The Computational Complexity of the ChordLink Model

Philipp Kindermann[1], **Jan Sauer**[2], **and Alexander Wolff**[2]

1 **Universität Trier, Trier, Germany**
`kindermann@uni-trier.de`

2 **Universität Würzburg, Würzburg, Germany**
`firstname.lastname@uni-wuerzburg.de`

Abstract

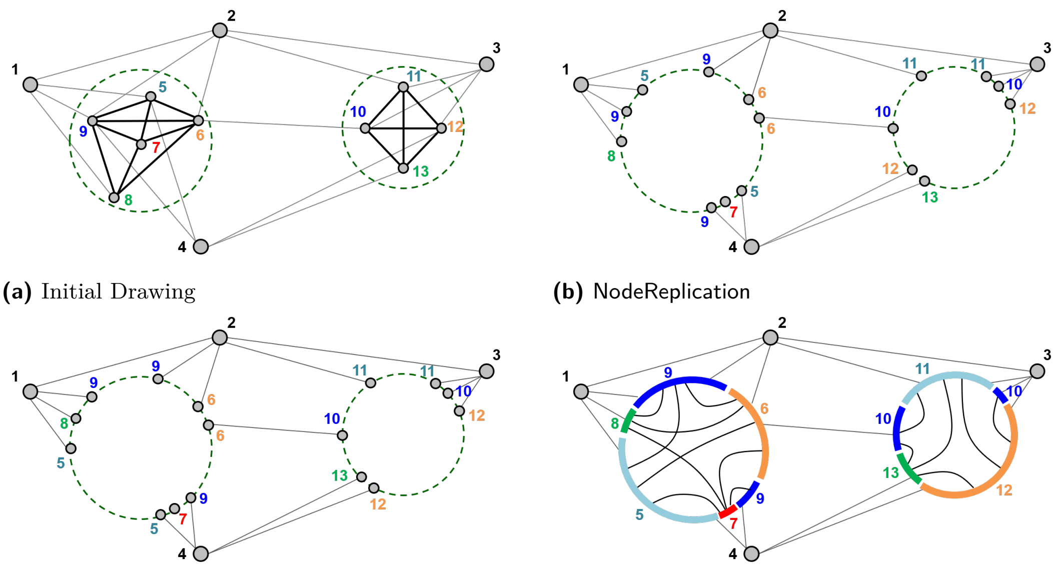
In order to visualize well-clustered graphs with many intra-cluster but few inter-cluster edges, hybrid approaches have been proposed. For example, ChordLink draws the clusters as chord diagrams and embeds these into a node-link diagram that represents the overall structure of the clustered graph. The ChordLink approach consists of four steps; node replication, node permutation, node merging, and chord insertion. In this paper, we focus on the optimization problems defined by two of these steps. We show that the decision version of the problem defined by node permutation is NP-complete and present an efficient algorithm for a special case. For chord insertion, we show that it is NP-complete to decide whether a crossing-free placement of the chords exists. Moreover, it is APX-hard to minimize the number of crossings among the chords. Our results answer an open question posed by Angori, Didimo, Montecchiani, Pagliuca, and Tappini, who introduced ChordLink [TVCG 2021].

1 Introduction

Node-link diagrams represent an intuitive tool for visualizing graphs. For dense graphs, however, node-link diagrams tend to degenerate into unintelligible hairballs. Less intuitive, but more robust visualization paradigms such as adjacency matrices can be a remedy. In practice, however, large graphs are often "globally sparse" and just "locally dense" [3]. This is the case, for example, in social networks such as collaboration and financial networks [4], but also in biological networks [9]. For visualizing such graphs, hybrid representations have been invented. In intersection-link representations, for example, each vertex is represented by a geometric object and each each is either represented by a curve connecting the two objects or, if it belongs to a dense subgraph, by a non-empty intersection of the two objects [2, 1]. Another example of a hybrid representation is NodeTrix [7], which uses matrices for dense subgraphs and links between the matrices for the global graph structure. ChordLink, recently introduced by Angori, Didimo, Montecchiani, Pagliuca, and Tappini [3], combines very effectively so-called chord diagrams [8] for dense subgraphs, again with links between them for the overall graph structure. In a chord diagram, each vertex is represented by one or several circular arcs, and each edge is a chord between any two arcs that represent the endpoints of the edge. Turning a given node-link diagram into a ChordLink visualization can, for example, be triggered by a user in an interactive system. We now formalize the ChordLink model and the four steps that are performed in order to compute a ChordLink visualization from a node-link diagram. To this end, for a graph G, let $V(G)$ be its vertex set and let $E(G)$ be its edge set. For a positive integer k, let $[k] = \{1, 2, \dots, k\}$.

