Drawing trees and triangulations with few geometric primitives

Philipp Kindermann
FernUniversität in Hagen

Joint work with
Gregor Hültenschmidt, Wouter Meulemans & André Schulz
Visual Complexity

Number of geometric objects
Visual Complexity

Number of geometric objects

21 edges
Visual Complexity

Number of geometric objects

21 edges

1 segment
Visual Complexity

Number of geometric objects

21 edges

2 segments
Visual Complexity

Number of geometric objects

21 edges

3 segments
Visual Complexity

Number of geometric objects

21 edges

4 segments
Visual Complexity

Number of geometric objects

21 edges

5 segments
Visual Complexity

Number of geometric objects

21 edges

9 segments
Visual Complexity

Number of geometric objects

21 edges

30 edges

9 segments
Visual Complexity

Number of geometric objects

- 21 edges
- 9 segments
- 30 edges
- 1 arc
Visual Complexity

Number of geometric objects

21 edges

9 segments

30 edges

2 arcs
Visual Complexity

Number of geometric objects

- 21 edges
- 9 segments

- 30 edges
- 5 arcs
Visual Complexity

Number of geometric objects

- 21 edges
- 9 segments
- 30 edges
- 6 arcs
Visual Complexity

Number of geometric objects

- 21 edges
- 9 segments
- 30 edges
- 9 arcs
Visual Complexity

Number of geometric objects

21 edges

9 segments

30 edges

12 arcs
## Known Results

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[3] Igamberdiev et al. 2015  
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|                 | Lower    | Upper         |
|                 |          |               |
| Grid?           |          | $3e/4$ [2]    |

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Trees on the grid

heavy path decomposition
Trees on the grid

heavy path decomposition
Trees on the grid

heavy path decomposition
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Trees on the grid

Heavy path decomposition
- draw each heavy path tree in a ”box”

Level 0:
Trees on the grid

heavy path decomposition
- draw each heavy path tree in a "box"
- boxes of same level are disjoint

level 0:
Trees on the grid

heavy path decomposition
- draw each heavy path tree in a "box"
- boxes of same level are disjoint

level 0:

level $k$:
Trees on the grid

heavy path decomposition
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level 0:

level $k$: 
Trees on the grid

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level 0:

level \( k \):
Trees on the grid

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level 0:

level k:
Trees on the grid

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level 0:

level \( k \):

level 3:
Trees on the grid

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level 0:

level k:

one vertex:
Trees on the grid

heavy path decomposition
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Trees on the grid

Heavy path decomposition:
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Level $k$:

One vertex:
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**heavy path decomposition**
- Draw each heavy path tree in a "box"
- Boxes of same level are disjoint

**level 0:**

**level k:**

**one vertex:**

**Tree \( \rightarrow \) 3e/4 arcs, \( O(n^2) \) \( \times \) \( O(n^{1.58}) \) grid
Trees on the grid

level $k$:

Tree $\rightarrow 3e/4$ arcs, $O(n^2) \times O(n^{1.58})$ grid
Trees on the grid

level $k$:

Tree $\rightarrow 3e/4$ arcs, $O(n^2) \times O(n^{1.58})$ grid
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Trees on the grid

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Tree $\rightarrow 3e/4$ arcs, $O(n^2) \times O(n^{1.58})$ grid
Trees on the grid

level $k$:

Tree $\rightarrow \frac{9}{2}$ arcs, quasipolynomial grid

Tree $\rightarrow 3e/4$ arcs, $O(n^2) \times O(n^{1.58})$ grid
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order

1

3

2
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order

1

4

7

8

2

3

5

6
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order

![Graph diagram with labeled vertices and edges]
Triangulations with circular arcs

canonical order

![Graph with circular arcs and canonical order](image)
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order
Triangulations with circular arcs

canonical order

red edges extend into bottom arc
Triangulations with circular arcs

canonical order

red edges extend into bottom arc
Triangulations with circular arcs

canonical order

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Triangulations with circular arcs

canonical order

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Triangulations with circular arcs

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Triangulations with circular arcs

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Triangulations with circular arcs

canonical order

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Triangulations with circular arcs

canonical order

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Triangulations with circular arcs

canonical order

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Triangulations with circular arcs

canonical order

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Triangulations with circular arcs

Canonical order

Red edges extend into bottom arc
Triangulations with circular arcs

canonical order

red edges extend into bottom arc

Triangulation $\rightarrow \frac{5n - 11}{3}$ arcs
Planar graphs with circular arcs

Canonical order

Red edges extend into bottom arc

Triangulation $\rightarrow (5n - 11)/3$ arcs
Planar graphs with circular arcs

Canonical order

- Red edges extend into bottom arc

Planar $\rightarrow (14n - 29)/3 - e$ arcs

Triangulation $\rightarrow (5n - 11)/3$ arcs
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- **Segments**: Lower $9/2$, Upper $9/2$; Circular Arcs: Lower $9/2$, Upper $9/2$.
- **3-trees**: Lower $2n$, Upper $2n$; Circular Arcs: Lower $e/6$, Upper $11e/8$.
- **3-connected**: Lower $2n$, Upper $5n/2$; Circular Arcs: Lower $e/6$, Upper $2e/3$.
- **cubic 3-conn.**: Lower $n/2$, Upper $n/2$; Circular Arcs: Lower $n/2$, Upper $n/2$.
- **Triangulation**: Lower $2n$, Upper $7n/3$; Circular Arcs: Lower $5n/3$.
- **Planar**: Lower $2n$, Upper $16n/3 - e$; Circular Arcs: Lower $5n/3$, Upper $14n/3 - e$.

[3] Igamberdiev et al. 2015