Finding Tutte Paths in Linear Time

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Universität Würzburg

joint work with Therese Biedl
University of Waterloo
Tutte Paths

Planar graph $G$
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Planar graph $G$

$X$ to $Y$ via $\alpha$

Tutte path: Path from $X$ to $Y$ via $\alpha$
Tutte Paths

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Tutte path: Path from $X$ to $Y$ via $\alpha$

Every comp. attached to $\leq 3$ vtc's of $P$
Tutte Paths

Planar graph $G$

Tutte path: Path from $X$ to $Y$ via $\alpha$

Every comp. attached to $\leq 3$ vtc's of $P$
Tutte Paths

Tutte path: Path from $X$ to $Y$ via $\alpha$

Every component attached to $\leq 3$ vertices of $P$

Every outer component attached to 2 vertices of $P$
Tutte Paths

Planar graph $G$

Tutte path: Path from $X$ to $Y$ via $\alpha$

Every comp. attached to $\leq 3$ vtcs of $P$
Every outer comp. attached to 2 vtcs of $P$
What is known?
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[Tutte ’77]

$G$ 2-conn., $X$, $Y$, $\alpha$ on outer face $\Rightarrow$ Tutte path
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[Tutte '77]
$G$ 2-conn., $X, Y, \alpha$ on outer face $\Rightarrow$ Tutte path

[Thomassen '83]
$G$ 2-conn., $X, X, \alpha$ on outer face $\Rightarrow$ Tutte path
What is known?

[Tutte '77]
G 2-conn., X, Y, α on outer face ⇒ Tutte path

[Thomassen '83]
G 2-conn., X, \(\times\), α on outer face ⇒ Tutte path

[Sanders '96]
G 2-conn., \(\times\), \(\times\), α on outer face ⇒ Tutte path
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G 2-conn., X, X, α on outer face ⇒ Tutte path

[Gao, Richter & Yu ‘95, ‘06]
G 3-conn., X, Y, α on outer face ⇒ T_{SDR}-path
Tutte Paths

Planar graph $G$

Tutte path: Path from $X$ to $Y$ via $\alpha$

Every comp. attached to $\leq 3$ vtc\$s of $P$

Every outer comp. attached to 2 vtc\$s of $P$
Planar graph $G$

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$T_{\text{SDR}}$-path: Tutte path + System of Distinct Representatives:
Tutte Paths

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Injective assignment of comp. to attachment pts
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[Chiba & Nishizeki ’89]
G 4-conn. ⇒ Tutte path in $O(n)$ time
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[**Chiba & Nishizeki ’89**] (= Hamil. path)
\[ G \text{ 4-conn.} \Rightarrow \text{Tutte path in } O(n) \text{ time} \]
What is known?

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G 2-conn., \(X, Y, \alpha\) on outer face \(\Rightarrow\) Tutte path

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G 3-conn., \(X, Y, \alpha\) on outer face \(\Rightarrow\) \(T_{SDR}\)-path

[Chiba & Nishizeki ’89]
(G = Hamil. path) 
G 4-conn. \(\Rightarrow\) Tutte path in \(O(n)\) time

[Schmid & Schmidt ’15]
… in \(O(n^2)\) time
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**What is known?**

<table>
<thead>
<tr>
<th>Reference</th>
<th>Result</th>
</tr>
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Tutte paths

Planar graph $G$

**Tutte path:** Path from $X$ to $Y$ via $\alpha$
- Every comp. attached to $\leq 3$ vtc of $P$
- Every outer comp. attached to 2 vtc of $P$

**$T_{SDR}$-path:** Tutte path + System of Distinct Representatives:
- Injective assignment of comp. to attachment pts
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- **Planar graph $G$**

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- $T_{\text{int}}$-path:
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  - visits all ext. vtc's

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Planar graph $G$
Tutte paths

Planar graph $G$

$T_{\text{int}}$-path:
- $T_{\text{SDR}}$-path
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Injective assignment of comp. to attachment pts
**Tutte paths**

