Drawing Graphs with Vertices at Specified Positions and Crossings at Large Angles

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Lehrstuhl für Informatik I
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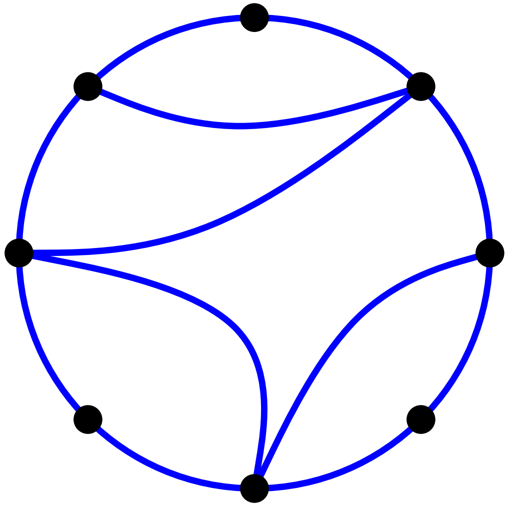
Joint work with
Jan Haunert, Tamara Mchedlidze, Joachim Spoerhase & Alexander Wolff

Vertices at Specified Positions

Gritzmann, Mohar, Pach, Pollack 1991:

graph



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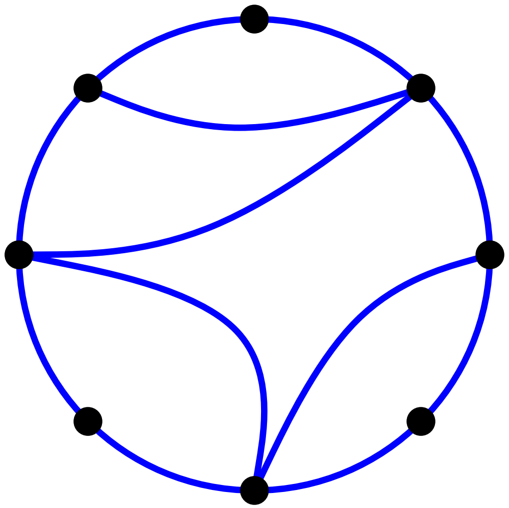
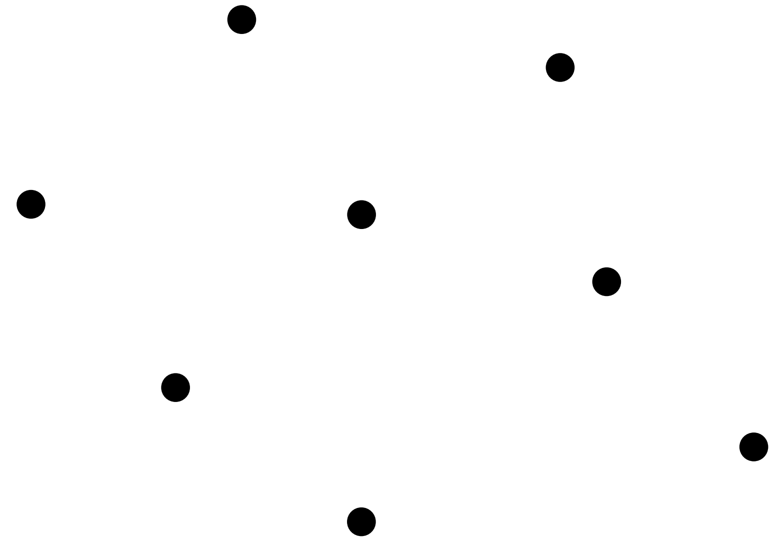


point-set embeddability

point set



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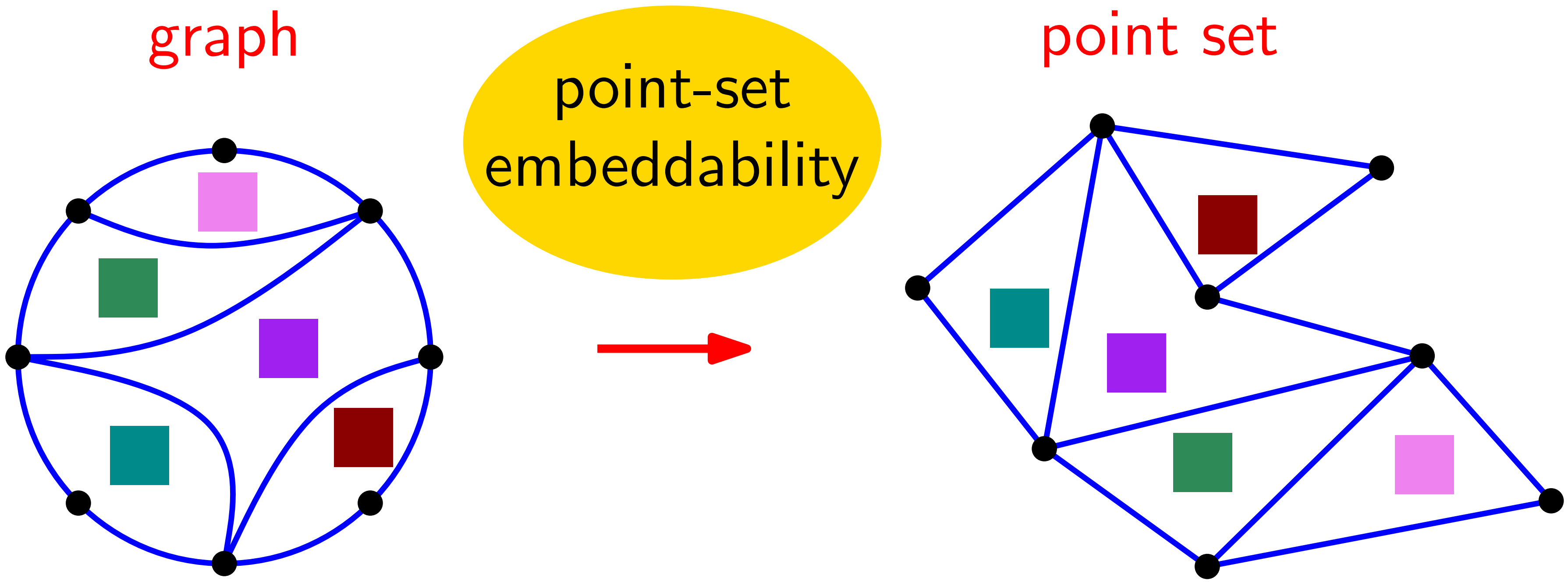

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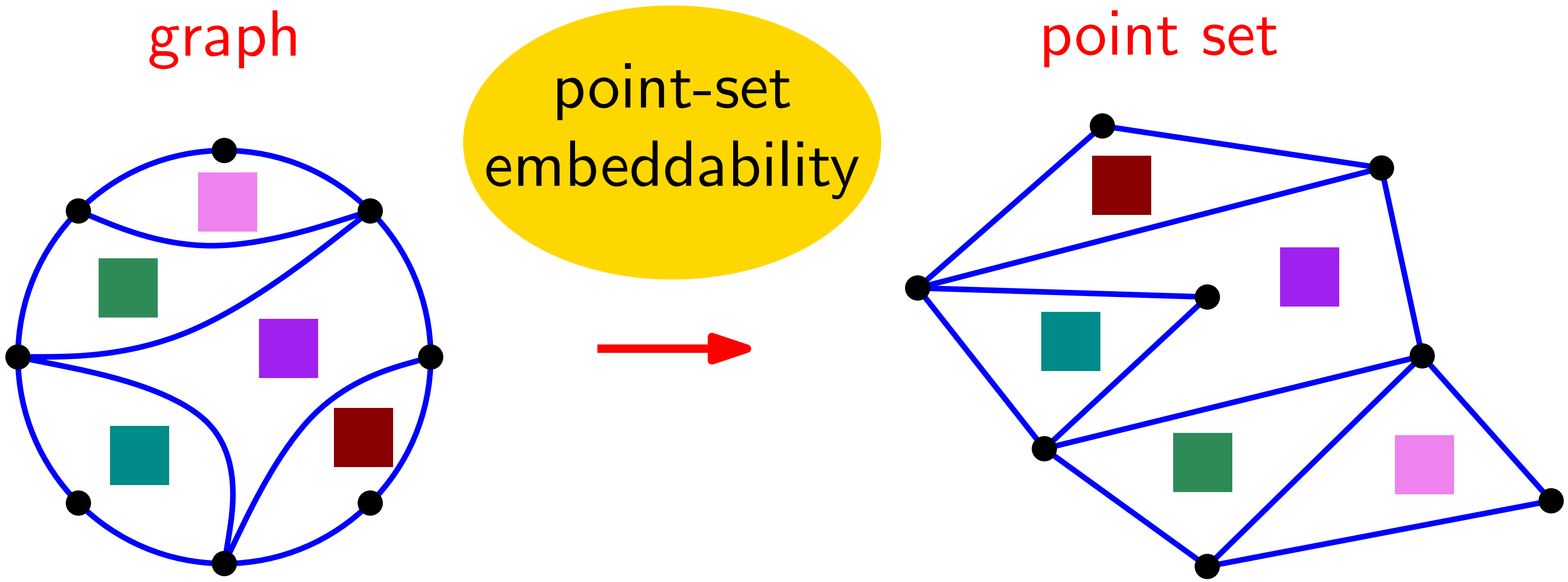

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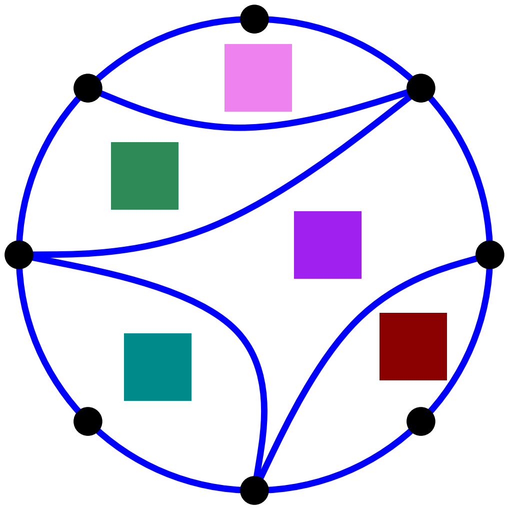

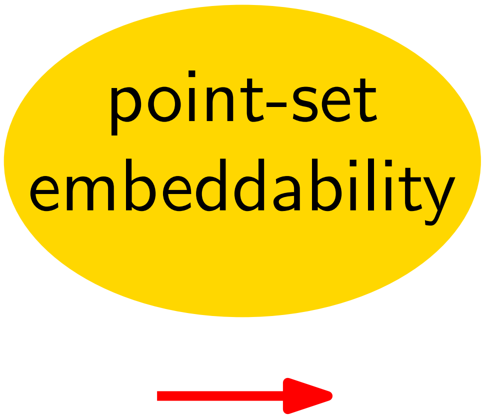

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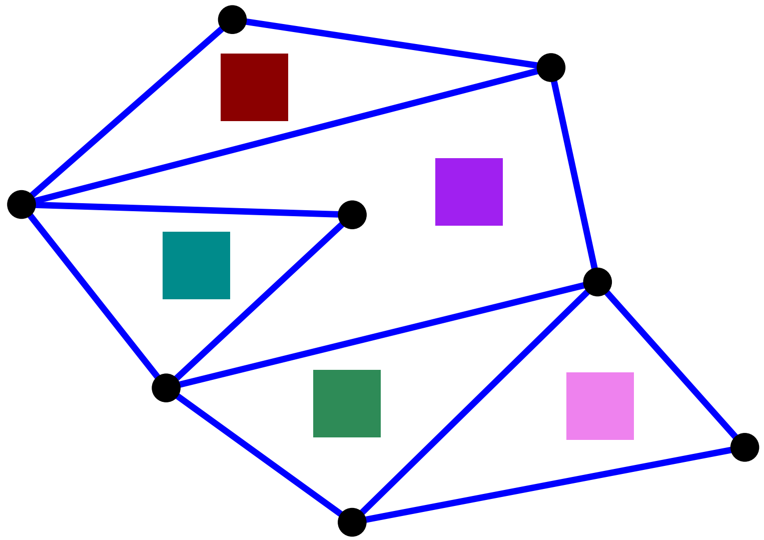

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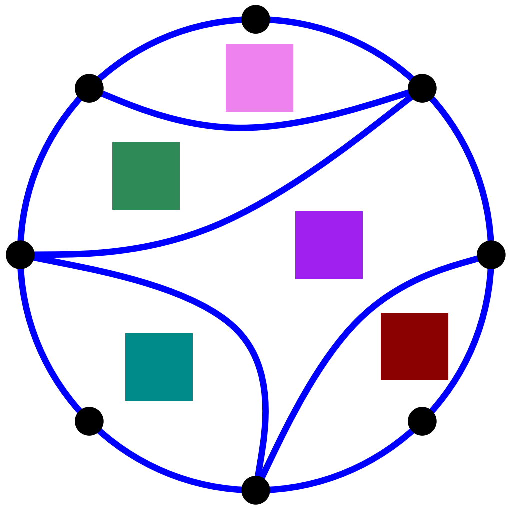

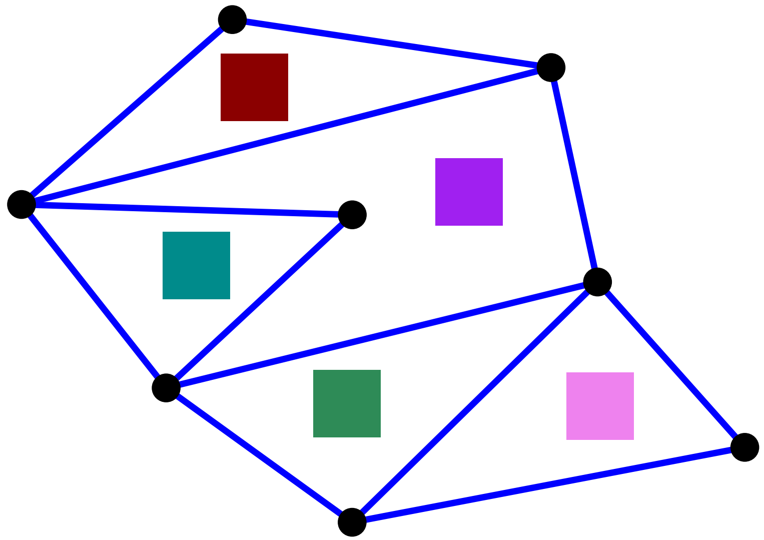

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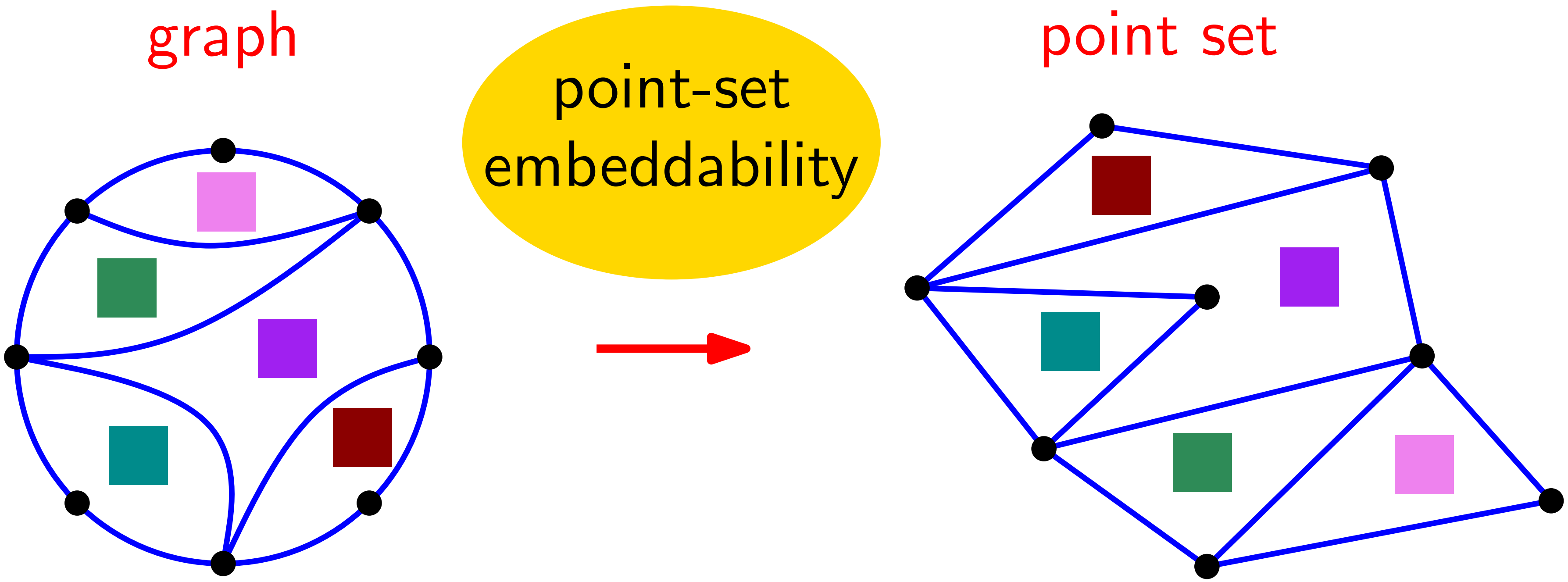

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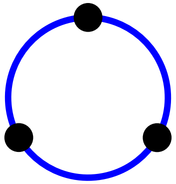

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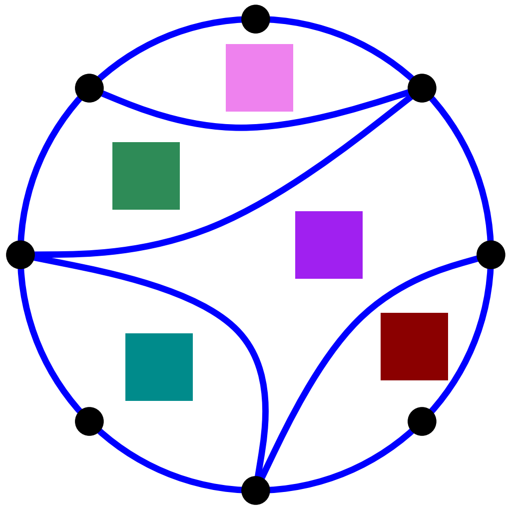

