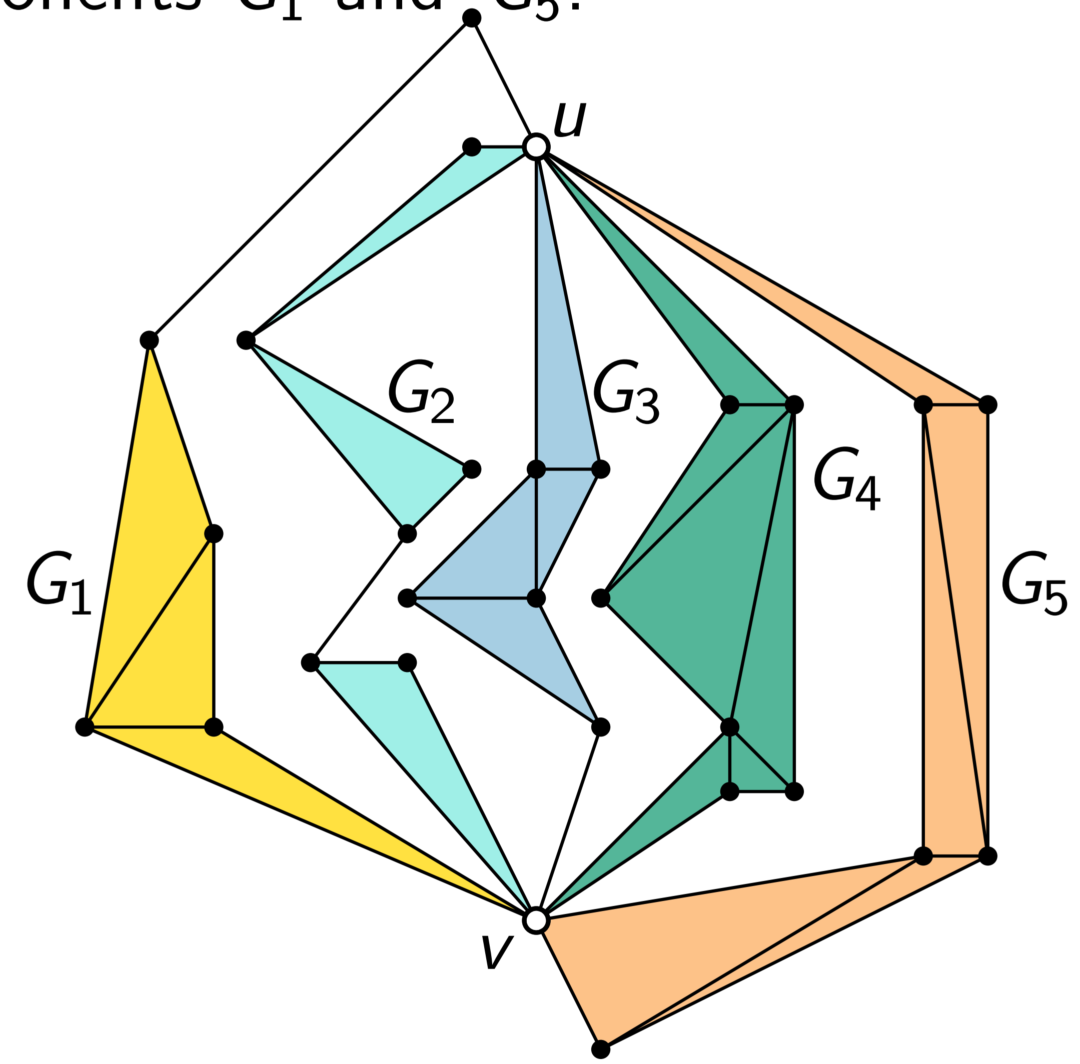
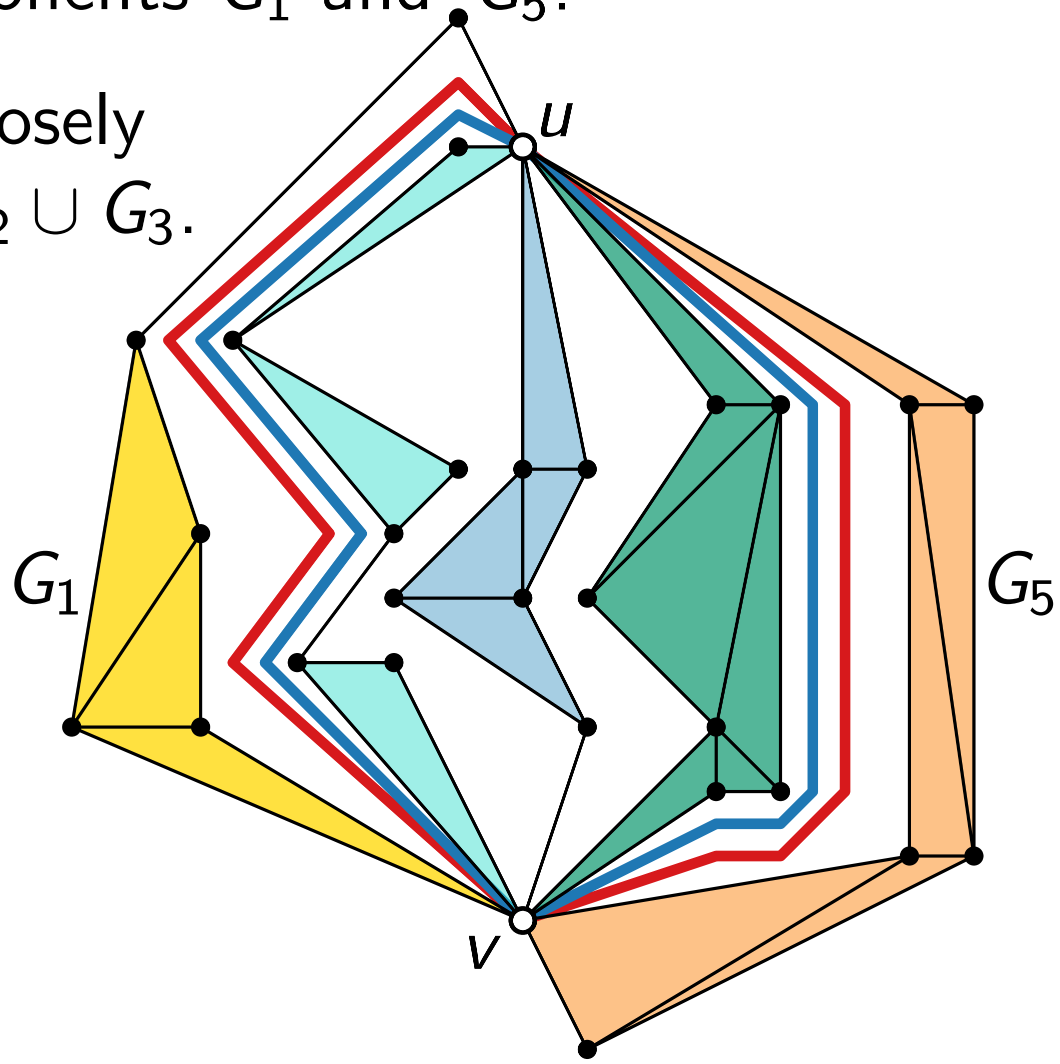