**Planar graph** $G$

- **$T_{\text{int}}$-path:**
  - $T_{\text{SDR}}$-path
  - visits all ext. vtcs
  - all comp. assigned to int. vtcs

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Tutte paths

Planar graph $G$

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- Injective assignment of comp. to attachment pts

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- all comp. assigned to int. vtc

Planar graph $G$
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... in \(O(n)\) time
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... in $O(n)$ time
Triangulated Graphs
4-conn. triangulation $\Rightarrow$ Hamiltonian path in $O(n)$ time.
[Asano, Kikuchi & Saito ’85]

4-conn. triangulation ⇒ Hamiltonian path in $O(n)$ time.
Triangulated Graphs

4-conn. triangulation $\Rightarrow$ Hamiltonian path in $O(n)$ time.

[Asano, Kikuchi & Saito '85]
Triangulated Graphs

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4-conn. triangulation ⇒ Hamiltonian path in $O(n)$ time.

triangulation ⇒ Tutte path in $O(n)$ time.

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[Asano, Kikuchi & Saito ’85]
Triangulated Graphs

$k$ vertices
$2k - 5$ int. faces

4-conn. triangulation $\Rightarrow$ Hamiltonian path in $O(n)$ time.

triangulation $\Rightarrow$ Tutte path in $O(n)$ time.

[Asano, Kikuchi & Saito '85]
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4-conn. triangulation $\Rightarrow$ Hamiltonian path in $O(n)$ time.

triangulation $\Rightarrow$ Tutte path in $O(n)$ time.

[Asano, Kikuchi & Saito '85]
Triangulated Graphs

$k$ vertices

$2k - 5$ int. faces

$k - 3$ int. vtcs

$k - 2$ int. edges in $P$

$\Rightarrow$

[Asano, Kikuchi & Saito '85]

4-conn. triangulation $\Rightarrow$ Hamiltonian path in $O(n)$ time.

triangulation $\Rightarrow$ Tutte path in $O(n)$ time.
Triangulated Graphs

4-conn. triangulation $\Rightarrow$ Hamiltonian path in $O(n)$ time.

triangulation $\Rightarrow$ Tutte path in $O(n)$ time.
Triangulated Graphs

\[ k \text{ vertices} \]
\[ 2k - 5 \text{ int. faces} \]
\[ k - 3 \text{ int. vtcs} \]
\[ k - 2 \text{ int. edges in } P \]

[Asano, Kikuchi & Saito ’85]

4-conn. triangulation ⇒ Hamiltonian path in \( O(n) \) time.

[Asano, Kikuchi & Saito ’85]

triangulation ⇒ Tutte path in \( O(n) \) time.
Substitution Trick
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Triangulated graphs

[Asano, Kikuchi & Saito ’85]
4-conn. triangulation ⇒ Hamiltonian path in $O(n)$ time.

triangulation ⇒ Tutte path in $O(n)$ time.
Triangulated graphs

- $k$ vertices
- $2k - 5$ faces
- $k - 2$ edges in $P - \alpha$
- $k - 3$ int. vtcs

4-conn. triangulation $\Rightarrow$ Hamiltonian path in $O(n)$ time.

triangulation $\Rightarrow$ Tutte path in $O(n)$ time.
Triangulated graphs

\[ k \text{ vertices} \]
\[ 2k - 5 \text{ faces} \]
\[ k - 2 \text{ edges in } P - \alpha \]
\[ k - 3 \text{ int. vtcs} \]

4-conn. triangulation ⇒ Hamiltonian path in \( O(n) \) time.

triangulation ⇒ Tutte path in \( O(n) \) time.

[Asano, Kikuchi & Saito '85]
Triangulated graphs

$\begin{align*}
\text{4-conn. triangulation } &\Rightarrow \text{ Hamiltonian path in } O(n) \text{ time.} \\
\text{Triangulation } &\Rightarrow \quad T_{\text{int}}\text{-path in } O(n) \text{ time.}
\end{align*}$

[Asano, Kikuchi & Saito ’85]
Corner-3-connectivity

int. 3-conn.
Corner-3-connectivity

int. 3-conn.
Corner-3-connectivity

int. 3-conn.

corner-3-conn.
Corner-3-connectivity

int. 3-conn.

corner-3-conn.
Corner-3-connectivity

int. 3-conn.

corner-3-conn.
Corner-3-connectivity

int. 3-conn.

corner-3-conn.

