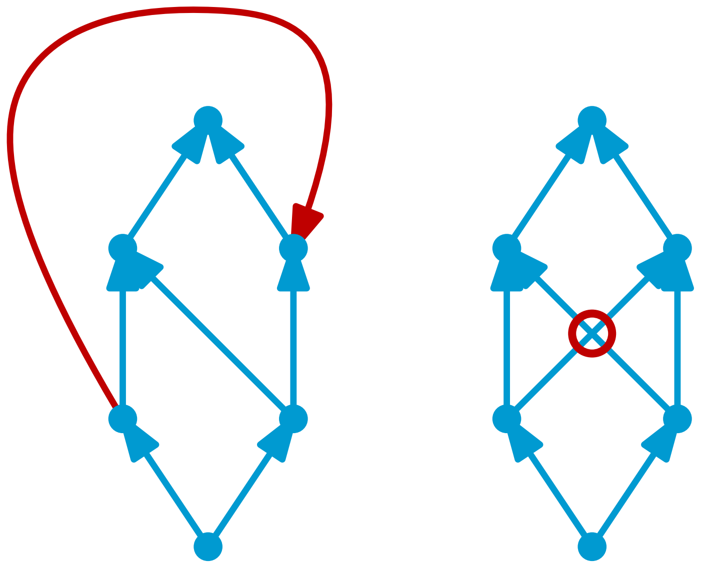
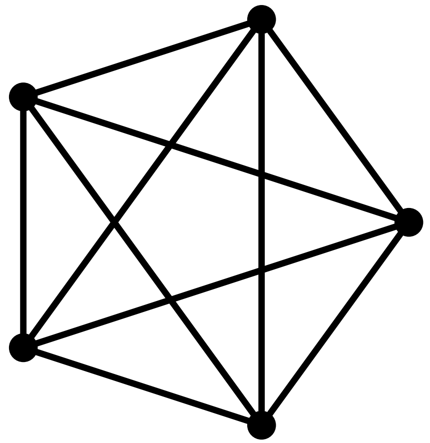
The Price of Upwardness



Patrizio Angelini (John Cabot)
Therese Biedl (Waterloo)
Markus Chimani (Osnabrück)
Sabine Cornelsen (Konstanz)
Giordano Da Lozzo (Roma III)
Seok-Hee Hong (Sydney)
Giuseppe Liotta (Perugia)
Maurizio Patrignani (Roma III)
Sergey Pupyrev (Meta)
Ignaz Rutter (Passau)
Alexander Wolff (Würzburg)

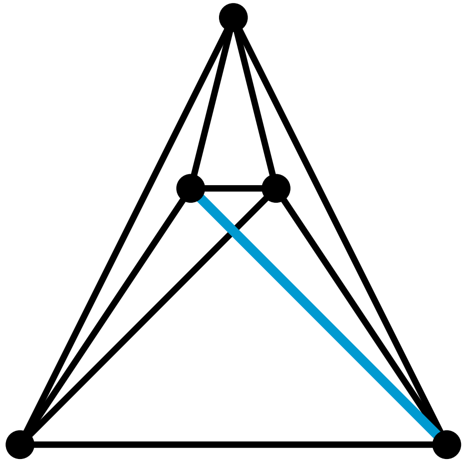
GD 2024, Vienna, September 18–20

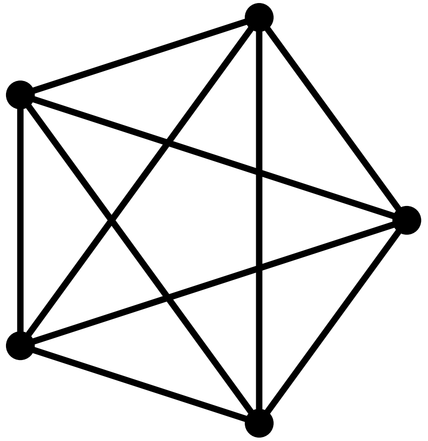
Beyond Planarity



K_5 is
not planar

Beyond Planarity





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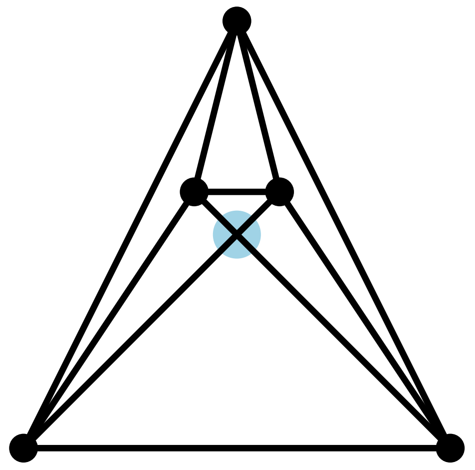

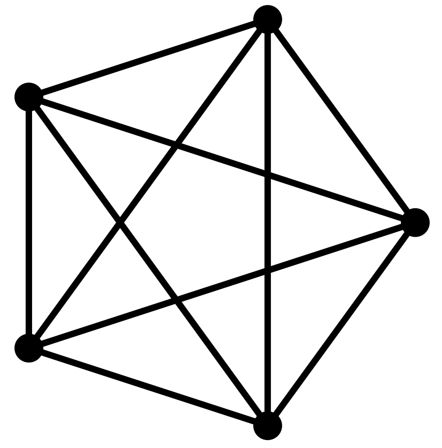

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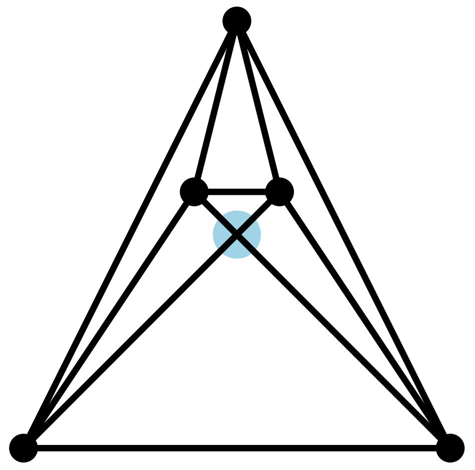

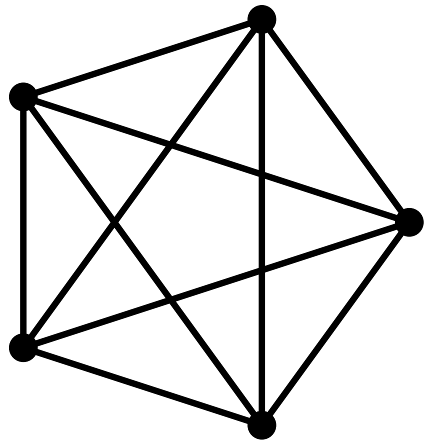

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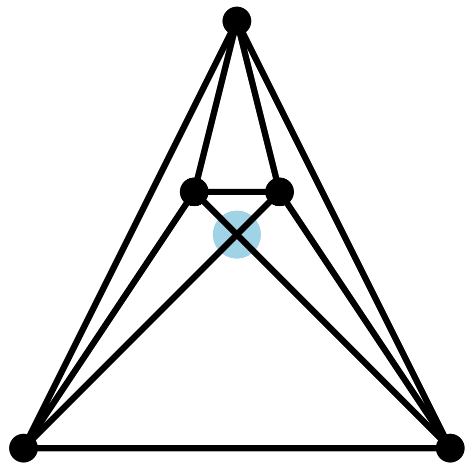

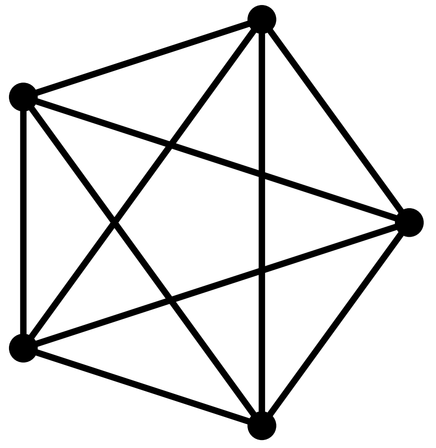

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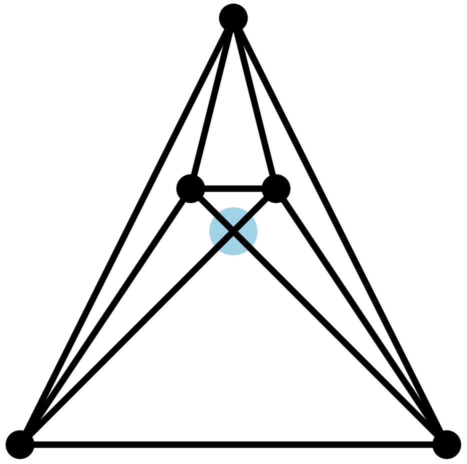

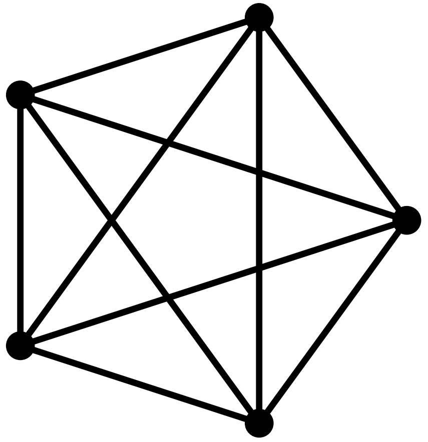

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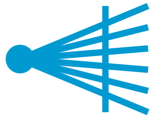
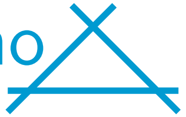
- fan planar 
- quasi planar, . . . no 

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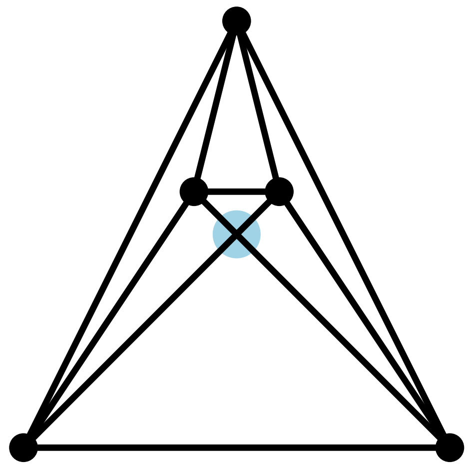

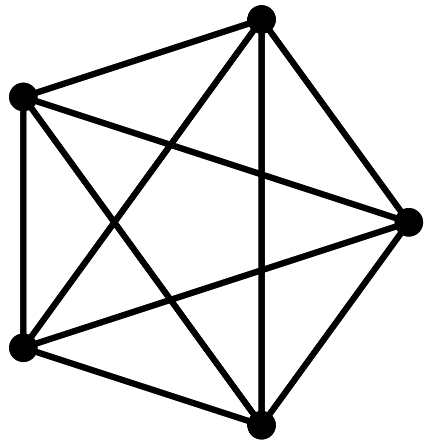

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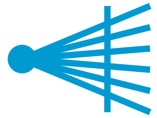
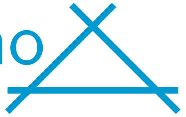
- fan planar 
- quasi planar, ... no 

What about beyond planarity for directed acyclic graphs?

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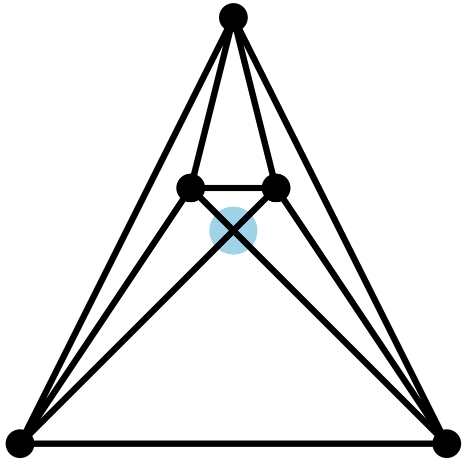

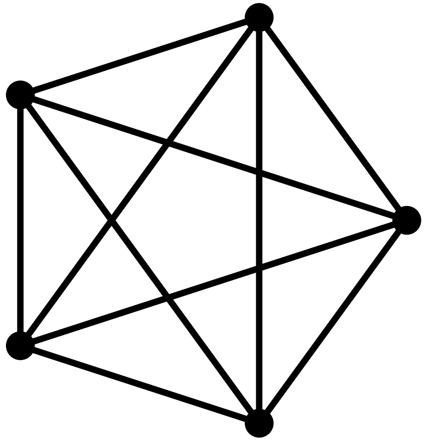

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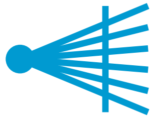
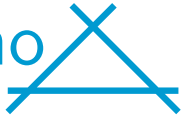
- fan planar 
- quasi planar, ... no 

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Problem Definition

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drawing in which each edge is

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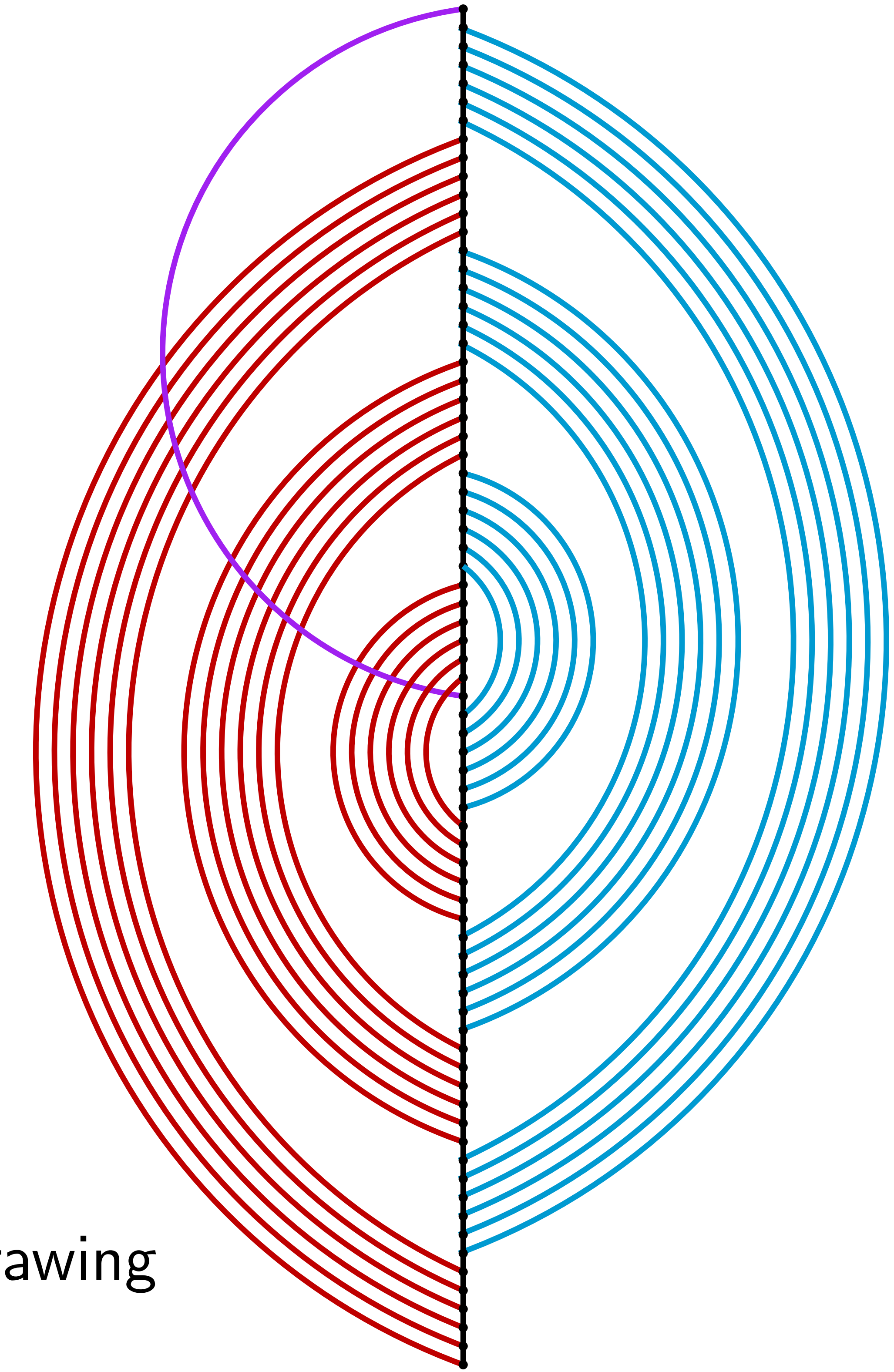

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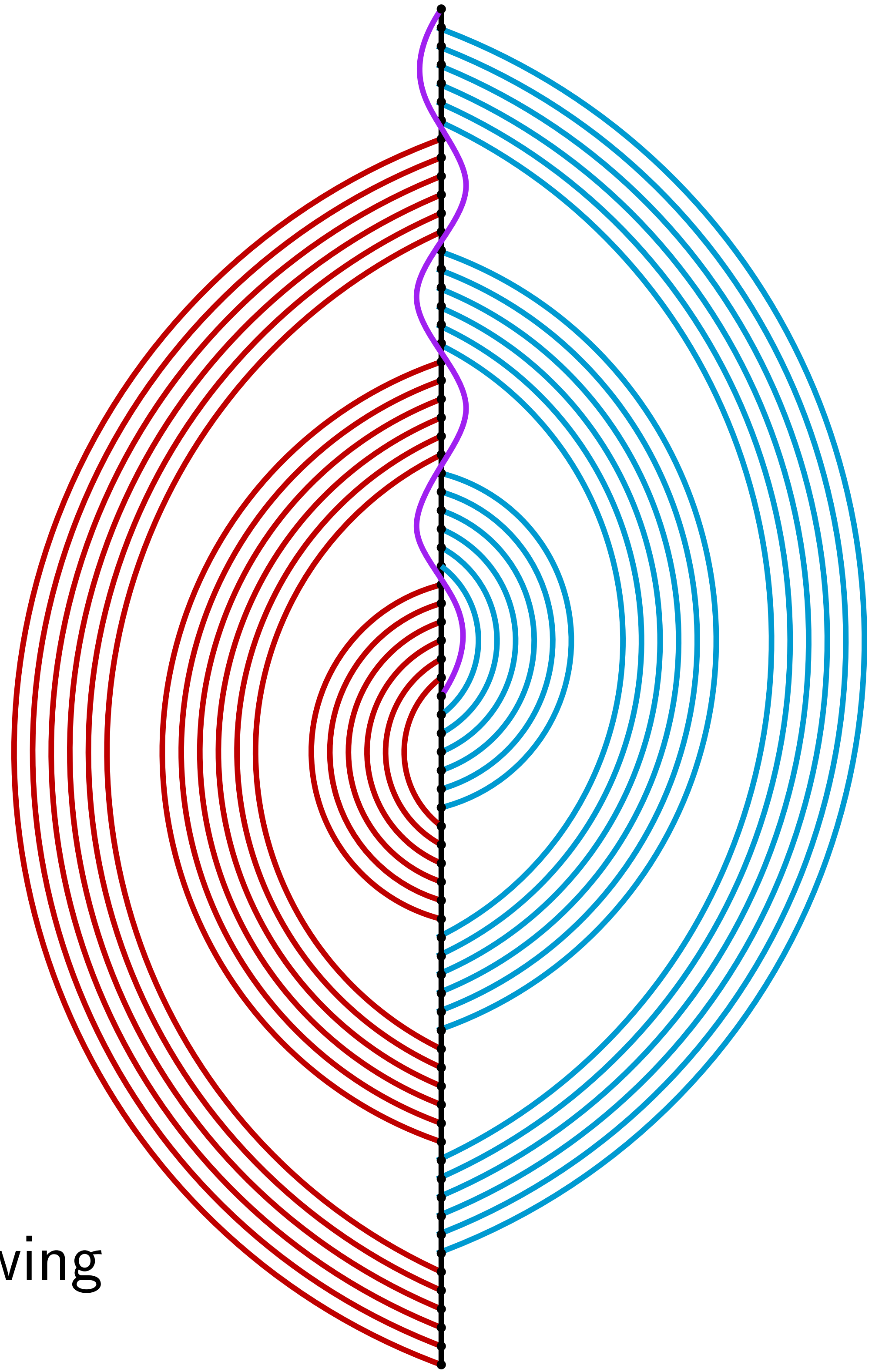

