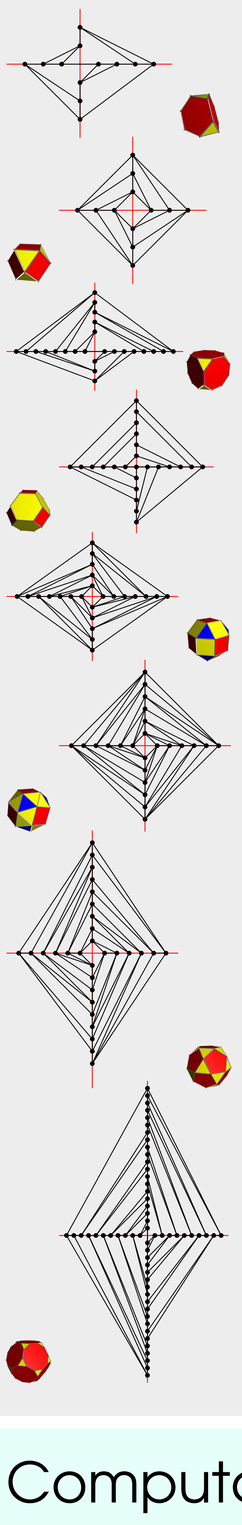
Weak Line Covers with Two Lines in the Plane

Oksana Firman Fabian Lipp Laura Straube Alexander Wolff

Lehrstuhl für Informatik I, Universität Würzburg



Problem

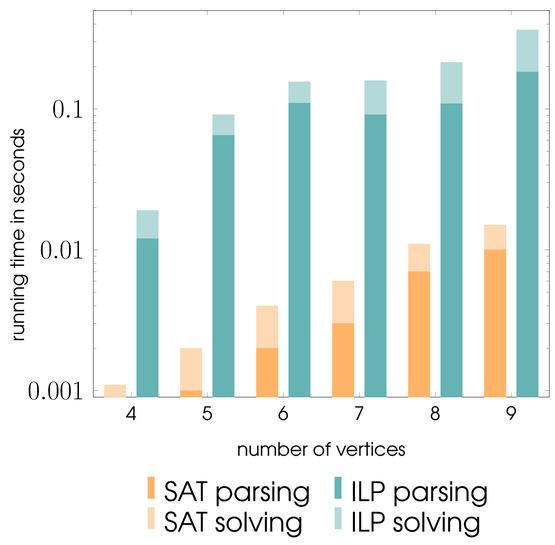
We examine the problem of drawing planar graphs in the plane such that they can be drawn on few lines, that is, every vertex must lie on one of the lines. At the same time, we insist that edges are drawn as (crossing-free) straight-line segments. Here, we want to identify classes of graphs that can be drawn on *two* lines.

Motivation

It is **NP-hard** to decide whether a given planar graph can be drawn on two lines (Biedl et al., Manuscript 2018).
We implemented an **ILP** and a **SAT** formulation in order to generate and verify hypotheses about classes of graphs that can be drawn on two lines.