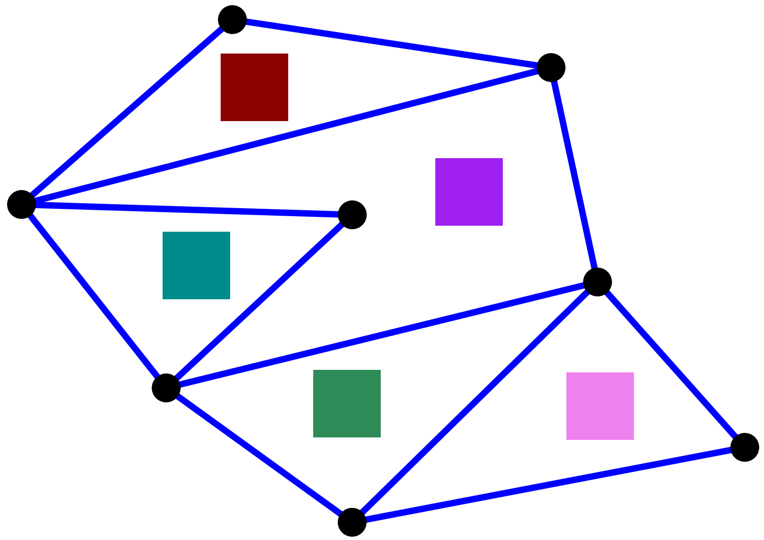

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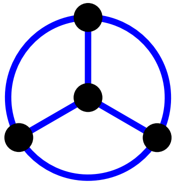

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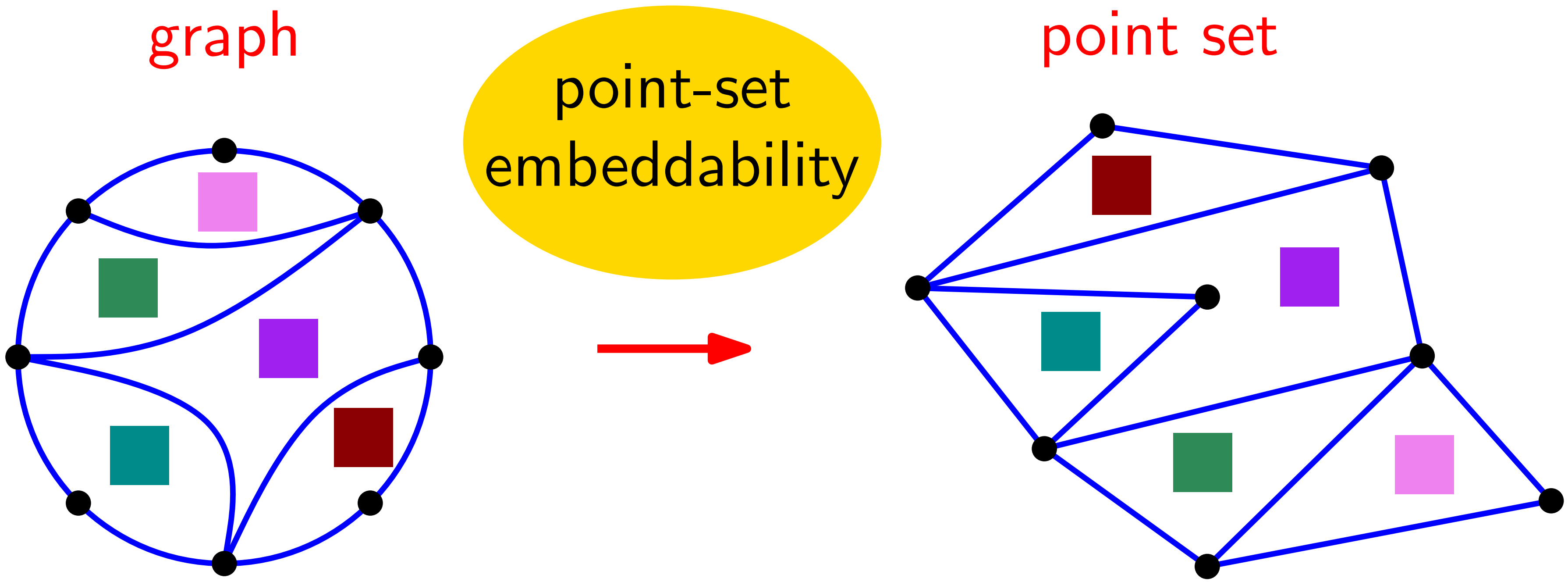

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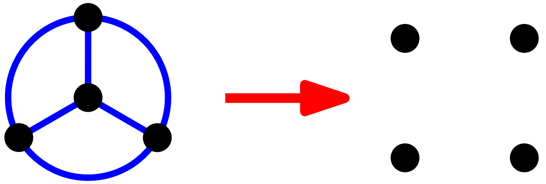

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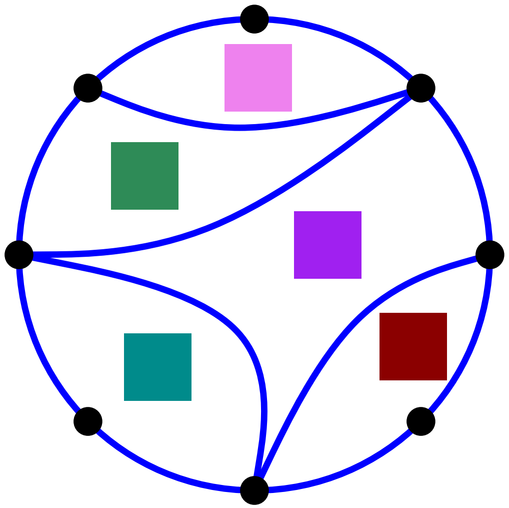

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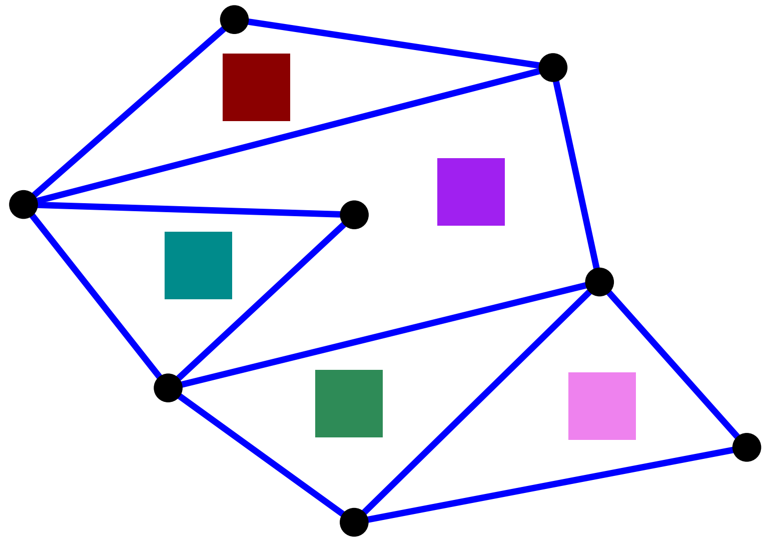

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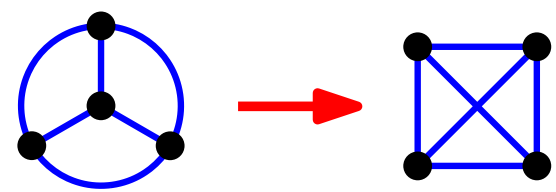

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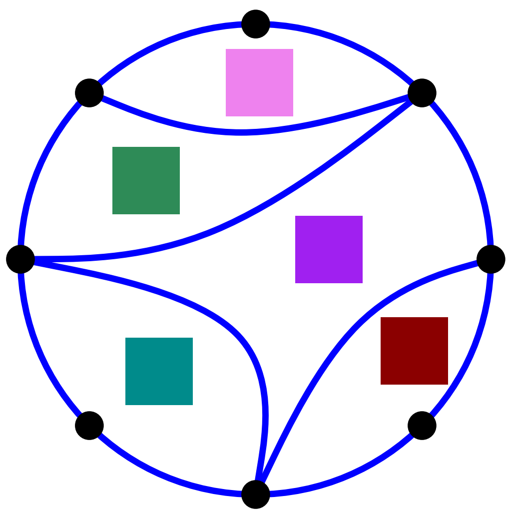

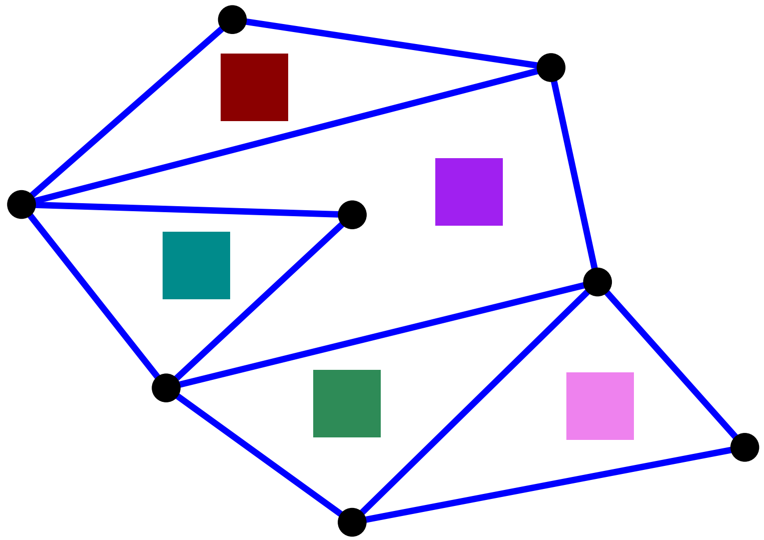

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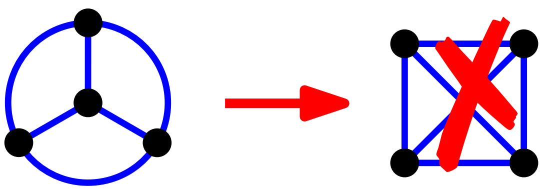

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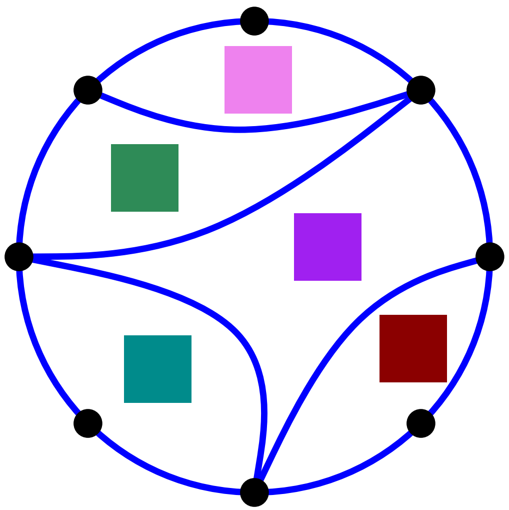

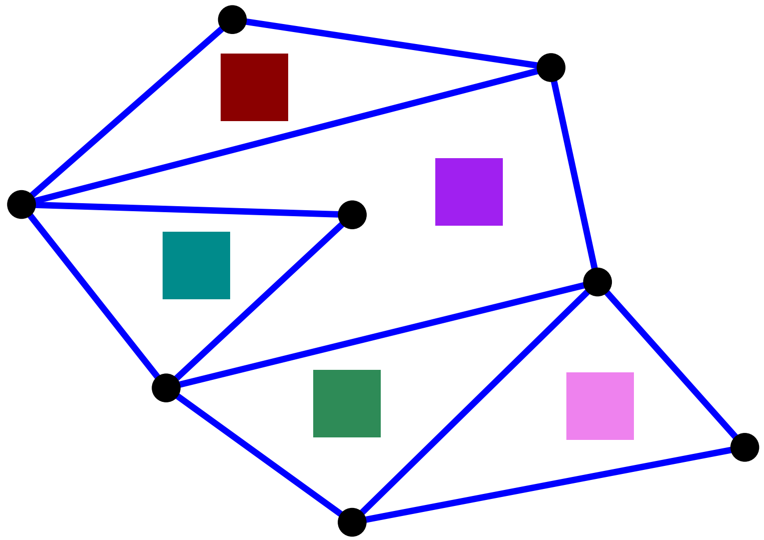

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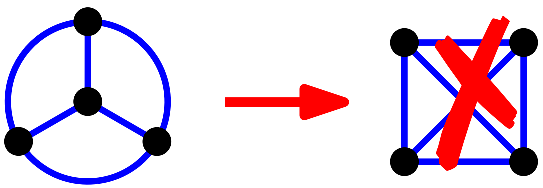

Cabello, 2006: NP-hard for general graphs

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Kaufmann & Wiese 2002: *Allow 2 bends per edge! Then...*

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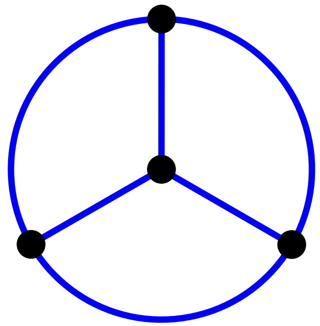

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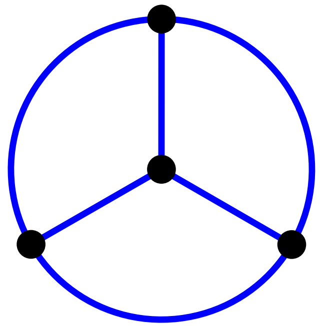

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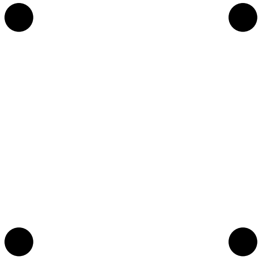

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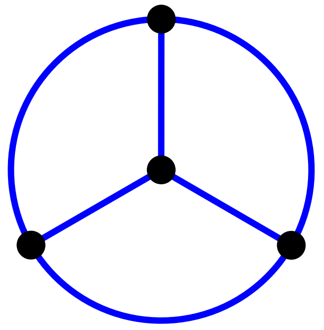

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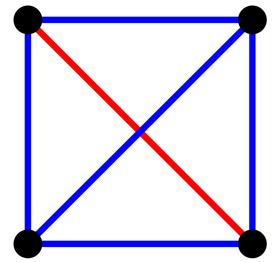

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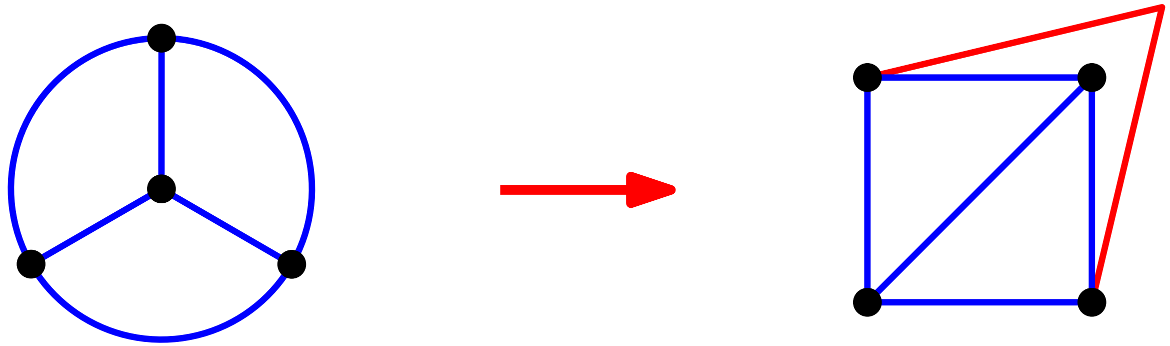

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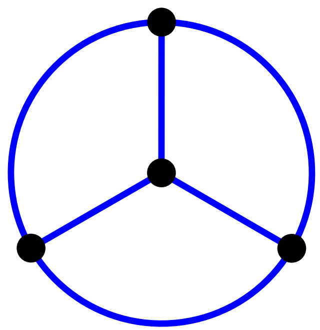