upward 5-planar drawing

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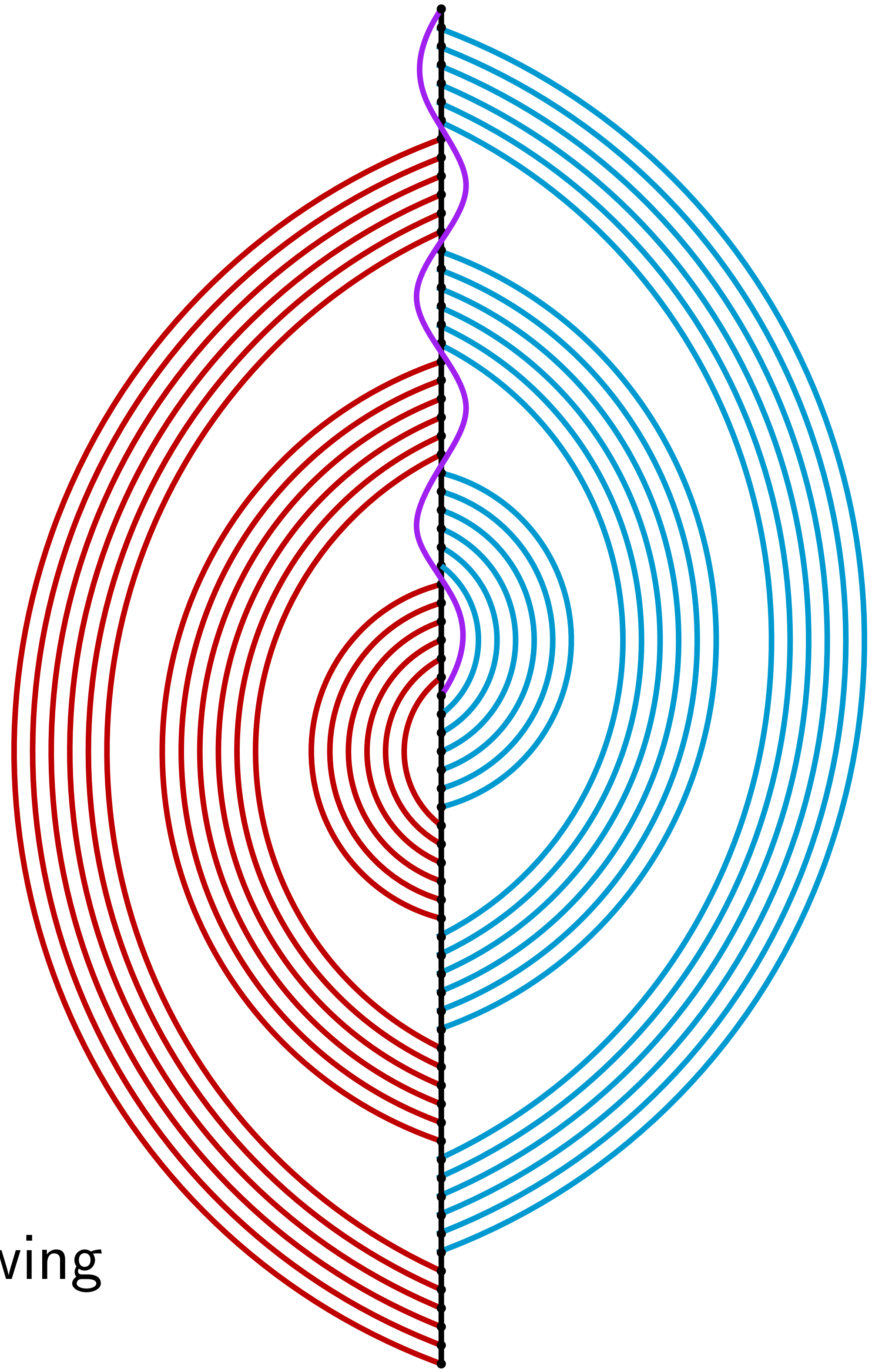
upward local crossing number of a DAG G:

minimum k such that G is upward k-planar



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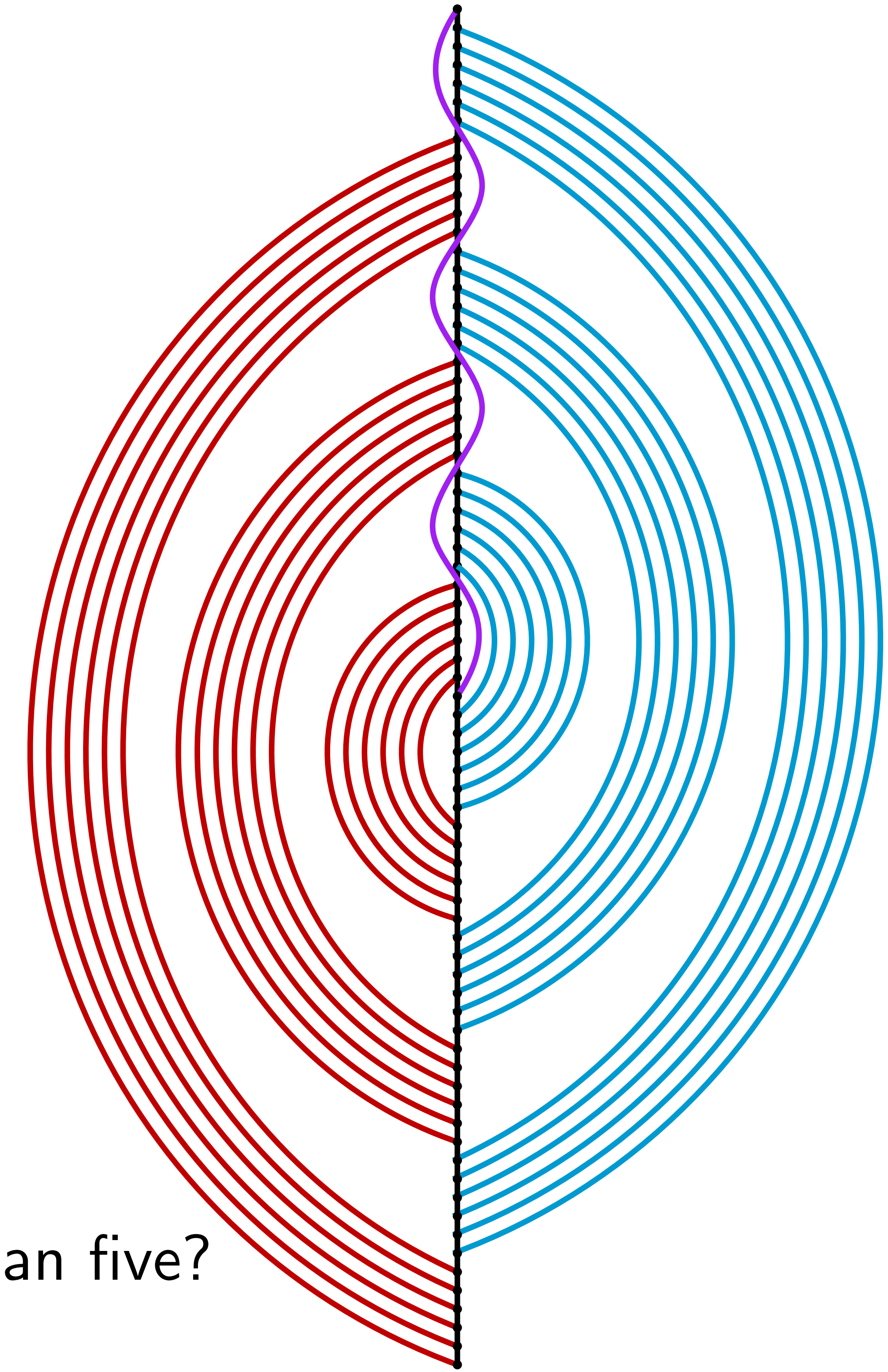

Can we do better than five?

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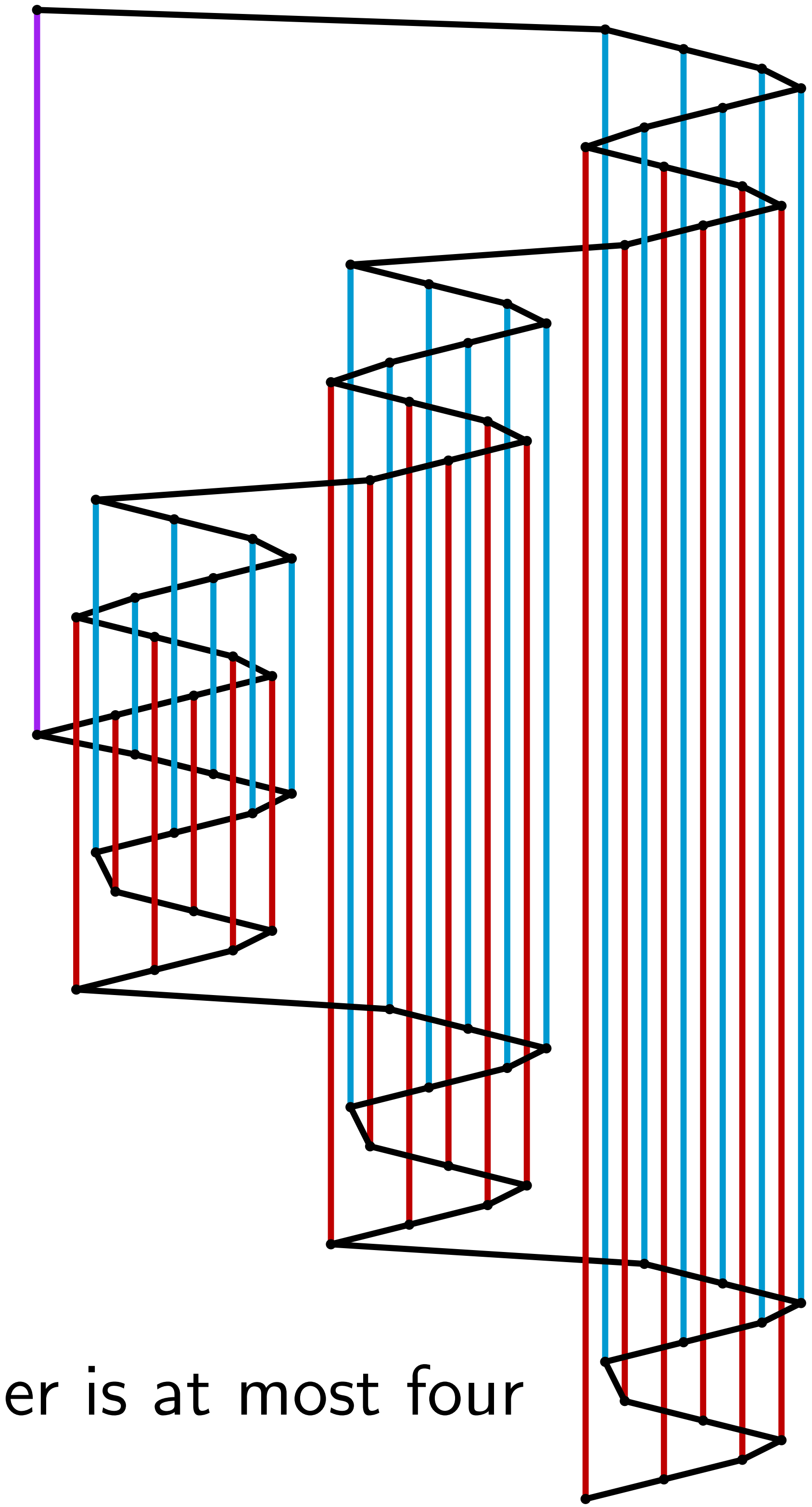

upward local crossing number is at most four

Previous Results

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- Linear layouts of directed graphs: draw vertices on a line in topological order
  - Upward book embeddings, minimize number of pages
    Frati, Fulek, Ruiz-Vargas GD'11
    Binucci et al. SoCG'19
    Bhore, Da Lozzo, Montecchiani, and Nöllenburg GD'21
    Bekos et al. GD'22
  - Stack and Queue Number
    Heath, Pemmaraju, and Trenk 1999
    Jungblut, Merker, Ueckerdt FOCS'23
    Nöllenburg and Pupyrev GD'23

Our Results: Upper and Lower Bounds

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Lower Bounds: In the worst case the upward local crossing number of

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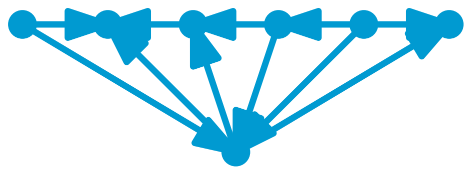

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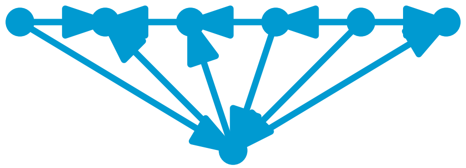

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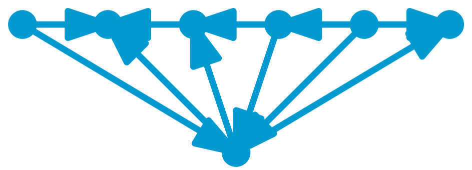

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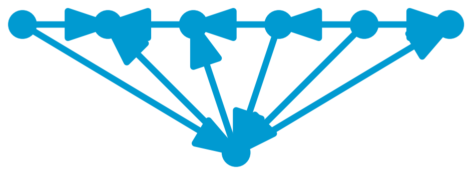

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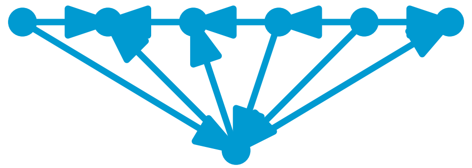
- fans (path + apex) is > 0. 
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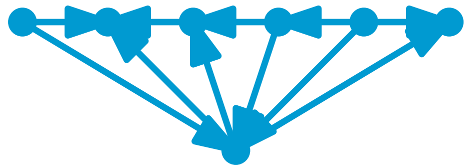

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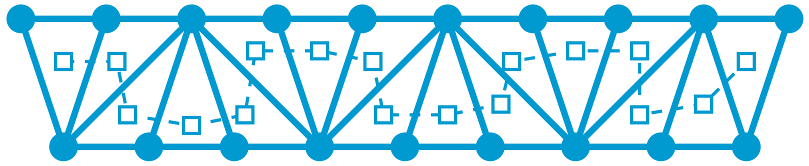

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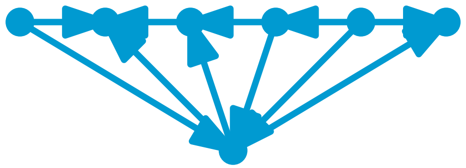

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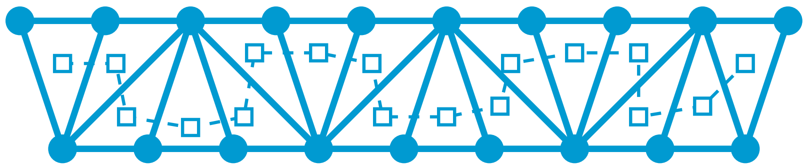

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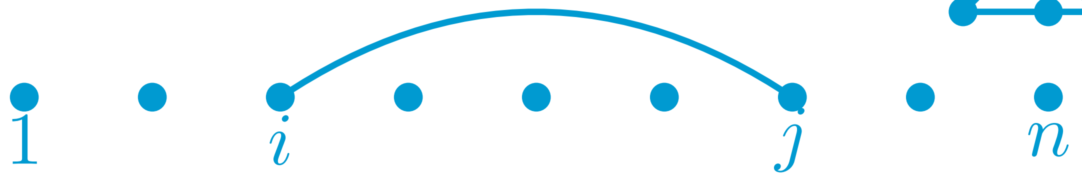

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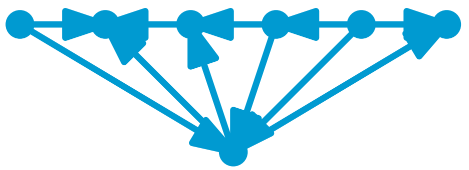

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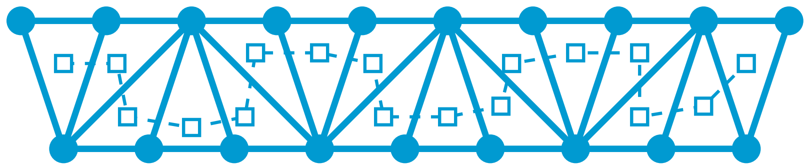

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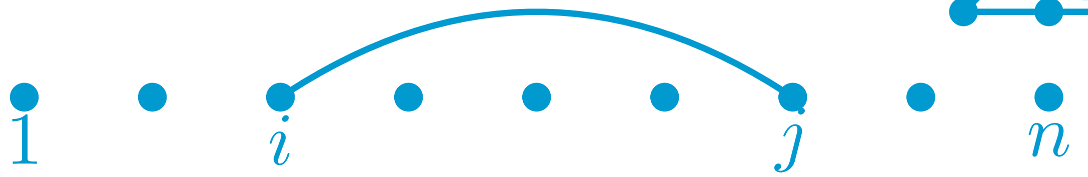

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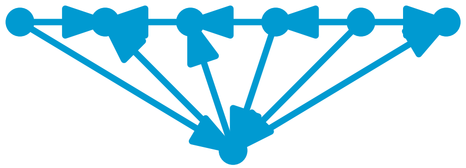

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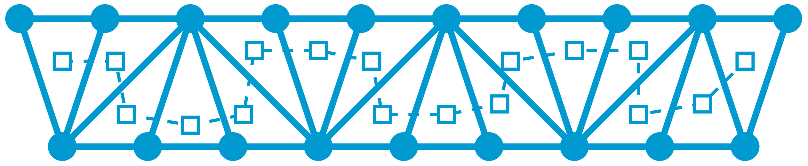

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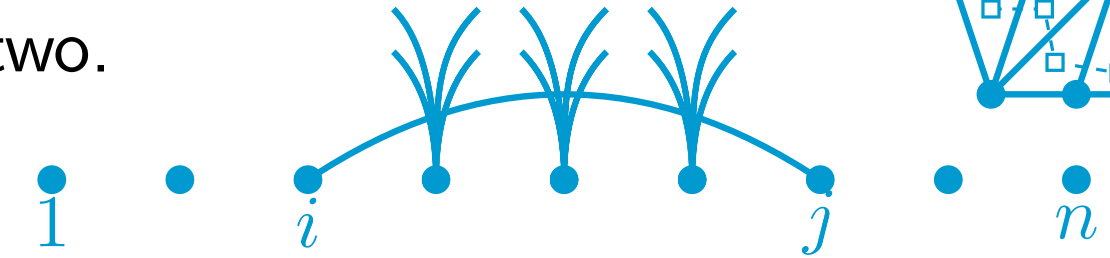

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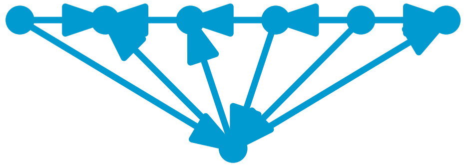

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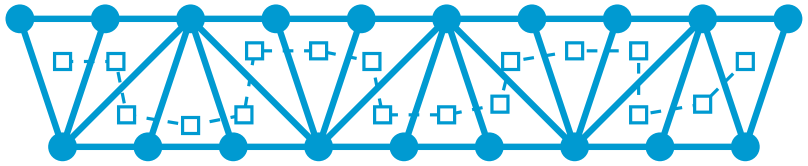

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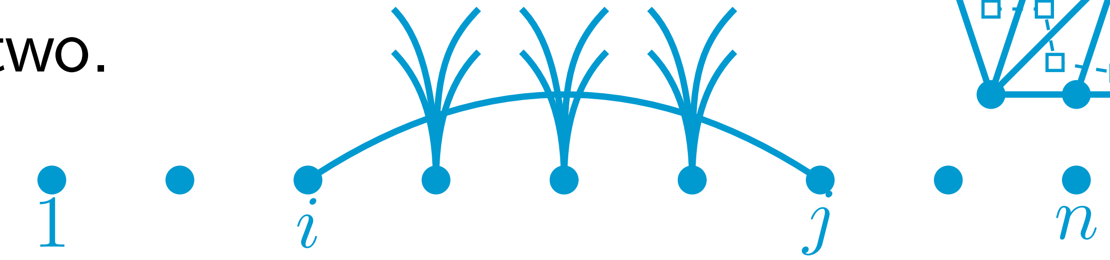

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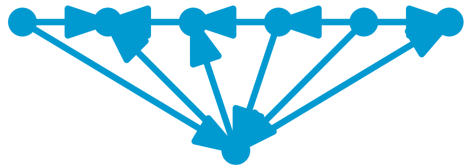

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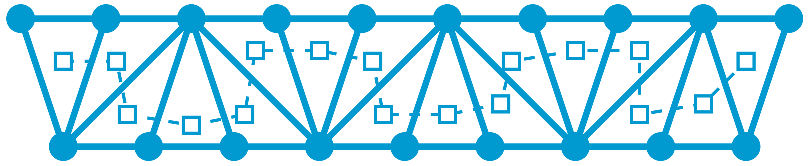

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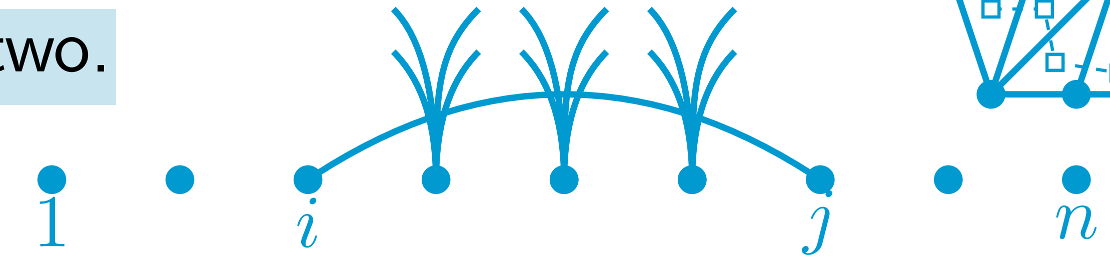