X

Y

α

side
Corner-3-connectivity

int. 3-conn.

corner-3-conn.
Corner-3-connectivity

int. 3-conn.
corner-3-conn.

side

X

Y

α
Corner-3-connectivity

int. 3-conn.

corner-3-conn.

side

X

Y

α
Corner-3-connectivity

int. 3-conn.

corner-3-conn.

side

X

Y

α
Corner-3-connectivity

int. 3-conn.

corner-3-conn.

side

X

Y

α
Corner-3-connectivity

int. 3-conn.

corner-3-conn.

X

Y

α

side
G is corner-3-conn., $X, Y, \alpha$ on outer face $\Rightarrow T_{\text{int-path}}$
Corner-3-connectivity

G is corner-3-conn., $X, Y, \alpha$ on outer face $\Rightarrow T_{\text{int-path}}$
Case 1: Outer Face is Triangle
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Case 1: Outer Face is Triangle
Case 1: Outer Face is Triangle
Case 2: left-right cutting pair
Case 2: left-right cutting pair
Case 2: left-right cutting pair
Case 2: left-right cutting pair
Case 2: left-right cutting pair

\[ G_b \]

\[ G_t \]
Case 2: left-right cutting pair
Case 2: left-right cutting pair
Case 2: left-right cutting pair
Case 2: left-right cutting pair
Case 2: left-right cutting pair
Case 3: top-right cutting pair
Case 3: top-right cutting pair
Case 3: top-right cutting pair
Case 3: top-right cutting pair

\[ G_b \]

\[ G_t \]

\[ \alpha \]
Case 3: top-right cutting pair
Case 3: top-right cutting pair

\[
\begin{array}{c}
X \quad \quad G_b \quad \quad \quad \quad Y \\
\alpha \\
G_t
\end{array}
\]
Case 3: top-right cutting pair
Case 3: top-right cutting pair
Case 3: top-right cutting pair
Case 3: top-right cutting pair
Case 3: top-right cutting pair
Case 3': top-left cutting pair
Case 3'': top-bottom cutting pair
Case 4: No cutting pair
Case 4: No cutting pair

Necklace $\langle Y_X = x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle$
Case 4: No cutting pair

Necklace $\langle Y_X=x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle$
Case 4: No cutting pair

Necklace $\langle Y_X = x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle$
Case 4: No cutting pair

Necklace $\langle Y_0 = x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle, x_i$ face-adj. to right side
Case 4: No cutting pair

Necklace $\langle Y_X=x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle$, $x_i$ face-adj. to right side
Case 4: No cutting pair

Necklace $\langle Y_X = x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle, x_i \text{ face-adj. to right side}$
Case 4: No cutting pair

Necklace $\langle Y_X = x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle, x_i$ face-adj. to right side
Case 4: No cutting pair

Necklace $\langle Y_X = x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle$, $x_i$ face-adj. to right side
Case 4: No cutting pair

Necklace $\langle Y_X=x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle$,
$x_i$ face-adj. to right side.
Case 4: No cutting pair

Necklace $\langle Y_X = x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle, x_i$ face-adj. to right side
Case 4: No cutting pair

Necklace $\langle Y_X = x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle$, $x_i$ face-adj. to right side, $G_1 = \emptyset$
Case 4: No cutting pair

Necklace $\langle Y_X=x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle$, $x_i$ face-adj. to right side, $G_1 = \emptyset$
Case 4: No cutting pair

Necklace $\langle Y_X=x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle, x_i$ face-adj. to right side, $G_1 = \emptyset$
Case 4: No cutting pair

