Solving Optimization Problems on Orthogonal Ray Graphs

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In this paper, we consider specialized subclasses of the intersection graphs of *rays* (or half-lines) in the plane. These were introduced by Kostocha and Nešetřil [5]. Formally, a graph $G = (V, E)$ is said to be a *ray graph* if each vertex $v \in V$ can be represented by a ray $r_v = (p_v, d_v)$ originating from the point p_v—its *anchor*—and continuing endlessly in direction d_v such that uv is an edge if and only if r_u and r_v intersect. We study *orthogonal ray graphs* (ORG) where the directions of the rays are restricted to be axis-parallel, that is, we require $d_v \in \{\uparrow, \downarrow, \leftarrow, \rightarrow\}$. This graph class was introduced by Shrestha et al. [6] in connection with defect tolerance schemes for nano-programmable logic arrays. We assume that no two anchors lie on the same horizontal or vertical line.

We distinguish two subclasses of ORG. In a *2D-orthogonal ray graph (2DORG)*, we restrict the ray directions to the set $\{\uparrow, \rightarrow\}$. Similarly, a *3D-orthogonal ray graph (3DORG)* is an ORG whose ray directions are restricted to $\{\uparrow, \rightarrow, \leftarrow\}$.

Shrestha et al. [6] have shown that a bipartite graph is a 2DORG if and only if its complement is a circular arc graph. Using this property, they gave an $O(n^2)$-time recognition algorithm and a minimal list of forbidden induced subgraphs for 2DORG. In Section 1, we show that all graphs of this list but C_6 are also forbidden induced subgraphs for 3DORG. Then, in Section 2, we present efficient algorithms for Maximum-Weight Independent Set (MIS), Minimum-Weight Independent Dominating Set (IDS), and Minimum-Weight Feedback Vertex Set (FVS) on ORG. These problems are known to be NP-complete on general graphs; MIS and FVS are on the list of Karp's 21 NP-complete problems [3]. Takaoka et al. [8] have already shown how to solve Minimum Weighted Dominating Set on 2DORGs in $O(n^4 \log n)$ time.

We give the proofs in detail in the full version [1].

1 Characterization of 3DORG

Shrestha et al. [6] have shown that $C_{\geq 6}$ are minimal forbidden induced subgraphs for 2DORG, while Kostochka and Nešetřil [5] have (implicitly) proved that $C_{\geq 14}$ are minimal forbidden induced subgraphs for ORG. We extend these results by showing that $C_{\geq 8}$ are minimal forbidden induced subgraphs for 3DORG. Note that these values are tight, that is, induced cycles of smaller lengths are representable; see Fig. 1.

We show that no 3DORG can have an independent triple of edges $\{e_1, e_2, e_3\}$, where each $G \setminus N(e_i)$ is connected. With this observation, we can prove that nearly all of the minimal forbidden induced subgraphs

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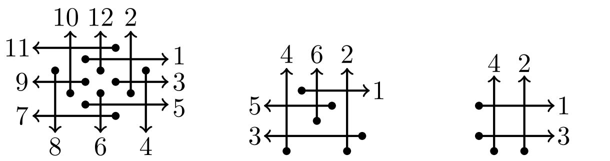


(a) ORG: C_{12} (b) 3DORG: C_6 (c) 2DORG: C_4

Fig. 1: Longest cycles in subclasses of ORG.

of 2DORG are also not 3DORGs. Only a few graphs need to be considered separately. In particular, C_6 is a 3DORG, $C_{\geq 8}$ are not 3DORGs, and there are two infinite families that are forbidden for 2DORG and do *not* have a "bad" independent triple of edges. However, analyzing the structure of the vertical rays, we can show that the graphs in these infinite families are not 3DORGs.

Theorem 1. *All forbidden induced subgraphs of 2DORG—except C_6—are also forbidden induced subgraphs of 3DORG.*

A complete characterization and the efficient recognition remain open for 3DORG.

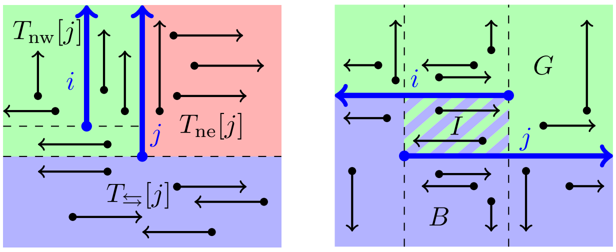
2 Optimization Problems on ORG

In this section, we consider the problems MIS, IDS, and FVS for subclasses of ORG.

Felsner et al. [2] gave an $O(n \log n)$-time algorithm for Maximum-Weight Clique (MC) on trapezoid graphs. The complement of a 2DORG is a trapezoid graph. Given a 2DORG G with representation, a trapezoid representation of the complement $\overline{G}$ of G can be computed in linear time. Since an MIS in G corresponds to an MC in $\overline{G}$, an MIS in an n-vertex 2DORG with given representation can be computed in $O(n \log n)$ time. We now turn to larger subclasses of ORG.