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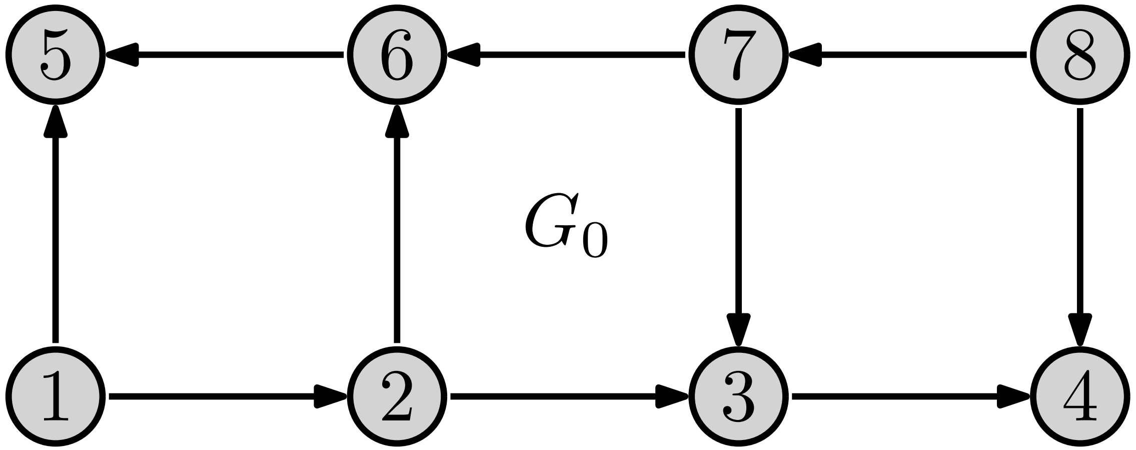
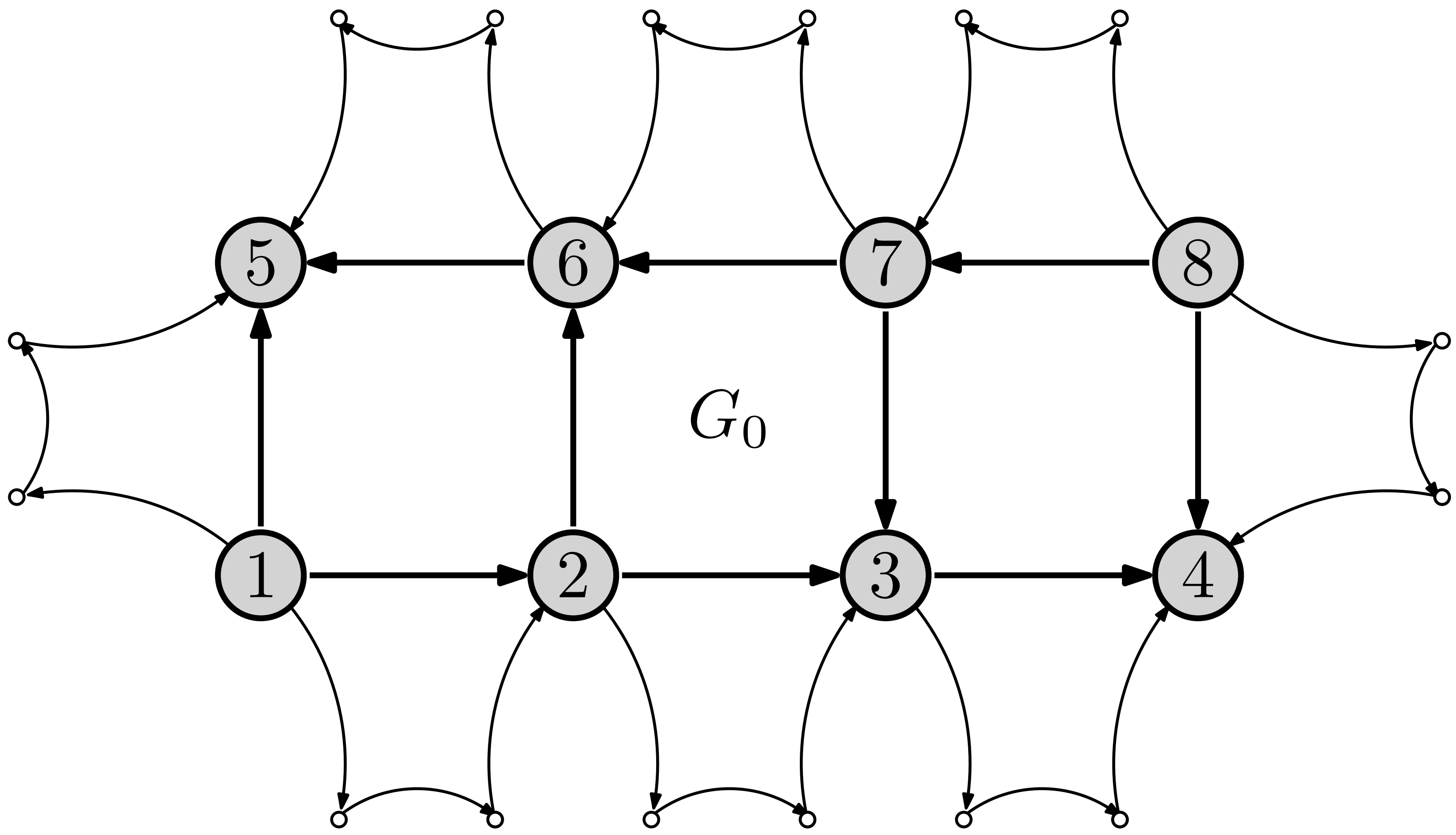
LB: Bipartite Outerplanar DAGs

- not upward-planar
(Papakostas GD'94)



LB: Bipartite Outerplanar DAGs



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- add to each outer edge a path of length 3 (iterate ℓ times)

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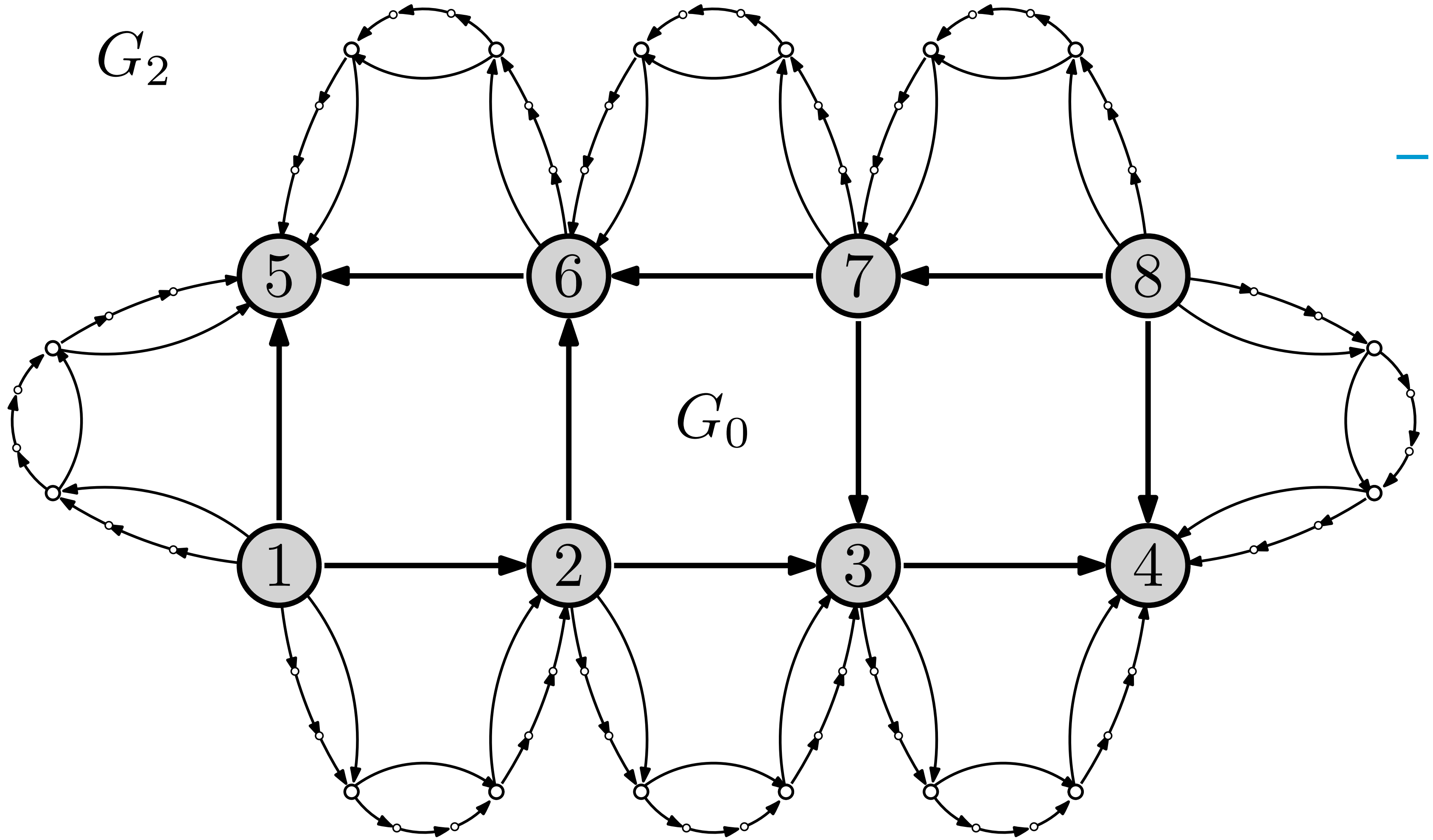

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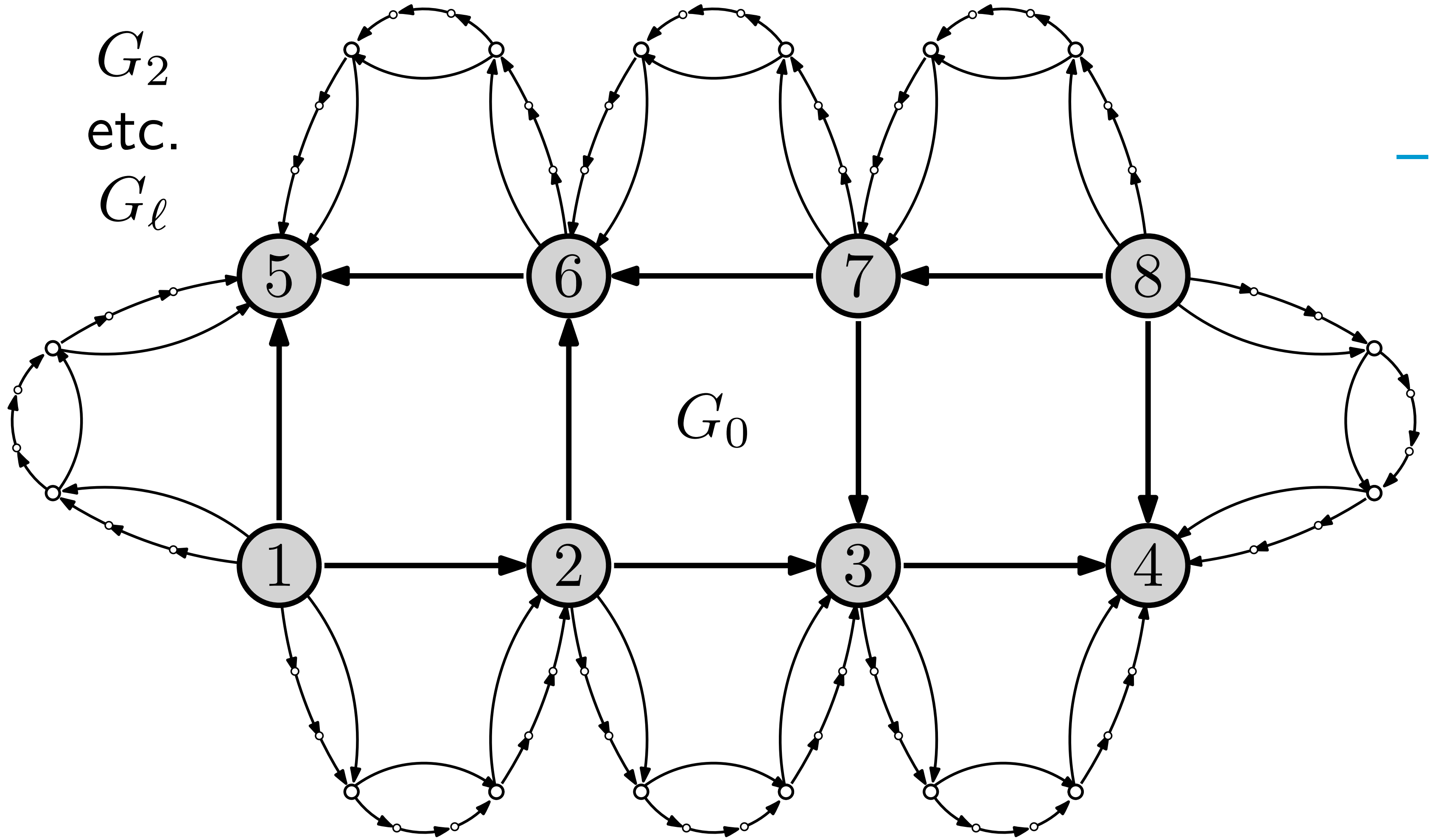

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$n_\ell = 8 + \sum_{i=1}^{\ell} 8 \cdot 3^{i-1} \cdot 2 = 8 \cdot 3^\ell$

$\Delta_\ell = 2\ell + 3$

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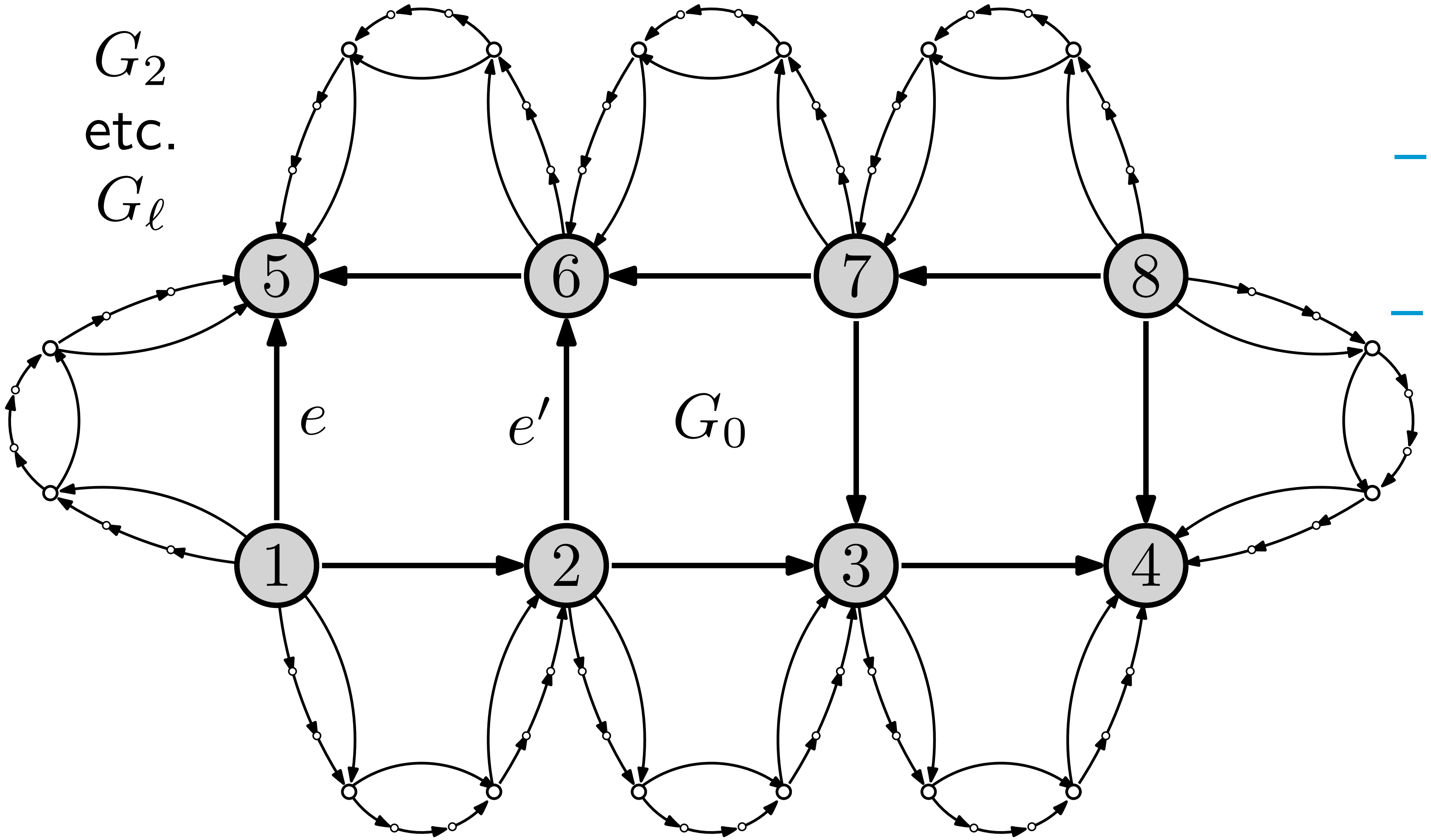

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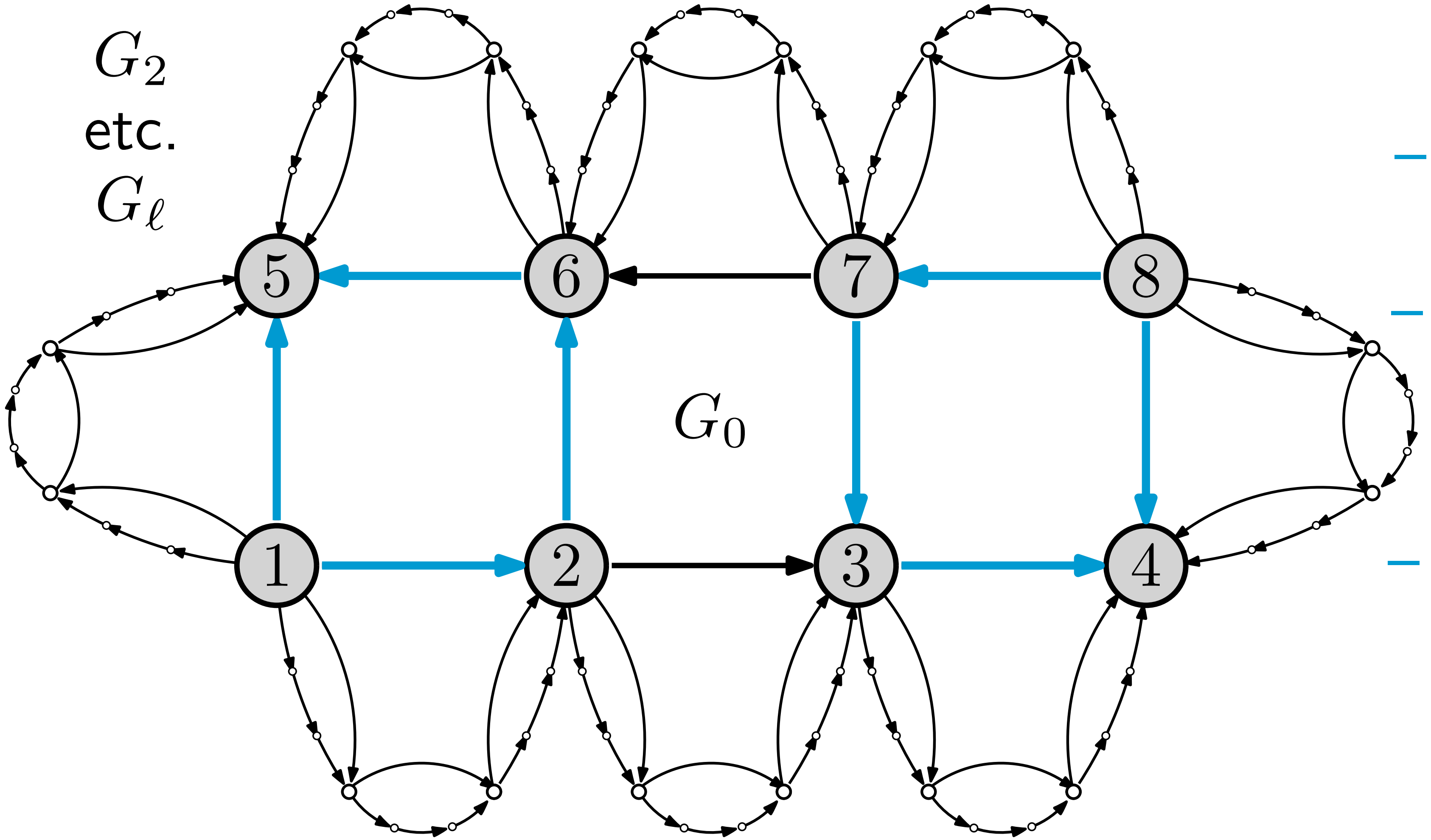