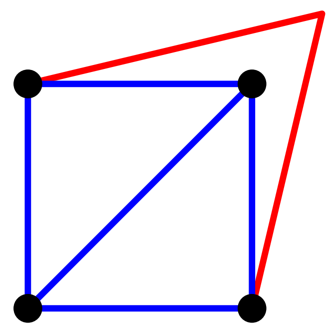

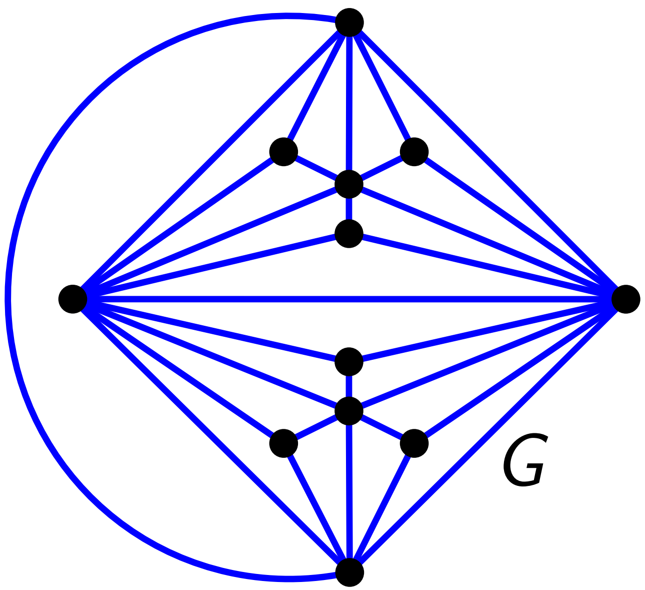

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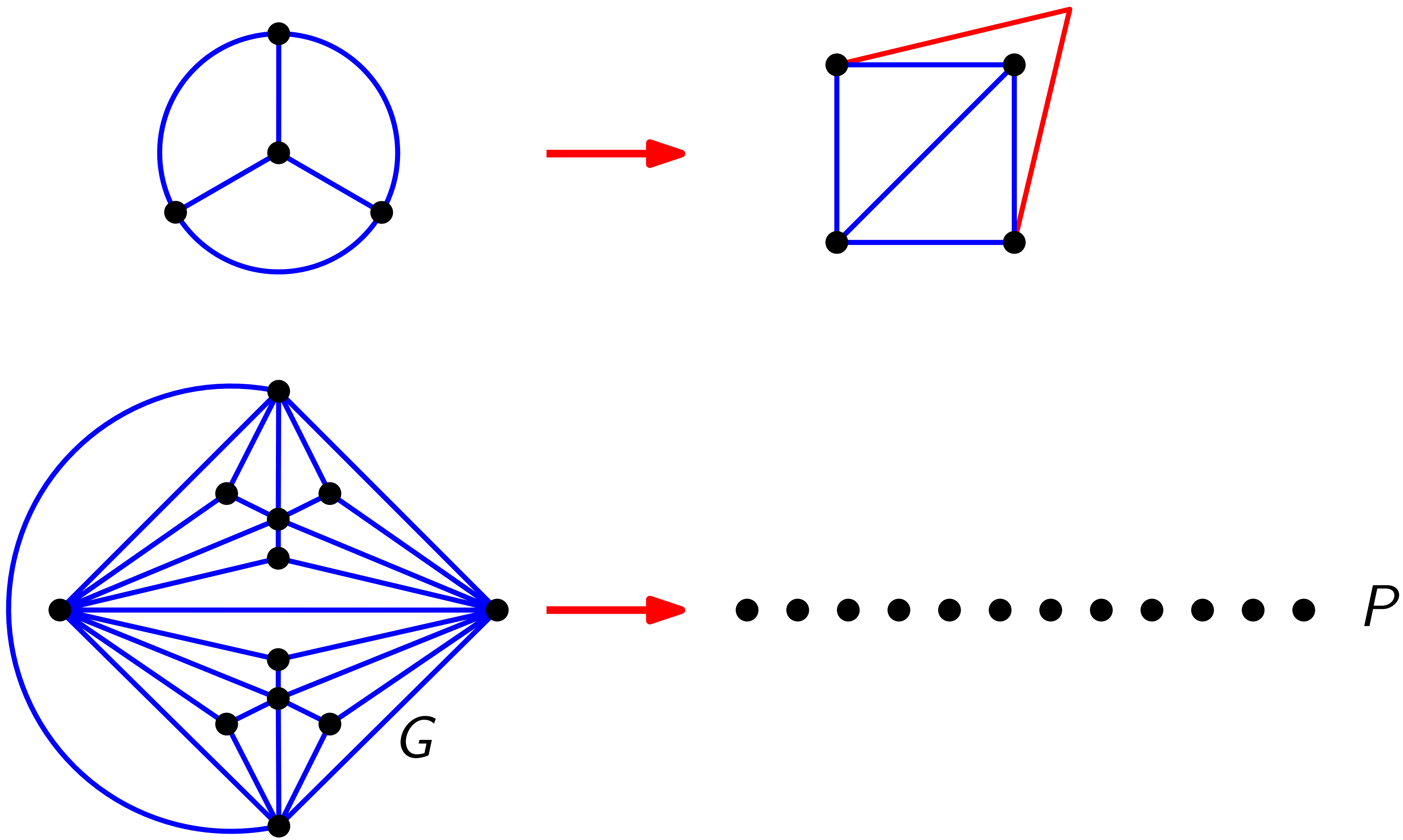

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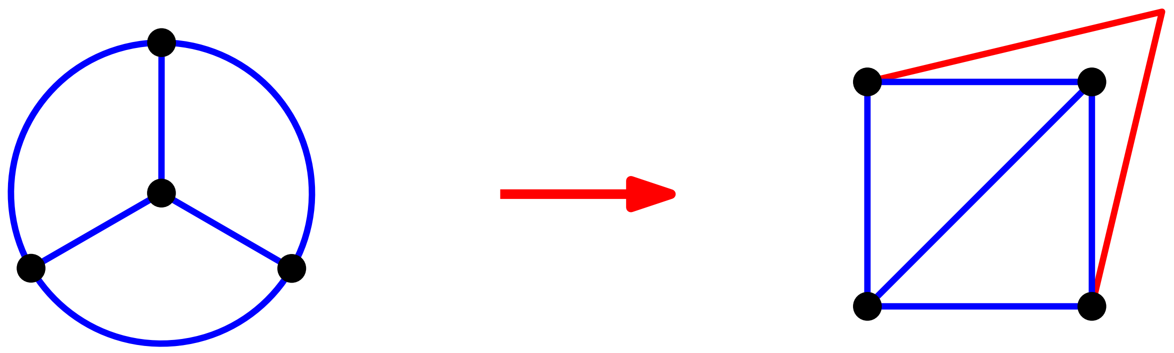
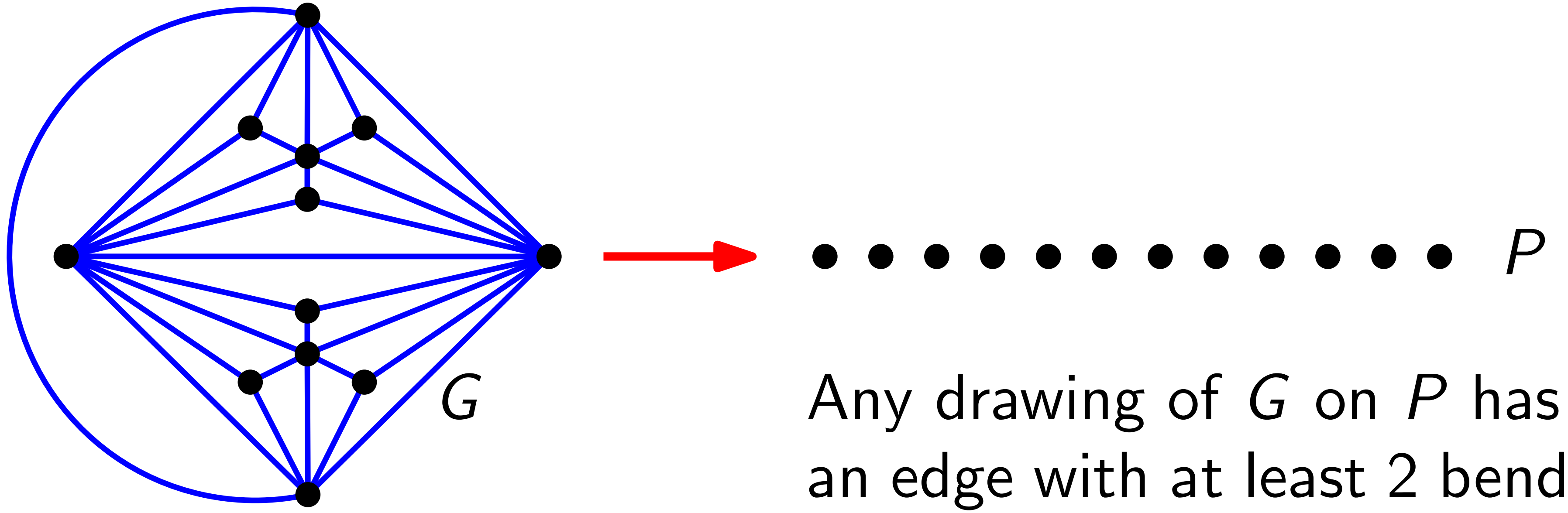


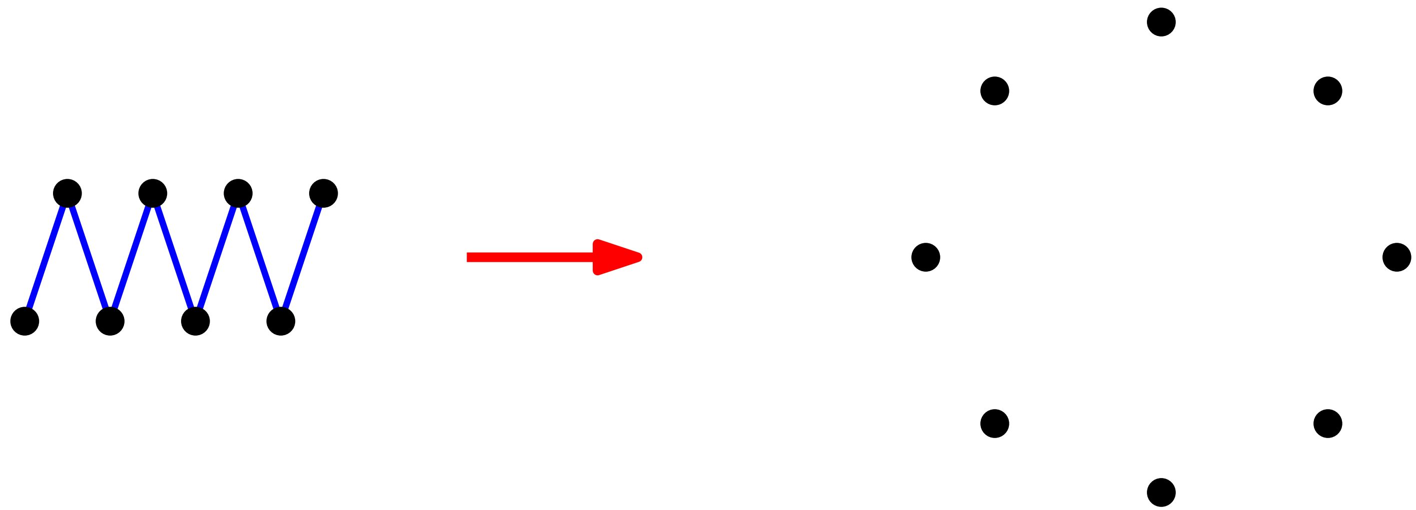


Any drawing of G on P has an edge with at least 2 bends!

The Bad News Is. . .

Pach & Wenger 2001:

a path → a pt set in convex position



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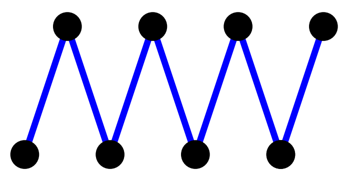

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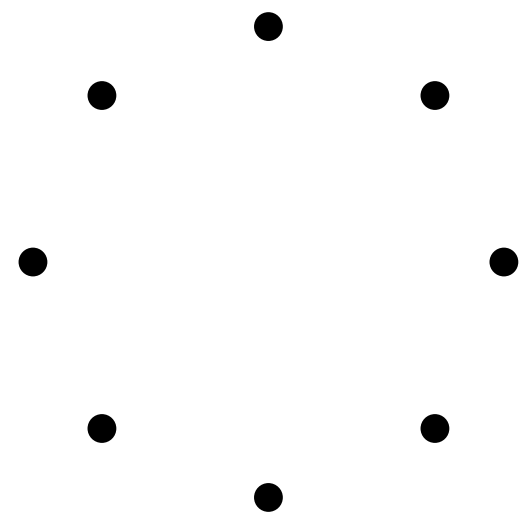

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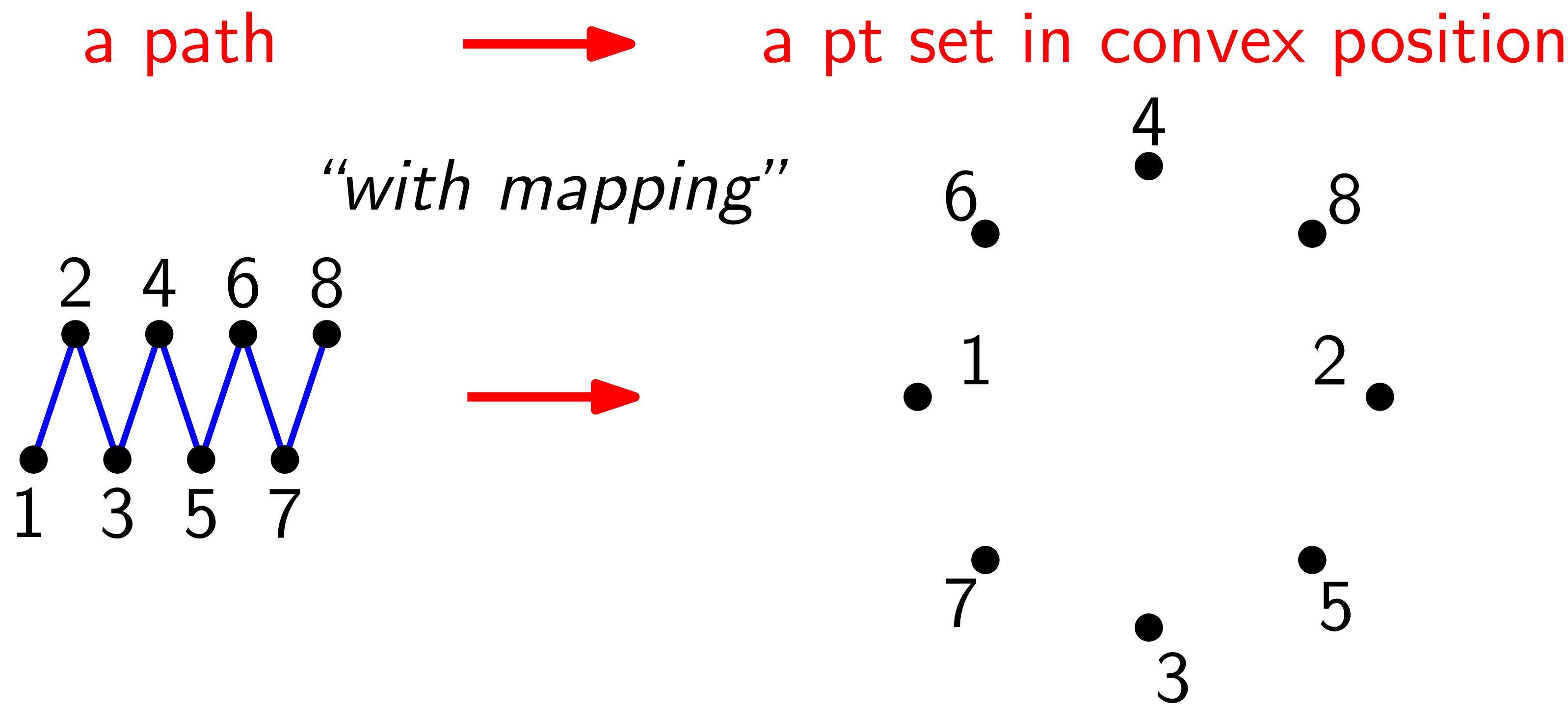

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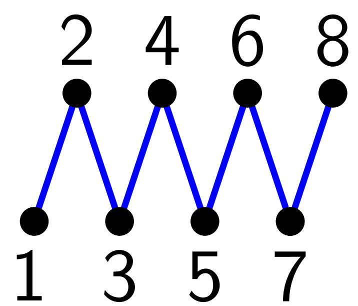

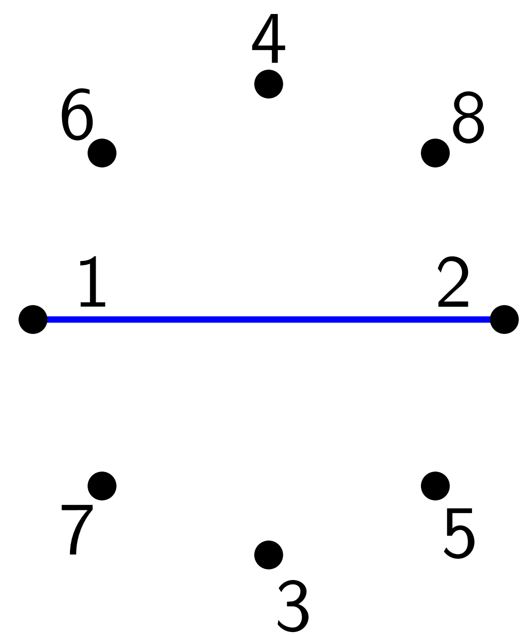

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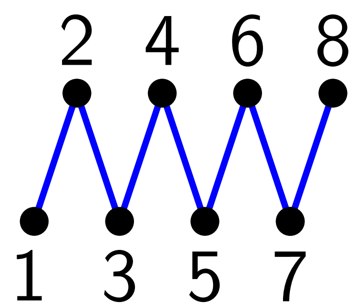

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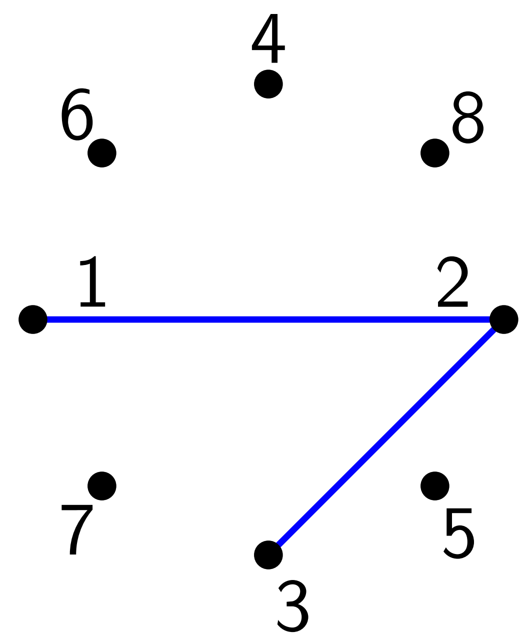

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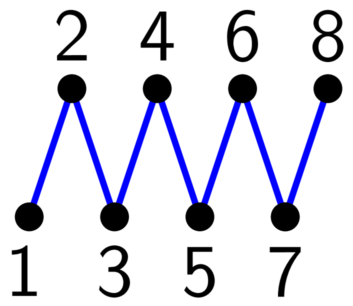

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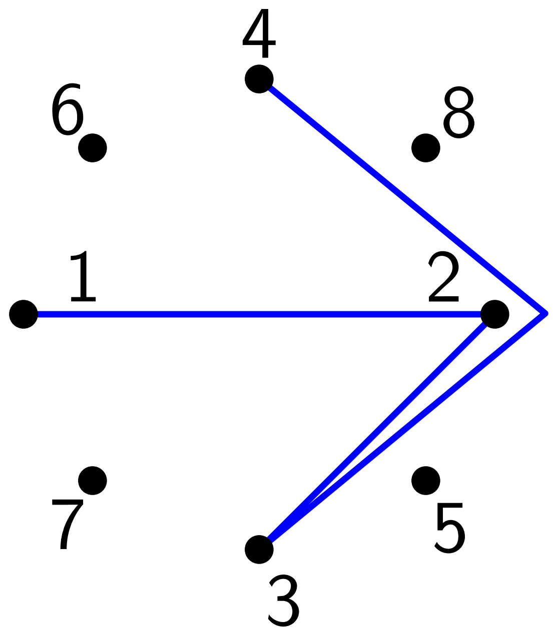