Algorithm for Operation 3

- We want to flip split components G_2, G_3, and G_4.
- But they may intersect split components G_1 and G_5.
- Insert a blue and a red polygon, closely following the outer face of $G_1 \cup G_2 \cup G_3$.



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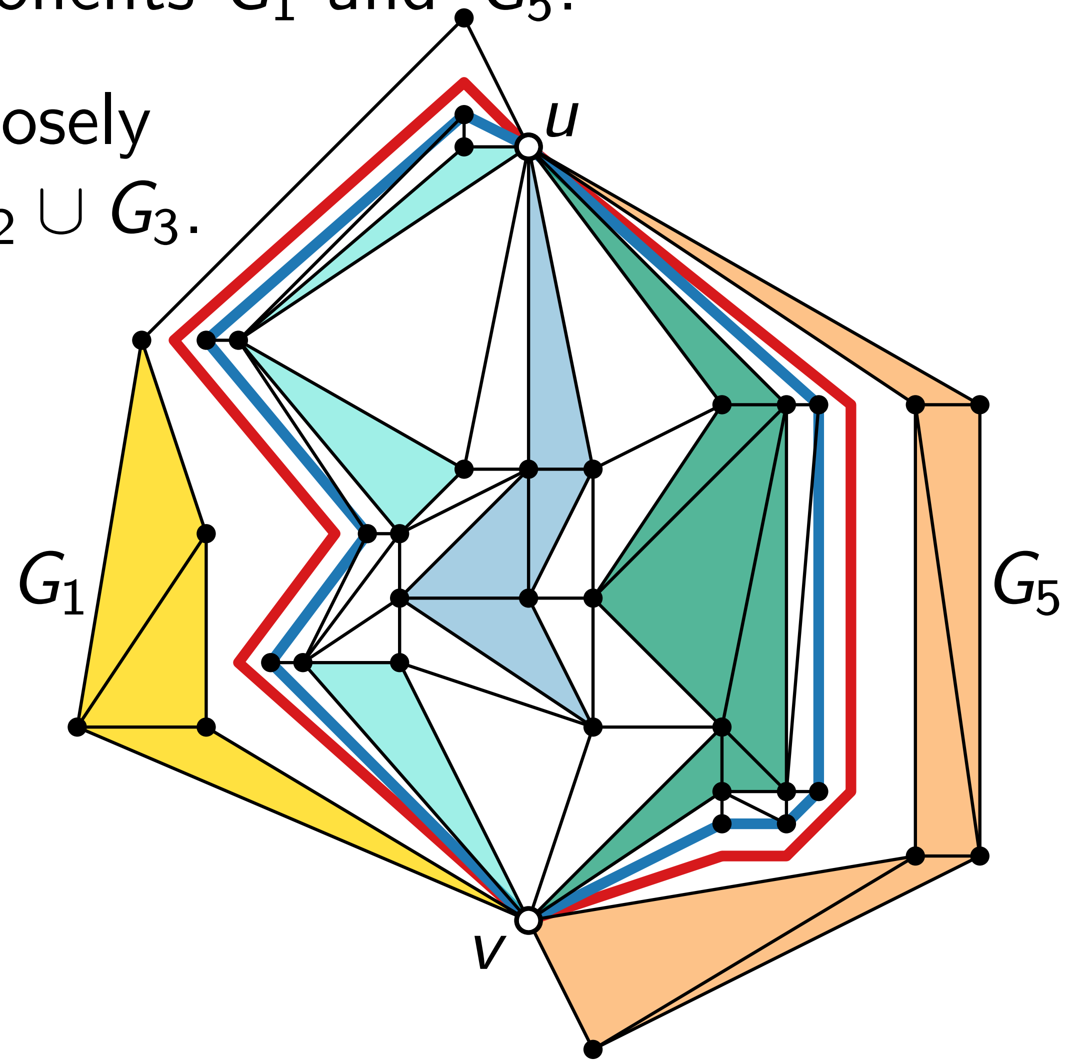
- Insert a blue and a red polygon, closely following the outer face of $G_1 \cup G_2 \cup G_3$.
- Triangulate everything inside the blue polygon.