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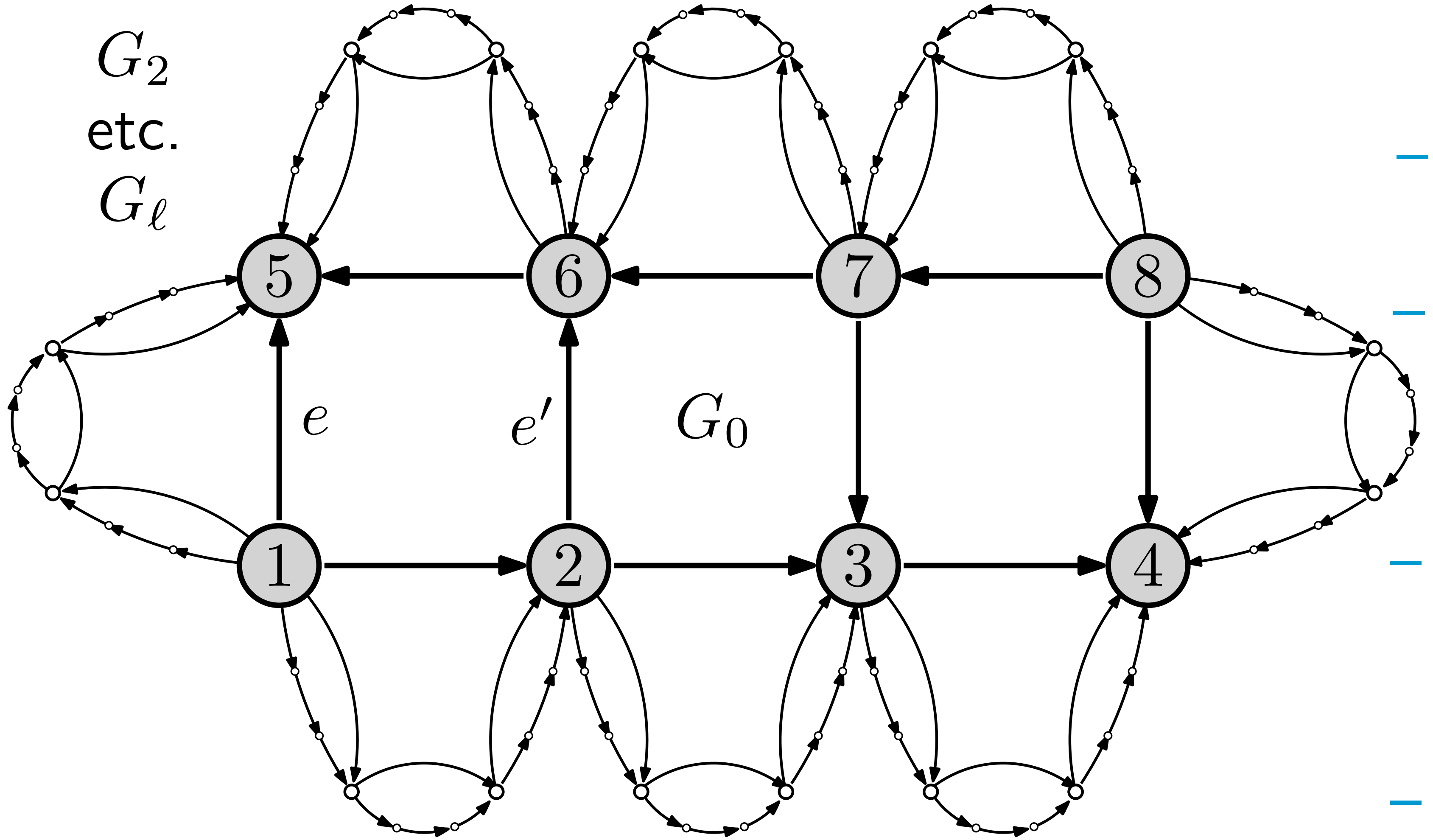

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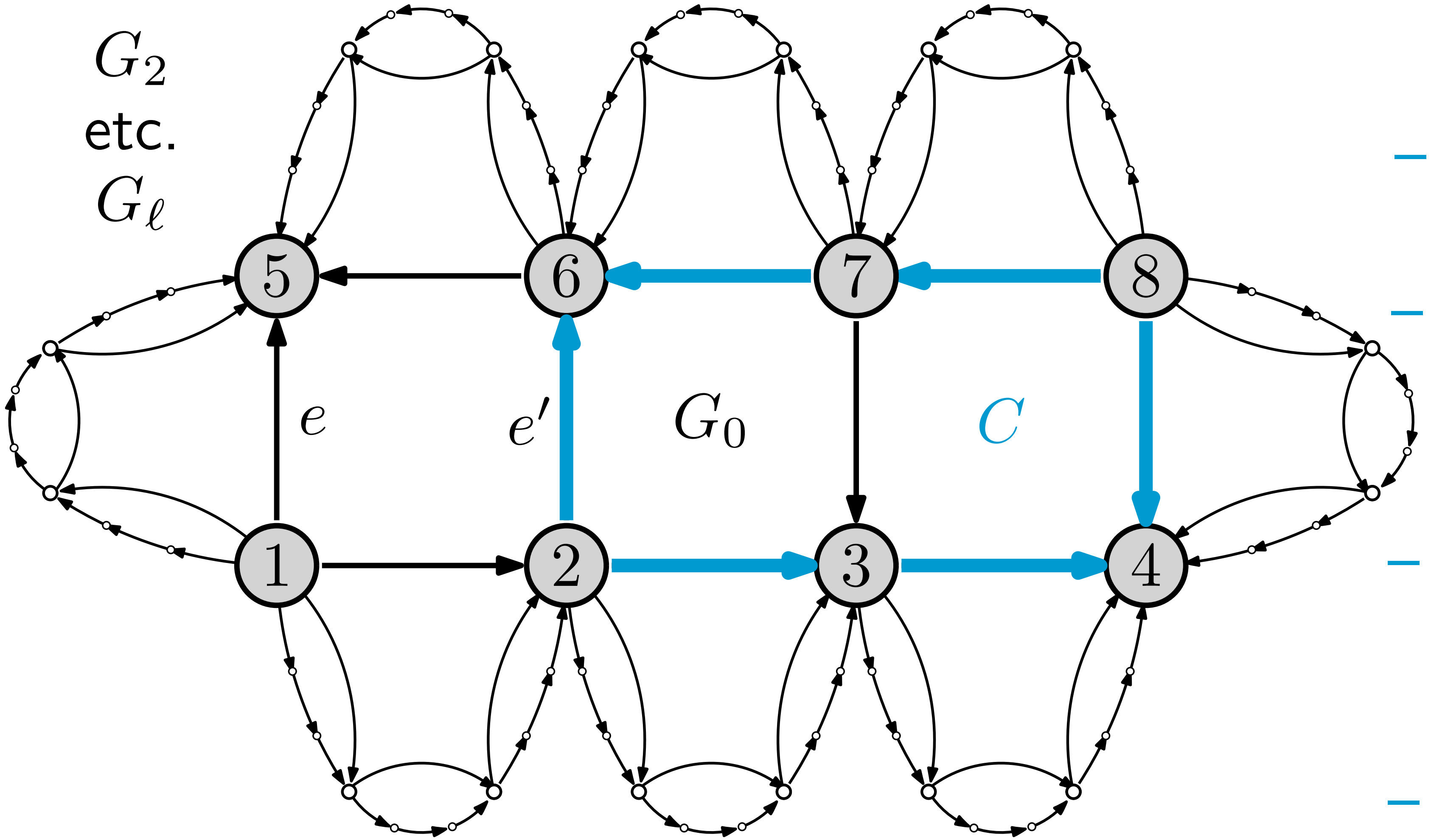

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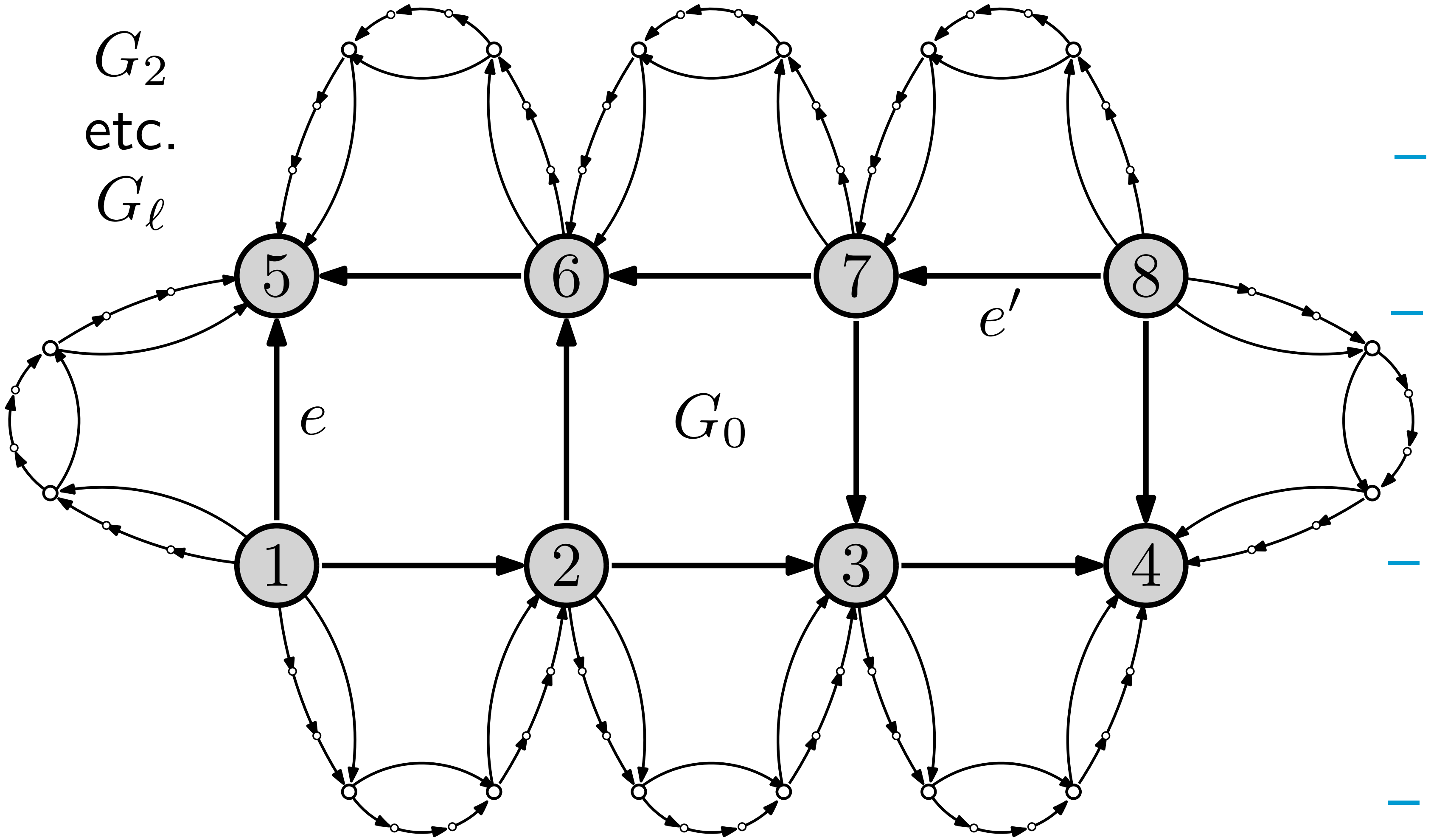

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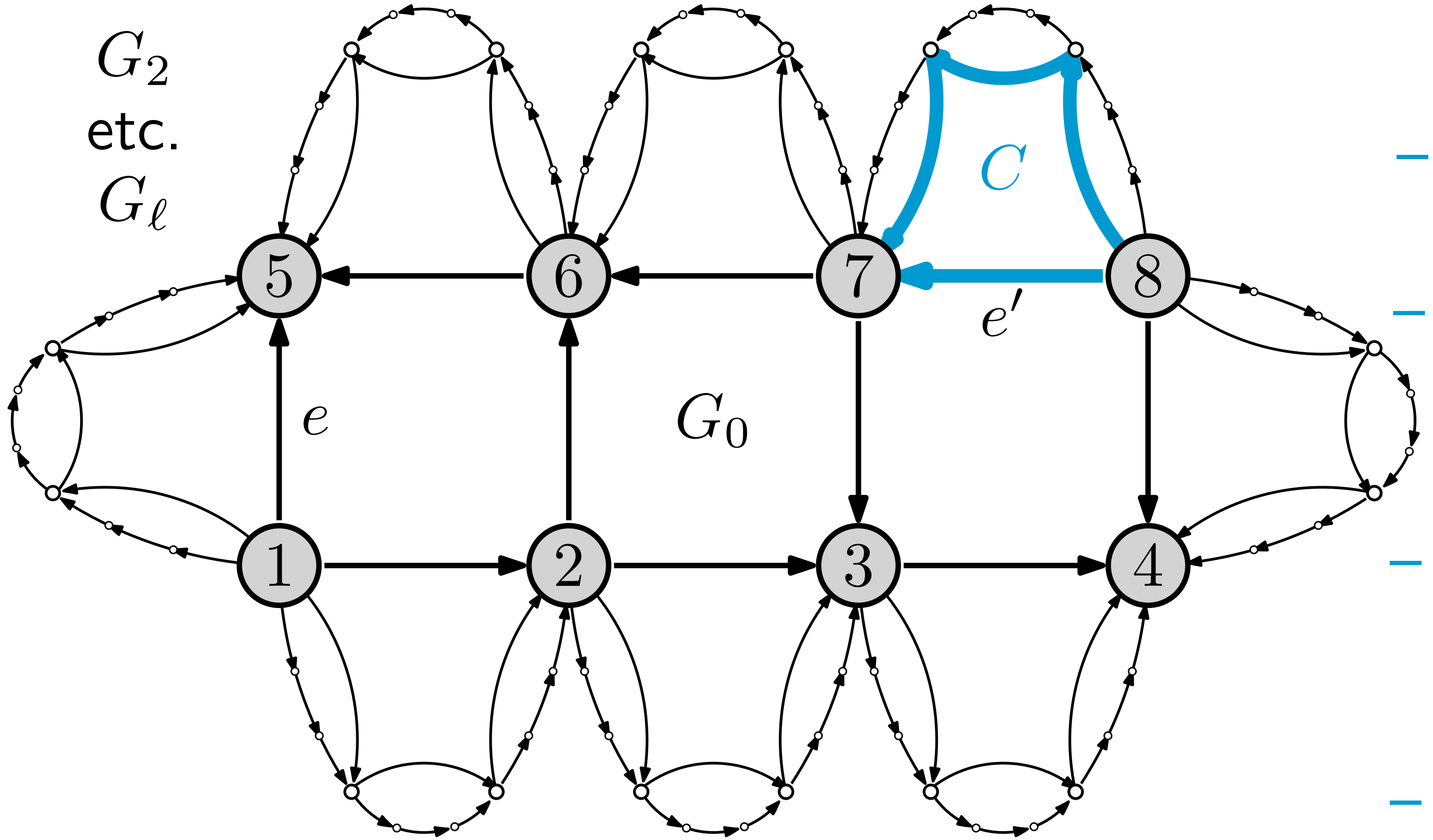

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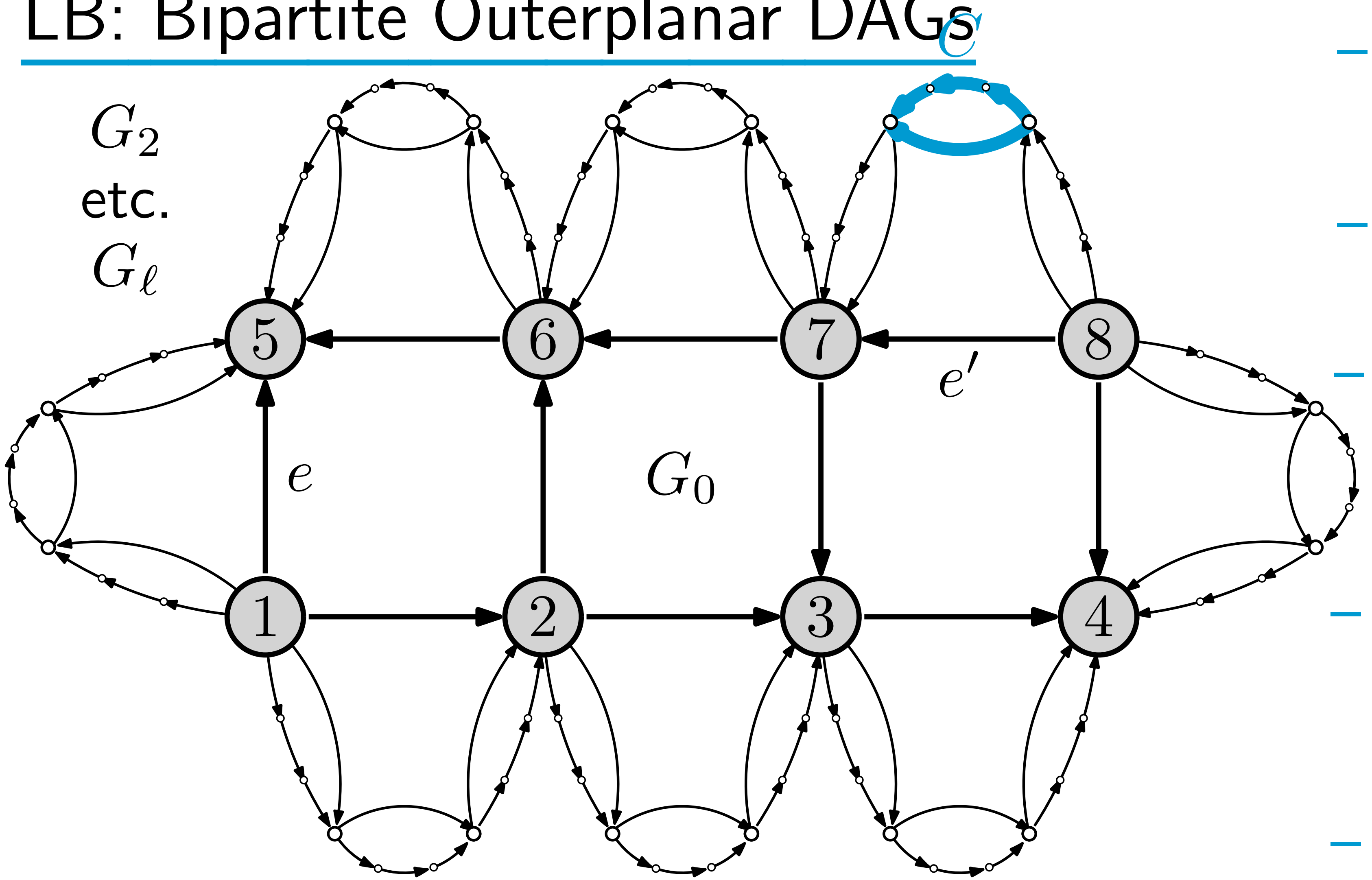

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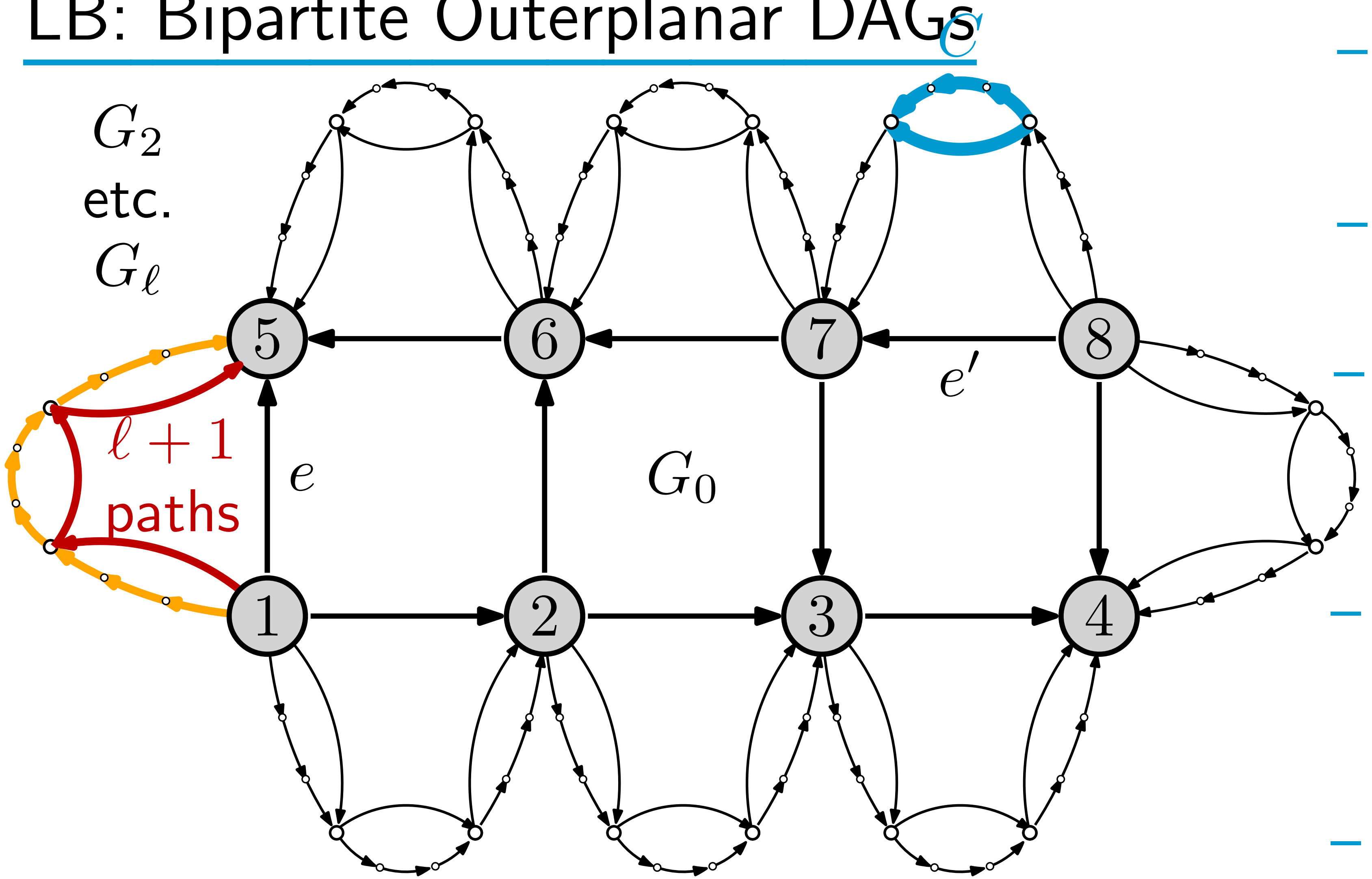

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- C crossed by $\ell + 1$ paths