Theorem 2. *Given a representation with n rays, we can solve MIS on (i) 3DORGs in $O(n^2)$ time and (ii) ORGs in $O(n^4)$ time.*

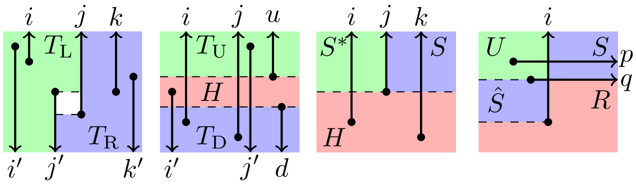
Proof. (i) We exploit that the bottommost vertical ray r_j in a 3DORG solution decomposes the plane into a green and a red quarterplane and a blue halfplane; see Fig. 2a. These regions can be solved independently. The subsolutions in the green and the red quarterplanes are both 2DORGs; the subsolution in the blue halfplane consists of all horizontal rays in that halfplane. We denote the weights of these subsolutions by $T_{\mathrm{nw}}[j]$, $T_{\mathrm{ne}}[j]$, and $T_{\leftrightarrows}[j]$. For two vertical rays r_i and r_j with $x_i < x_j$ and $y_i > y_j$, let $T_{\uparrow}[i, j]$ be the total weight of all vertical rays in the strip $(x_i, x_j) \times (y_i, \infty)$, and let $T_{\leftarrow}[i, j]$ be the total weight of all left-pointing rays in the strip $(-\infty, x_j) \times (y_j, y_i)$. It is easy to compute the tables $T_{\leftrightarrows}$, $T_{\uparrow}$, and $T_{\leftarrow}$ in $O(n^2)$ total time. Then,



(a) The bottommost vertical ray r_j in a MIS of a 3DORG yields three regions (shaded green, red, and blue) that can be solved independently.

(b) The rightmost leftward ray r_i and the leftmost rightward ray r_j in a MIS of an ORG yield two regions G and B that can be solved independently.

Fig. 2: Partitions of the plane used for computing the MIS of 3DORGs and ORGs.



(a) (b) (c) (d)

Fig. 3: Ideas behind our efficient algorithm for FVS.

$T_{\mathrm{nw}}[j] = \max_i T_{\mathrm{nw}}[i] + T_{\uparrow}[i,j] + T_{\leftarrow}[i,j]$. Similarly, we can compute tables $T_{\rightarrow}$ and T_{ne}. Then, the optimal solution has weight $w^\star = \max_j T_{\mathrm{nw}}[j] + T_{\mathrm{ne}}[j] + T_{\leftrightarrows}[j]$ and can be computed in $O(n)$ time.

(ii) We exploit that the rightmost leftward ray r_i and the leftmost rightward ray r_j in an ORG solution yield a covering of the plane into a green region G above at least one of the two rays and into a blue region B below at least one of the two rays; see Fig. 2b. The case that the two rays don't overlap horizontally is simple. Otherwise, let $I = G \cap B$ (hatched in Fig. 2b). The rightward (leftward) rays starting in I belong to G (B). Since I is the rectangle spanned by p_i and p_j, G and B contain independent 3DORG instances, which we can solve in $O(n^2)$ time due to (i). As there are $O(n^2)$ candidates for (r_i, r_j), our algorithm runs in $O(n^4)$ time. □

We can use a similar DP as in Thm. 2(i) to solve IDS on 3DORGs. The main difference is that the subsolutions in the green and red quarterplane have to dominate some vertical rays of the blue halfplane, which increases the dimension of the tables by one.

Theorem 3. *Given a representation with n rays, IDS on 3DORGs can be solved in $O(n^3)$ time.*

FVS is NP-hard for planar bipartite graphs of maximum degree 4 [7]. Such graphs are a subclass of the grid intersection graphs, but not of ORG. Kloks et al. [4] gave an efficient algorithm for chordal bipartite graphs, which contain 2DORG, but not 3DORG. Their approach does not generalize to 3DORG, and does not appear to handle the weighted case. We show:

Theorem 4. *Given a representation, FVS on ORGs can be solved efficiently.*

Proof sketch. Our approach reduces an ORG instance of FVS to polynomially many 3DORG instances, and then to polynomially many 2DORG instances. Note that FVS is equivalent to finding a maximum-weight induced forest F, which is more convenient to describe.

The idea for ORGs is based on finding the three *closest overlapping pairs*, that is, the three disjoint pairs of upward/downward rays that vertically overlap and have the least horizontal distance. The middle pair and its at most 5 neighbors partition an instance into two 3DORG subinstances T_L and T_R; see Fig. 3a. If F contains less than three overlapping pairs, then we guess the bottommost upward ray u and the topmost downward ray d to split the instance into three subinstances: two 3DORG instances T_U above u and T_D below d, and an instance H of only horizontal rays between u and d; see Fig. 3b.

To reduce a 3DORG instance to 2DORG instances, we similarly consider the three bottommost upward rays (r_i, r_j, r_k) and the neighborhood of the middle ray r_j; see Fig. 3c. Here, j can have at most two common neighbors with i and k in F. Once we know these neighbors, the instance decomposes into two 2DORG subinstances S^* and S on the left and right of r_j and a subinstance H of only horizontal rays below the highest anchor of r_i, r_j, and r_k.

We further decompose each 2DORG instance into smaller 2DORG instances by considering the bottemmost upward ray i and its two highest neighbors p and q; see Fig. 3d. We obtain four subinstances: S and $\hat{S}$ are 2DORG subinstances, whereas in the regions U and R, we select precisely the upward and rightward rays, respectively. □

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