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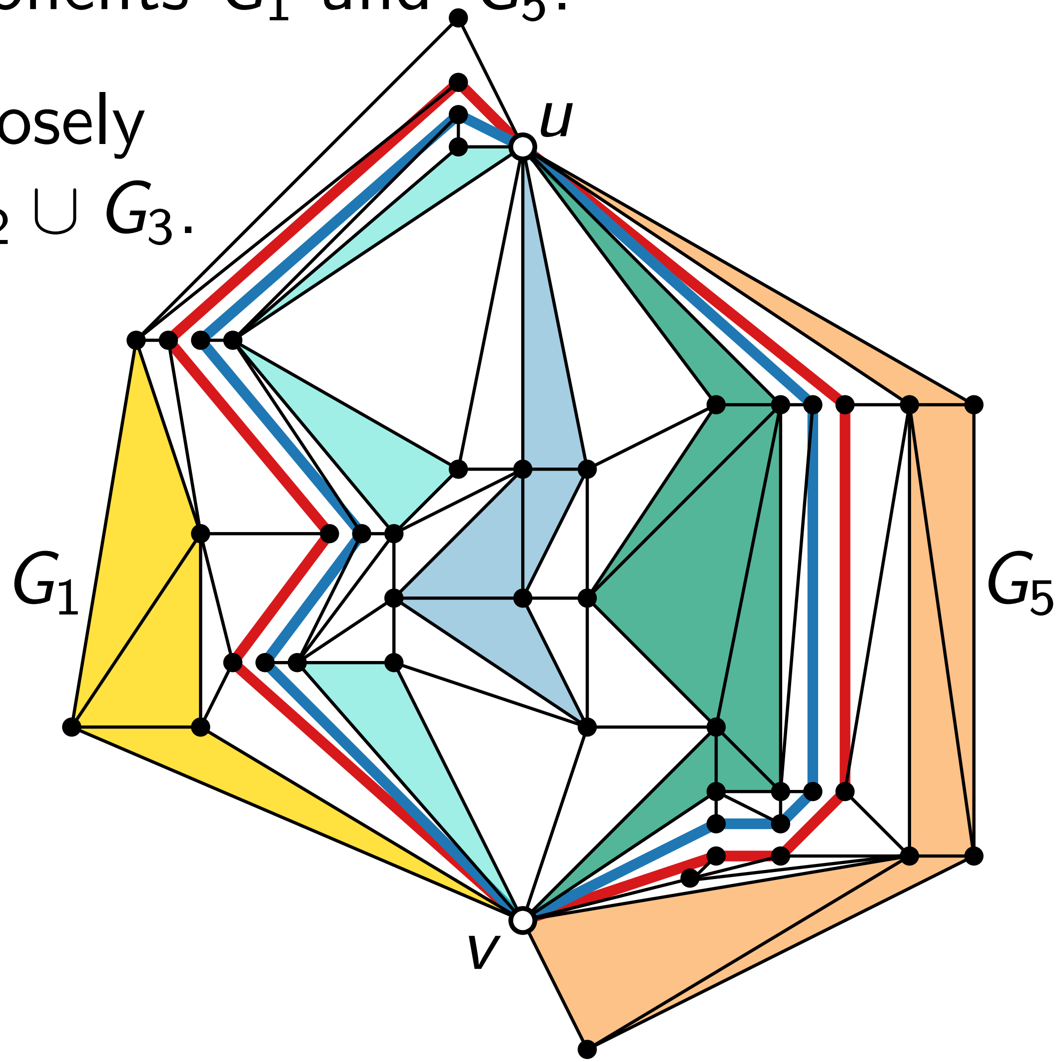
- Triangulate the outside of the red polygon, subdividing potential chords.