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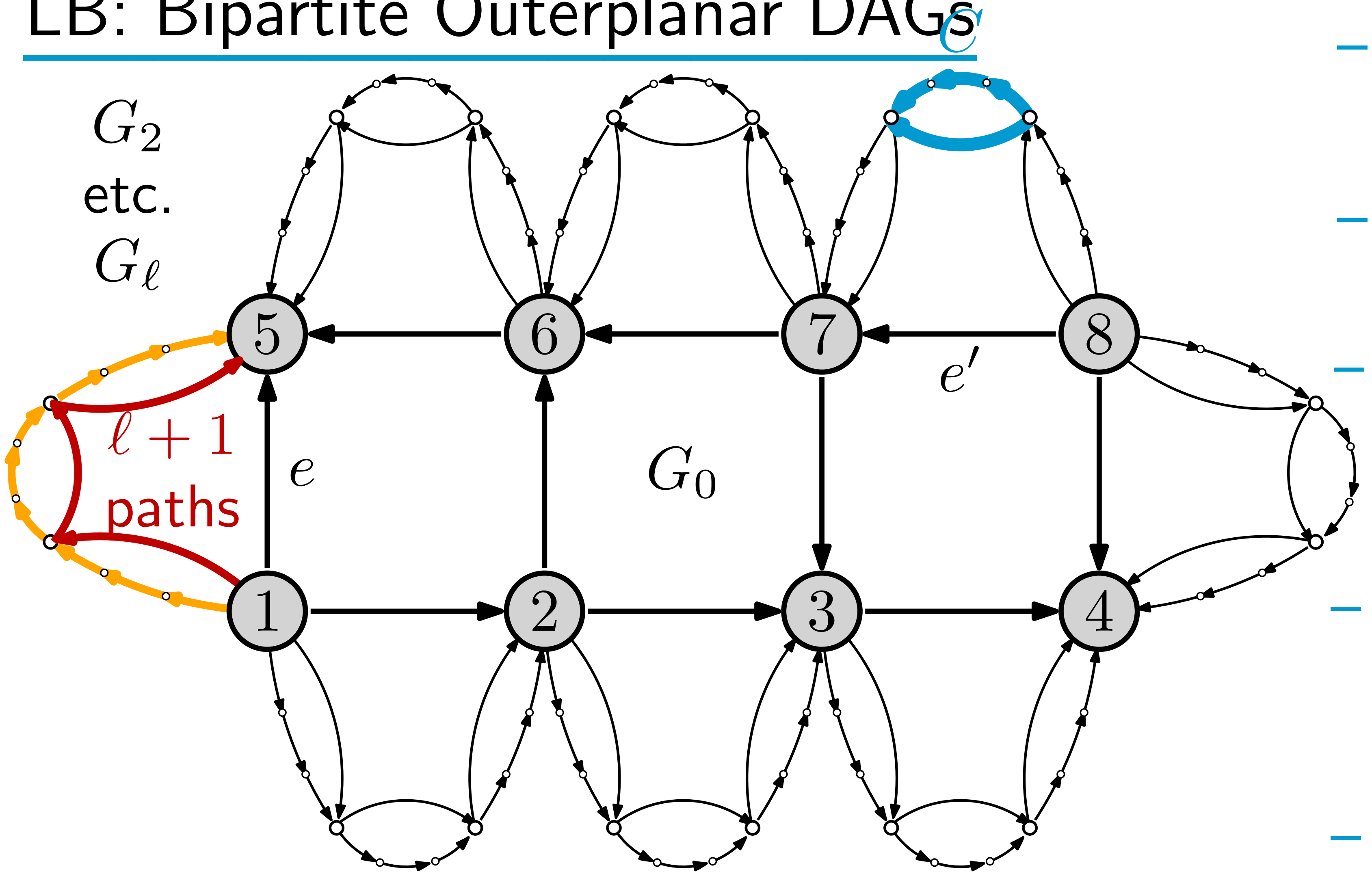

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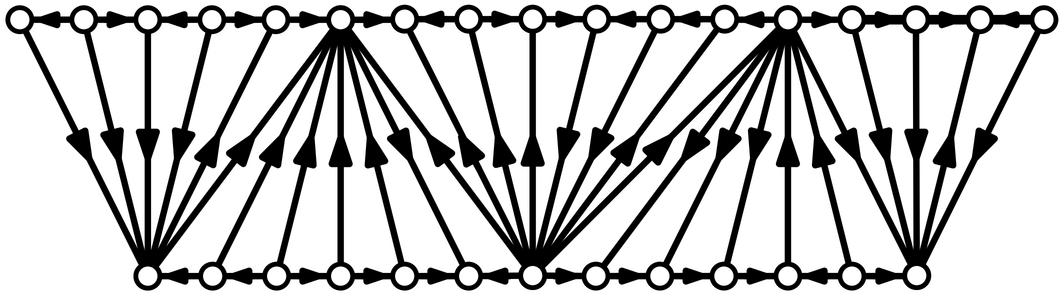
upward local crossing number $> \ell/6 \in \Omega(\log n_\ell) \cap \Omega(\Delta_\ell)$

ths

Upper Bounds

UP: Outer Paths



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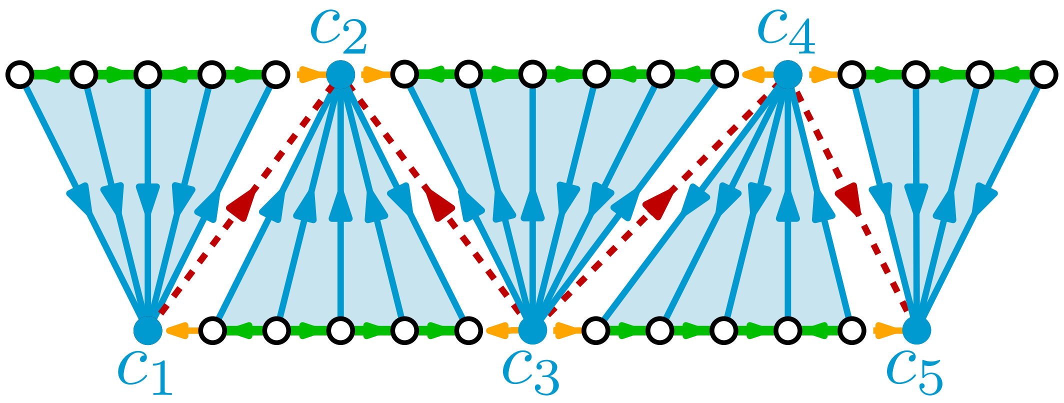
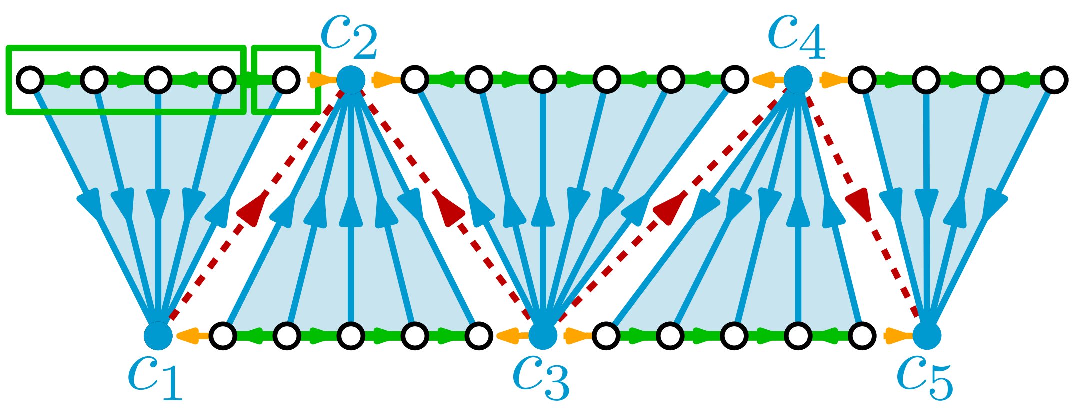
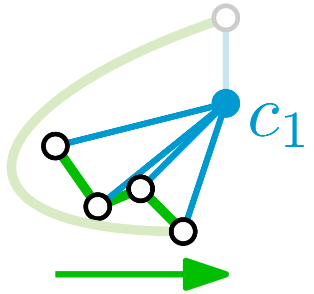


- Split G into fans.

UP: Outer Paths



- Split G into fans.
- Draw fans respecting edge direction
  - apex rightmost within fan
  - no crossings on internal edges
  - external edges: ≤ 2 crossings



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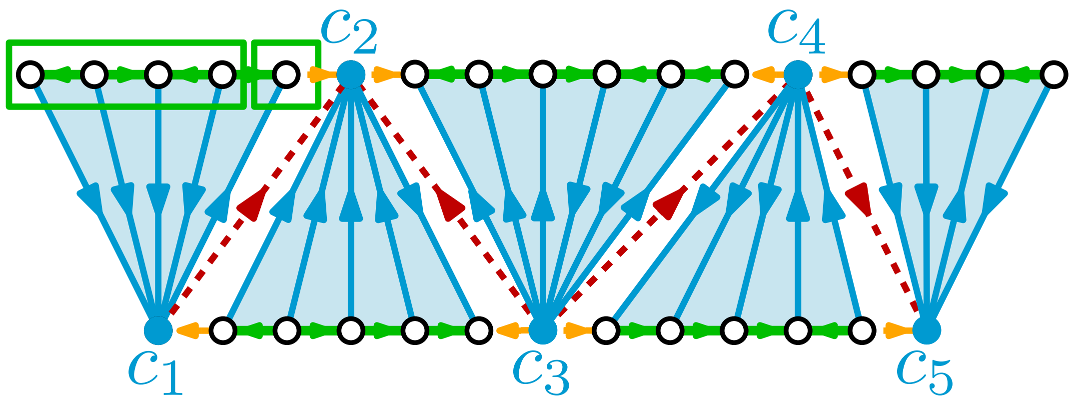

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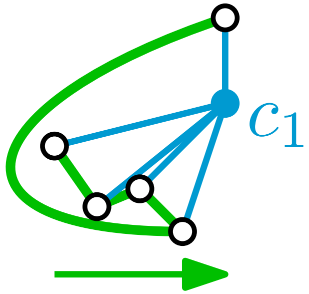

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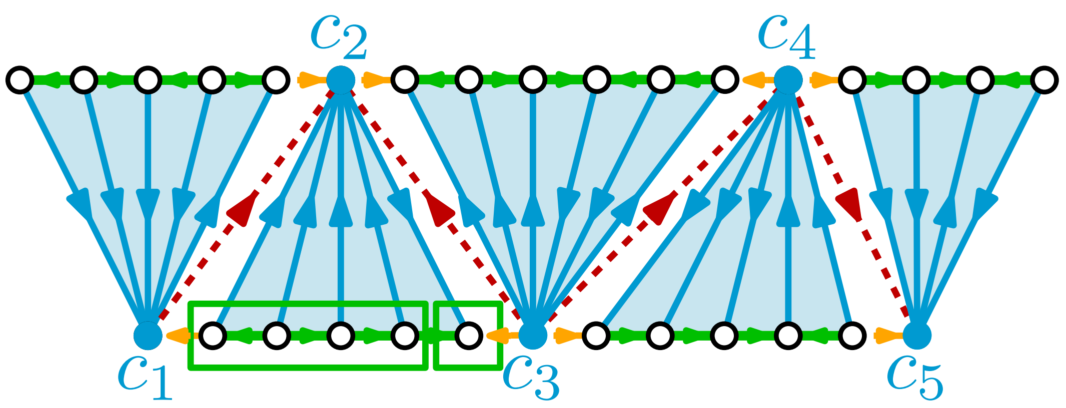

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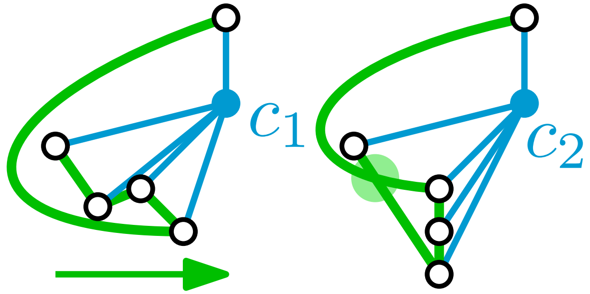

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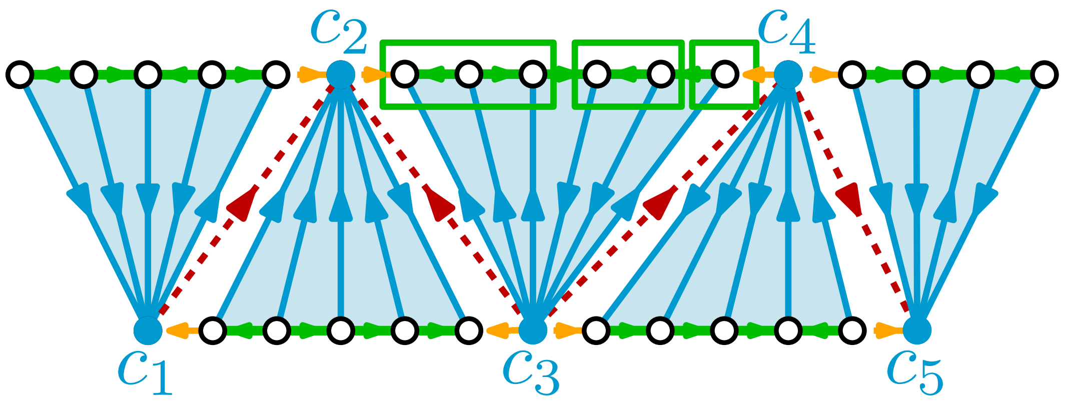

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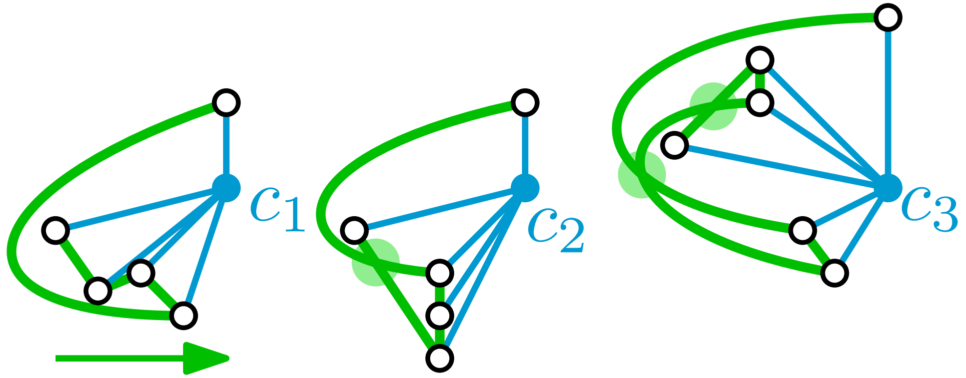

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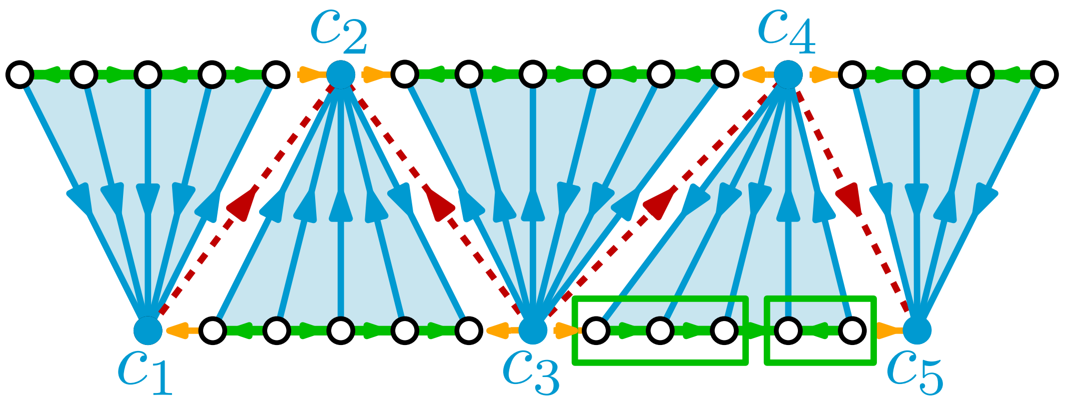

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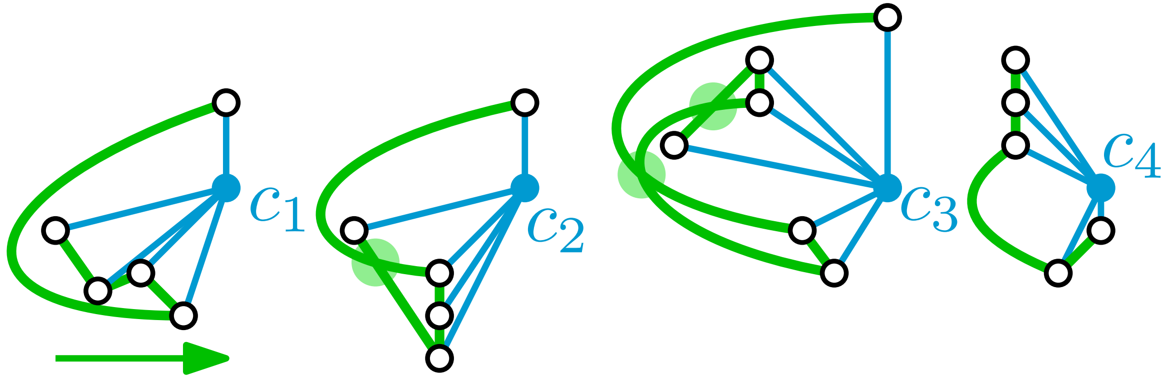

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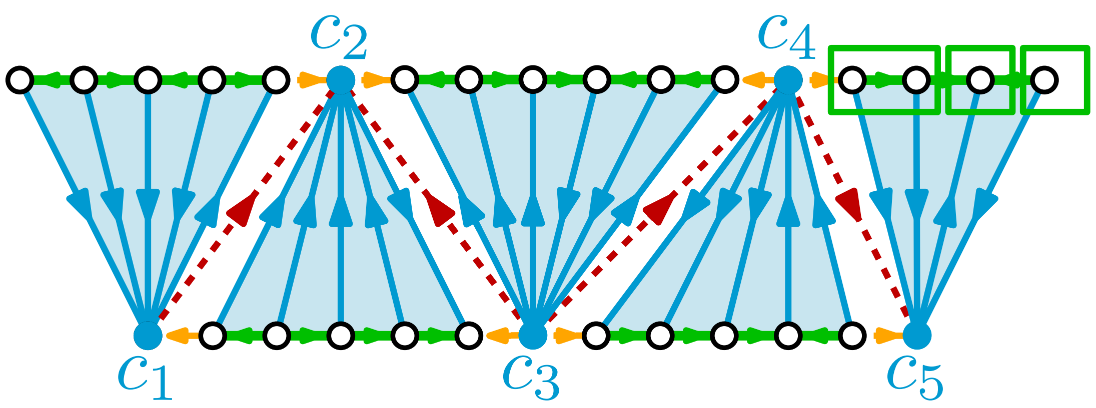

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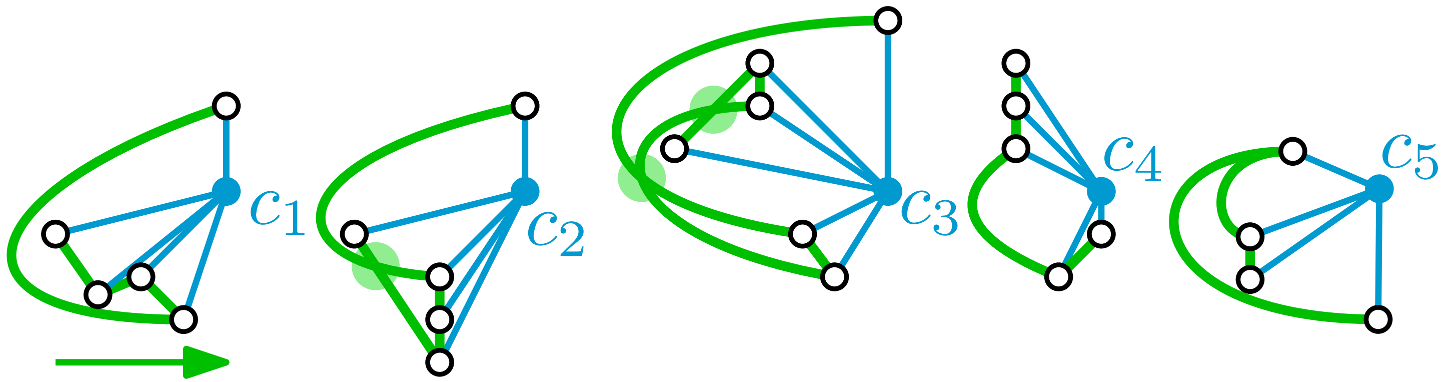

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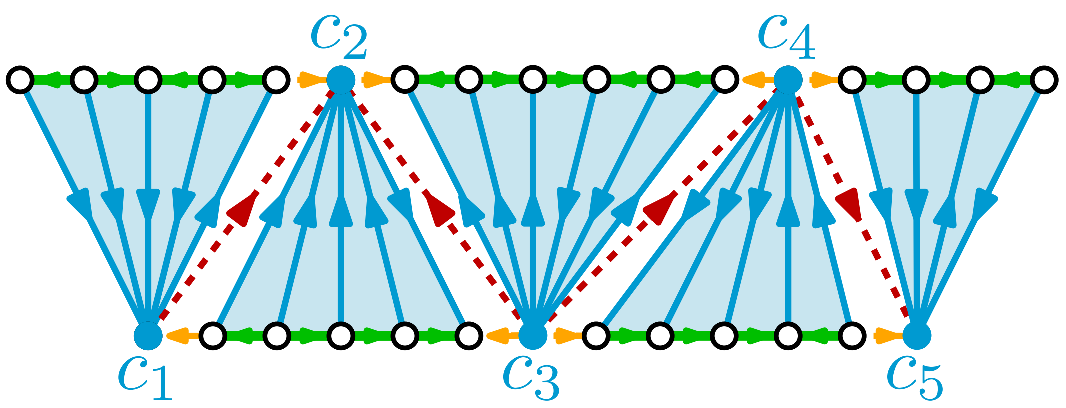