Necklace \( \langle Y_X = x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle, x_i \) face-adj. to right side, \( G_1 = \emptyset \)
Case 4: No cutting pair

Necklace $\langle Y_X = x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle, x_i \text{ face-adj. to right side}, G_1 = \emptyset$
Case 4: No cutting pair

Necklace $\langle Y_X=x_0, f_1, x_1, \ldots , x_{s-1}, f_s, x_s \rangle, x_i$ face-adj. to right side, $G_1 = \emptyset$
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Necklace $\langle Y_X = x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle$, $x_i$ face-adj. to right side, $G_1 = \emptyset$
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Necklace $\langle Y_X = x_0, f_1, x_1, \ldots, x_{s-1}, f_s, x_s \rangle$,
$x_i$ face-adj. to right side, $G_1 = \emptyset$
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Running Time
5 Linear-time complexity for 3-connected graphs

Running Time

Algorithm 1: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 2: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 3: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 4: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 5: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 6: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 7: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 8: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 9: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 10: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 11: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 12: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 13: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 14: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

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Algorithm 15: Running Time

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Running Time

Algorithm 16: Running Time

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Algorithm 17: Running Time

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Algorithm 18: Running Time

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Algorithm 19: Running Time

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Algorithm 20: Running Time

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Algorithm 21: Running Time

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Algorithm 22: Running Time

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Algorithm 23: Running Time

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Algorithm 24: Running Time

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Algorithm 25: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 26: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 27: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 28: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 29: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.

Running Time

Algorithm 30: Running Time

For each non-trivial component of the graph, run a linear-time algorithm to compute the running time. Then, sum the running times of all the components.
Running Time

Store:
Running Time

Store:

• corners
Running Time

Store:

- corners
- faces: all vtcs on each side
Running Time

Store:

• corners

• faces: all vtcs on each side

• vtcs: all face-incidences to each side
Running Time

Store:

• corners

• faces: all vtcs on each side

• vtcs: all face-incidences to each side

• sides: all cutting pairs
Necklace scan
Necklace scan
Necklace scan
Necklace scan
Necklace scan
Necklace scan
Necklace scan
Necklace scan

X

Y

\alpha

\gamma_Y \gamma_X
Necklace scan
Necklace scan
Necklace scan
Necklace scan

face gets scanned:
Necklace scan

face gets scanned:
Necklace scan

⇒ one vtx becomes outer

face gets scanned:

⇒ one vtx becomes outer
Necklace scan

face gets scanned:

⇒ one vtx becomes outer
Necklace scan

face gets scanned:
⇒ one vtx becomes outer
⇒ $O(1)$ times
Necklace scan

- face gets scanned:
  - \( \Rightarrow \) one vtx becomes outer
  - \( \Rightarrow O(1) \) times
  - \( \Rightarrow O(\sum_f \text{deg}(f)) = O(n) \) time
Necklace scan

face gets scanned:
⇒ one vtx becomes outer
⇒ $O(1)$ times
⇒ $O(\sum_f \deg(f)) = O(n)$ time

**Theorem.**
$G$ int. 3-conn., $X, Y, \alpha$ on outer face ⇒ $T_{\text{int-path}}$ in $O(n)$ time.
Applications
Theorem.
G int. 3-conn. \(\Rightarrow\) binary spanning tree in \(O(n)\) time.
Theorem.
$G$ int. 3-conn. $\Rightarrow$ binary spanning tree in $O(n)$ time.
Applications

\[ \text{Theorem.} \]
\[ G \text{ int. 3-conn.} \implies \text{binary spanning tree in } O(n) \text{ time.} \]
Applications

Theorem.
$G$ int. 3-conn. $\implies$ binary spanning tree in $O(n)$ time.

Theorem.
$G$ int. 3-conn. $\implies$ 2-circuit in $O(n)$ time.
**Theorem.**
$G$ int. 3-conn. $\Rightarrow$ binary spanning tree in $O(n)$ time.