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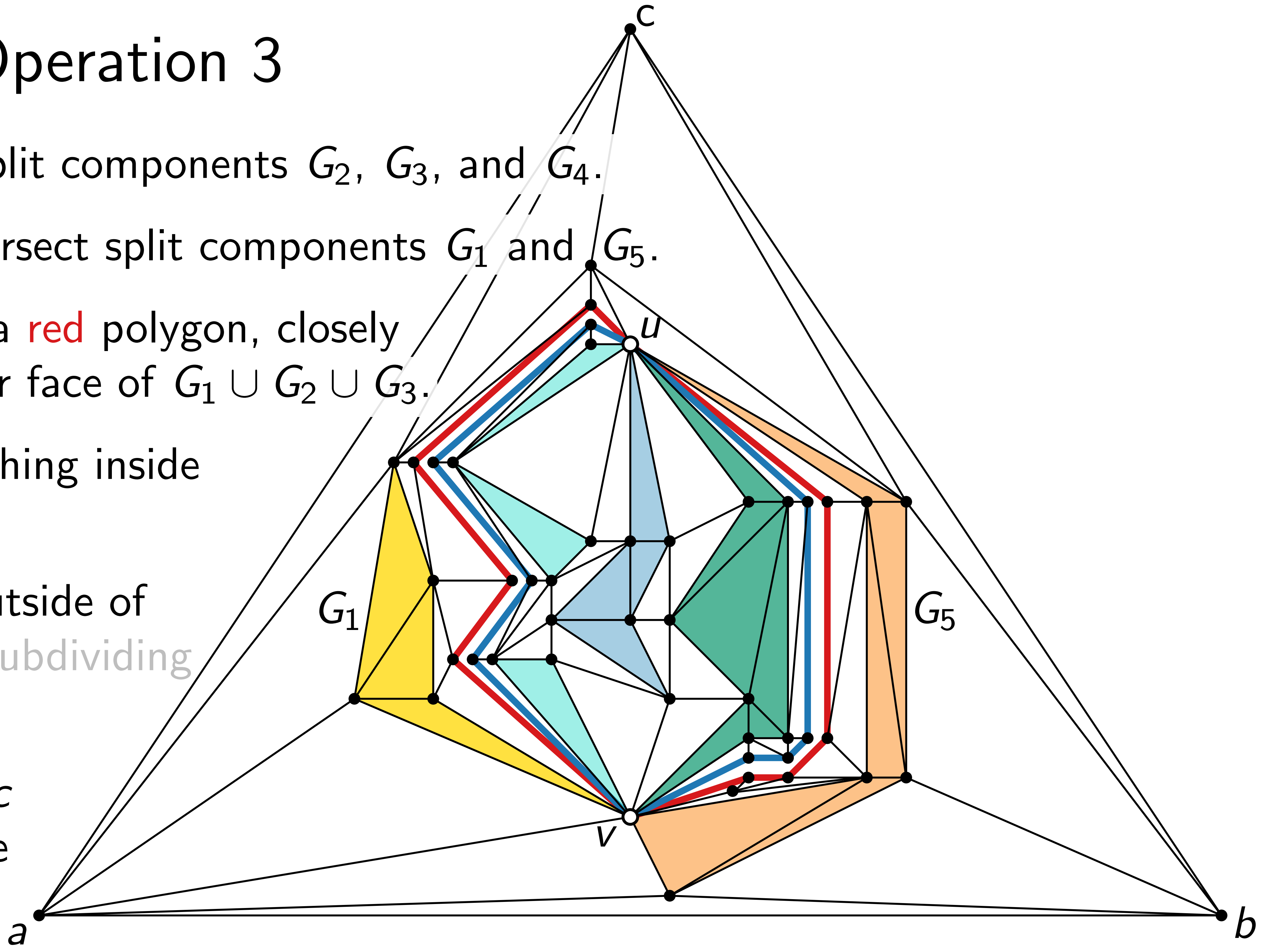
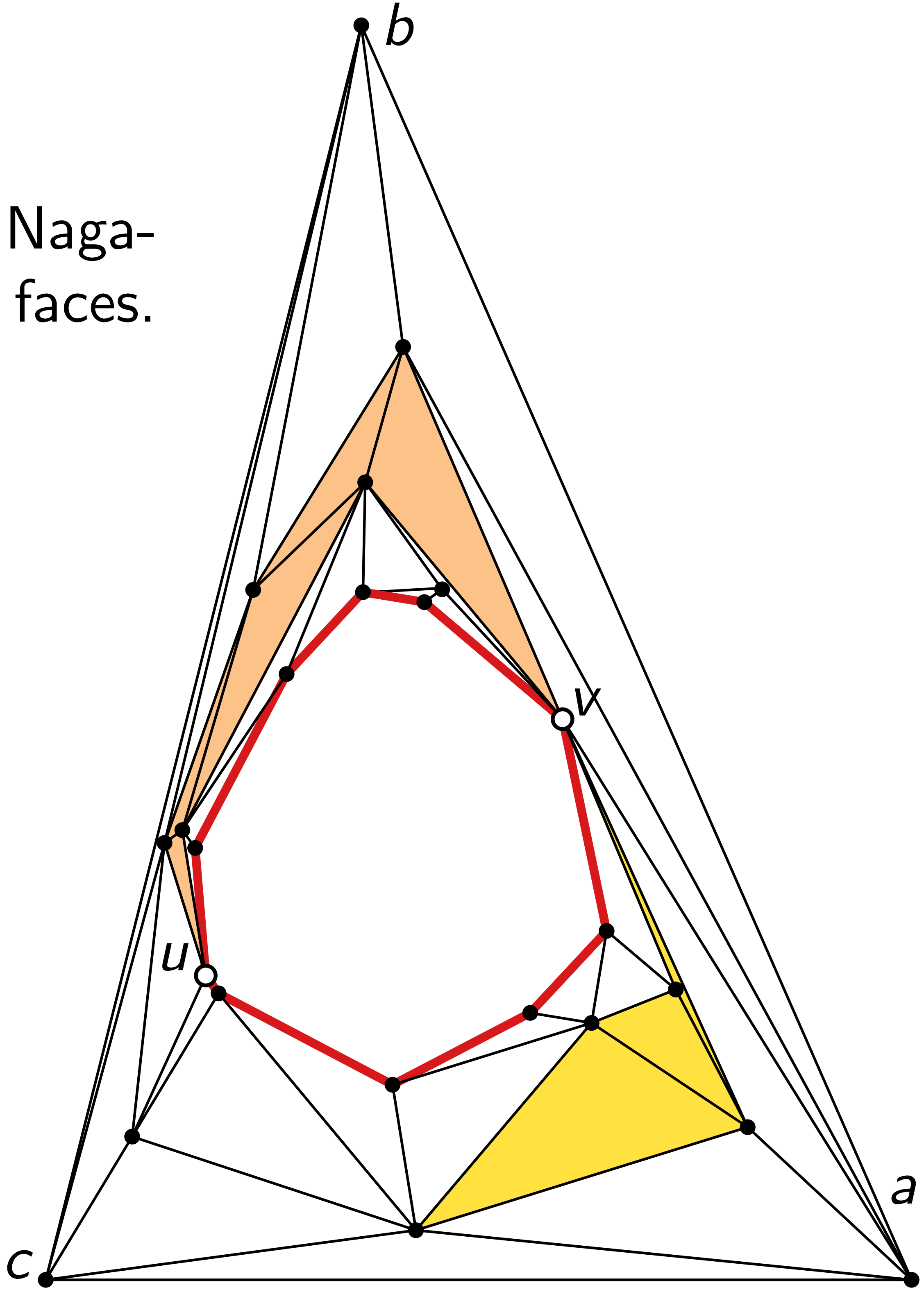
- Triangulate the outside of the red polygon, subdividing potential chords.
- Add a triangle abc around the outside and triangulate.