The ChordLink Model. Given a node-link diagram Γ of a graph G, a cluster $C \subseteq V(G)$, and a circle R that contains only the vertices in C (at their positions in Γ), a ChordLink

38th European Workshop on Computational Geometry, Perugia, Italy, March 14–16, 2022.



(a) Initial Drawing **(b)** NodeReplication

(c) NodePermutation **(d)** NodeMerging+ChordInsertion

Figure 1 The steps of the ChordLink approach (drawings from Angori et al. [3]).

visualization of G locally modifies Γ such that $G[C]$ is drawn as a chord diagram with the vertices of C on R. There are four steps; see Fig. 1, which treats two clusters simultaneously.

NodeReplication: For each node $w \in C$ connected to a node $u \notin C$, create a copy w_u of w on the intersection of the edge (w, u) with R, and add the edge (w_u, u); see Fig. 1b.

NodePermutation: Copies v_u and w_u of different nodes v and w that are connected to the same node $u \notin C$ can be exchanged in the order of the node copies on R. This step naturally defines the optimization problem NodePermutation, where the aim is to find a permutation of the node copies on R that exchanges only copies connected to the same node outside of R and maximizes the total number of pairs of consecutive copies of the same node; see Fig. 1c. (The number of such pairs increases by 3 when going from Fig. 1b to Fig. 1c.)

NodeMerging: Replace each maximal subsequence of consecutive copies of a node w along R by a circular arc c_w; see Fig. 1d.

ChordInsertion: For each edge $(v, w) \in G[C]$, select an arc c_v representing v and an arc c_w representing w, and insert a chord that connects c_w and c_z in the interior of R; see Fig. 1d. Angori et al. [3] suggest to minimize the total number of crossings among the chords. (They also suggest maximizing the smallest angle formed by any pair of crossing chords, but we do not consider this here.) This defines the optimization problem CrossingMinimal ChordInsertion.

For NodePermutation, Angori et al. [3] describe a dynamic program that yields optimal solutions if the node copies that are adjacent to the same external node form intervals along R. If this condition does not hold, they simply split R into maximal pieces where the condition does hold and treat each piece seperately, which yields a heuristic overall solution.

For CrossingMinimal ChordInsertion, Angori et al. suggest a greedy algorithm that first draws the chords whose endpoints are both represented by unique arcs. Then it adds the other chords one by one, making the currently best choice in terms of crossings (and, with lower weight, in terms of crossing angles). It draws chords as Bézier curves; see Fig. 1d.

Contribution. First, we prove that NodePermutation is NP-complete; see Section 2. Then, we give an efficient algorithm for NodePermutation for the special case that the neighborhood of C contains only two vertices; see Section 3. Finally, we show that (even a rather special case of) CrossingMinimal ChordInsertion is APX-hard; see Section 4.

2 NP-Completeness of NodePermutation

Above, we have stated NodePermutation as an optimization problem. We now formally define the corresponding decision problem. In the ChordLink model, for every vertex c in the cluster C and each neighbor $g \notin C$ of c, a copy of c is placed on the circle R. Since G is simple, each copy can be described as a unique pair (c, g). Abstracting from the original problem, we call c the *color* and g the *group* of the pair (c, g). This leads to the following formulation of the problem, where we associate every vertex of C with a distinct color.

