Necklace Splitting and Multiobjective Optimization [1]

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[1] joint with Christian Glaßer and Maximilian Witek at Würzburg, Germany

Splitting Golden Necklaces

Problem Statement

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gold density:	integrable $f\colon \mathbb{S}^1 \to \mathbb{R}^+$
necklace part:	"interval" (connected subset) in $\mathbb{S}^1$
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Observation

intermediate value theorem: two cuts suffice
even works for arbitrary fractions

Splitting More Complex Necklaces

Problem Extension

Necklace consists of k materials (gold, silver, diamond, ...).
1.: Each thief wants exactly half of each material.
2.: A fraction of α of each material for A and $1 - \alpha$ for B.

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Theorem (Stromquist, Woodall 1985)

Let $k \geq 2$ and $f_1, \ldots, f_k : \mathbb{S}^1 \to \mathbb{R}^+$ integrable. For each $0 \leq \alpha \leq 1$ there are $k-1$ intervals $I_1, I_2, \ldots, I_{k-1} \subseteq \mathbb{S}^1$ such that for all $i \in \{1, \ldots, k\}$

$$\int_{I_1 \cup I_2 \cup \cdots \cup I_{k-1}} f_i(x)dx = \alpha \cdot \int_{\mathbb{S}^1} f_i(x)dx.$$

Proof ingredients: Ham-Sandwich- or Borsuk-Ulam-Theorem for $\alpha \Rightarrow \frac{1}{2}\alpha$, closedness of solution space

Combining Necklaces

We want to combine r necklaces consisting of k materials:

Corollary (Convex Combinations of Necklaces)

For each $r, k \geq 1$ there is a (polynomial) bound $t \in \mathbb{N}$ such that the following holds: For $j = 1, \ldots, r$ and $i = 1, \ldots, k$ let $f_{j,i} \colon \mathbb{S}^1 \to \mathbb{R}$ be integrable and let $\alpha \in [0,1]^r$ such that $||\alpha||_1 = 1$. There is a function $c \colon \mathbb{S}^1 \to \{1, \ldots, r\}$ with at most t jumps such that for all $i \in \{1, \ldots, k\}$

$$\int_{\mathbb{S}^1} f_{c(x),i}(x)dx = \sum_{j=1}^{r} \alpha_j \cdot \int_{\mathbb{S}^1} f_{j,i}(x)dx.$$

Multiobjective Optimization - Motivation

Typical Problem: Buying a Car

Objectives can be:

- price
- gas consumption
- maximal speed
- capacity
- age

Most real-world-problems have multiple conflicting objectives that cannot be easily converted into a single measure like money

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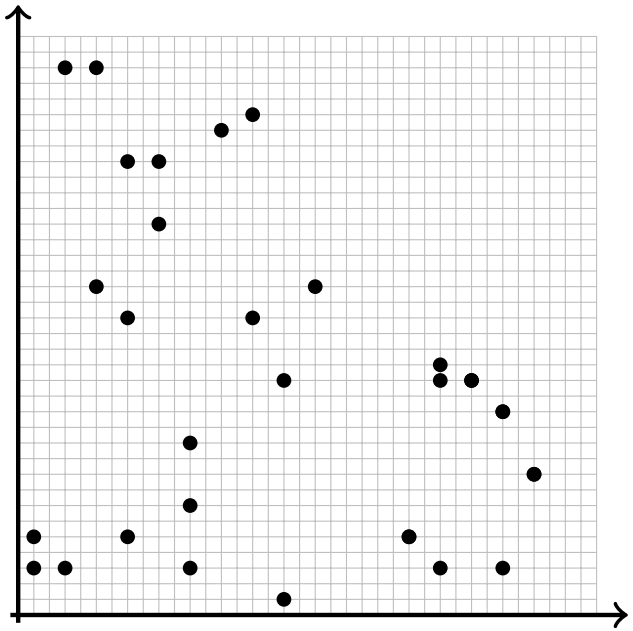
Multiobjective Optimization - Algorithmic View