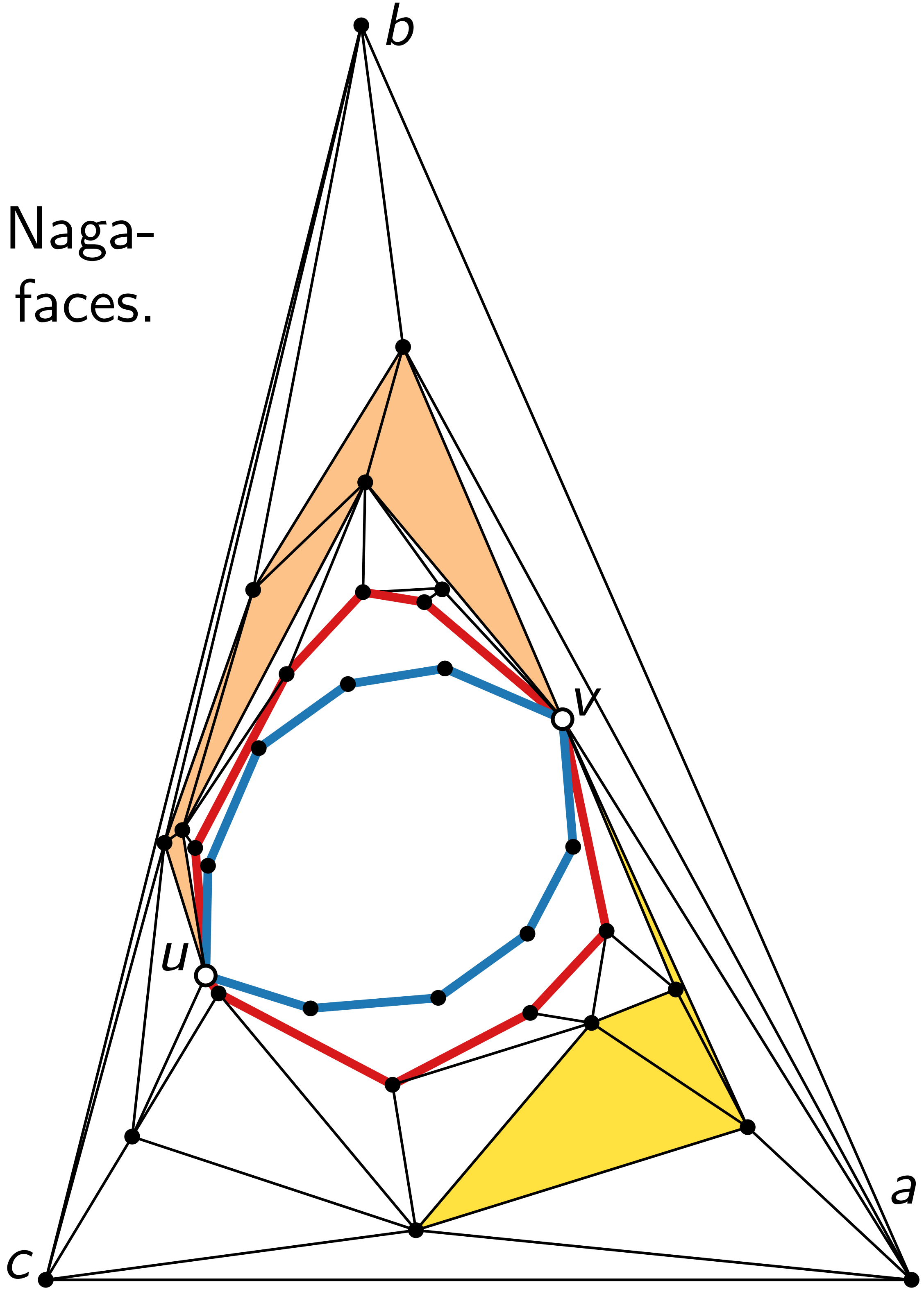
Algorithm for Operation 3 (cont'd)

- Use the algorithm of Tutte [1963] or of Hong & Nagamochi [2010] to draw the "outside" with convex faces.



Algorithm for Operation 3 (cont'd)

- Use the algorithm of Tutte [1963] or of Hong & Nagamochi [2010] to draw the "outside" with convex faces.
- Draw the blue polygon inside the red polygon such that the blue is symmetric to the line uv.



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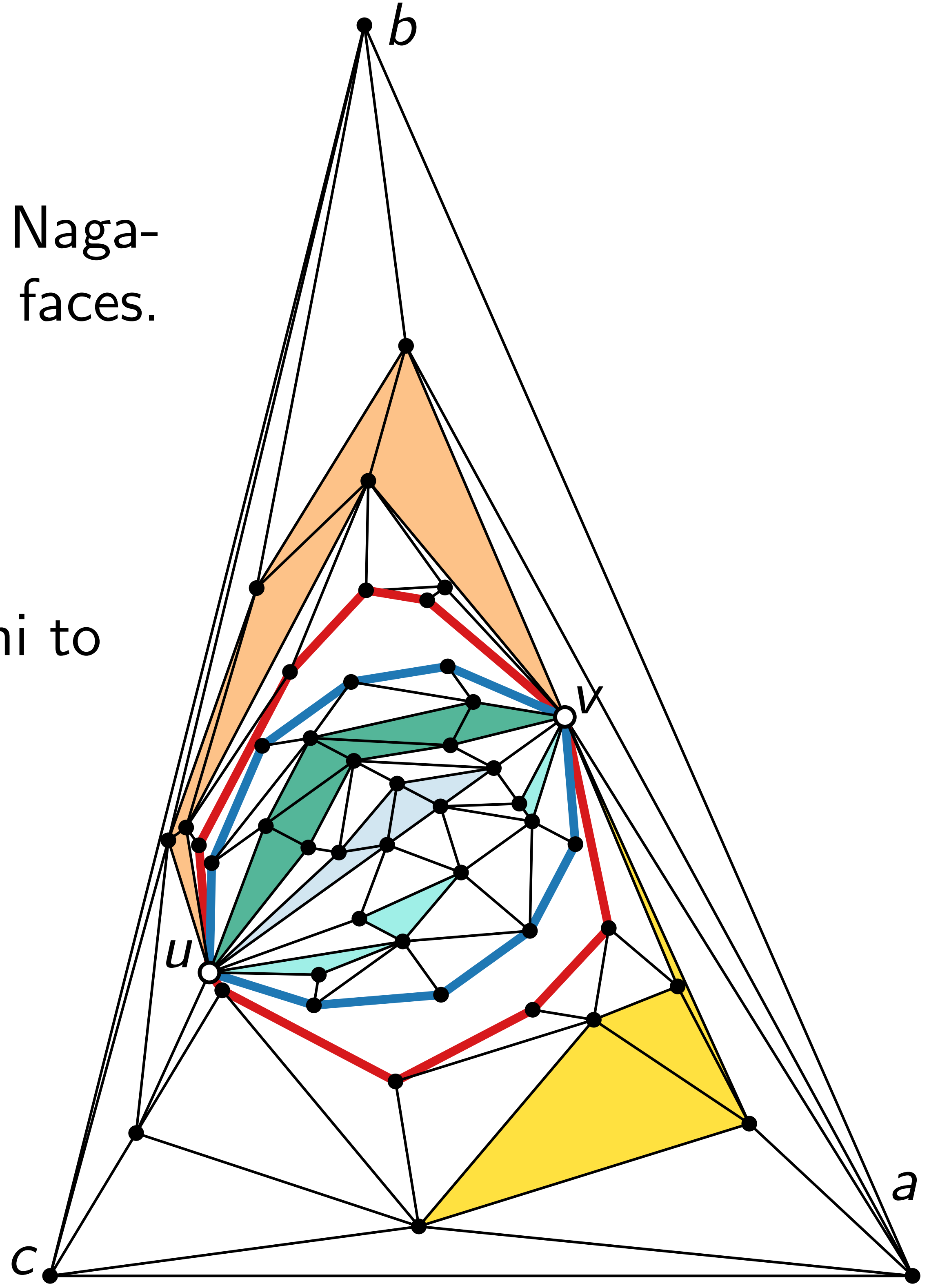
- Use the algorithm of Tutte [1963] or of Hong & Nagamochi [2010] to draw the "outside" with convex faces.
- Draw the blue polygon inside the red polygon such that the blue is symmetric to the line uv.
- Use the algorithm of Tutte / Hong & Nagamochi to draw the "inside" into the blue polygon.