Let $\mathcal{C}$ be a set of colors, let $\mathcal{G}$ be a set of groups, and let $L = (L_1, \dots, L_n, L_1)$ be a circular list of *distinct* pairs where, for $i \in [n]$, $L_i = (c_i, g_i) \in \mathcal{C} \times \mathcal{G}$. Define $\mathcal{G}(L) = (g_1, \dots, g_n, g_1)$, $\mathcal{C}(L) = (c_1, \dots, c_n, c_1)$, and let $N(L)$ be the number of pairs of consecutive equal entries of $\mathcal{C}(L)$. Given $\mathcal{C}$, $\mathcal{G}$, L, and an integer $K > 0$, find a permutation π of L such that (i) $\mathcal{G}(\pi(L)) = \mathcal{G}(L)$ and (ii) $N(\pi(L)) \geq K$. Note that requirement (i) ensures that we can permute only elements of L that belong to the same group.

▶ **Theorem 1.** *NodePermutation is NP-complete.*

Proof. The problem is in NP since we can verify a permutation easily.

To show hardness, we reduce from 3SetCover. This problem generalizes VertexCover in cubic graphs, which is NP-hard [5]. In the decision version of 3SetCover, given a finite universe U (the edge set of the cubic graph), a family $\mathcal{S}$ of size-3 subsets of U (for each vertex, its three incident edges), and an integer $k > 0$, the task is to find a subfamily $\mathcal{S}'$ (corresponding to a vertex cover) of $\mathcal{S}$ of size at most k that covers U. (In the special case of VertexCover, each element of the universe appears in exactly two elements of $\mathcal{S}$.)

Given an instance $(U, \mathcal{S}, k)$ of 3SetCover, we construct an instance $(\mathcal{G}, \mathcal{C}, L, K)$ of NodePermutation such that one is a yes-instance if and only if the other is a yes-instance. Let $U = \{u_1, \dots, u_n\}$, $\mathcal{S} = \{S_1, \dots, S_m\}$, and, for $i \in [m]$, let $S_i = \{u_{i_1}, u_{i_2}, u_{i_3}\}$ with $i_1, i_2, i_3 \in [n]$. To construct an instance of NodePermutation, we use only a single color $c^\star$ that appears in more than one entry; all other entries have a unique color. Furthermore, we have one group for every $u_i \in U$, and there is exactly one entry $(c^\star, u_i)$, and we have one additional group z such that there is no entry $(c^\star, z)$; the entries with group z basically serve as *blockers* between the gadgets. The full reduction is as follows:

$$
\begin{aligned}
K &= n - k, \\
\mathcal{G} &= U \cup \{z\}, \text{where } z \notin U, \text{ and} \\
\mathcal{C} &= \{c^\star\} \cup \{c_1, \dots, c_n\} \cup \textstyle\bigcup_{i=1}^{m} \{c_{i,1}, c_{i,2}, c_{i,3}, c_{i,4}, c_{i,5}\}, \text{and} \\
L^0 &= \big((c^\star, u_1), (c_1, z), (c^\star, u_2), (c_2, z), \dots, (c^\star, u_n), (c_n, z)\big) \\
L^i &= \big((c_{i,1}, u_{i_1}), (c_{i,2}, u_{i_2}), (c_{i,3}, u_{i_3}), (c_{i,4}, u_{i_1}), (c_{i,5}, z)\big) \text{ for each } i \in [m] \\
L &= L^0 \oplus L^1 \oplus \dots \oplus L^m, \text{where } \oplus \text{ concatenates lists.}
\end{aligned}
$$

Clearly, this reduction can be performed in polynomial time. Intuitively, every sublist L^i that contains a color-$c^\star$ entry corresponds to a set S_i in a solution $\mathcal{S}'$ of $\mathcal{S}$. If these sublists contain K consecutive color-$c^\star$ entries, then there are $2K$ elements that are covered by K sets in $\mathcal{S}'$, so $|\mathcal{S}'| \leq n - K = k$.