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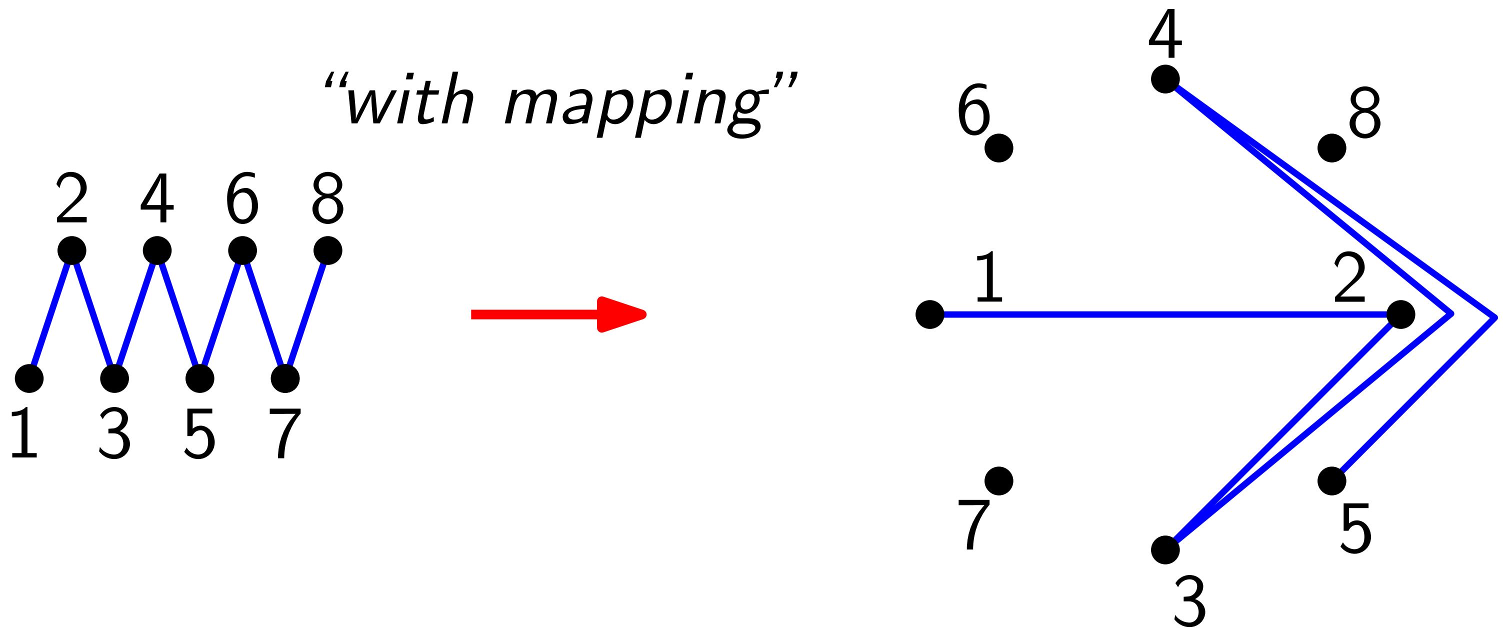

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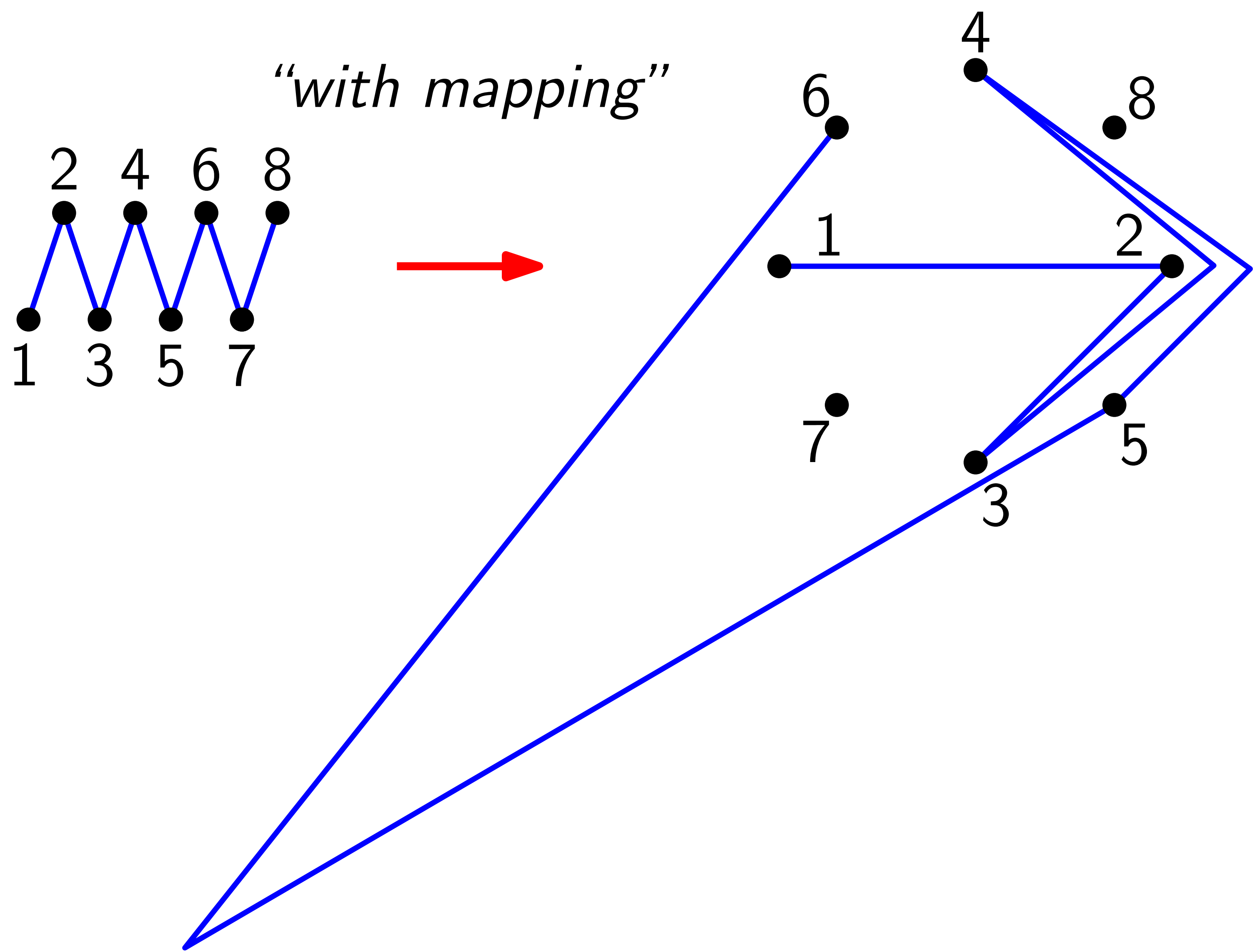

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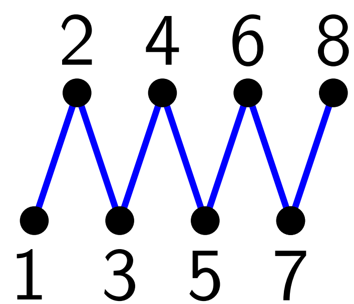

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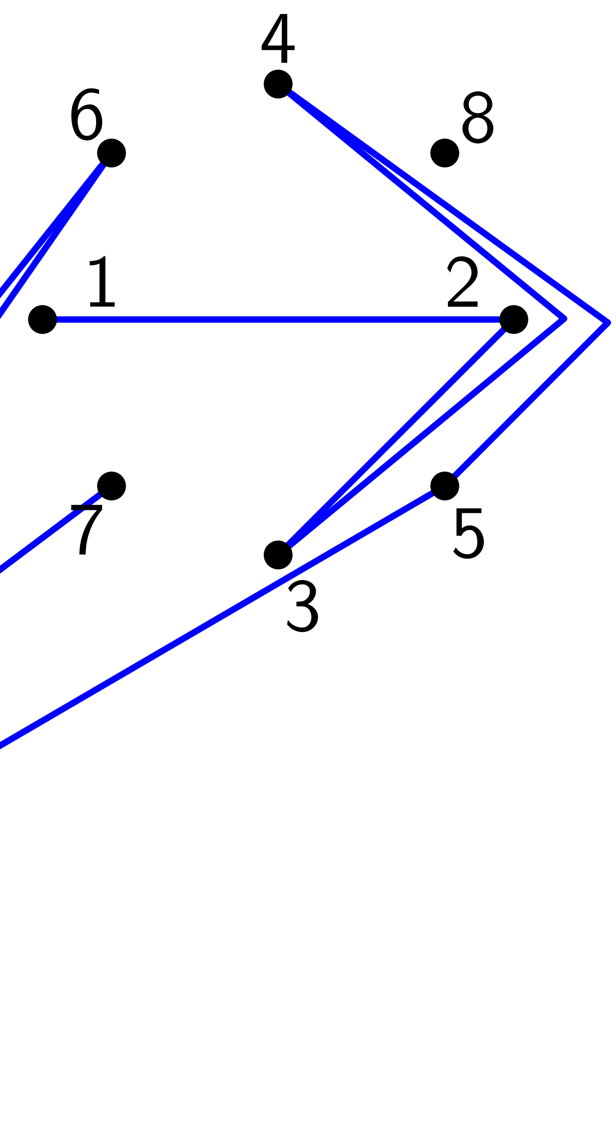

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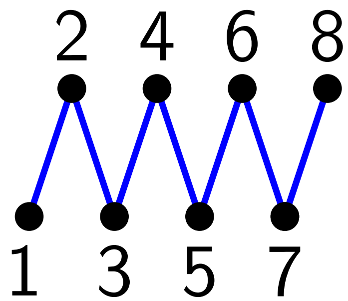

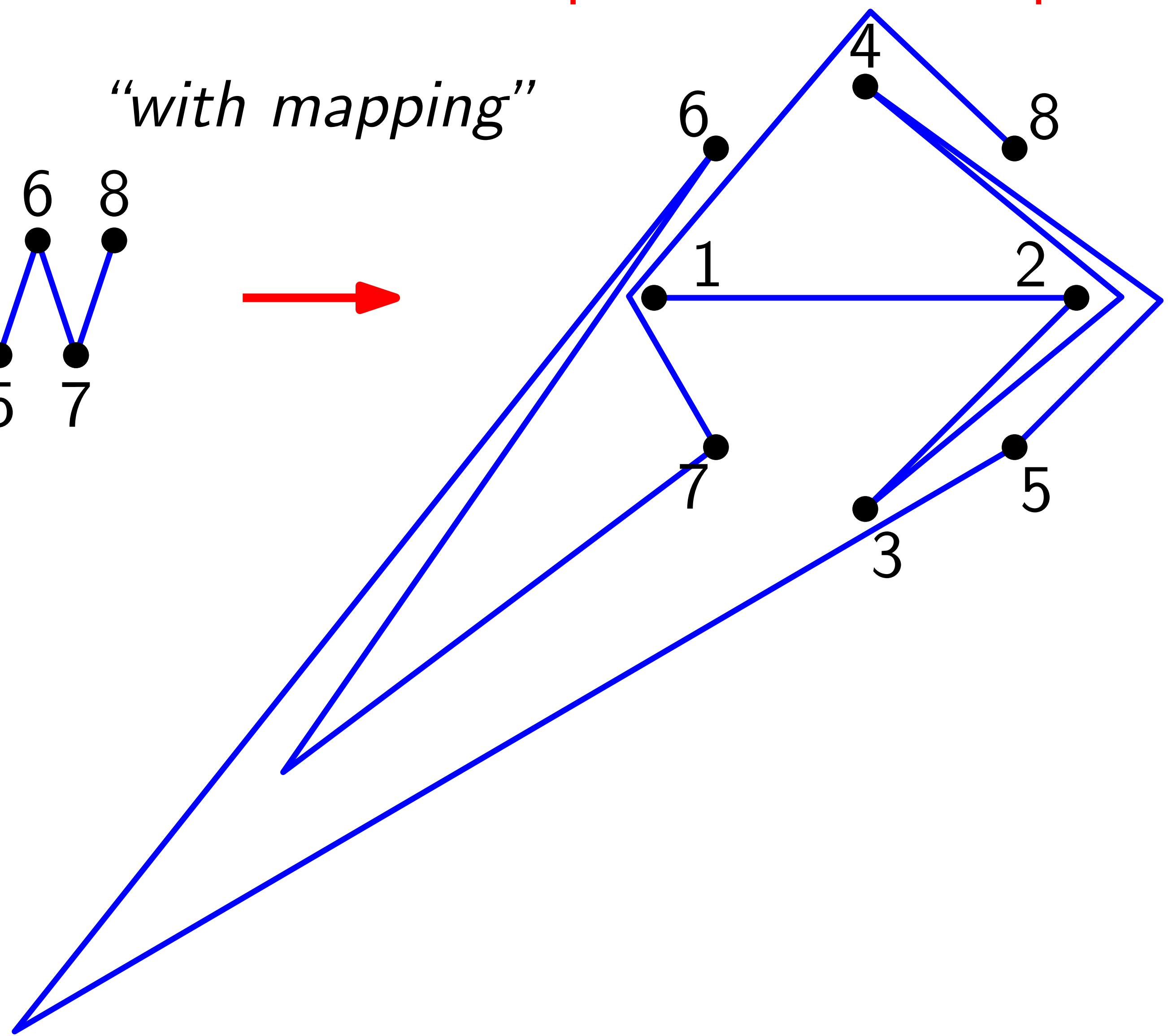

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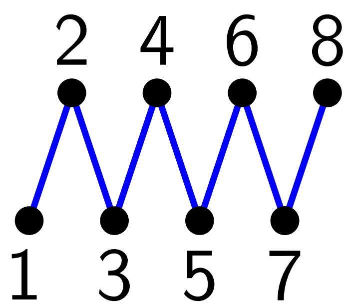

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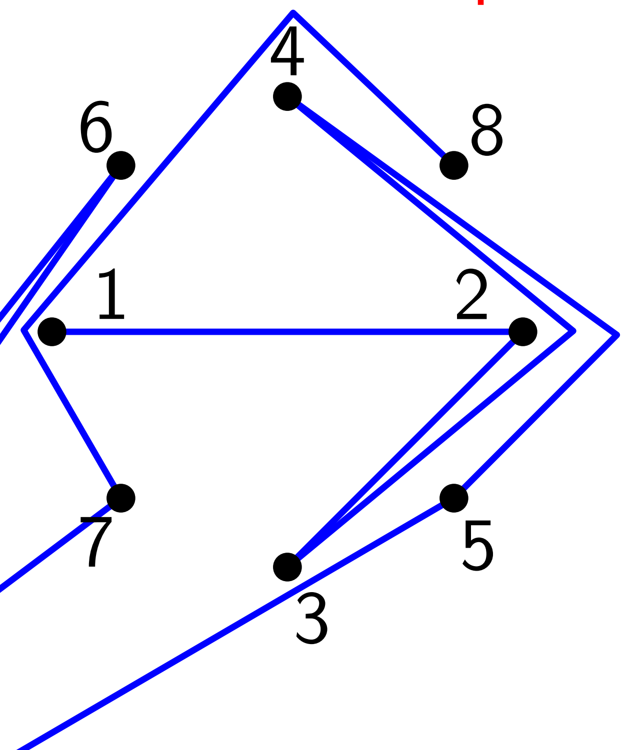

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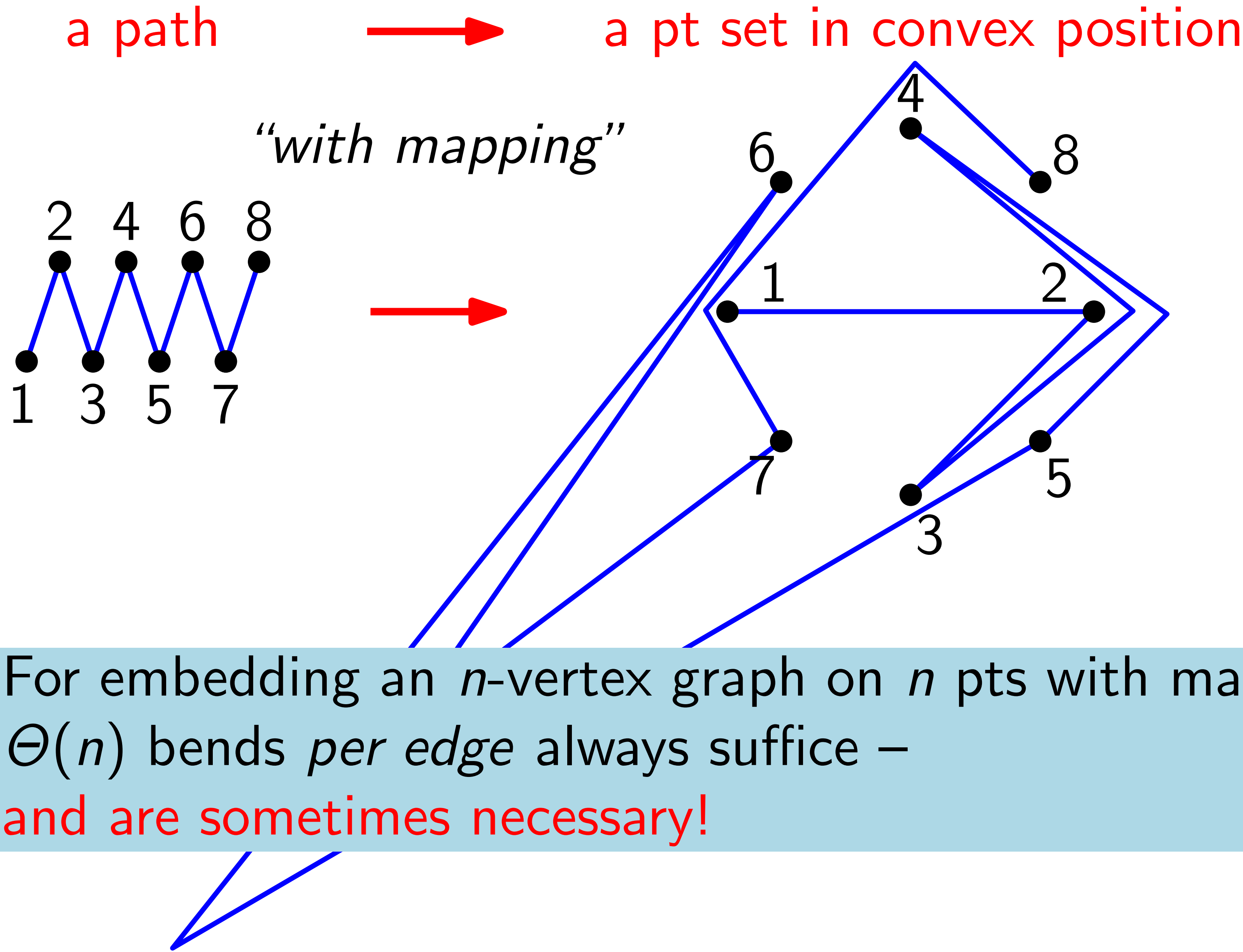