- Split G into fans.
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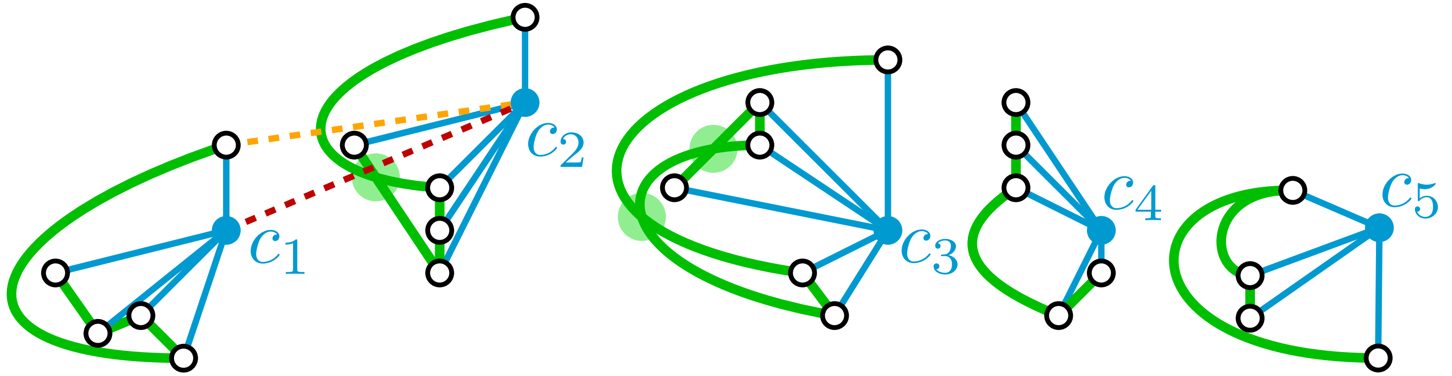
- Adjust height such that inter-fan edges are upward



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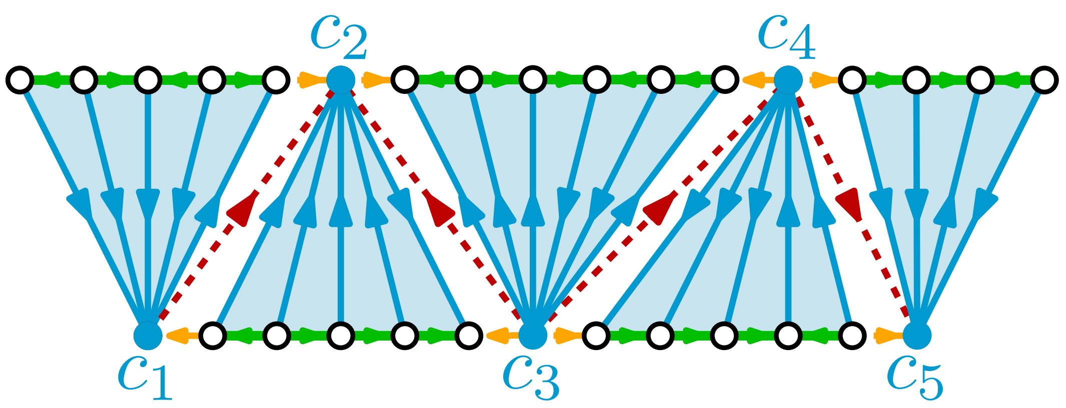

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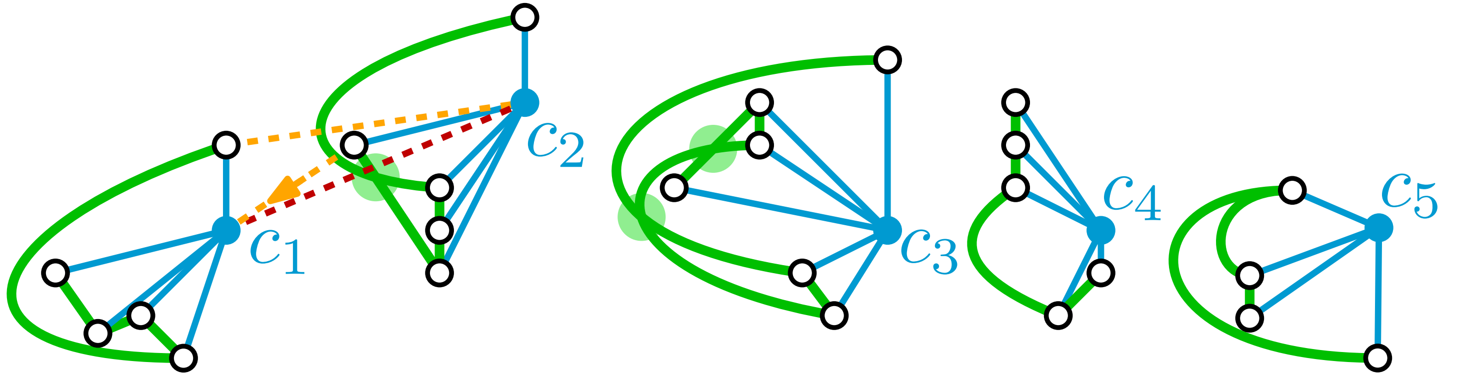

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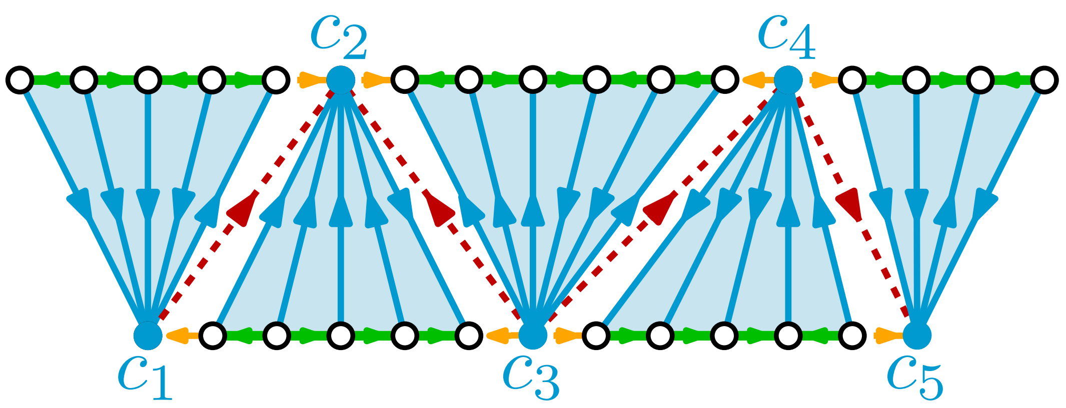

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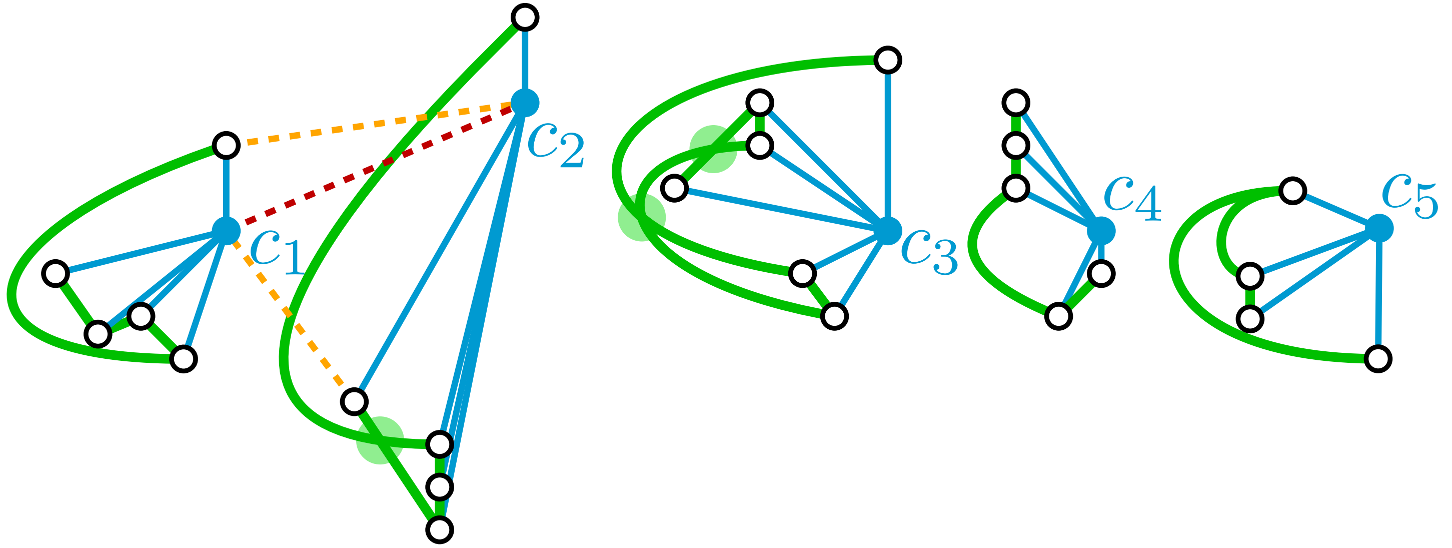

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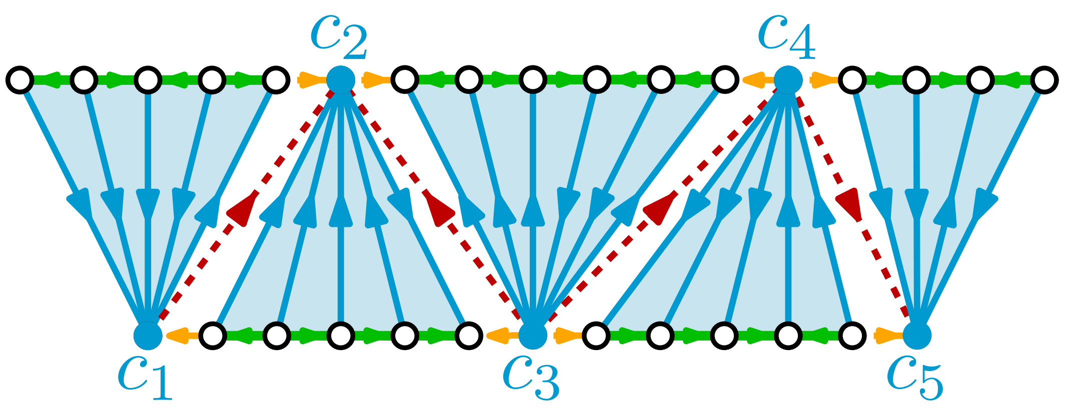

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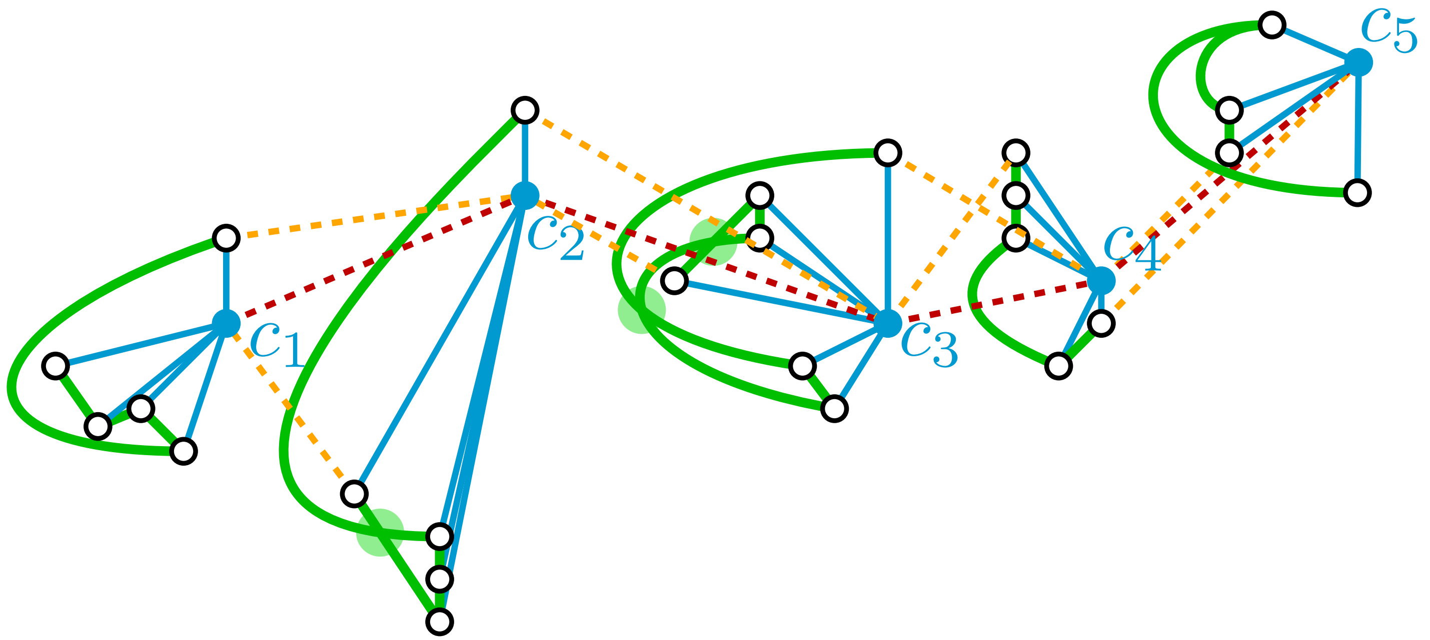

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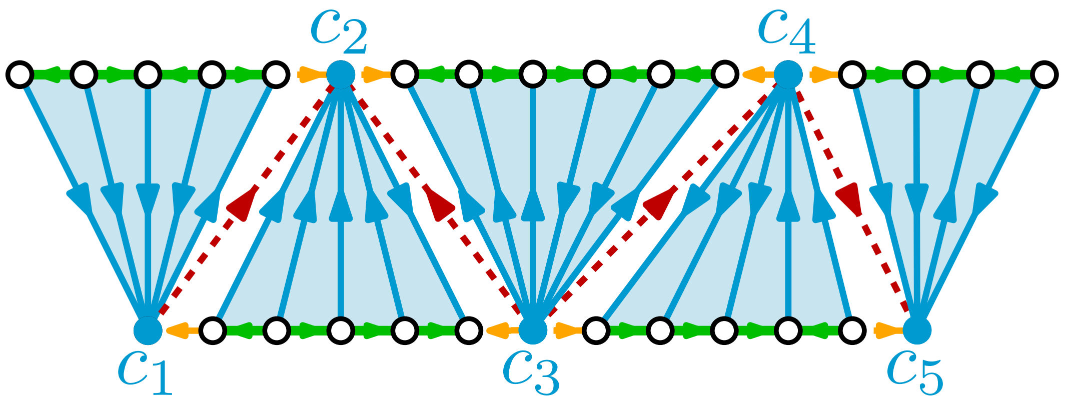

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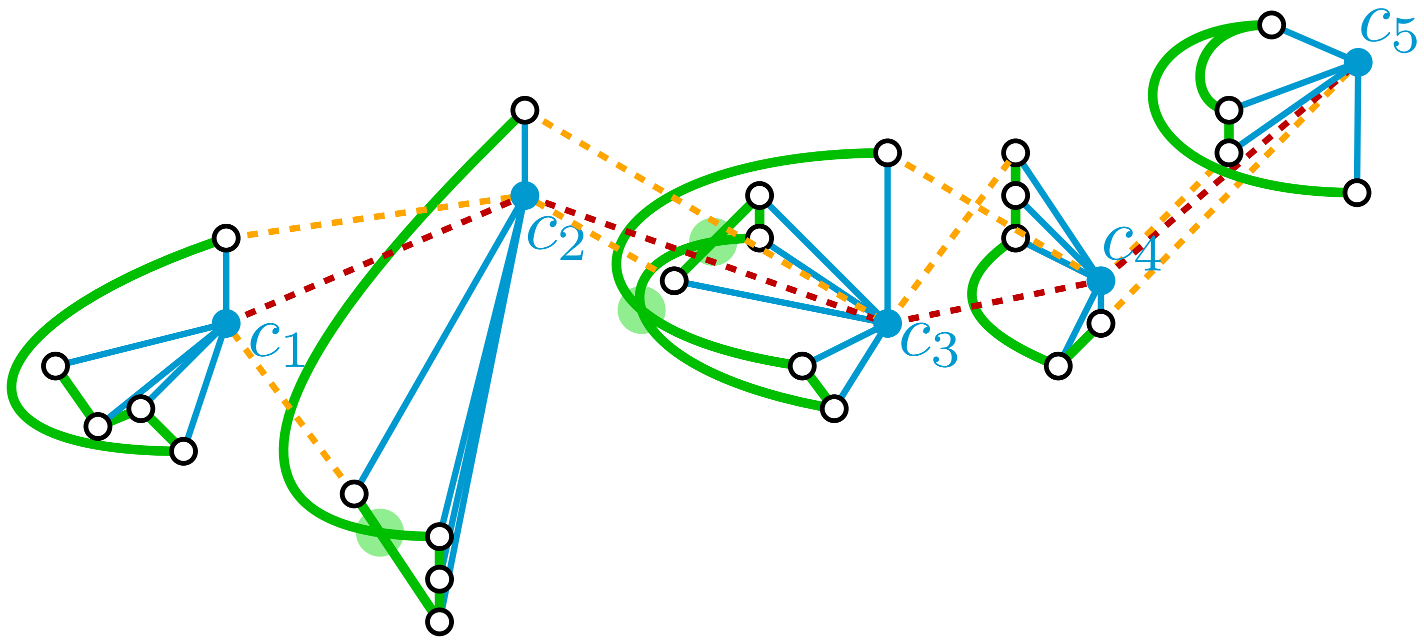

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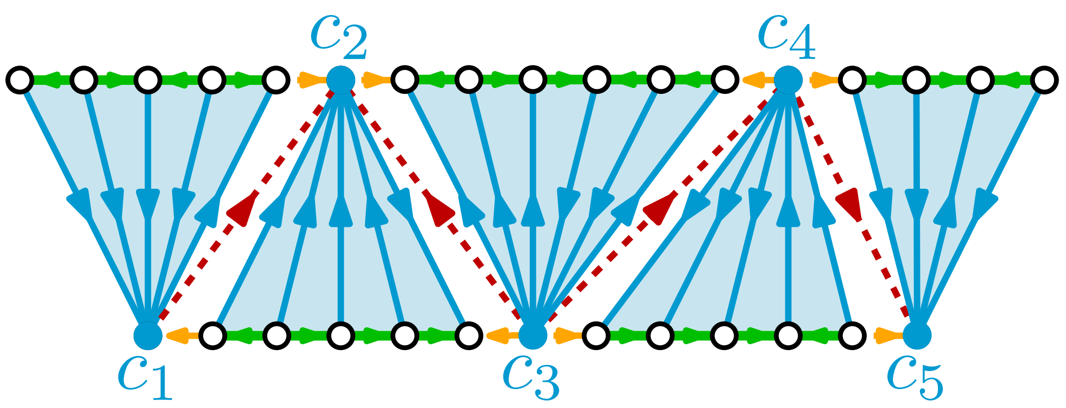

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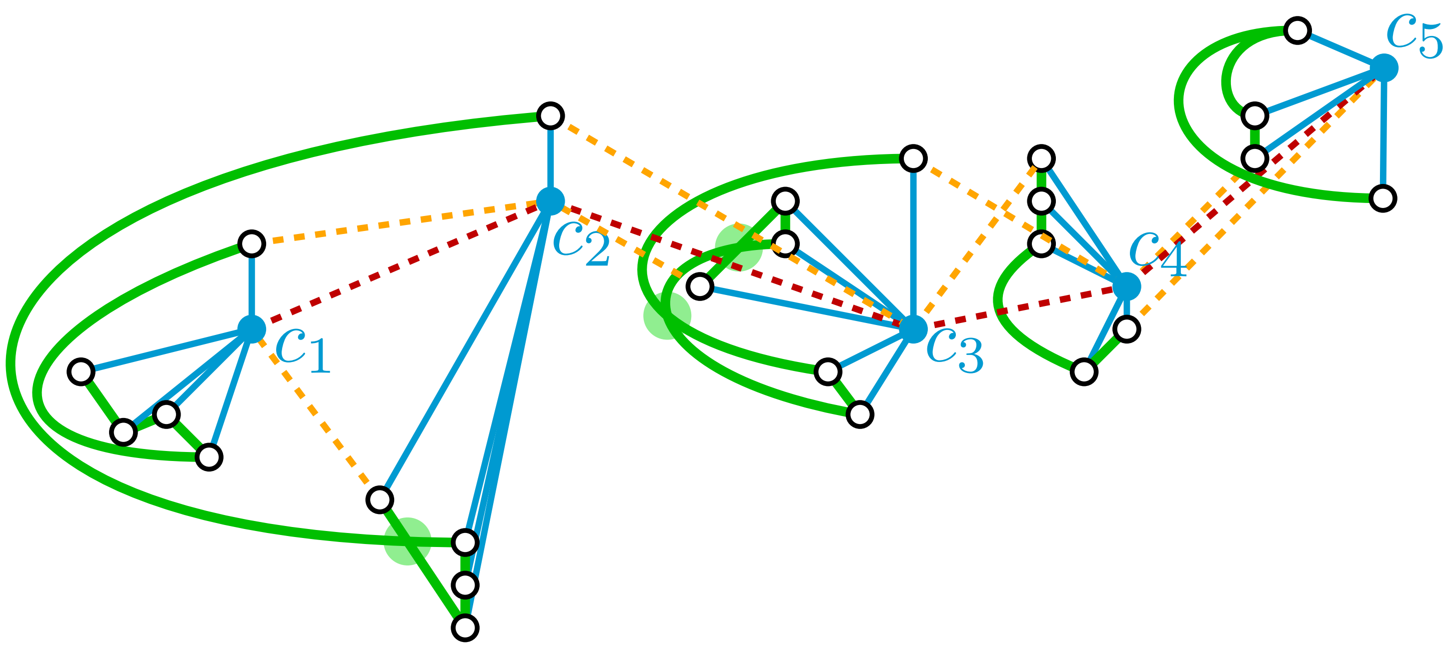