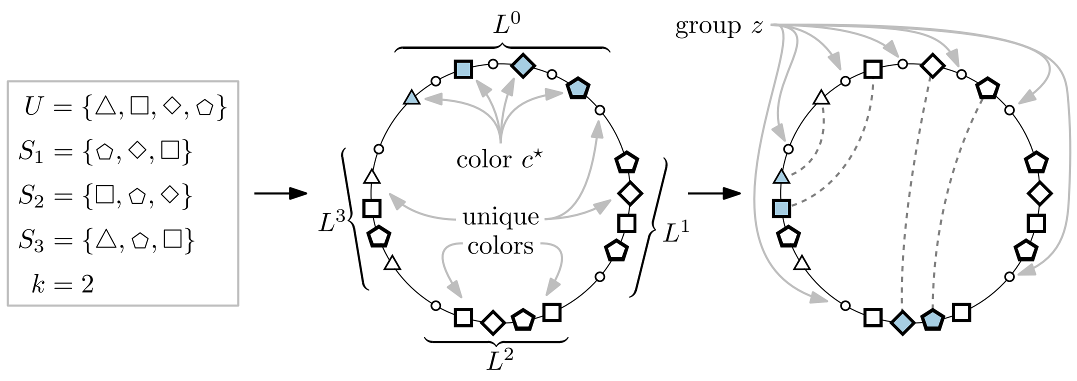

$U = \{\triangle, \square, \diamond, ⬠\}$
$S_1 = \{⬠, \diamond, \square\}$
$S_2 = \{\square, ⬠, \diamond\}$
$S_3 = \{\triangle, ⬠, \square\}$
$k = 2$
L^0
L^1
L^2
L^3
color $c^\star$
unique colors
group z

Figure 2 Here, the reduction from 3SetCover to NodePermutation yields the cover $\{S_2, S_3\}$.

First, we assume that $(U, \mathcal{S}, k)$ is a yes-instance of 3SetCover, that is, there is a size-k subfamily $\mathcal{S}'$ of $\mathcal{S}$ that covers U. We need to construct a permutation π of L such that (i) $\mathcal{G}(\pi(L)) = \mathcal{G}(L)$ and (ii) $N(\pi(L)) \geq K$.

For each $j \in [n]$, let i be the index of an arbitrary set in $\mathcal{S}'$ that contains u_j. Swap $(c^\star, u_j)$ in L^0 with an entry in L^i with group u_j. There is a choice only if $u_j = u_{i_1}$. If $(c_{i,2}, u_{i_2})$ has been or will be swapped with another entry, too, then swap $(c^\star, u_j)$ with $(c_{i,1}, u_{i_1})$; otherwise, swap $(c^\star, u_j)$ with $(c_{i,4}, u_{i_1})$. This makes sure that all swapped entries in L^i are consecutive. In total, we make n swaps. In the resulting permutation of L, the elements of color u form at most k groups of consecutive entries (as $\mathcal{S}'$ might contain "unnecessary" sets in the decision version). Hence, the number of pairs of consecutive entries with the same color $c^\star$ is at least $n - k = K$.

Now assume that $(\mathcal{G}, \mathcal{C}, L, K)$ is a yes-instance of NodePermutation, that is, there is a permutation π of L such that (i) $\mathcal{G}(\pi(L)) = \mathcal{G}(L)$ and (ii) $N(\pi(L)) \geq K$. We have to show that then $(U, \mathcal{S})$ admits a set cover of size $k = n - K$. Without loss of generality, we can assume that π swaps *each* color-$c^\star$ element of L^0 with a color-$c^\star$ element of $L^1 \oplus \cdots \oplus L^m$ (because such a swap does not decrease $N(\pi(L))$) and that π does not swap any other entries of L (because swapping group-z elements does not change $N(\pi(L))$). Consider the family $\mathcal{S}'$ of those sets $S_i \in \mathcal{S}$ such that π modifies L^i. We claim that (i) $\mathcal{S}'$ covers U and (ii) $|\mathcal{S}'| \leq k$.

Property (i) holds due to our assumption that π swaps all color-$c^\star$ entries of L^0 with a sublist L^i with $i \in [m]$. Thus, every element of U is contained in an element of $\mathcal{S}'$. For property (ii), note that the only pairs of consecutive entries with equal color in $\pi(L)$ are pairs of color-$c^\star$ entries in sublists L^i. Among the n color-$c^\star$ entries, at least K pairs are consecutive. Let K_1, K_2 and K_3 be the number of sublists L^i that contain one, two and three color-$c^\star$ entries, respectively. Then we have $K_2 + 2K_3 \geq K$ and $K_1 = n - 2K_2 - 3K_3$. Hence, the total number of sublists L^i that contain at least one color-$c^\star$ entry is $K_1 + K_2 + K_3 = n - K_2 - 2K_3 \leq n - K = k$, so $|\mathcal{S}'| \leq k$. ◀

3 An Algorithm for a Special Case of NodePermutation

In this section, we describe a linear-time algorithm to solve the optimization version of NodePermutation for the special case that there are only two groups (but an arbitrary number of colors). In ChordLink, this means that the cluster C has a neighborhood of size 2.