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Huang, Hong, Eades 2008: *Forget planarity!*

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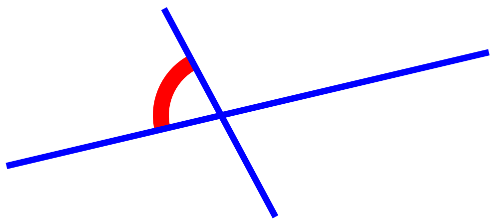

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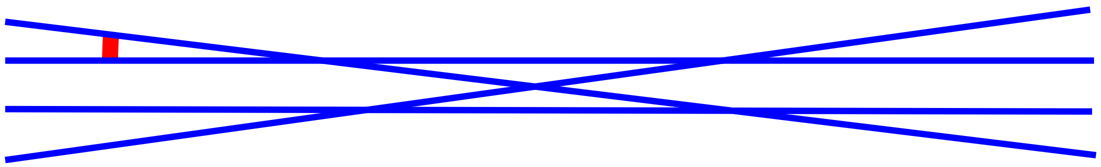

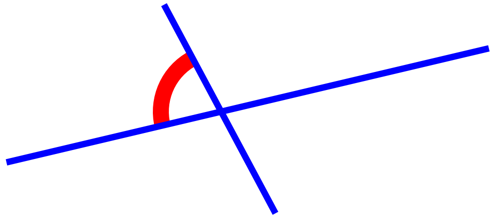

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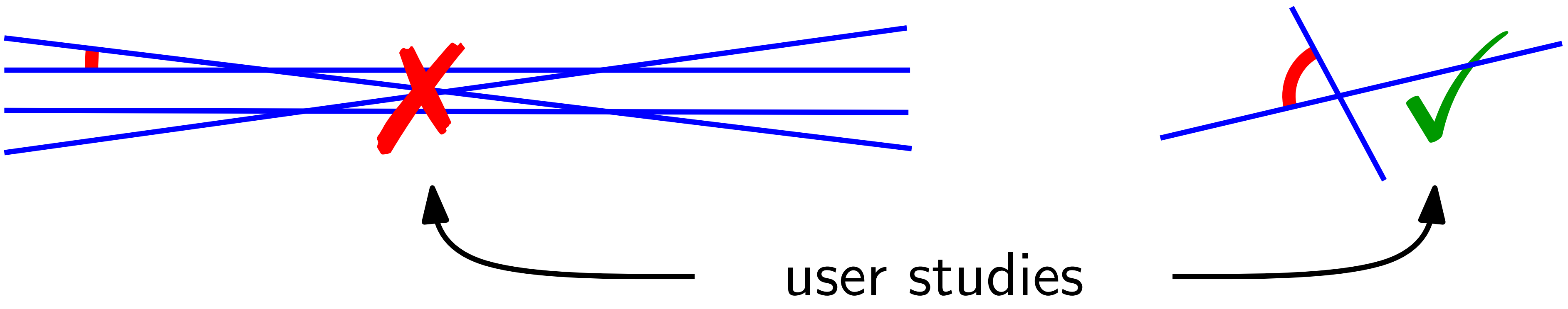

A New Trick!

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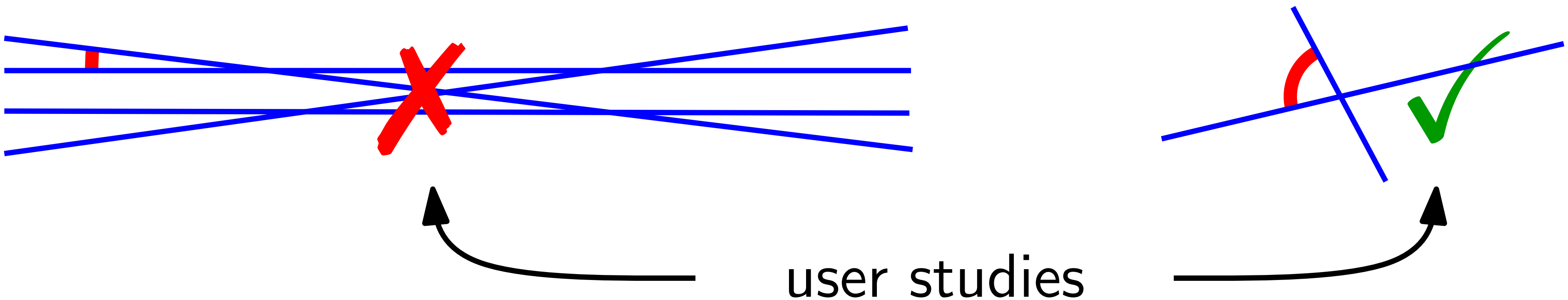

Didimo, Eades, Liotta 2009: *90° crossings & 3 bends per edge*

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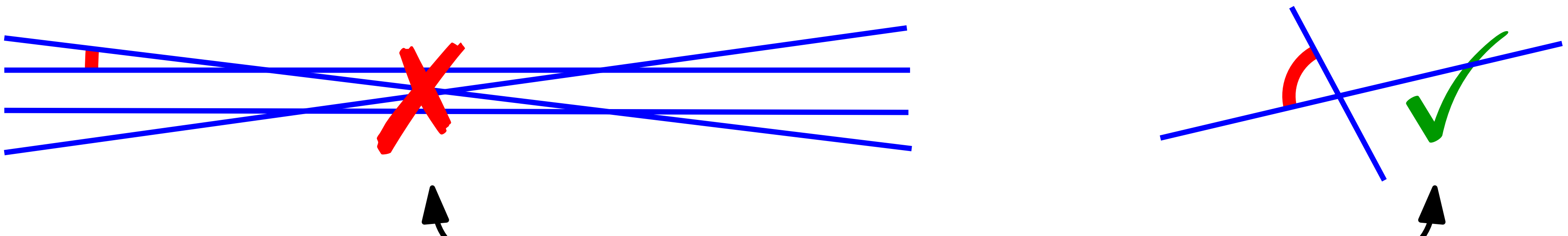

user studies

RAC_3 drawings

Didimo, Eades, Liotta 2009: *90° crossings & 3 bends per edge*

A New Trick!

Huang, Hong, Eades 2008: *Forget planarity!*

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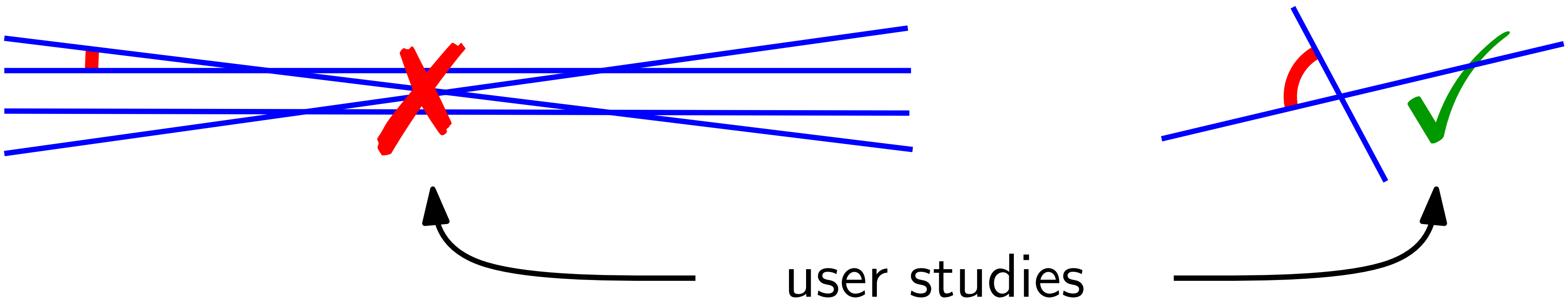

RAC_3 drawings

Didimo, Eades, Liotta 2009: *90° crossings & 3 bends per edge*

1. *any* ~~planar~~ graph $\xrightarrow{\text{RAC}_3}$ grid of size $O(m) \times O(m)$

A New Trick!

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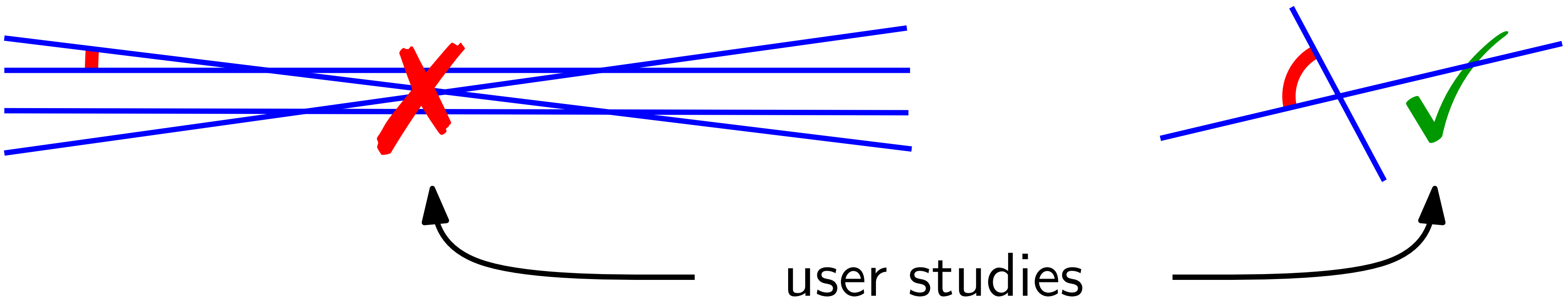

RAC_3 drawings

Didimo, Eades, Liotta 2009: *90° crossings & 3 bends per edge*

1. *any* pla~~n~~ar graph $\xrightarrow{RAC_3}$ grid of size $O(m) \times O(m)$

bends on grid points!

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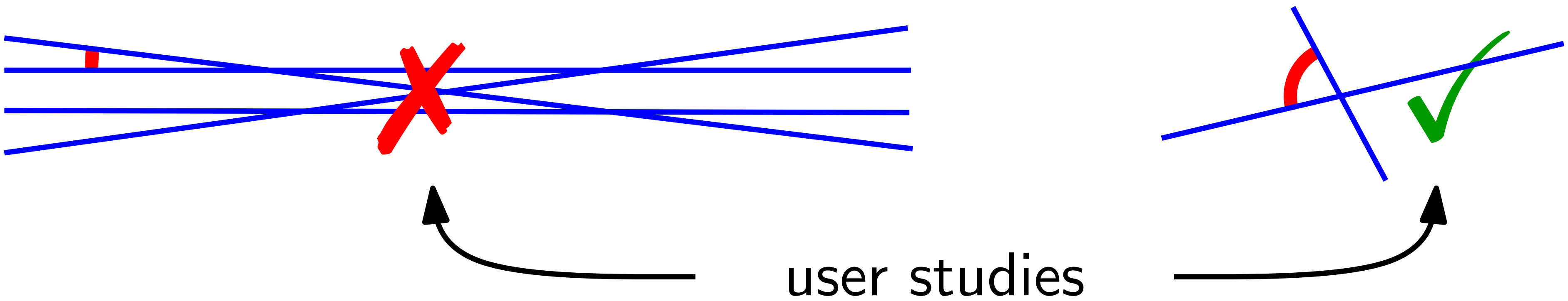

RAC$_3$ drawings

Didimo, Eades, Liotta 2009: *90° crossings & 3 bends per edge*

1. *any* ~~planar~~ graph $\xrightarrow{\text{RAC}_3}$ grid of size $O(m) \times O(m)$
2. some graphs need grid size $\Omega(n^2)$ for RAC$_3$ drawings

A New Trick!

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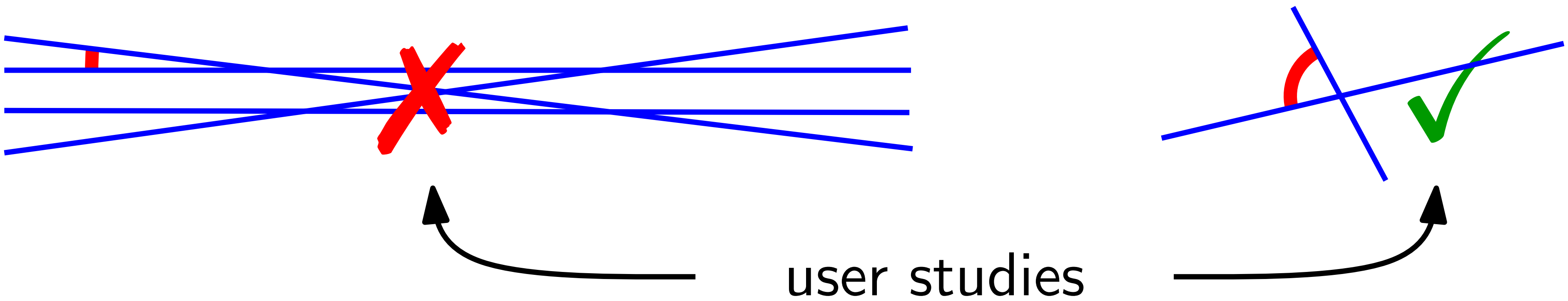

RAC_3 drawings

Didimo, Eades, Liotta 2009: *90° crossings & 3 bends per edge*

1. *any* planar graph $\xrightarrow{RAC_3}$ grid of size $O(m) \times O(m)$
2. some graphs need grid size $\Omega(n^2)$ for RAC_3 drawings
3. only graphs with $O(n)$ edges can be drawn RAC_2

...

Our contribution

Combine point-set embeddability & RAC.

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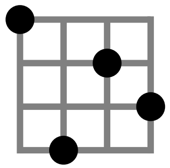

our general position: We assume that our n input points lie on an $n \times n$ grid and that no two points lie on the same horizontal or vertical line.

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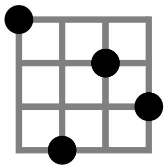

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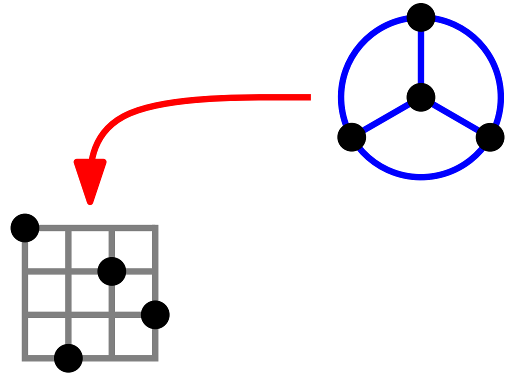
A first result:

any graph $\xrightarrow{\text{RAC}_3}$ $O(m) \times O(m)$ grid

Our contribution



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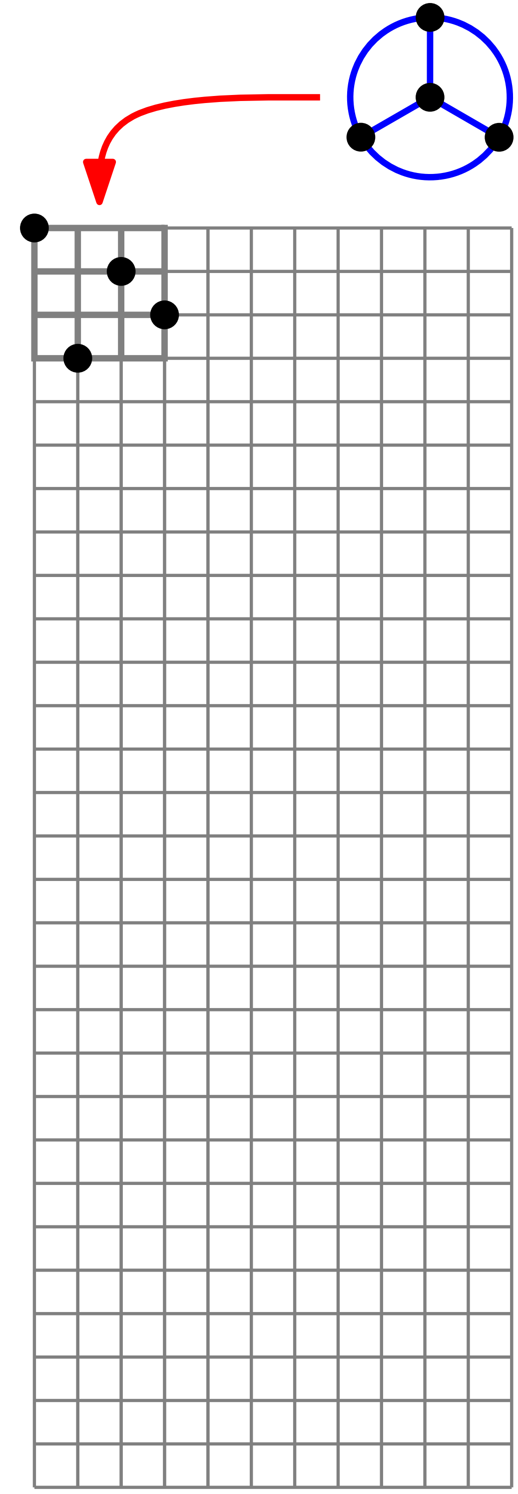