Algorithmic Tasks

- find all Pareto-optimal solutions (unnecessarily hard if there are many Pareto-optimal solutions)
- find some solution that satisfies a given constraint
- find approximation of such a solution



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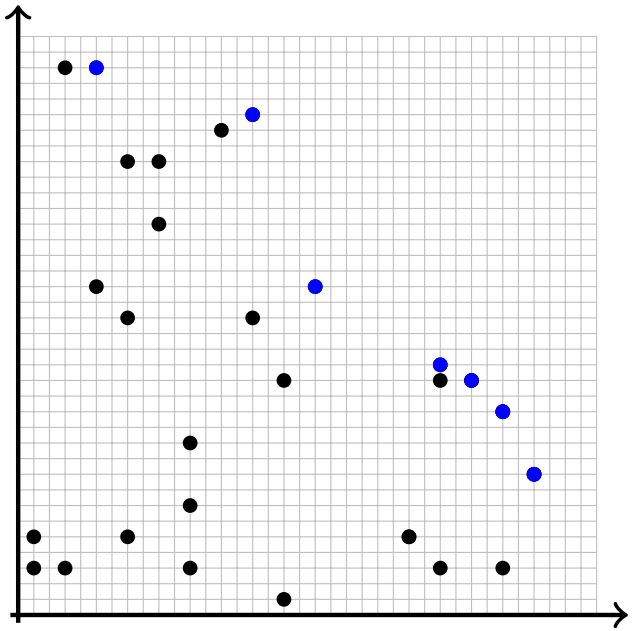

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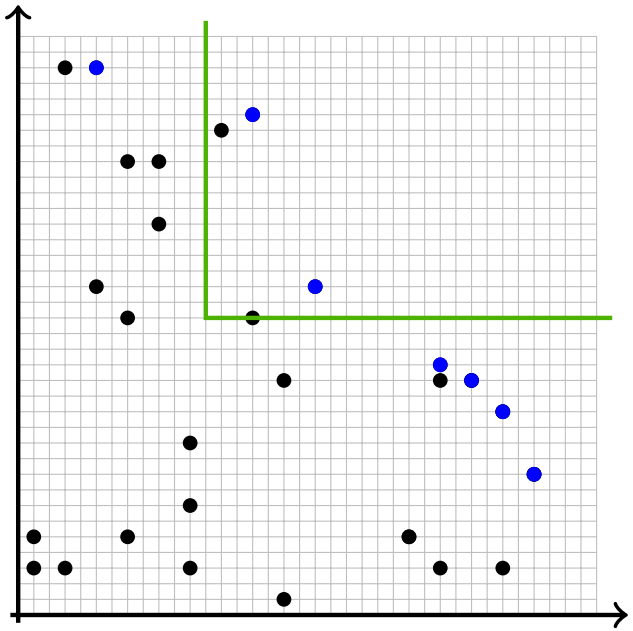

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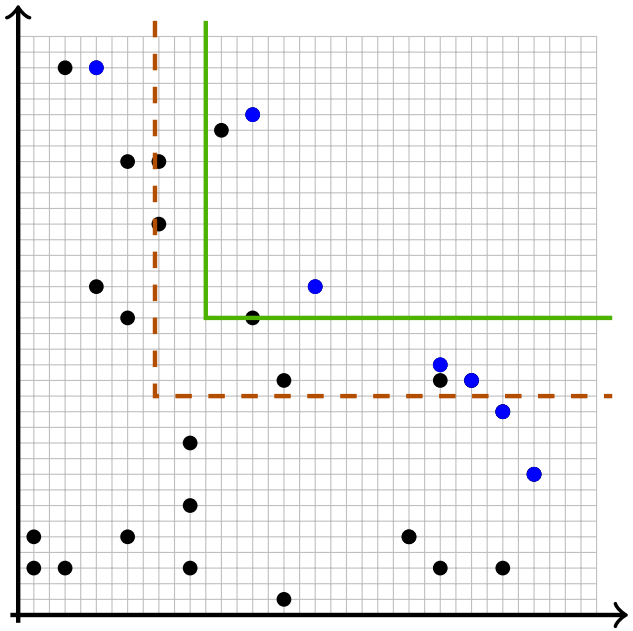