▶ **Theorem 2.** *NodePermutation can be solved in $O(n)$ time for two groups.*

Proof. Let $\mathcal{G} = \{x, y\}$. The goal is to find a permutation π of L that maximizes $N(\pi(L))$.

Recall that the elements of L are pairwise disjoint. Thus, $c_{\pi^{-1}(i)} = c_{\pi^{-1}(i+1)}$ implies that $g_i \neq g_{i+1}$. Since there are only two groups, we have either $g_{i+2} = g_i$ or $g_{i+2} = g_{i+1}$, so $c_{\pi^{-1}(i+2)} \neq c_{\pi^{-1}(i+1)}$. Hence, we cannot have $c_{\pi^{-1}(i)} = c_{\pi^{-1}(i+1)} = c_{\pi^{-1}(i+2)}$.

For $\circ \in \{x, y\}$, let $\mathcal{C}_\circ$ be the set of colors c_i such that there exists some list element $(c_i, \circ)$, and let $S = \mathcal{C}_x \cap \mathcal{C}_y$. We say that the color c_j is *assigned* to index i if $c_{\pi^{-1}(i)} = c_j$.

We can formulate this problem as a maximum independent set problem on a graph G'. The graph contains a vertex v'_i for every pair of consecutive indices $i, i+1$ with $g_i \neq g_{i+1}$, and a vertex v'_n if $g_n \neq g_1$. If $v'_i, v'_{i+1} \in V(G')$, then $E(G')$ contains the edge (v'_i, v'_{i+1}). Any assignment of colors to K pairs of consecutive indices $(i_1, i_1+1), \dots, (i_K, i_K+1)$ with $g_{i_1} \neq g_{i_1+1}, \dots, g_{i_K} \neq g_{i_K+1}$ induces an independent set $v'_{i_1}, \dots, v'_{i_K}$ in G'.

Hence, if we find an independent set $(v'_{i_1}, \dots, v'_{i_k})$ of size k in G', then we can find an assignment of $K = \min\{|S|, k\}$ colors in S to consecutive pairs of indices $(i_1, i_1 + 1), \dots, (i_K, i_K + 1)$. By construction, G' is either a cycle or a linear forest, so we can find a maximum independent set of G' in linear time with a simple greedy algorithm. To obtain a permutation π of L with $N(\pi(L)) = K$, we arbitrarily assign the remaining colors of $\mathcal{C}_x$ and $\mathcal{C}_y$ to the remaining list elements of types (c_i, x) and (c_i, y), respectively. ◀

4 APX-Hardness of CrossingMinimal ChordInsertion

In this section, we focus on the optimization problem CrossingMinimal ChordInsertion: Given a graph G with at least one copy of each of its vertices placed on a circle R, insert every edge between a copy of each of its endvertices such that the total number of crossings between the edges is minimized. Since the number of crossings only depends on the order of the vertex copies along R, we can also assume them to be drawn as points (rather than circular arcs) on R. We first prove that finding a crossing-free solution is NP-complete.

▶ **Theorem 3.** *It is NP-complete to decide whether a given graph (with node copies on R) admits a crossing-free solution for ChordInsertion.*

Proof. Membership in NP is obvious since we can verify an assignment easily.

To show hardness, we reduce from SAT. Let $(\mathcal{X}, \mathcal{C})$ be a SAT instance with variables $\mathcal{X} = X_0, \dots, X_n$ and clauses $\mathcal{C} = C_0, \dots, C_m$. We create a graph G as follows. The vertex set $V(G)$ consists of (i) a vertex s; (ii) for every clause C_i a *clause vertex* c_i; and (iii) for every variable X_j two *variable vertices* x_j and y_j. The edge set $E(G)$ consists of (i) for every clause C_i a *clause edge* (s, c_i); and (ii) for every variable X_j a *variable edge* (x_j, y_j). For every variable X_j, let T_j be the set of clauses that contain the literal X_j, and let F_j be the set of clauses that contain the literal $\neg X_j$.