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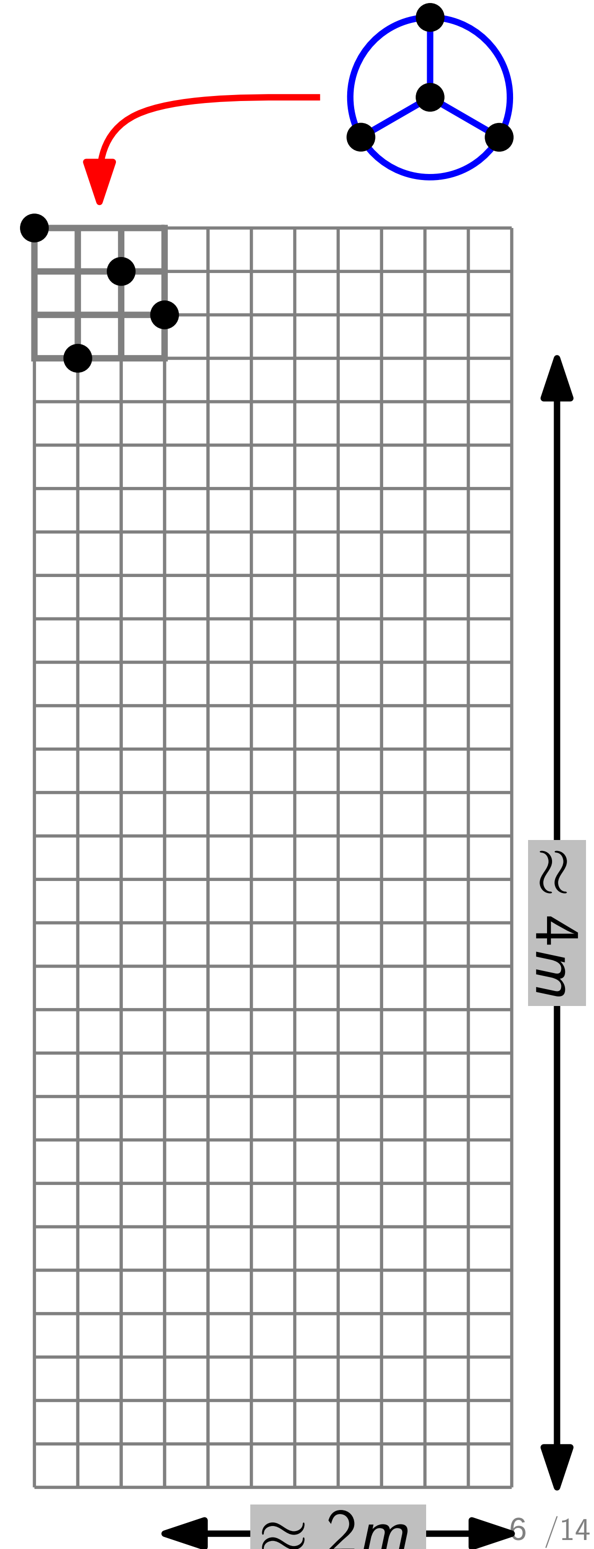

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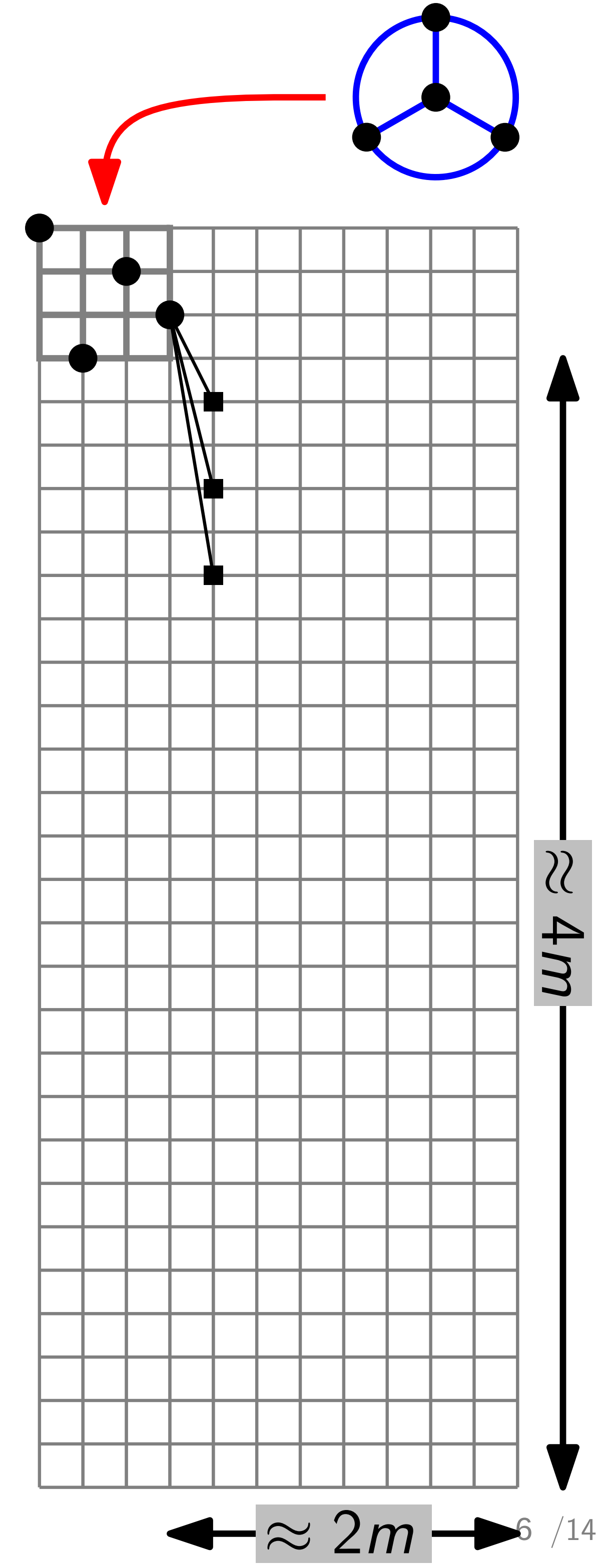

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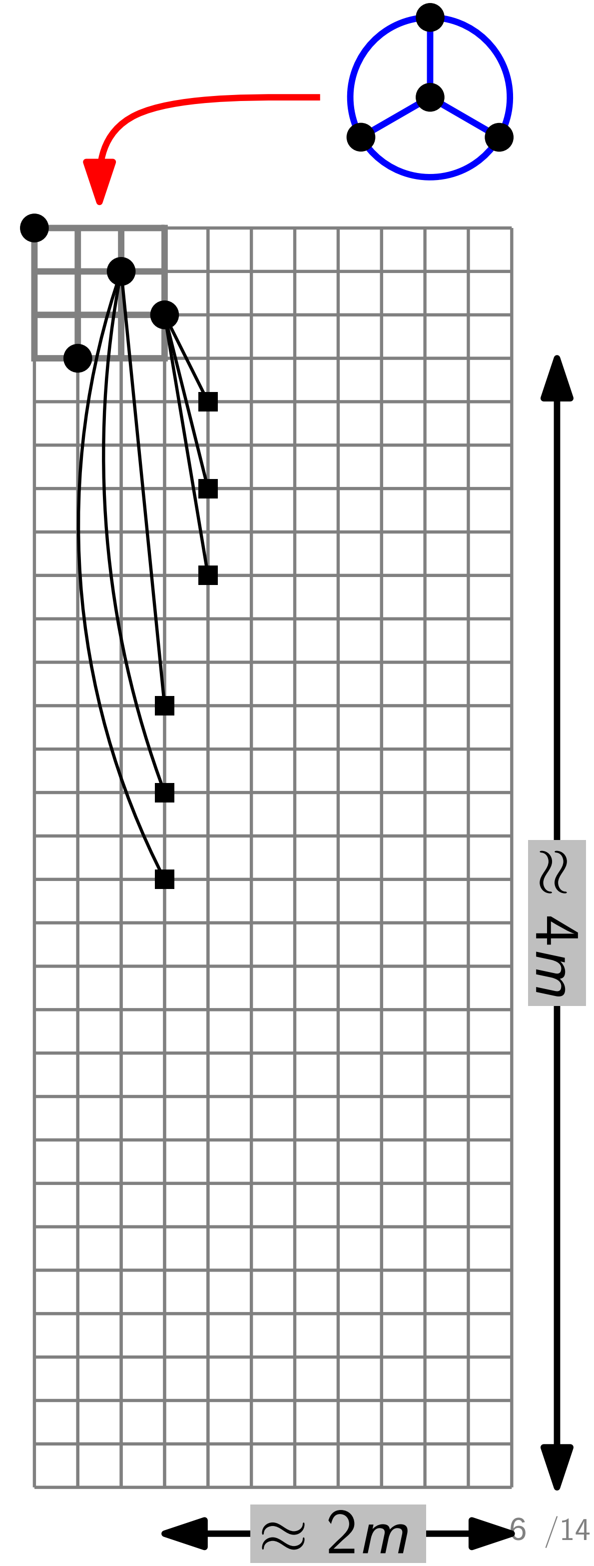

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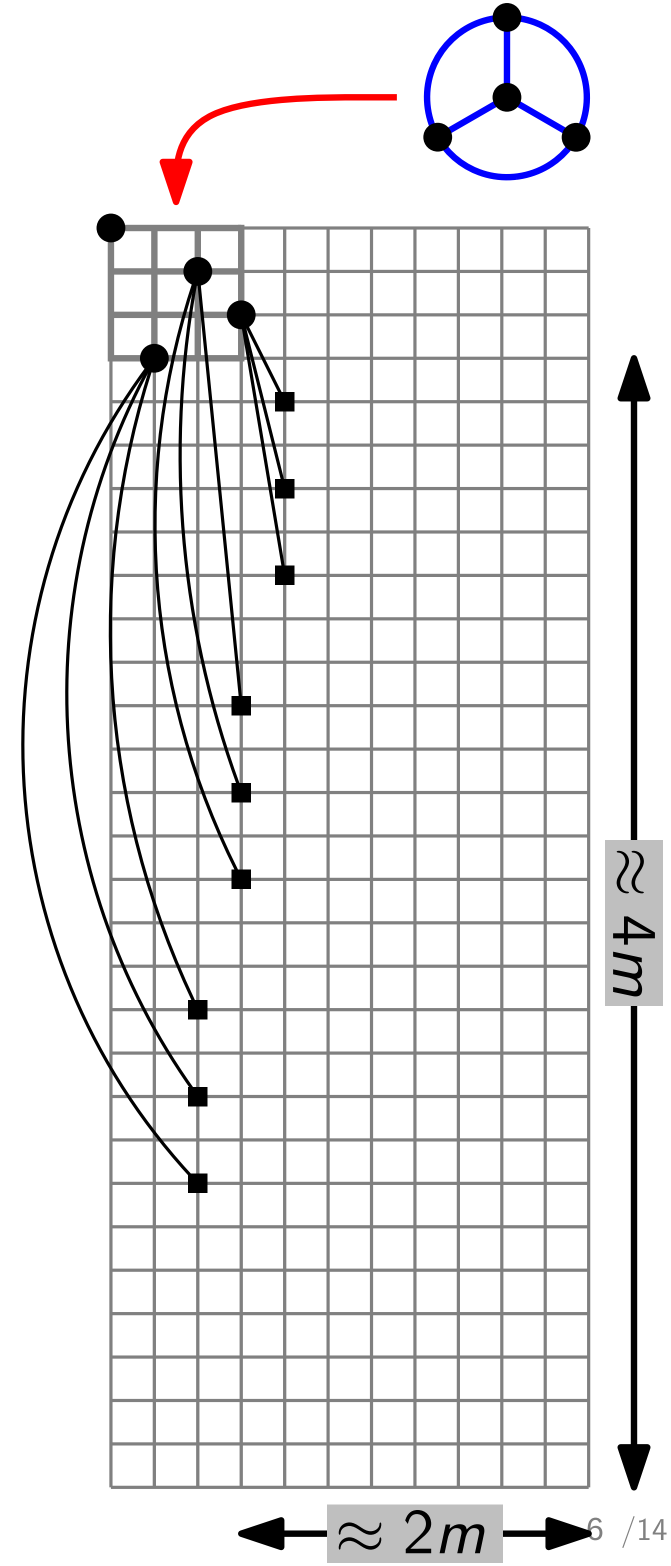

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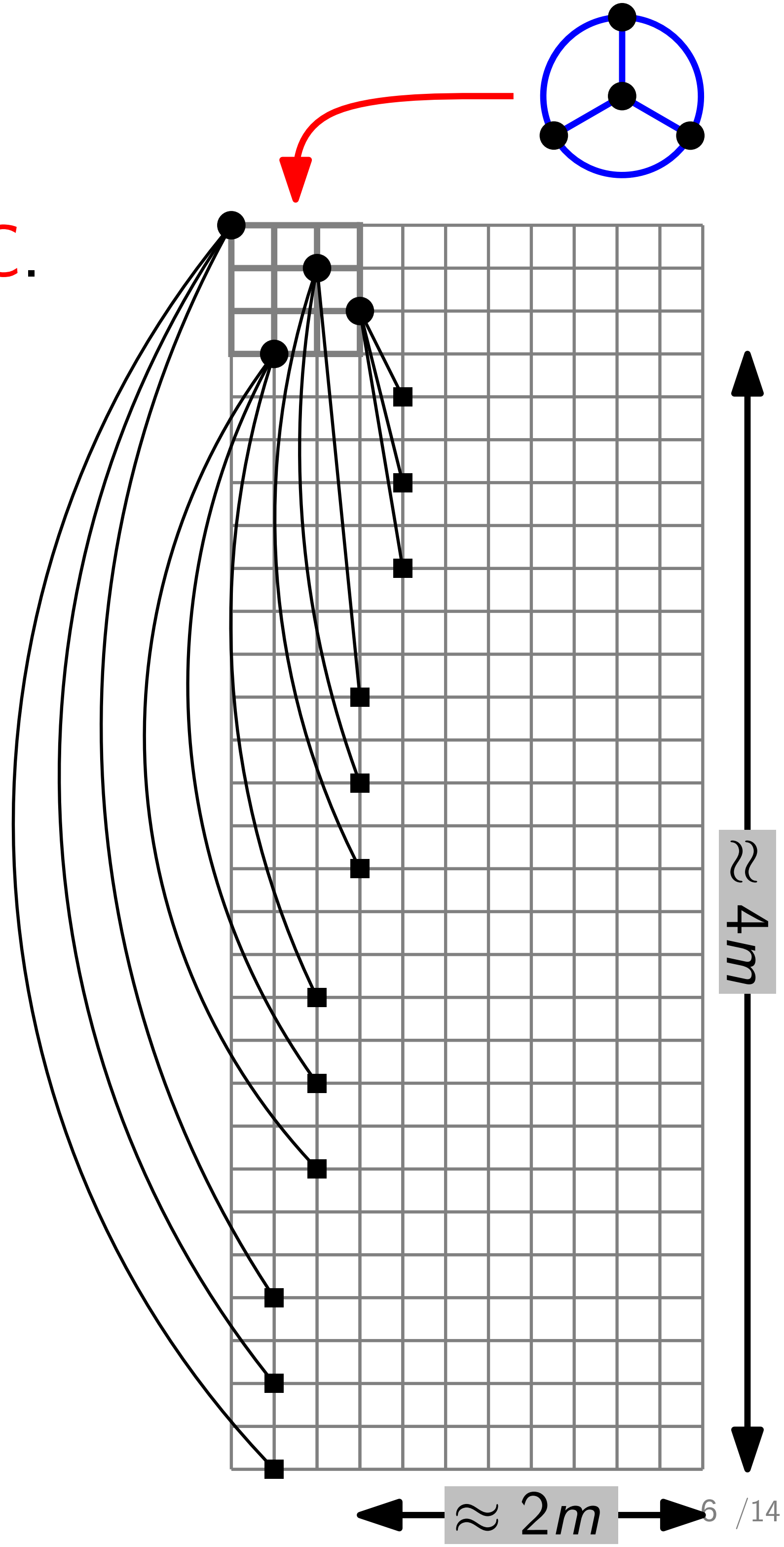

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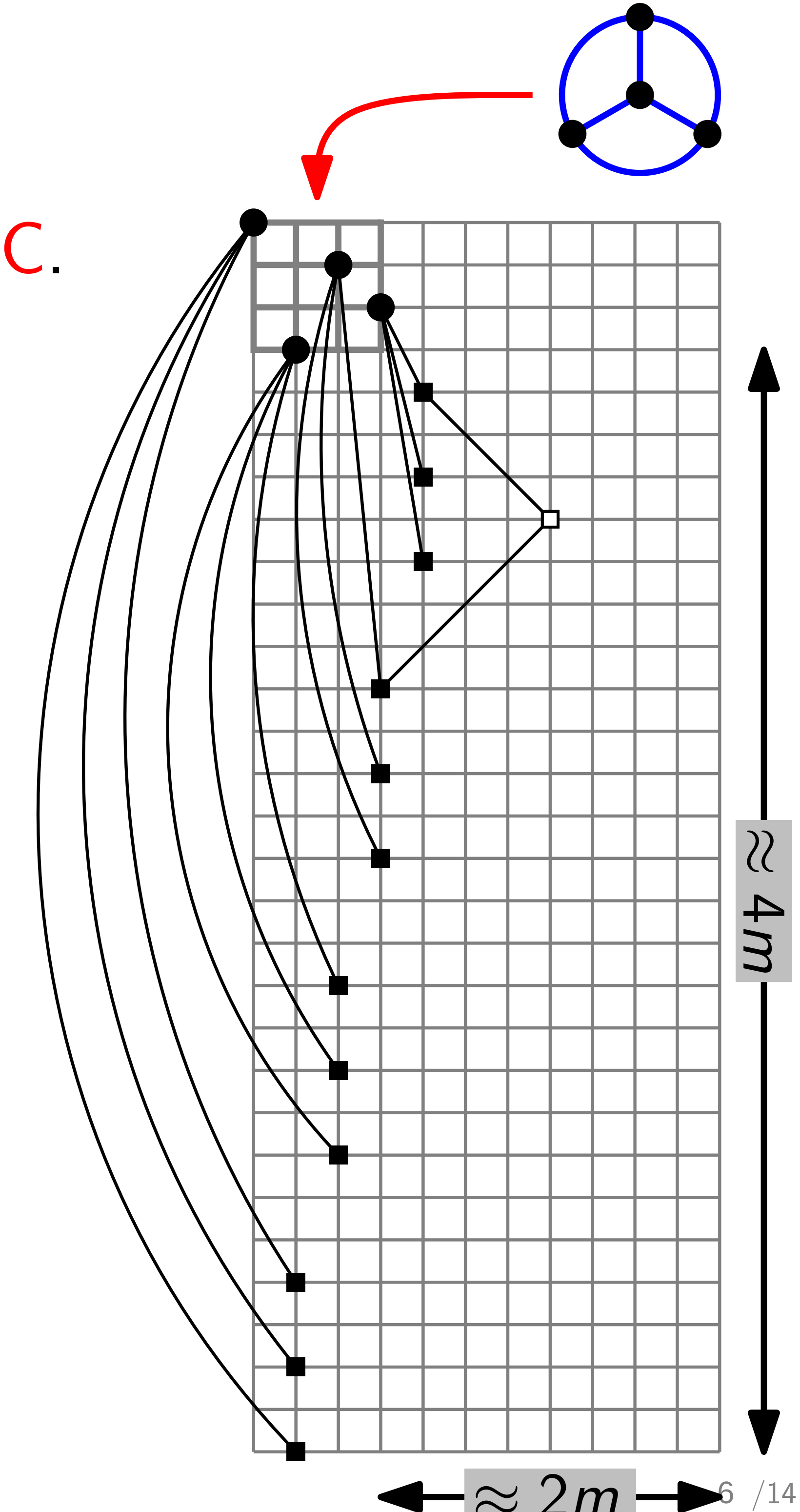