Multiobjective Optimization - Definitions

Definition (k-Objective Maximization Problem)

A k-objective maximization problem has the form $(S, f_1, \ldots, f_k)$, where

- $S(x)$: set of solutions for instance x
- $f_i(x, s)$: value of solution $s \in S(x)$ in i-th objective

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Example (Buying a Car)

- x: car dealer
- $S(x)$: set of cars sold by dealer x
- $f_1(x, s)$: maximal speed of car s sold by dealer x
- $f_2(x, s)$: capacity of car s sold by dealer x

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Example (Maximum k-Weighted Matching)

- $x = (V, E, w_1, \ldots, w_k)$, where (V, E) is a complete undirected graph and $w_i \colon E \to \mathbb{N}$
- $S(x) = \{M \subseteq E \mid M \text{ is matching of } (V, E)\}$
- $f_i(x, M) = w_i(M) = \sum_{e \in M} w_i(e)$

Known Approximation-Results about Matching

Theorem (Papadimitriou, Yannakakis 2000)

For every k there is an FPRAS (fully polynomial-time randomized approximation scheme) for maximum k-weighted matching.

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For every k there is an FPRAS (fully polynomial-time randomized approximation scheme) for maximum k-weighted matching.

This means, for every k there is a randomized algorithm A such that on input $(x, \varepsilon, c_1, \ldots, c_k)$, with probability at least $\frac{1}{2}$ it

- finds some $M \in S(x)$ s. t. $f_i(x, M) \geq (1 - \varepsilon)c_i$ for all i or
- states that there is no $M \in S(x)$ with $f_i(x, M) \geq c_i$ for all i

and the runtime of A is polynomial in $|x| + \frac{1}{\varepsilon}$.
Proof: Long chain of reductions, boils down to isolation lemma by Mulmuley, Vazirani, Vazirani.

PTAS for Maximum k-Weighted Matching

Theorem (Chekuri, Vondrák, Zenklusen 2011)

For every k there is a (deterministic) PTAS (polynomial-time approximation scheme) for maximum k-weighted matching.

We present an easier proof using necklace splitting results, although these results are non-constructive and in a continuous setting.
Main feature of necklace splittings: Constant number of cuts or "jump points"

1. rounding error is small (constant times largest element)
2. rounded splitting can be found by exhaustive search

PTAS for Maximum k-Weighted Matching (cont. 1)

Lemma (Convex Combination of Matchings)

Let $k, r \in \mathbb{N}$. On input of $(V, E, w_1, \ldots, w_k)$, matchings $M_1, \ldots, M_r$ and $\alpha \in [0,1]^r$, $||\alpha||_1 = 1$ we can compute a matching M in polynomial time such that for all i

$$w_i(M) \geq \left(\sum_j \alpha_j w_i(M_j) \right) - 2rt \max_{e \in E} w_i(e)$$

where t is the "jump bound" in the necklace combination result.

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$$w_i(M) \geq \left(\sum_j \alpha_j w_i(M_j) \right) - 2rt \max_{e \in E} w_i(e)$$

where t is the "jump bound" in the necklace combination result.

Proof idea: Successively combine two matchings by bringing the edges in order and combining them.