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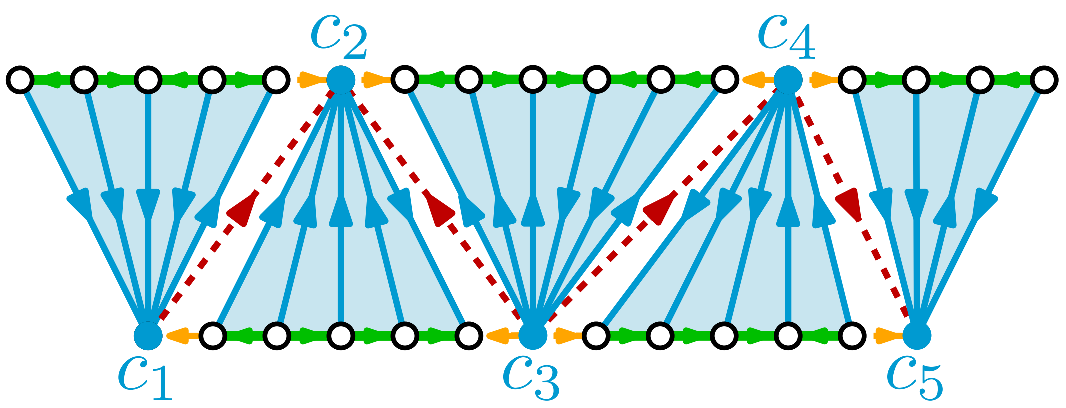

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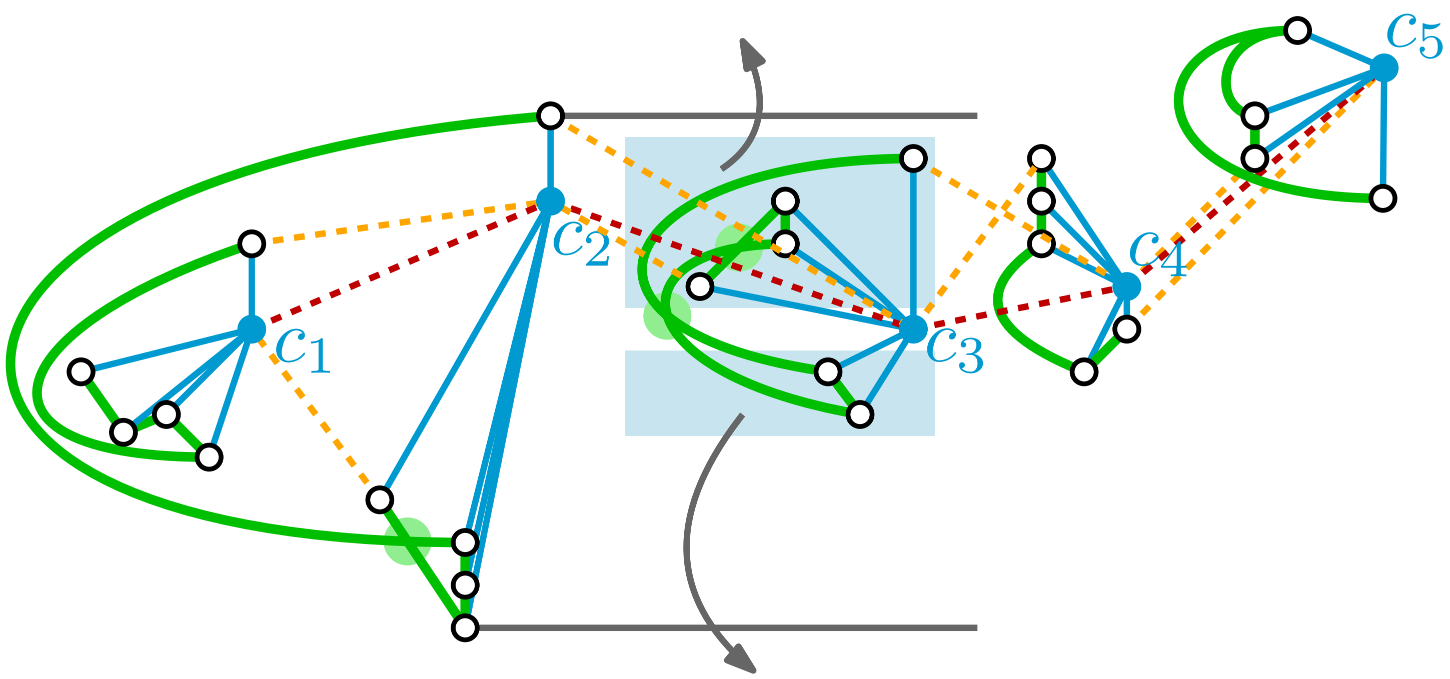

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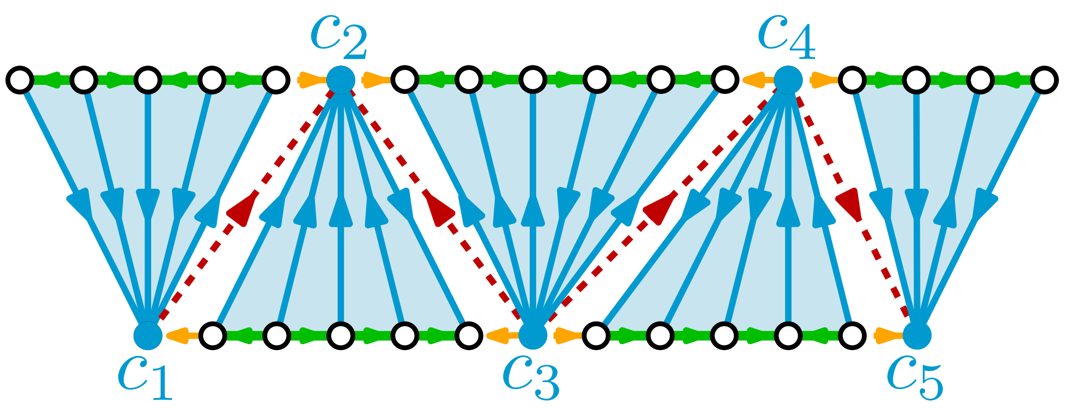

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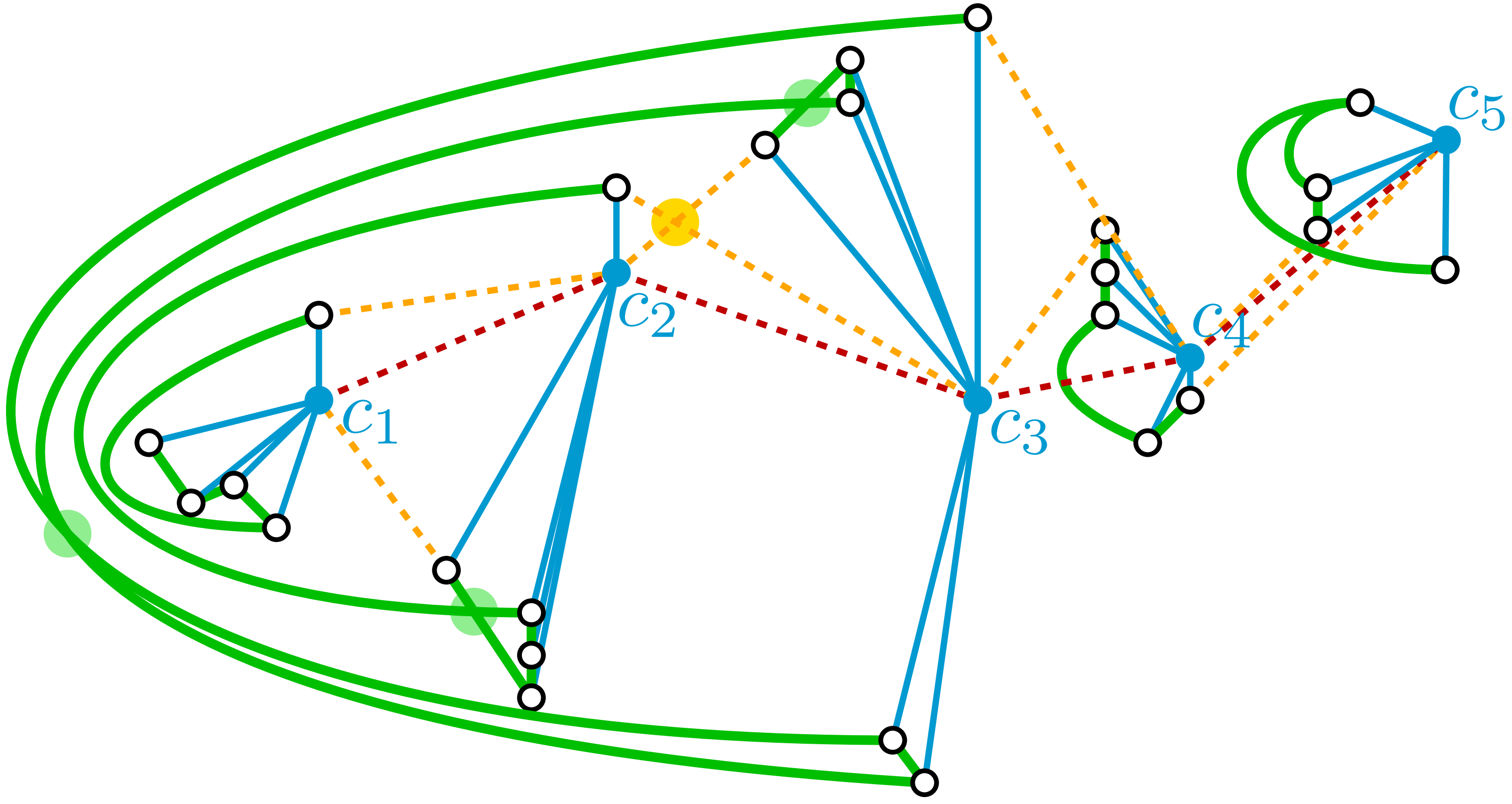

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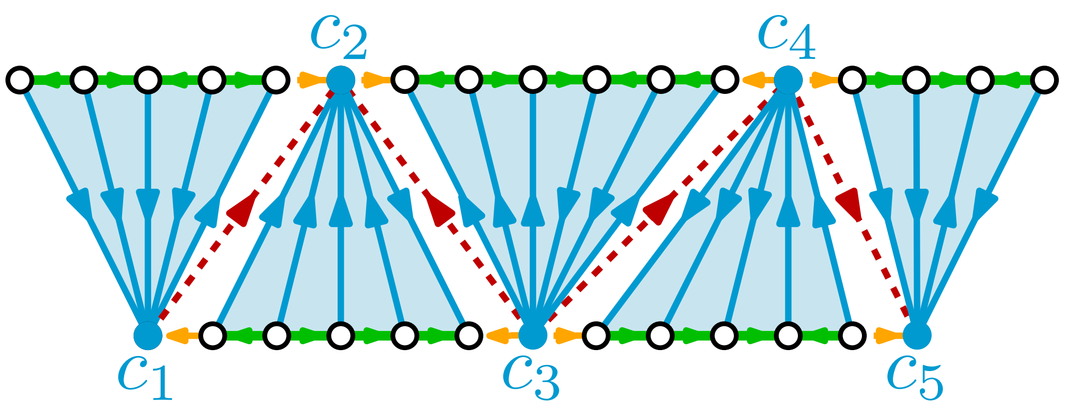

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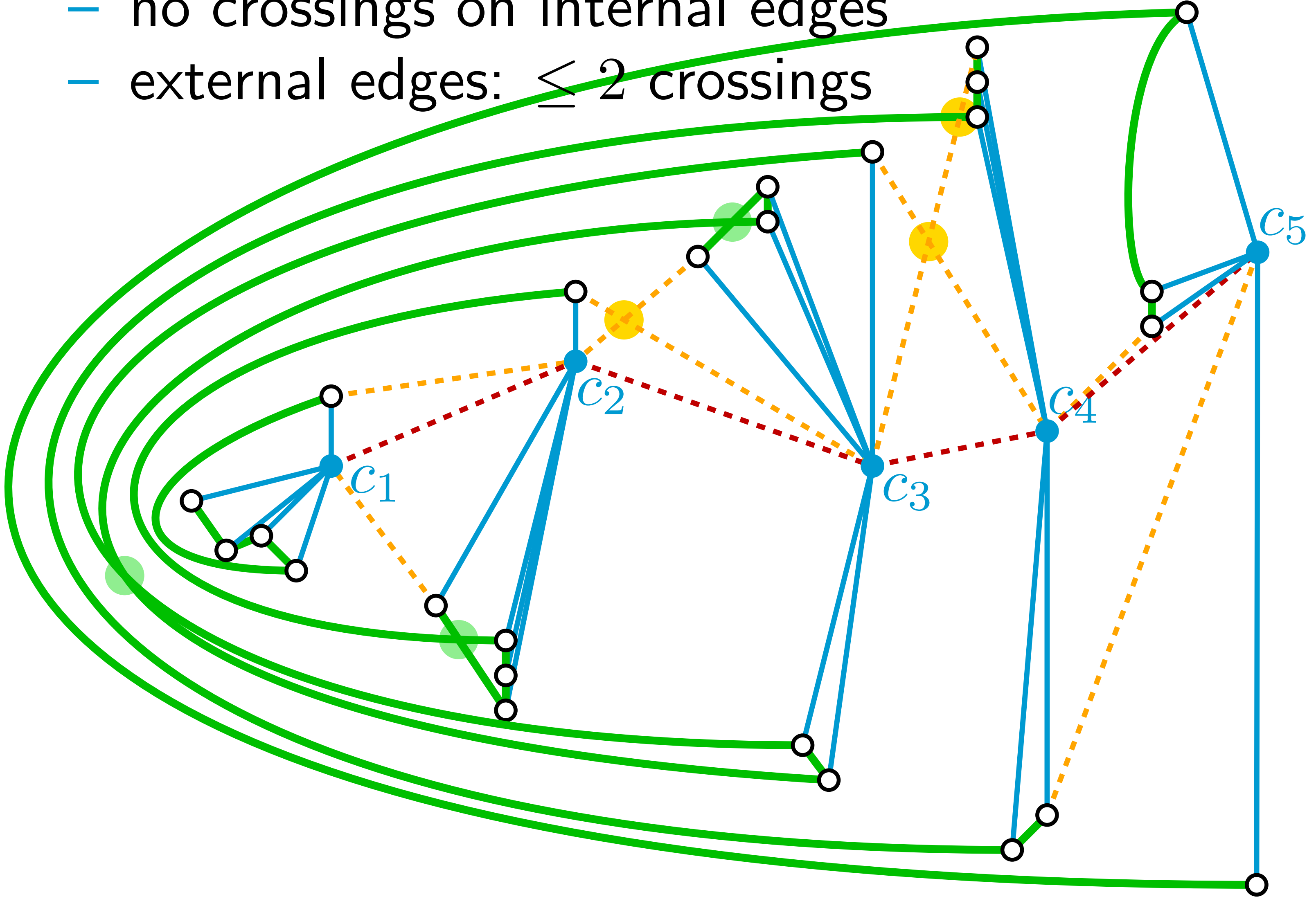

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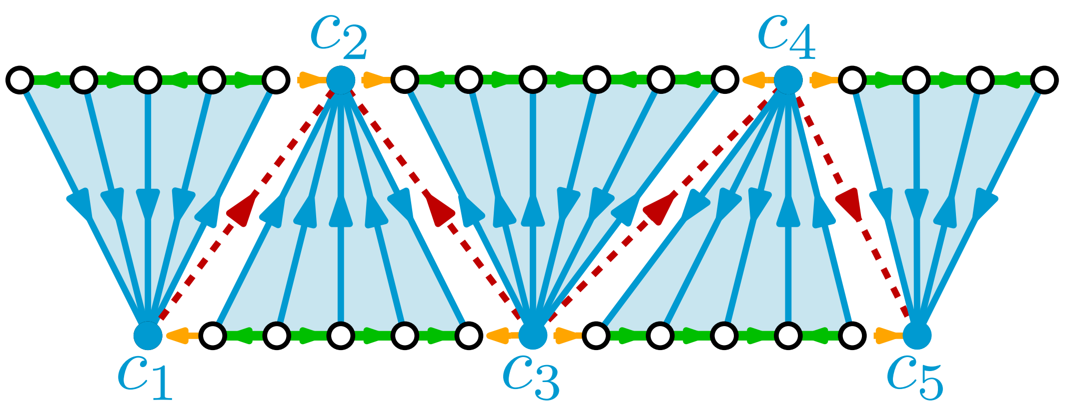

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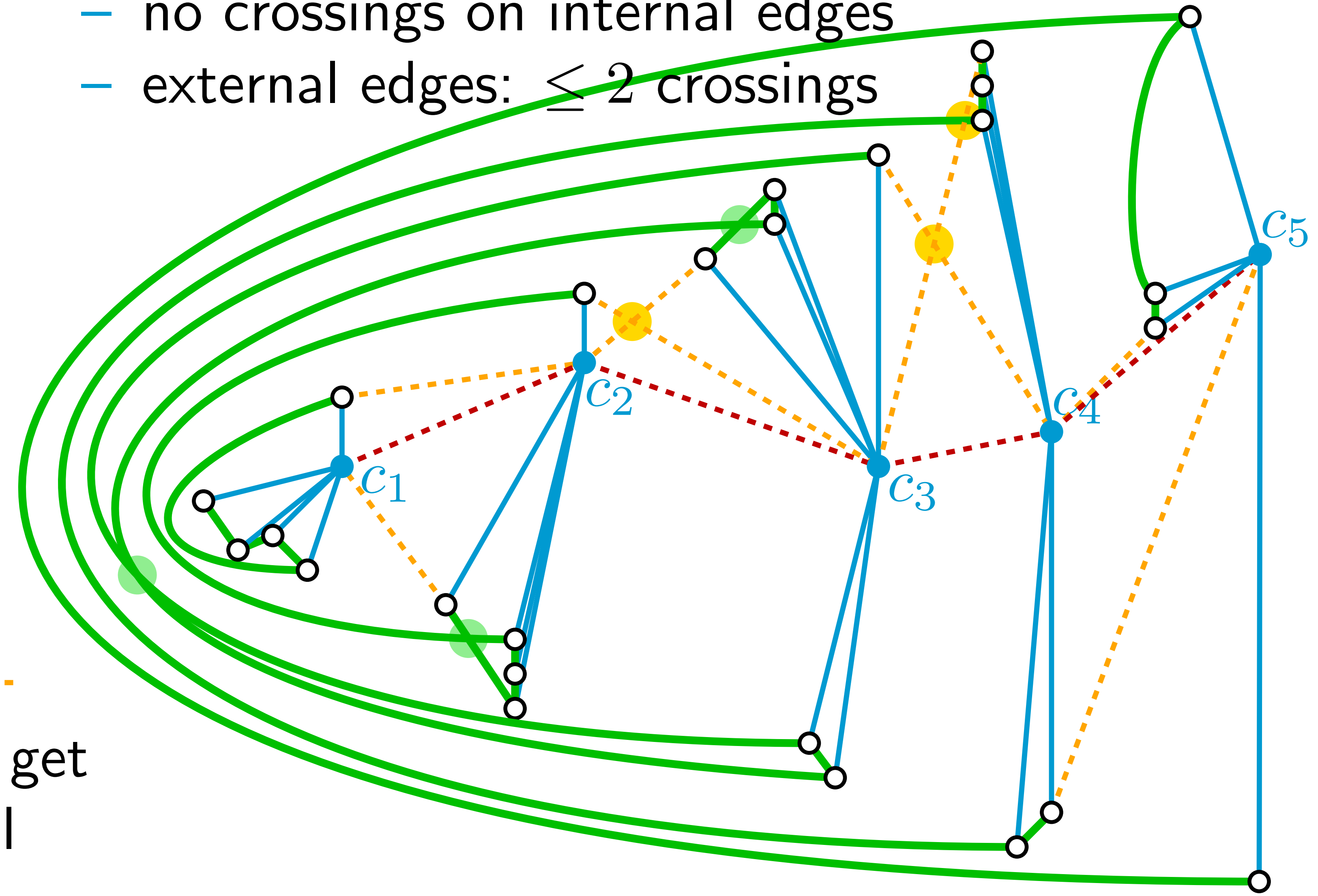
- Adjust height such that inter-fan edges are upward
- nest fans
- Internal inter-fan edges are not crossed
- At most two crossings on external inter-fan edges
- External intra-fan edges do not get more than two crossings in total



Complexity Results

Our Results: Complexity of Recognition

Problem: Given a digraph G, test whether it admits an upward 1-planar drawing

		Upward Planarity		Upward 1-Planarity	
Underlying planar graph	Acyclic orientation	Fixed embedding	Variable embedding	Fixed rot. system	Variable rot. system
Series-parallel	Multi-source Multi-sink	P	P		
	Single-source Single-sink	P	P		
General	Multi-source Multi-sink	P	NPC		
	Single-source Single-sink	P	P		