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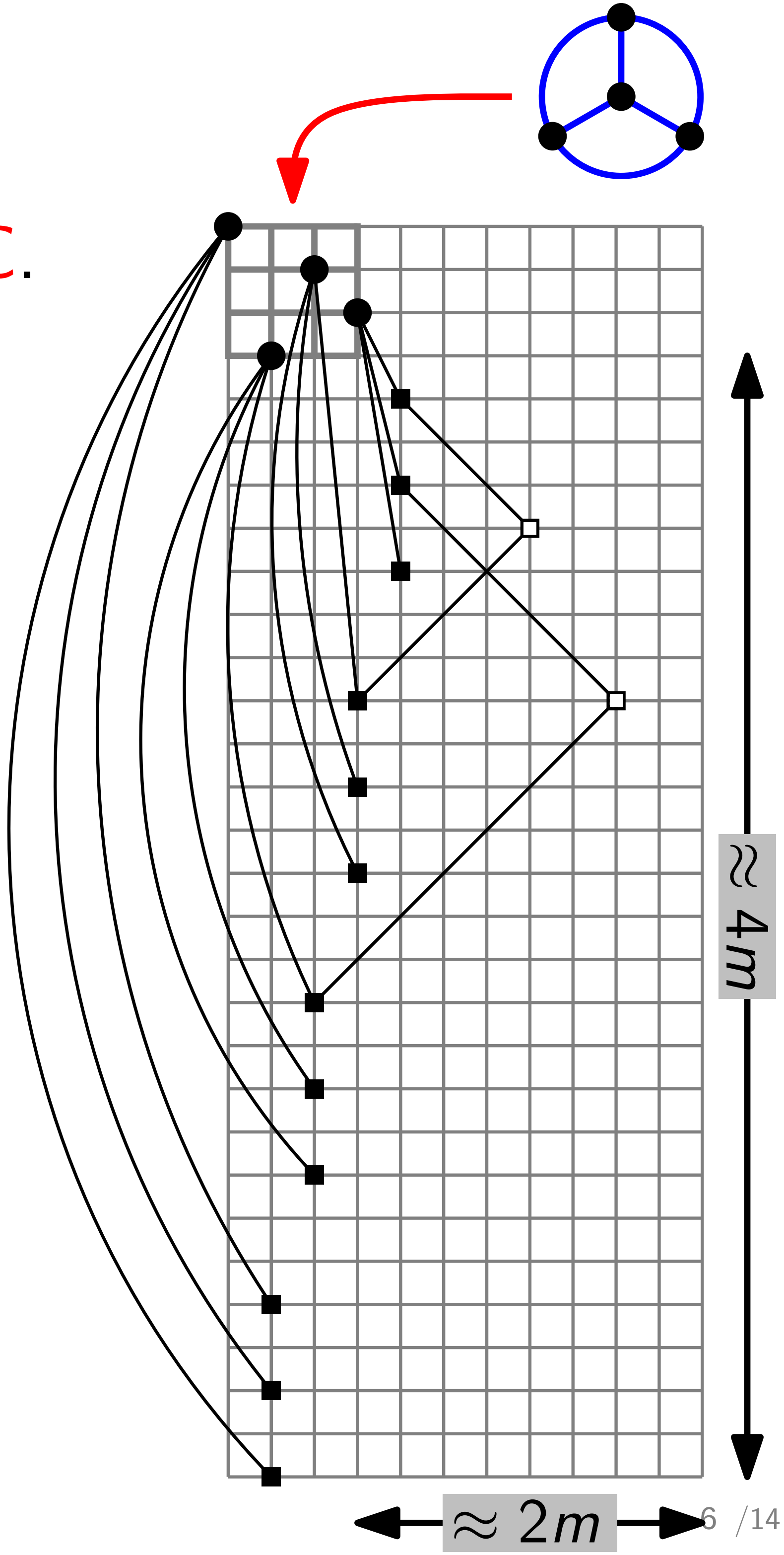

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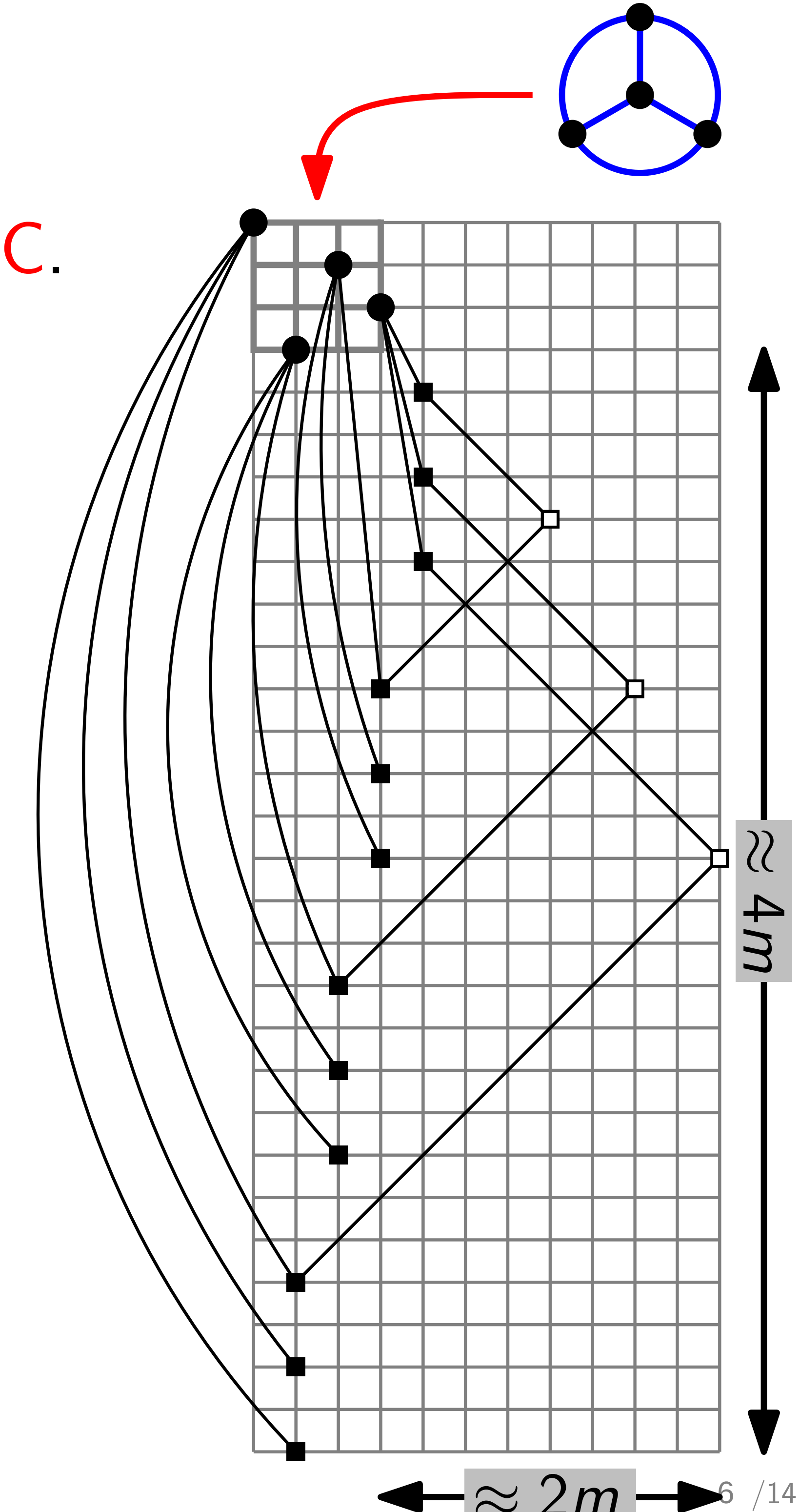

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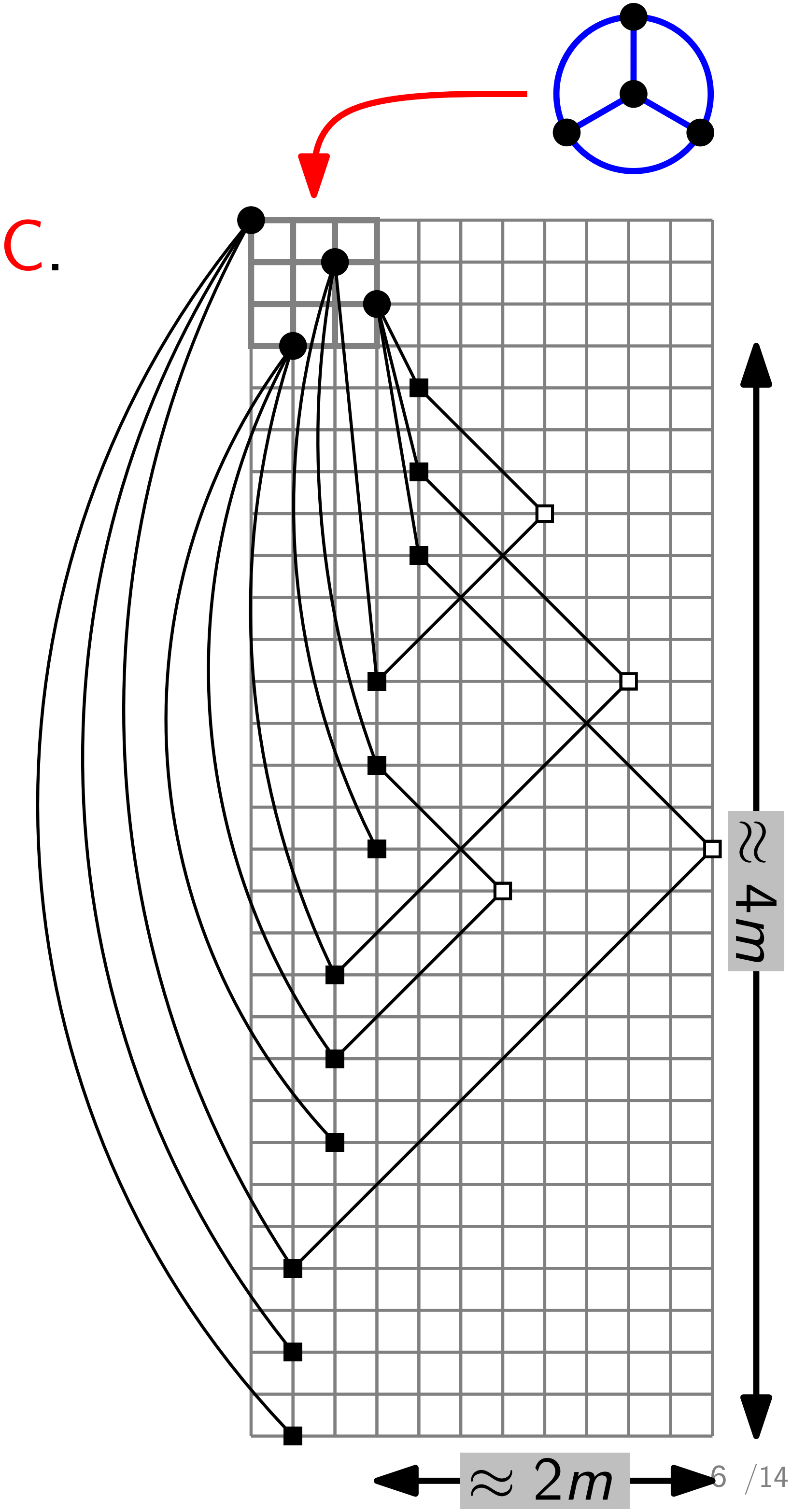

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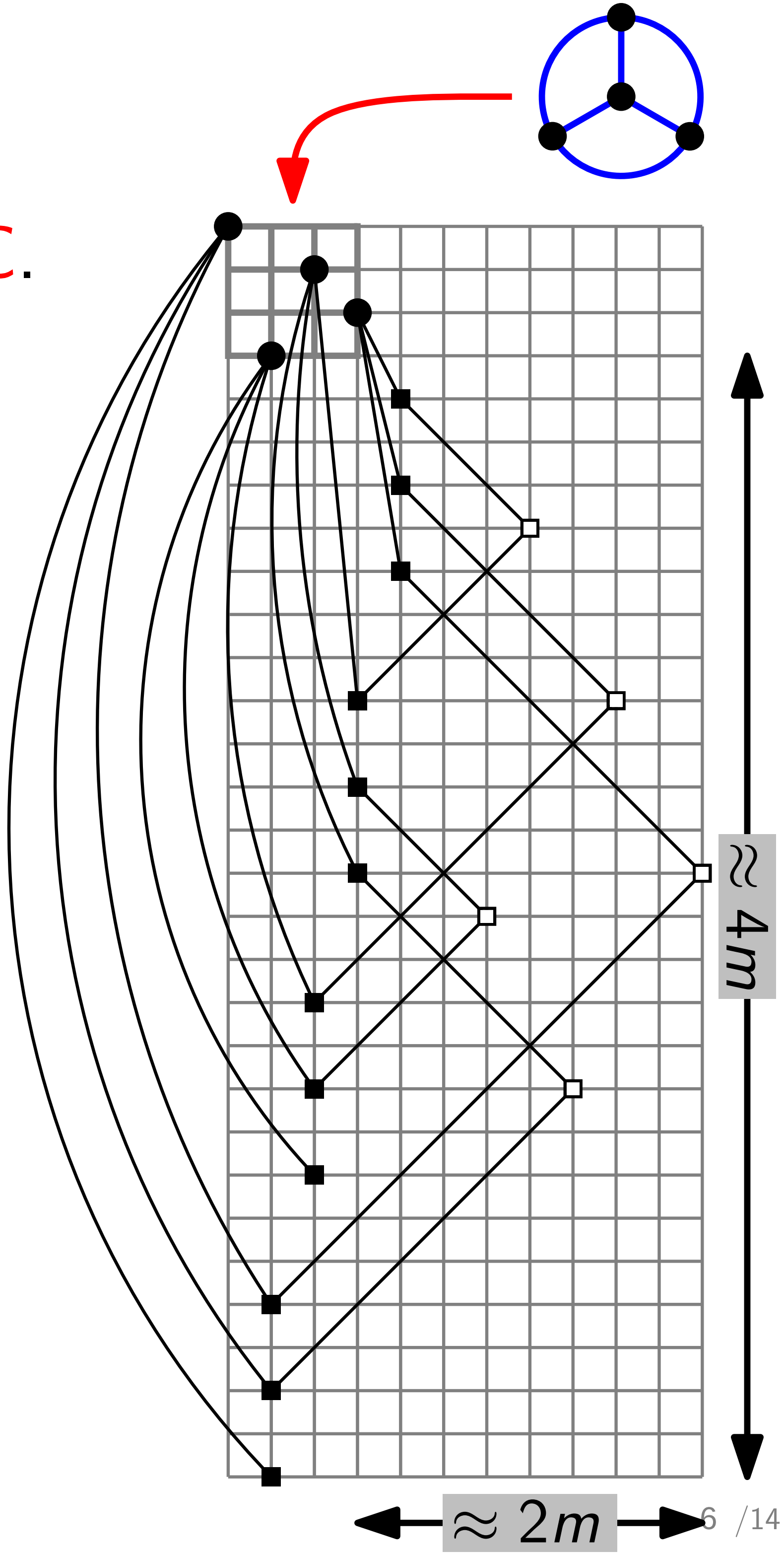

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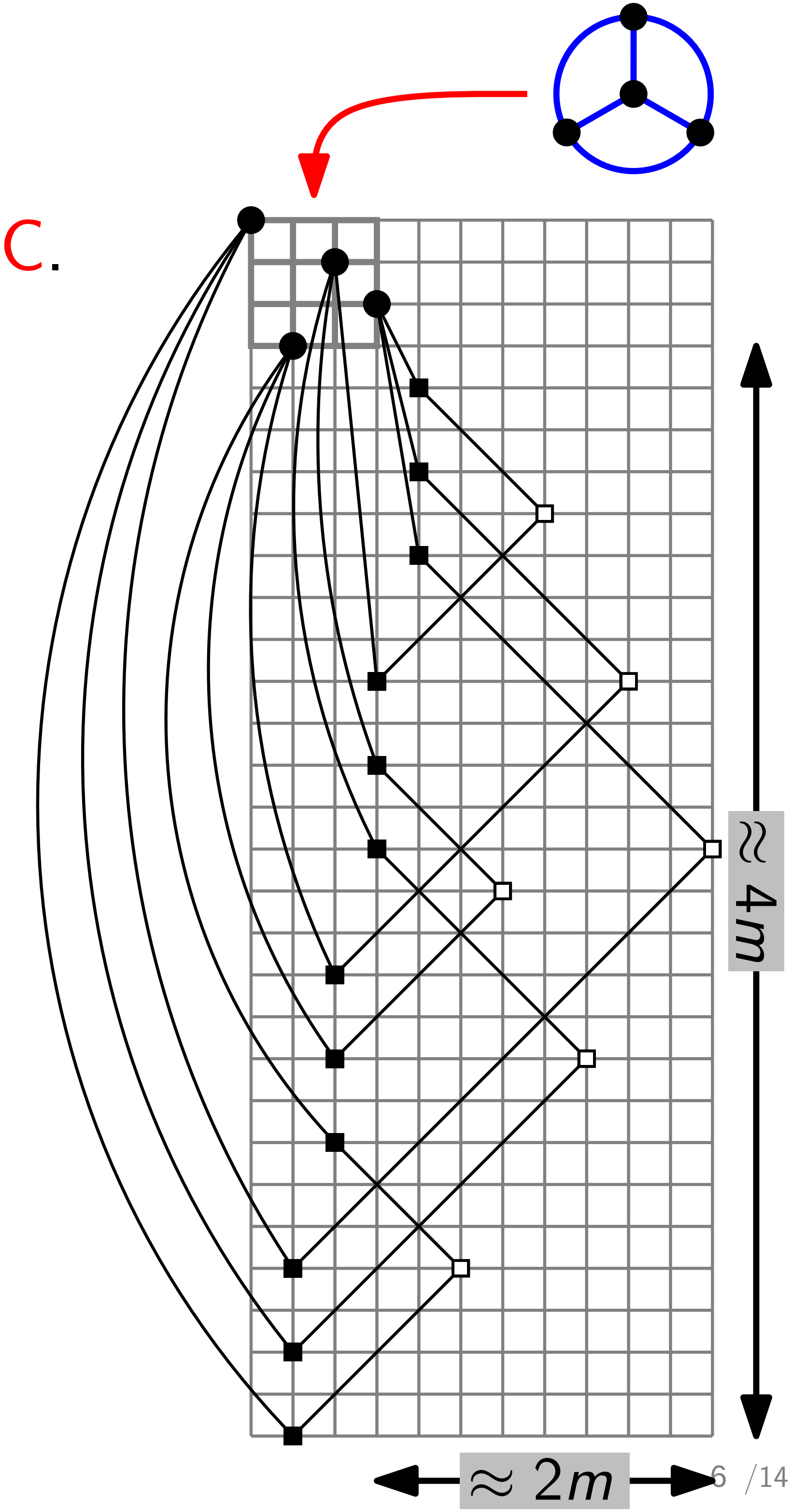