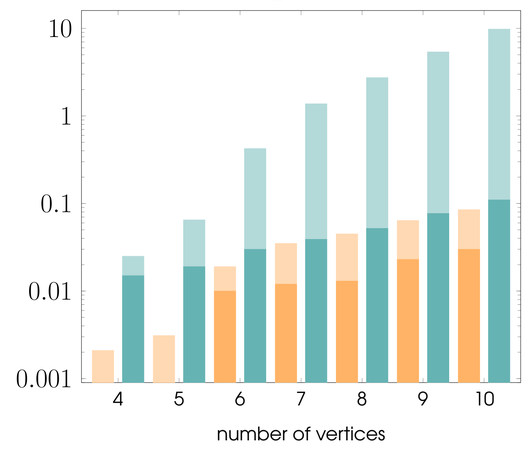
Running Time Comparison

Maximum-Degree-3 Graphs



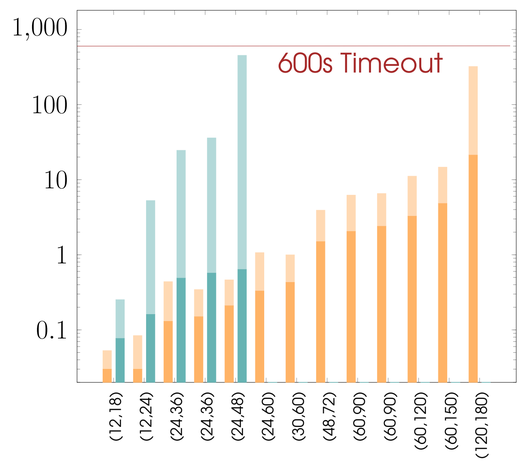
All tested maximum-degree-3 graphs can be drawn on two lines.

Triangulations



None of the tested triangulations – with one exception for each number of vertices – can be drawn on two lines.

Archimedean Graphs



Archimedean graphs are highly symmetric 3-regular graphs. All can be drawn on two lines. See the margins for the Archimedean solids and drawings on two lines. Each graph is specified by a pair (n, m), where n is the number of vertices and m is the number of edges.

Note: The time scale is logarithmic and therefore special care must be taken when comparing the parsing to the solving time.
It is easy to see that SAT is much faster than ILP and even the parsing time of ILP is longer than the complete computing time of SAT.

Computational Results

Graph class number of vertices ("✓" means "all")	# drawable use of the line crossing permitted	 prohibited	# graphs planar, non-isomorphic
MAX-DEG-3			
4	✓	5	6
5..11	✓	✓	7658
BICONNECTED			
12..13	✓	✓	11050
3-REGULAR			
14..21	✓	✓	29516
MAX-DEG-4			
6	73	63	74
8	1659	1489	1663
MAX-DEG-5			
5	✓	16	20
6	98	71	99
7	555	383	566
TRIANGLE-FREE			
4..9	✓	✓	1375
BIPARTITE			
10	✓	✓	2749
QUADRANGULATIONS			
11..16	✓	✓	94
17	138	138	139
18	448	448	451
TRIANGULATIONS			
4..11	8	0	1555
SPECIAL GRAPHS			
Archimedean	✓	✓	13
Catalan	6	6	13
Platonic	3	2	5

⇓

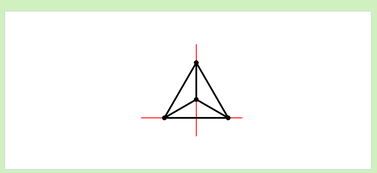
Conjecture.
Any planar graph of maximum degree 3 can be drawn on two lines.

Future Work

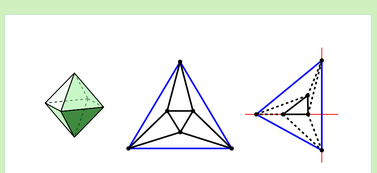
- *Prove the conjecture!*

Conclusions

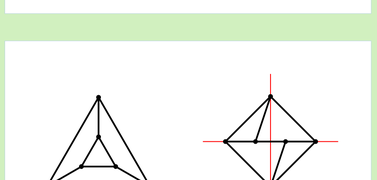
The complete graph K_4 can only be drawn on two lines if the drawing is permitted to use the **crossing** of the lines (see the differences between columns 2 and 3 in the table to the left).



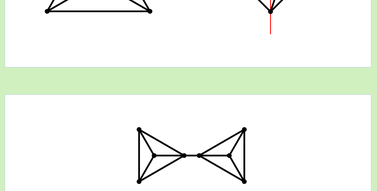
A **nested triangles** graph cannot be drawn on two lines if the outer face of the drawing is a triangle. For example, in the octahedron are faces are triangles; therefore it cannot be drawn on two lines (Firman et al., EuroCG 2018).