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General	Multi-source Multi-sink	P	NPC	NPC	NPC
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Our Results: Complexity of Recognition

Problem: Given a digraph G, test whether it admits an upward 1-planar drawing

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Underlying planar graph	Acyclic orientation	Fixed embedding	Variable embedding	Fixed rot. system	Variable rot. system
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1 source, 2 sinks

One K_4 minor

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1 source, 2 sinks

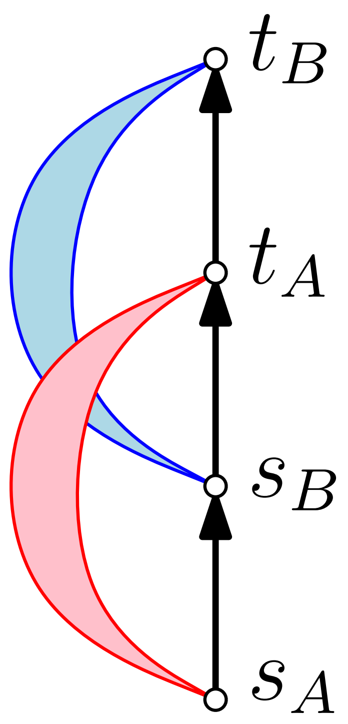
Always upward planar

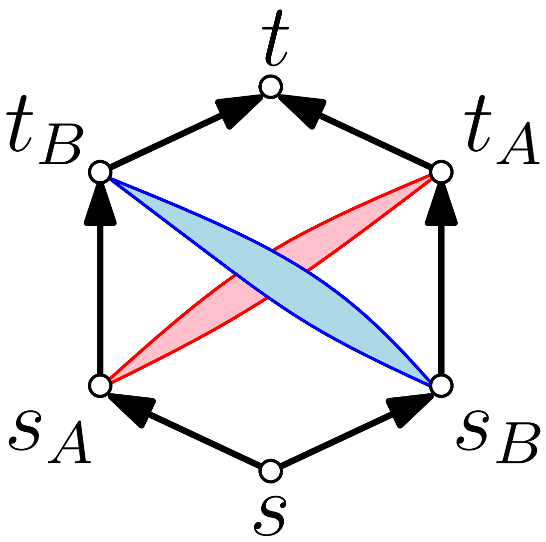
One K_4 minor

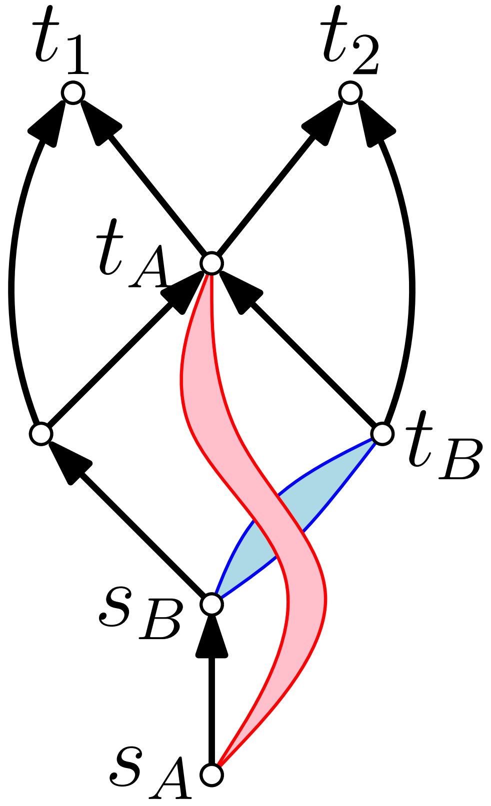
NP-hardness

Based on the different settings, we identify two subgraphs that must cross each other

- Every source-sink path in a subgraph crosses every source-sink path in the other
- Both subgraphs have a single source and a single sink, and their underlying graph is series-parallel







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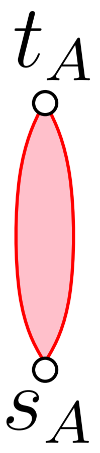
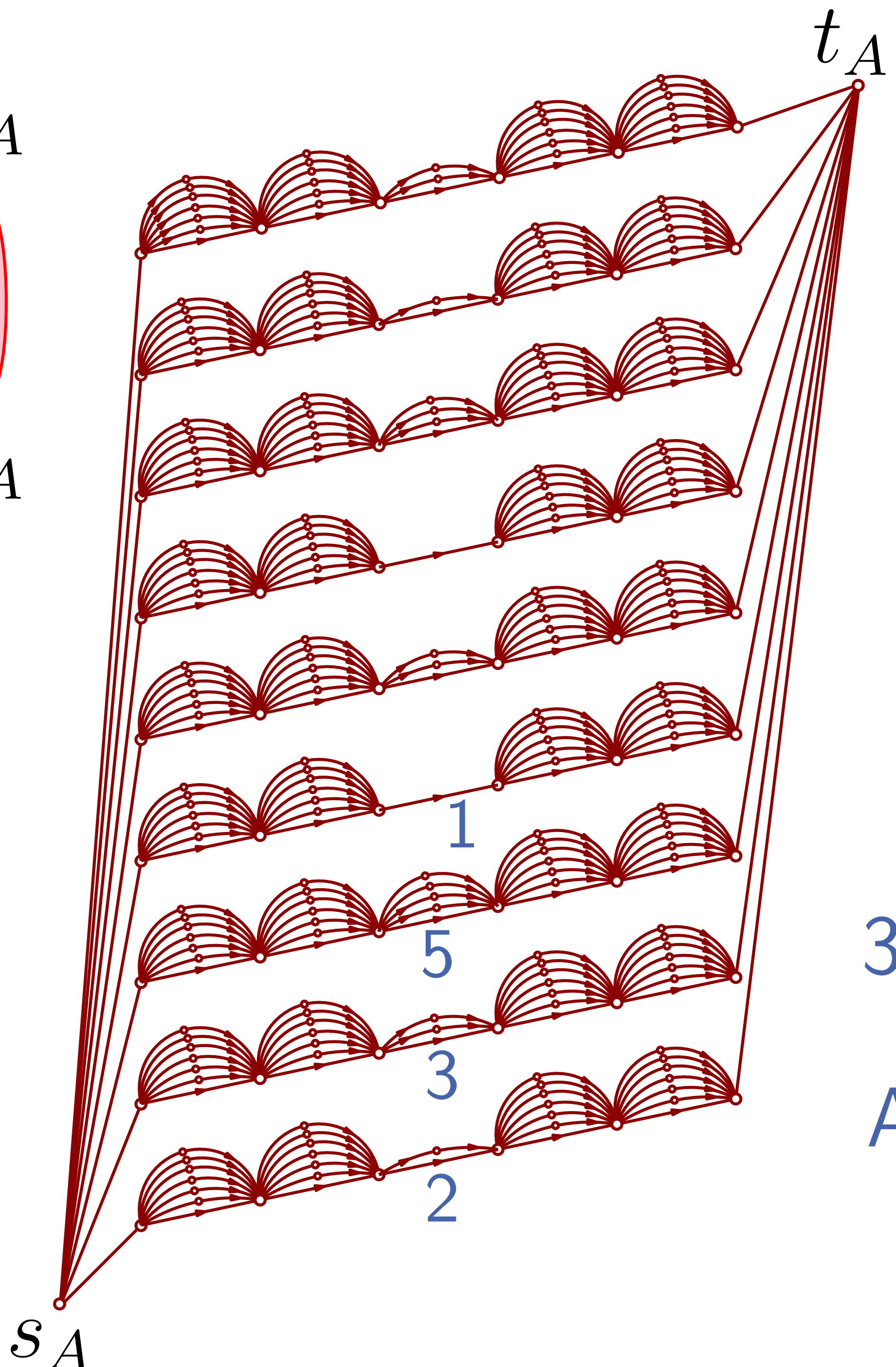
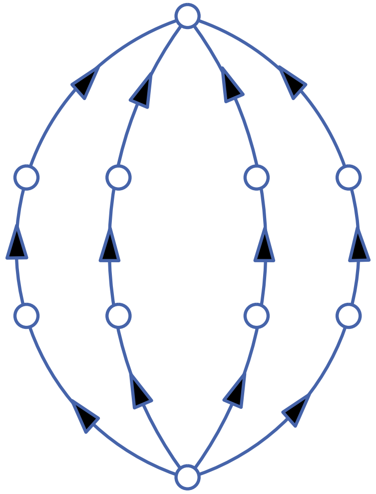
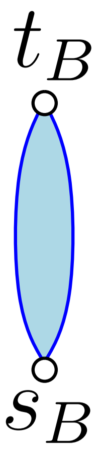




3-Partition instance:

$A = \{2, 3, 5, 1, \ldots\}$





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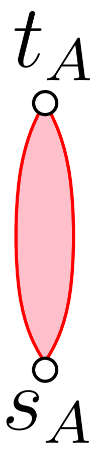

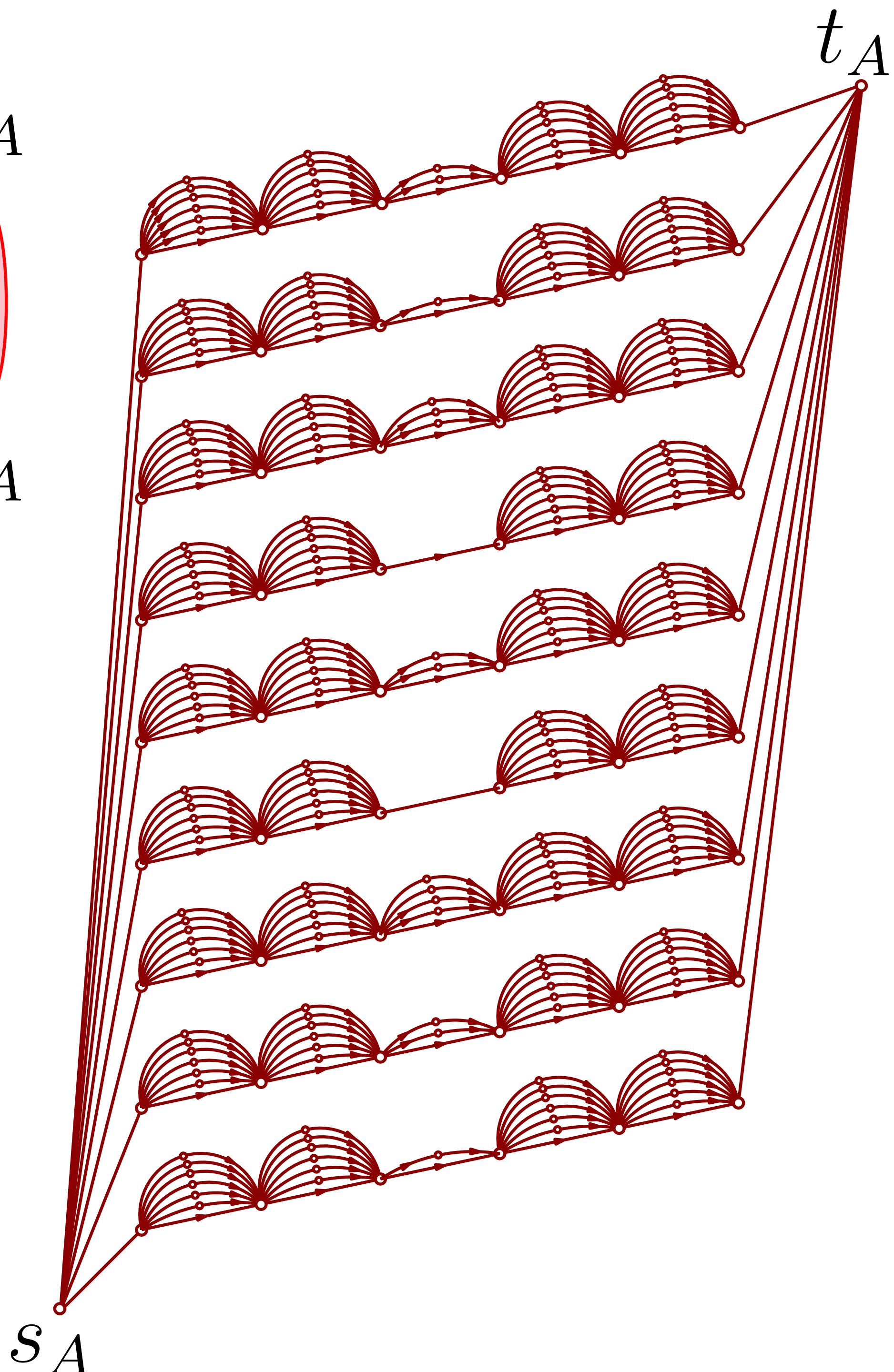

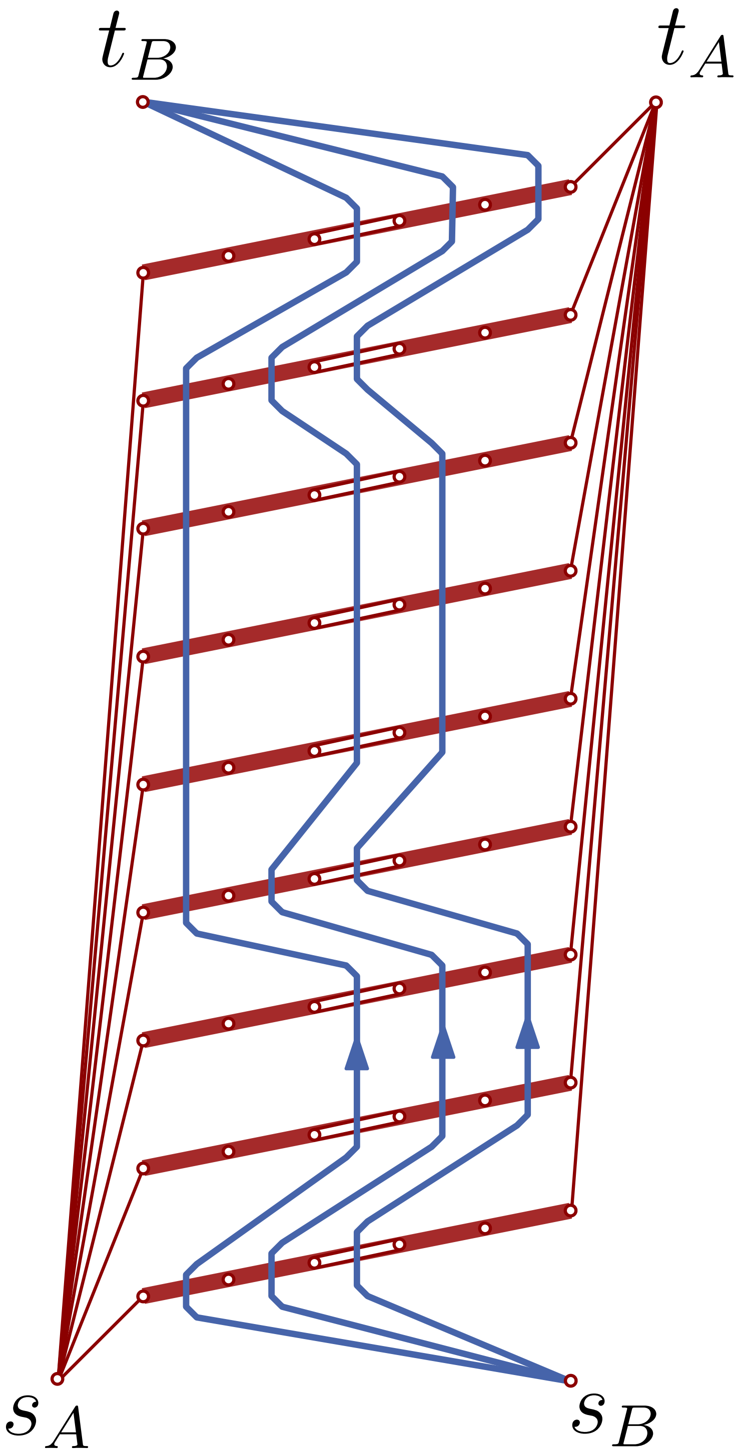

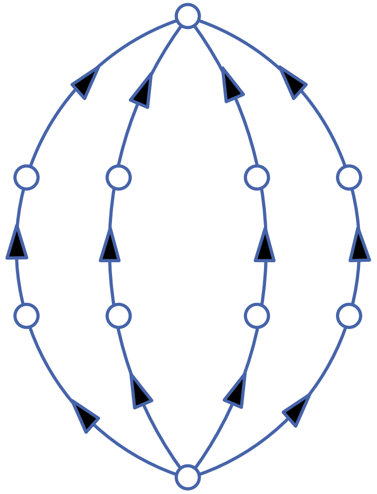

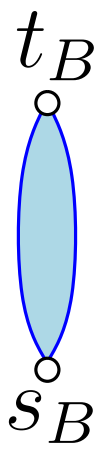

A Positive Result

Problem: Given a digraph G, test whether it admits an upward 1-planar drawing

Theorem: If all vertices are required to lie on the outer face, Upward 1-planarity can be tested in linear time for single-source DAGs

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In particular, in this *outer* setting, upward 1-planarity can be tested in linear time

- S.-H. Hong, P. Eades, N. Katoh, G. Liotta, P. Schweitzer, and Y. Suzuki. *A linear-time algorithm for testing outer-1-planarity*. Algorithmica, 2015.
- C. Auer, C. Bachmaier, F. J. Brandenburg, A. Gleißner, K. Hanauer, D. Neuwirth, and J. Reislhuber. *Outer 1-planar graphs.* Algorithmica, 2016.

Linear-time algorithm for upward outer 1-planarity: Main ingredients

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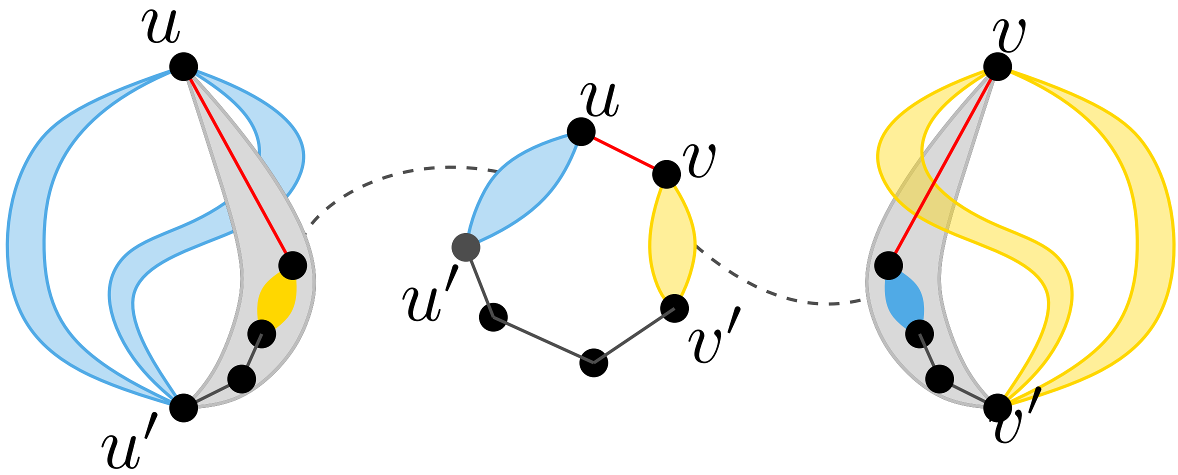

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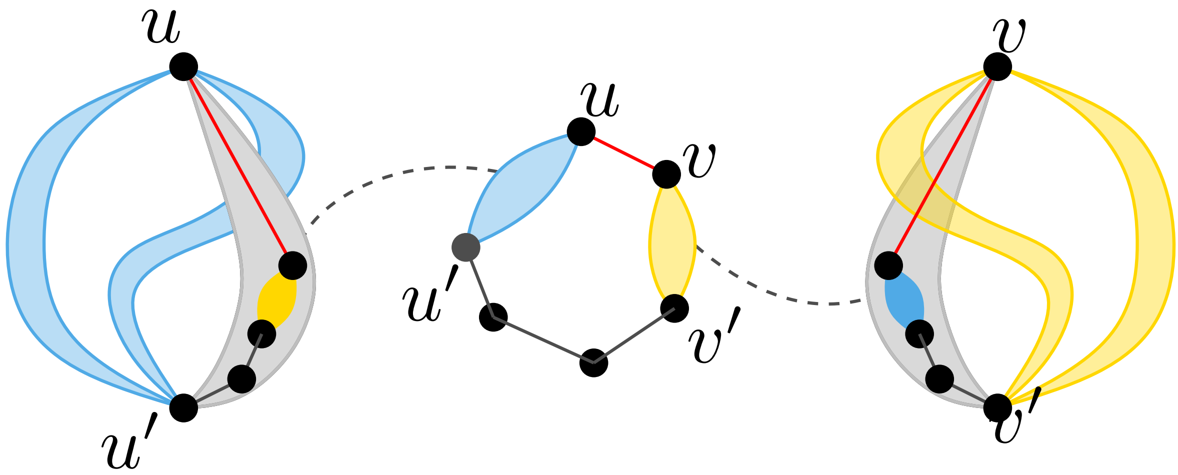

Each skeleton has a constant number of embeddings, with acyclic planarizations, satisfying the local properties $\rightsquigarrow$ enumerate and check the global property!

Summary

- We defined upward k-planarity and upward local crossing number of DAGs
- We gave upper and lower bounds for various graph classes
- Upper 1-planarity testing is NP-complete

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