**Theorem.**
$G$ int. 3-conn. $\Rightarrow$ 2-circuit in $O(n)$ time.
Theorem.
\( G \text{ int. 3-conn.} \Rightarrow \text{binary spanning tree in } O(n) \text{ time.} \)

Theorem.
\( G \text{ int. 3-conn.} \Rightarrow 2\text{-circuit in } O(n) \text{ time.} \)
Theorem.
G int. 3-conn. ⇒ binary spanning tree in \( O(n) \) time.

Theorem.
G int. 3-conn. ⇒ 2-circuit in \( O(n) \) time.
Applications

**Theorem.**

\( G \text{ int. 3-conn.} \Rightarrow \text{binary spanning tree in } O(n) \text{ time.} \)

**Theorem.**

\( G \text{ int. 3-conn.} \Rightarrow 2\text{-circuit in } O(n) \text{ time.} \)
Conclusion
Conclusion

Theorem.

$G$ int. 3-conn., $X, Y, \alpha$ on outer face $\Rightarrow T_{int}$-path in $O(n)$ time.
Conclusion

**Theorem.**

$G$ int. 3-conn., $X, Y, \alpha$ on outer face $\Rightarrow T_{\text{int}}$-path in $O(n)$ time.

**Theorem.**

$G$ 2-conn., $X, Y, \alpha$ on outer face $\Rightarrow$ Tutte path in $O(n)$ time.
Theorem.
\[ G \text{ int. 3-conn.}, X, Y, \alpha \text{ on outer face} \Rightarrow T_{\text{int}}\text{-path in } O(n) \text{ time.} \]

Theorem.
\[ G \text{ 2-conn.}, X, Y, \alpha \text{ on outer face} \Rightarrow \text{Tutte path in } O(n) \text{ time.} \]

Theorem.
\[ G \text{ int. 3-conn.} \Rightarrow \text{binary spanning tree in } O(n) \text{ time.} \]
Conclusion

Theorem.
$G$ 2-conn., $X, Y, \alpha$ on outer face $\Rightarrow$ Tutte path in $O(n)$ time.

Theorem.
$G$ int. 3-conn. $\Rightarrow$ binary spanning tree in $O(n)$ time.

Theorem.
$G$ int. 3-conn. $\Rightarrow$ 2-circuit in $O(n)$ time.
Conclusion

**Theorem.**
G int. 3-conn., $X, Y, \alpha$ on outer face $\Rightarrow$ $T_{\text{int}}$-path in $O(n)$ time.

**Theorem.**
G 2-conn., $X, Y, \alpha$ on outer face $\Rightarrow$ Tutte path in $O(n)$ time.

**Theorem.**
G int. 3-conn. $\Rightarrow$ binary spanning tree in $O(n)$ time.

**Theorem.**
G int. 3-conn. $\Rightarrow$ 2-circuit in $O(n)$ time.

$X, Y, \alpha$ on different faces?
Conclusion

Theorem.
\( G \) int. 3-conn., \( X, Y, \alpha \) on outer face \( \Rightarrow \) \( T_{\text{int}} \)-path in \( O(n) \) time.

Theorem.
\( G \) 2-conn., \( X, Y, \alpha \) on outer face \( \Rightarrow \) Tutte path in \( O(n) \) time.

Theorem.
\( G \) int. 3-conn. \( \Rightarrow \) binary spanning tree in \( O(n) \) time.

Theorem.
\( G \) int. 3-conn. \( \Rightarrow \) 2-circuit in \( O(n) \) time.

\( X, Y, \alpha \) on different faces?

Non-planar graphs?
Conclusion

**Theorem.**
G int. 3-conn., X, Y, α on outer face ⇒ $T_{\text{int}}$-path in $O(n)$ time.

**Theorem.**
G 2-conn., X, Y, α on outer face ⇒ Tutte path in $O(n)$ time.

**Theorem.**
G int. 3-conn. ⇒ binary spanning tree in $O(n)$ time.

**Theorem.**
G int. 3-conn. ⇒ 2-circuit in $O(n)$ time.

X, Y, α on different faces?

Non-planar graphs?