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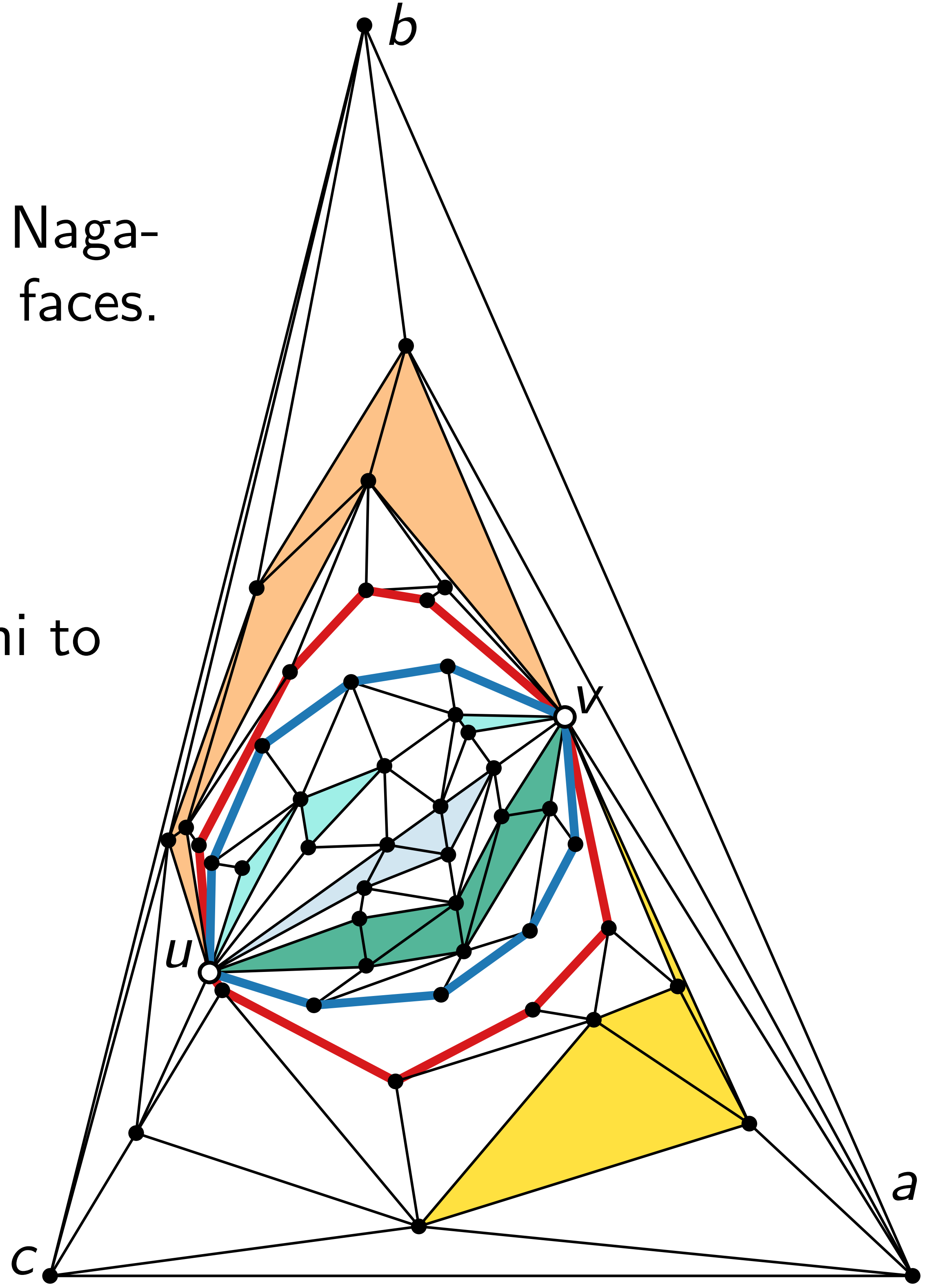
- Use the algorithm of Tutte [1963] or of Hong & Nagamochi [2010] to draw the "outside" with convex faces.
- Draw the blue polygon inside the red polygon such that the blue is symmetric to the line uv.
- Use the algorithm of Tutte / Hong & Nagamochi to draw the "inside" into the blue polygon.
- Mirror the blue polygon with its inside at uv.