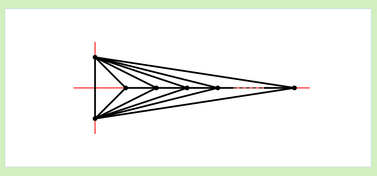
A **maximum degree of 3** limits the connectivity of triangles which seem to make it always possible to remove the nesting of triangles by changing the embedding.



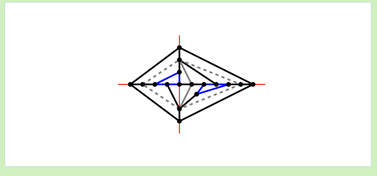
A graph with **degree greater than 3** can contain connected tetrahedra and then it is not drawable on two lines.



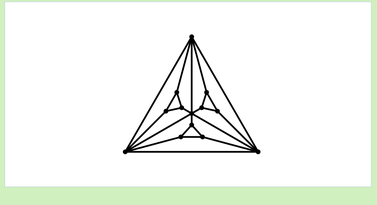
Triangulations need more than two lines except for an extension of K_4 where each triangle shares an edge with the outer face and thus has no nested triangles.

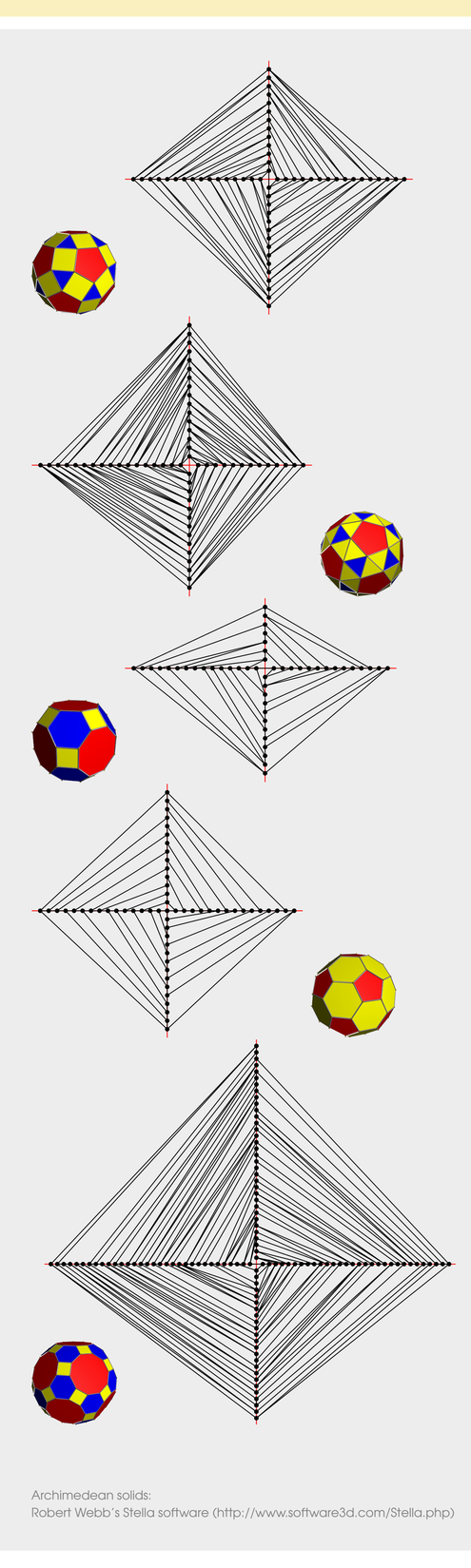


There are **triangle-free** graphs that cannot be drawn on two lines such as this **quadrangulation**. Other than triangles, quadrilaterals can easily be nested in drawings on two lines. Our triangulation, however, cannot be drawn.



All outerplanar graphs can be drawn on two lines (Chaplick et al., GD 2016). We found a **2-outerplanar** graph that cannot be drawn on two lines.





Archimedean solids:
Robert Webb's Stella software (http://www.software3d.com/Stella.php)