Reminder (Convex Combinations of Necklaces):

$\exists c \colon \mathbb{S}^1 \to \{1, \ldots, r\}$ with at most t jumps s.t. $\forall i \in \{1, \ldots, k\}$

$$\int_{\mathbb{S}^1} f_{c(x),i}(x)dx = \sum_{j=1}^{r} \alpha_j \cdot \int_{\mathbb{S}^1} f_{j,i}(x)dx.$$

PTAS for Maximum k-Weighted Matching (cont. 2)

PTAS for Maximum k-Weighted Matching

Input: $(V, E, w_1, \ldots, w_k)$, $(c_1, \ldots, c_k)$, ε

1. eliminate error by guessing heavy edges in solution
2. setup linear program for matching on (V, E)
3. add constraints $w_i(x) \geq c_i$
4. find solution x (fractional matching)
5. find matchings $M_1, \ldots, M_{k+1}$ and $\alpha \in [0,1]^{k+1}$, $||\alpha||_1 = 1$ s.t. $w_i(x) = \sum_{j=1}^{k+1} \alpha_j w_i(M_i)$ for all i
6. use previous lemma to find matching M with $w_i(M) \gtrsim w_i(x)$

Max-k-TSP

Definition (Maximum k-TSP)

- Instances: $(V, E, w_1, \ldots, w_k)$, where (V, E) is a complete undirected graph and $w_i \colon E \to \mathbb{N}$
- $S(x) = \{T \subseteq E \mid T \text{ is a Hamiltonian cycle of } (V, E)\}$
- $f_i(x, T) = w_i(T)$

General idea for approximation:

1. Compute maximum cycle cover
2. Remove one edge from each cycle
3. Connect paths to Hamiltonian cycle

For $k = 1$: Removing lightest edge results in $\frac{2}{3}$-approximation (optimal tour is cycle cover, a cycle has at least three edges)
Not so easy for larger k: There is no “lightest” edge.

Known Results for Max-k-TSP

Approximability of Max-k-TSP

- Bläser, Manthey, Putz, 2008: randomized $\frac{1}{k} - \varepsilon$
- Manthey, 2009: randomized $\frac{2}{3} - \varepsilon$
- Manthey, 2011: deterministic $\frac{1}{2k} - \varepsilon$

All obtained with refinements of the basic cycle-cutting idea.

Necklace splitting shows us how to choose the edges perfectly and thus improve randomized and deterministic result to $\frac{2}{3}$ and $\frac{2}{3} - \varepsilon$, respectively.

Approximating Max-k-TSP using Necklace Splitting

Theorem

For every k and every $\varepsilon > 0$, there is a randomized $\frac{2}{3}$-approximation and a deterministic $(\frac{2}{3} - \varepsilon)$-approximation for Max-$k$-TSP.

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Theorem

For every k and every $\varepsilon > 0$, there is a randomized $\frac{2}{3}$-approximation and a deterministic $(\frac{2}{3} - \varepsilon)$-approximation for Max-$k$-TSP.

1. eliminate error by guessing heavy edges in solution
2. compute maximum cycle cover (via matching)
3. from each cycle C_i select three edges $e_{i,1}, e_{i,2}, e_{i,3}$
4. combine the three necklaces $\{e_{1,1}, \ldots, e_{m,1}\}$, $\{e_{1,2}, \ldots, e_{m,2}\}$ and $\{e_{1,3}, \ldots, e_{m,3}\}$ in ratios $\frac{1}{3} : \frac{1}{3} : \frac{1}{3}$
5. remove resulting edges from cycles and join obtained paths to single cycle

Similarly: $\frac{1}{2}$ / $(\frac{1}{2} - \varepsilon)$ for Max-k-TSP on directed graphs (smallest cycles have two edges)

Conclusion

Applications of Necklace Splitting in Multiobjective Optimization

- PTAS for k-objective matching and cycle cover
- $\frac{2}{3}$ / $(\frac{2}{3} - \varepsilon)$-appr. for Max-$k$-TSP
- $\frac{1}{2}$ / $(\frac{1}{2} - \varepsilon)$-appr. for Max-$k$-ATSP (on directed graphs)
- $\frac{1}{2}$-approximation for k-objective weighted Max-SAT

More to come?