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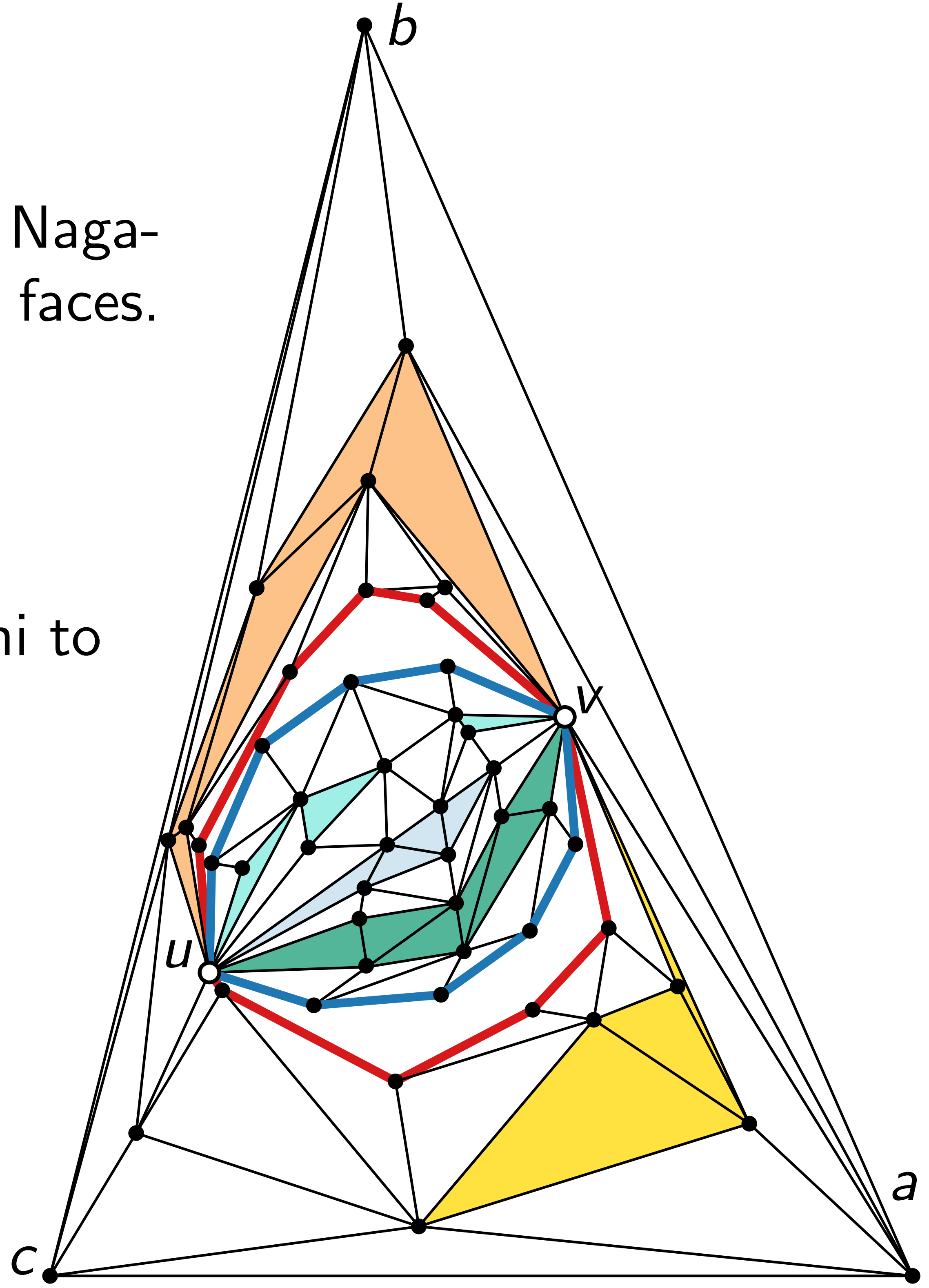
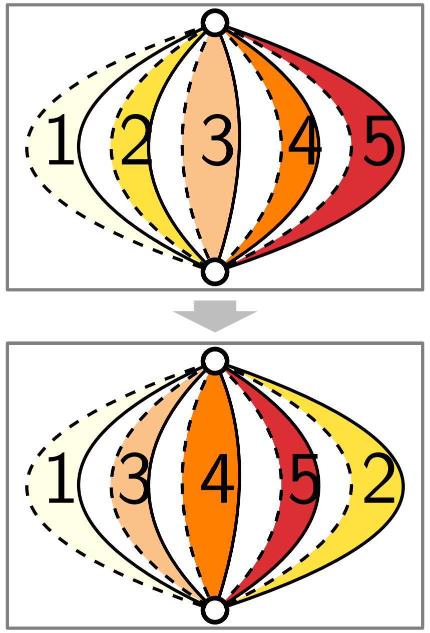
- Mirror the blue polygon with its inside at uv.

Done, using $O(n)$ steps.



Operation 4: Skipping a Split Component

Lemma 4. Let G, $G_1, \ldots, G_k$, $\{u, v\}$, and Γ be as before. If $uv \in E(G)$, $G_1 = \{uv\}$. There exists an $O(n)$-step 3D crossing-free morph from Γ to a planar straight-line drawing Γ' in which $G_1, \ldots, G_k$ have the same embedding as in Γ and their cw order around u is $G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_k, G_i$ (and G_1 and G_i are incident to the outer face).



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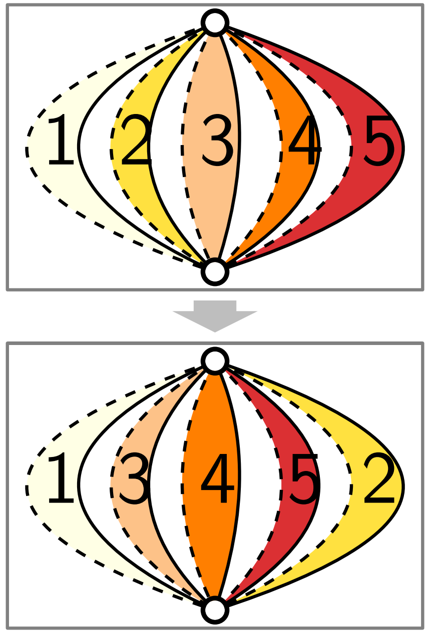
. . . even more complicated :-(



Putting Things Together

Starting from a planar graph drawing Γ, one can obtain the rotation system of any other planar drawing Φ of the same graph by:

(i) suitably changing the permutation of the components in some parallel compositions; that is, for some split pairs $\{u, v\}$ that define ≥ 3 split components, changing the clockwise (circular) ordering of such components; and

(ii) flipping the embedding for some rigid compositions; that is, for some split pairs that define a maximal split component that is biconnected, flipping the embedding of the component.

[Di Battista, Tamassia: SIAM J. Comput. 1996]

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Total # split components is $O(n)$ [DT'96] $\Rightarrow O(n^2)$ steps in total. $\square$

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- Is there a family of plane graphs $(G_n)_n$ such that $\Omega(n)$ steps are needed to 3D morph between two planar drawings of G_n with the same embedding?