To create an instance I of ChordInsertion, we place copies of the vertices of $V(G)$ along R as follows. We start with the vertex s, and then append for every variable X_j (i) one copy y_j^{F} of y_j, (ii) for every clause $C_i \in T_j$ a copy c_i^j of c_i, (iii) the vertex x_j, (iv) for every clause $C_i \in F_j$ a copy c_i^j of c_i, (v) a second copy y_j^{T} of y_j; see Fig. 3. Observe that, by the placement of the vertices along R, the only crossings that can occur are between a clause edge (s, c_i) and a variable edge (x_j, y_j) such that clause C_i contains variable X_j.

Assume that I is a yes-instance, i.e., it admits a crossing-free drawing Γ. For every variable X_i, Γ must contain either the chord (x, y_i^{F}) or the chord (x, y_i^{T}); see Fig. 4.

Consider a clause C_i. In Γ, there has to be one chord between s and one of the copies of c_i. Consider any literal X_j contained in C_i. Then, the edge (s, c_i) can be drawn as the chord (s, c_i^j) only if Γ contains the chord (x_i, y_i^{T}); otherwise, there would be a crossing

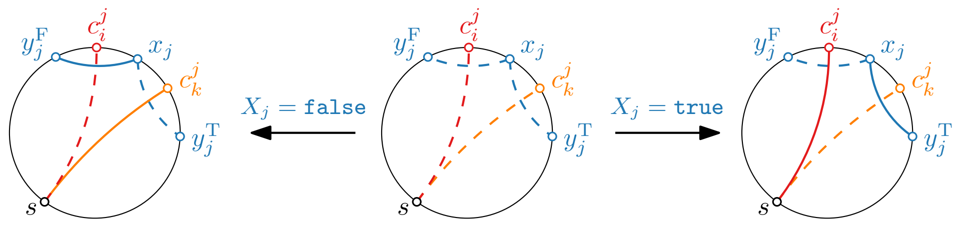


Figure 3 Illustration for the part corresponding to variable X_j in the reduction from SAT to ChordInsertion for the formula $c_i \wedge c_k \wedge \cdots = (X_j \vee \ldots) \wedge (\overline{X_j} \vee \ldots) \wedge \ldots$.

$$C_1 = X_1 \vee \overline{X_2} \vee X_3$$
$$C_2 = \overline{X_1} \vee X_3 \vee X_4$$
$$C_3 = \overline{X_2} \vee \overline{X_4} \vee X_5$$
$$C_4 = X_2 \vee \overline{X_3} \vee X_5$$

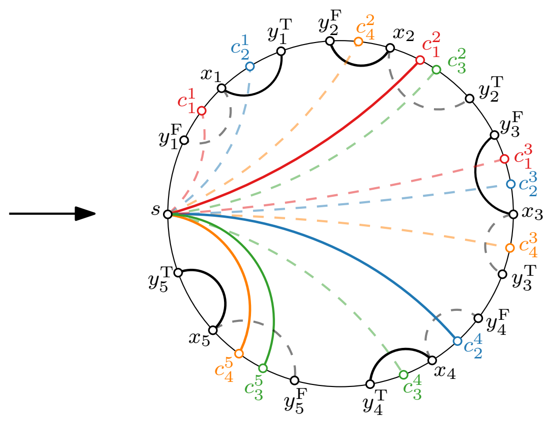


Figure 4 Example of the reduction from SAT to ChordInsertion

between (s, c_i^j) and (x_i, y_i^{F}). Conversely, for any literal $\neg X_j$ contained in C_i, the edge (s, c_i) can be drawn as the chord (s, c_i^j) only if Γ contains the chord (x_i, y_i^{F}). Hence, for the edge (s, c_i) to be drawn in Γ, for at least one of its literals X_j there must be the chord (x_j, y_j^{T}), or for at least one of its literals $\neg X_j$ there must be the chord (x_j, y_j^{F}).