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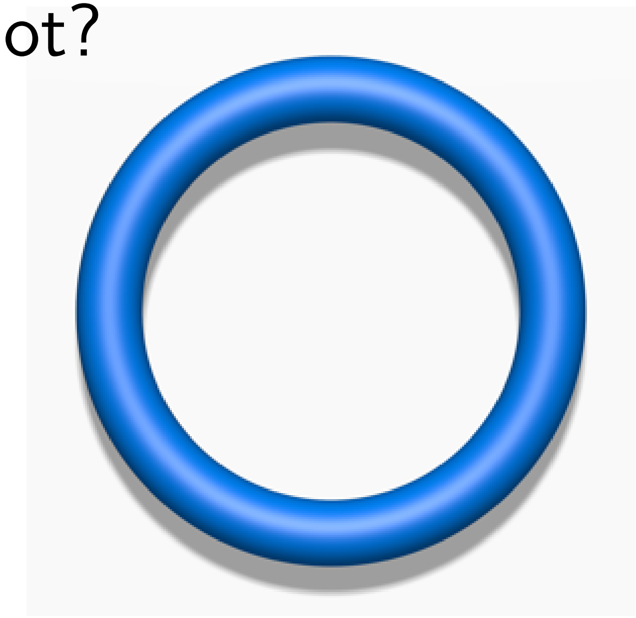
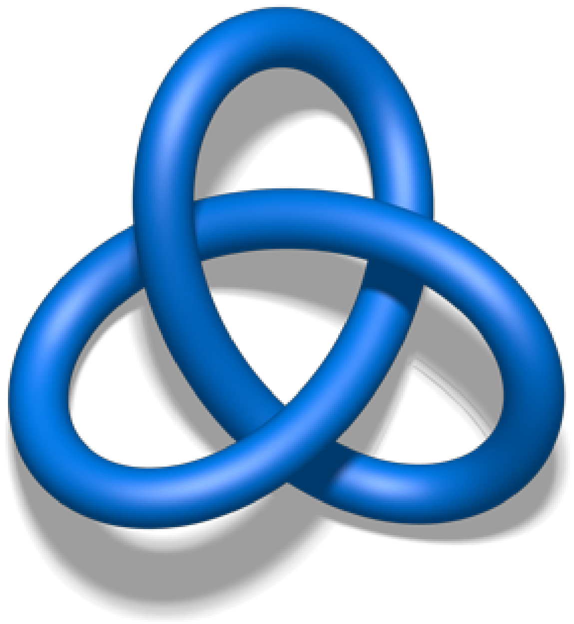
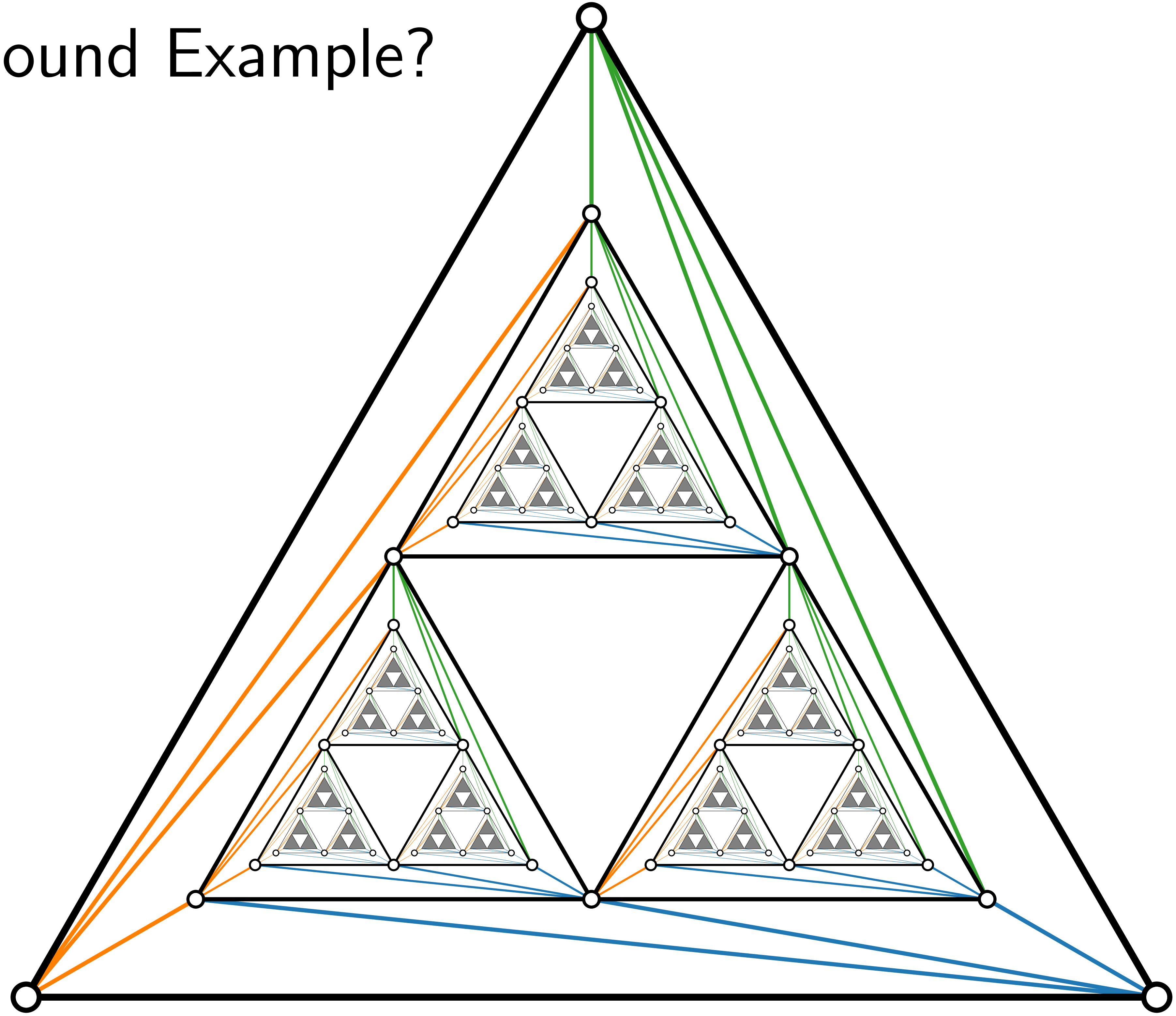
- Complexity of recognizing unknot vs. knot?





A Lower Bound Example?



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