Thus, we can obtain a feasible solution for the SAT instance $(\mathcal{X}, \mathcal{C})$ as follows: if Γ contains the chord (x_j, y_j^{T}), then set $X_j = \texttt{true}$, otherwise (Γ contains the chord (x_j, y_j^{F})), set $X_j = \texttt{false}$. Since every edge (s, c_i) is drawn as a chord in Γ, at least one literal in every clause C_i must be true, so the solution is feasible.

Conversely, if $(\mathcal{X}, \mathcal{C})$ is a yes-instance, then we can obtain a crossing-free drawing Γ for I by using the chord $(x_j, y_j)^{\mathrm{T}}$ for every `true` variable X_j, the chord $(x_j, y_i)^{\mathrm{F}}$ for every `false` variable X_j, and a chord (s, c_i^j) for every clause C_i with satisfied literal X_j. ◀

In the above proof we reduced from SAT. If we instead reduce from Max-2-SAT, which is APX-hard [6], we can show that our problem is even APX-hard.

▶ **Theorem 4.** *CrossingMinimal ChordInsertion is APX-hard even if there are at most two possible choices for every edge.*

References

1 Patrizio Angelini and Giordano Da Lozzo. Beyond clustered planar graphs. In Seok-Hee Hong and Takeshi Tokuyama, editors, *Beyond Planar Graphs: Communications of NII Shonan Meetings*, pages 211–235. Springer, 2020. doi:10.1007/978-981-15-6533-5_12.

2 Patrizio Angelini, Giordano Da Lozzo, Giuseppe Battista, Fabrizio Frati, Maurizio Patrignani, and Ignaz Rutter. Intersection-link representations of graphs. *J. Graph Algorithms Appl.*, 21:731–755, 2017. doi:10.7155/jgaa.00437.

3 Lorenzo Angori, Walter Didimo, Fabrizio Montecchiani, Daniele Pagliuca, and Alessandra Tappini. Hybrid graph visualizations with ChordLink: Algorithms, experiments, and applications. *IEEE Trans. Vis. Comput. Graphics*, 28(2):1288–1300, 2020. URL: https://arxiv.org/abs/1908.08412, doi:10.1109/TVCG.2020.3016055.

4 Punam Bedi and Chhavi Sharma. Community detection in social networks. *Wiley Interdiscip. Rev. Data Min. Knowl. Discov.*, 6(3):115–135, 2016. doi:10.1002/widm.1178.

5 Michael R. Garey, David S. Johnson, and Larry J. Stockmeyer. Some simplified NP-complete problems. In Robert L. Constable, Robert W. Ritchie, Jack W. Carlyle, and Michael A. Harrison, editors, *Proc. 6th Ann. ACM Symp. Theory Comput. (STOC)*, pages 47–63, 1974. doi:10.1145/800119.803884.

6 Johan Håstad. Some optimal inapproximability results. *J. ACM*, 48(4):798–859, 2001. doi:10.1145/502090.502098.

7 Nathalie Henry Riche, Jean-Daniel Fekete, and Michael McGuffin. Nodetrix: A hybrid visualization of social networks. *IEEE Trans. Vis. Comput. Graphics*, 13(6):1302–1309, 2007. doi:10.1109/TVCG.2007.70582.

8 Martin Krzywinski, Jacqueline Schein, Inanç Birol, Joseph Connors, Randy Gascoyne, Doug Horsman, Steven J. Jones, and Marco A. Marra. Circos: An information aesthetic for comparative genomics. *Genome Res.*, 19(9):1639–1645, 2009. doi:10.1101/gr.092759.109.

9 Hassan Mahmoud, Francesco Masulli, Stefano Rovetta, and Giuseppe Russo. Community detection in protein-protein interaction networks using spectral and graph approaches. In Enrico Formenti, Roberto Tagliaferri, and Ernst Wit, editors, *Proc. 10th Int. Meeting Comput. Intell. Methods for Bioinf. Biostat. (CIBB)*, volume 8452 of *Lect. Notes Comput. Sci.*, pages 62–75. Springer, 2013. doi:10.1007/978-3-319-09042-9_5.