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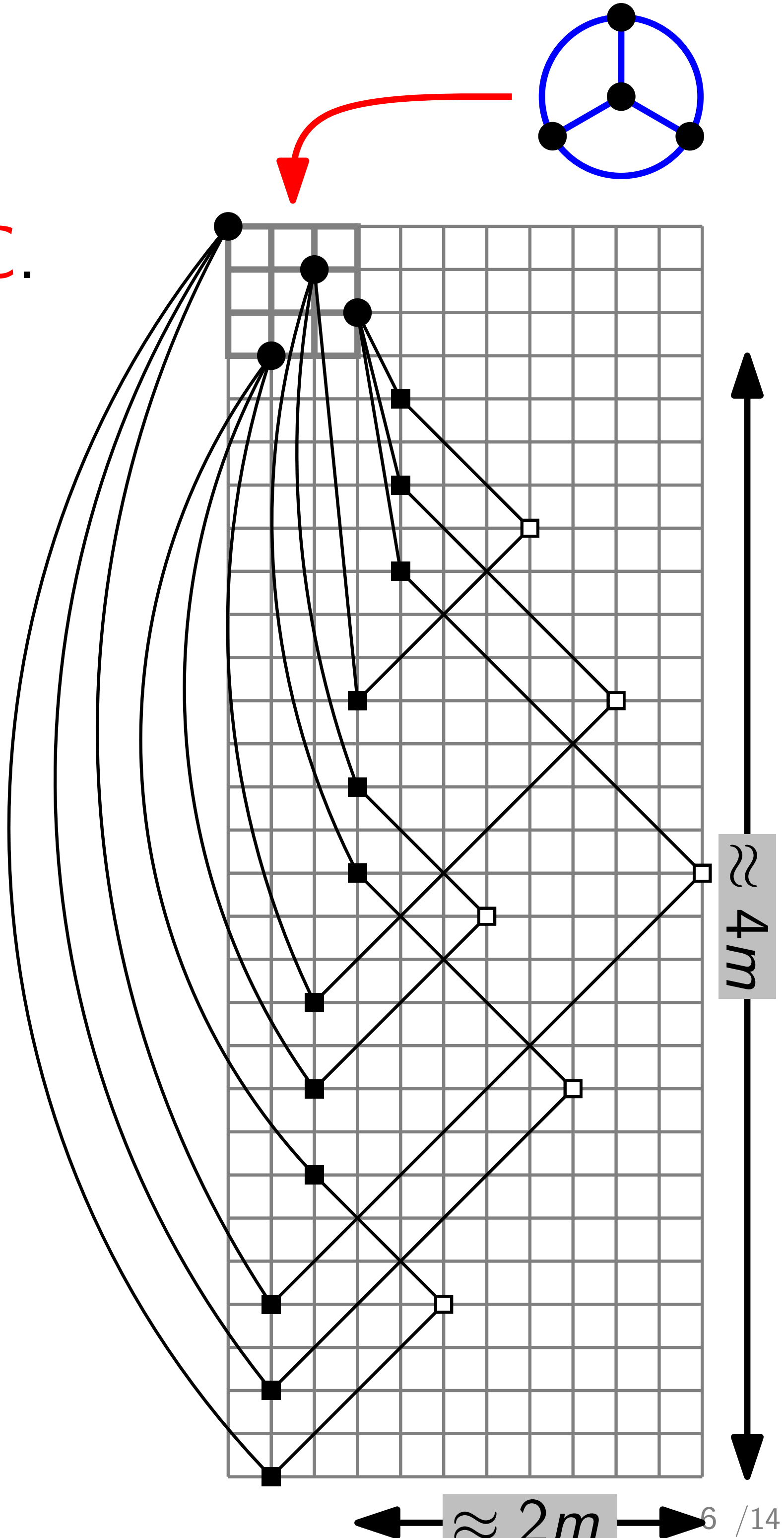
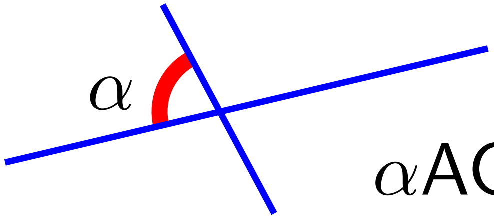
even with given *mapping*

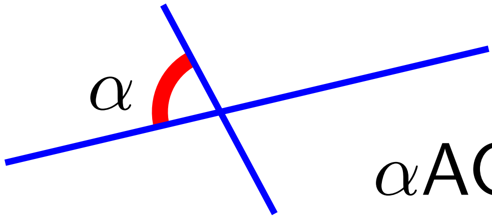


Loosening RAC to LAC



αAC: all crossing angles $\geq \alpha$

Loosening RAC to LAC

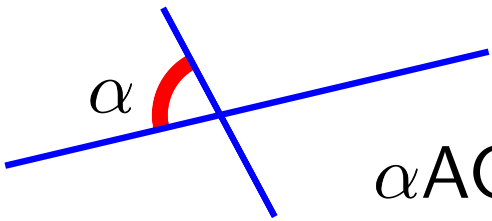


αAC: all crossing angles $\geq \alpha$

any graph with *mapping*

$\xrightarrow{\alpha\text{AC}_2}$ $O(m) \times n + 1$ grid

Loosening RAC to LAC



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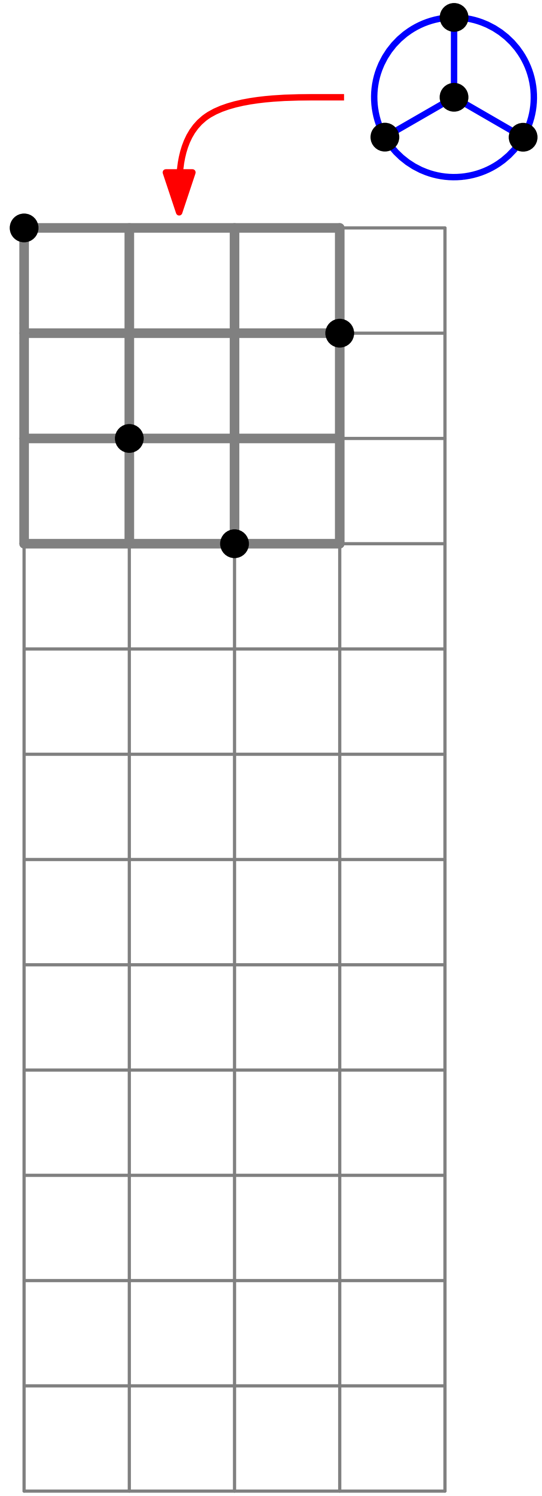
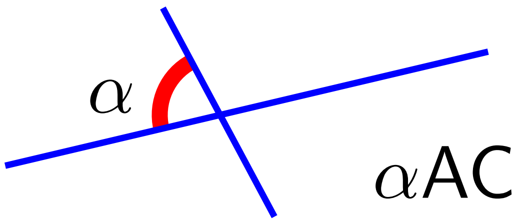


Loosening RAC to LAC



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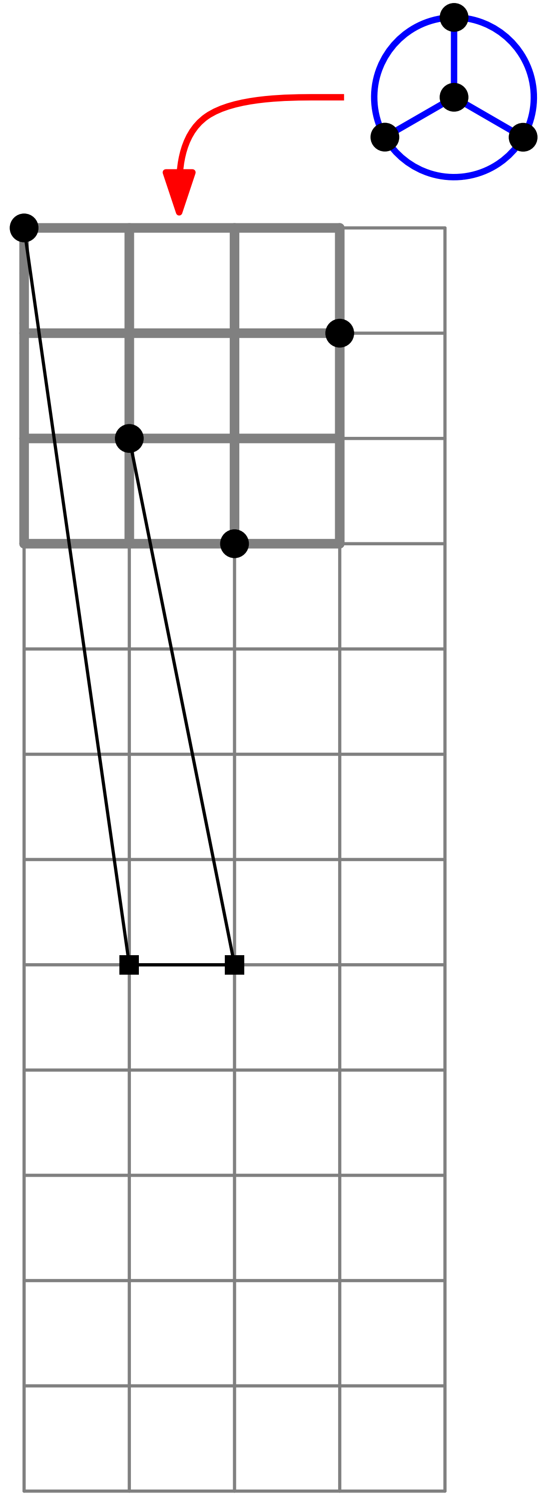
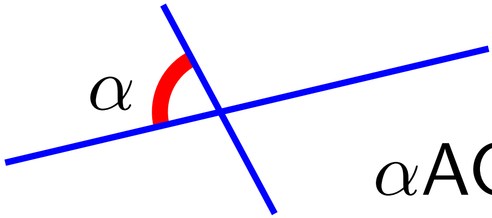


Loosening RAC to LAC



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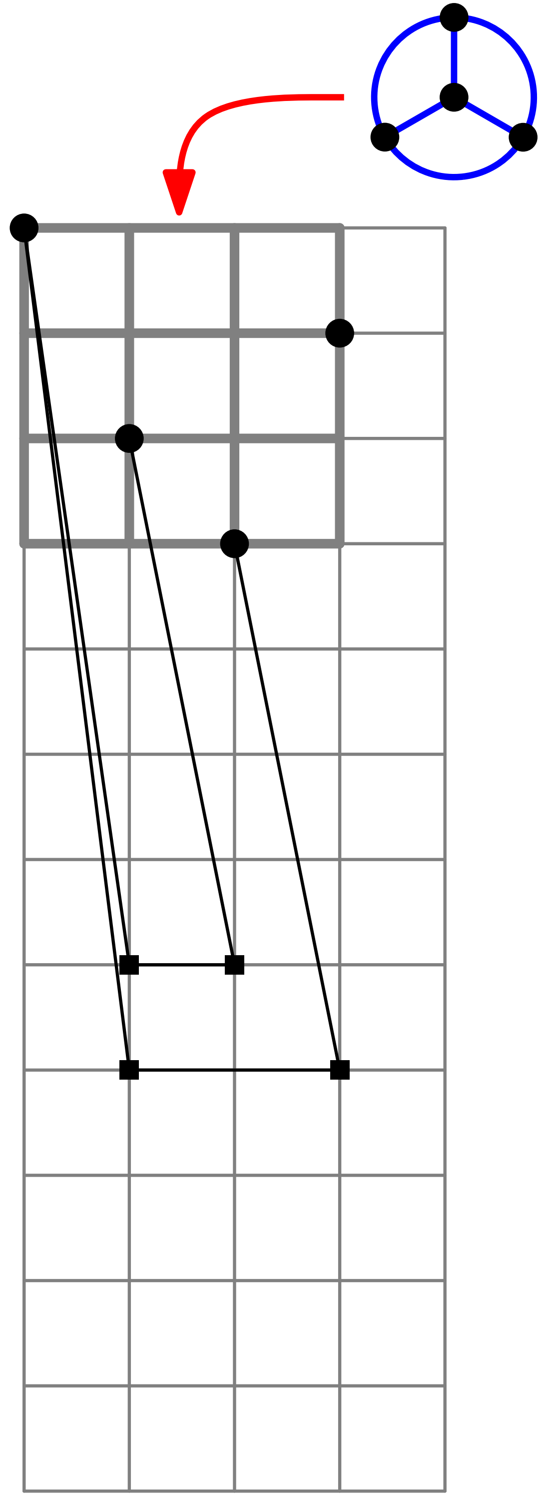
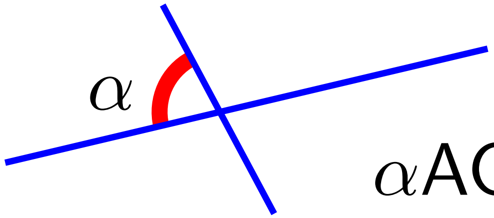


Loosening RAC to LAC



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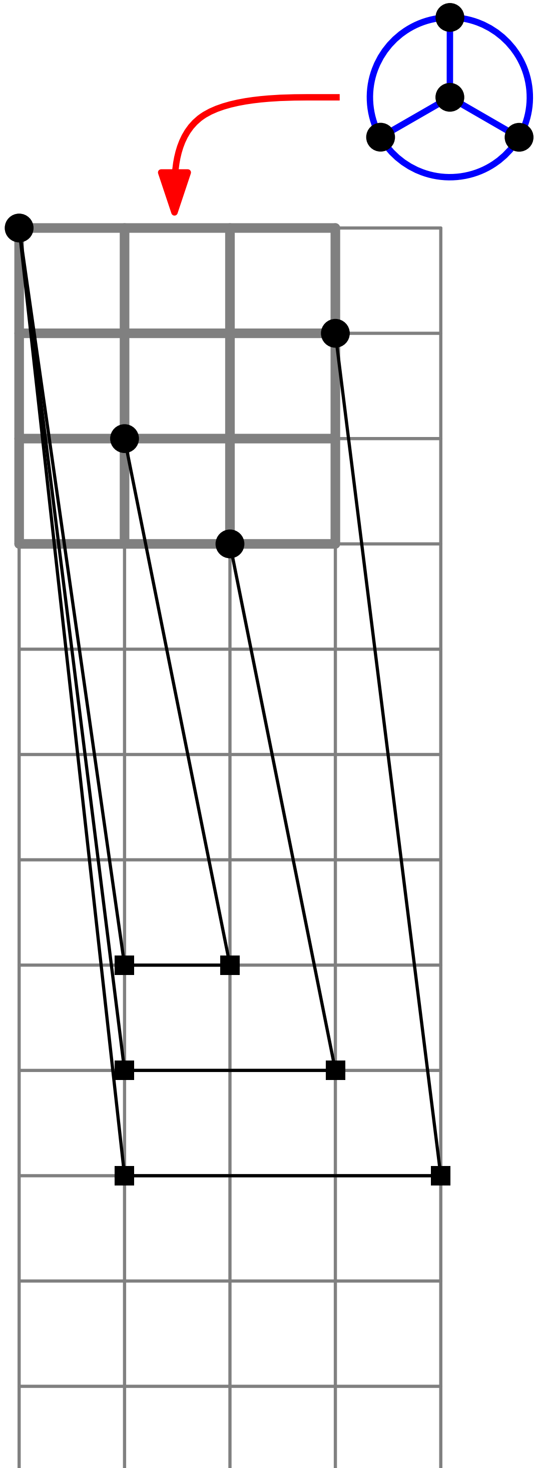

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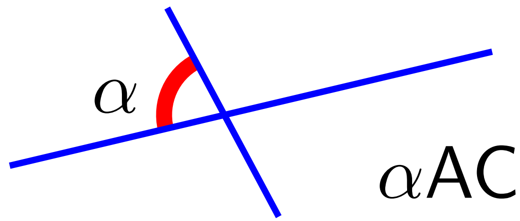

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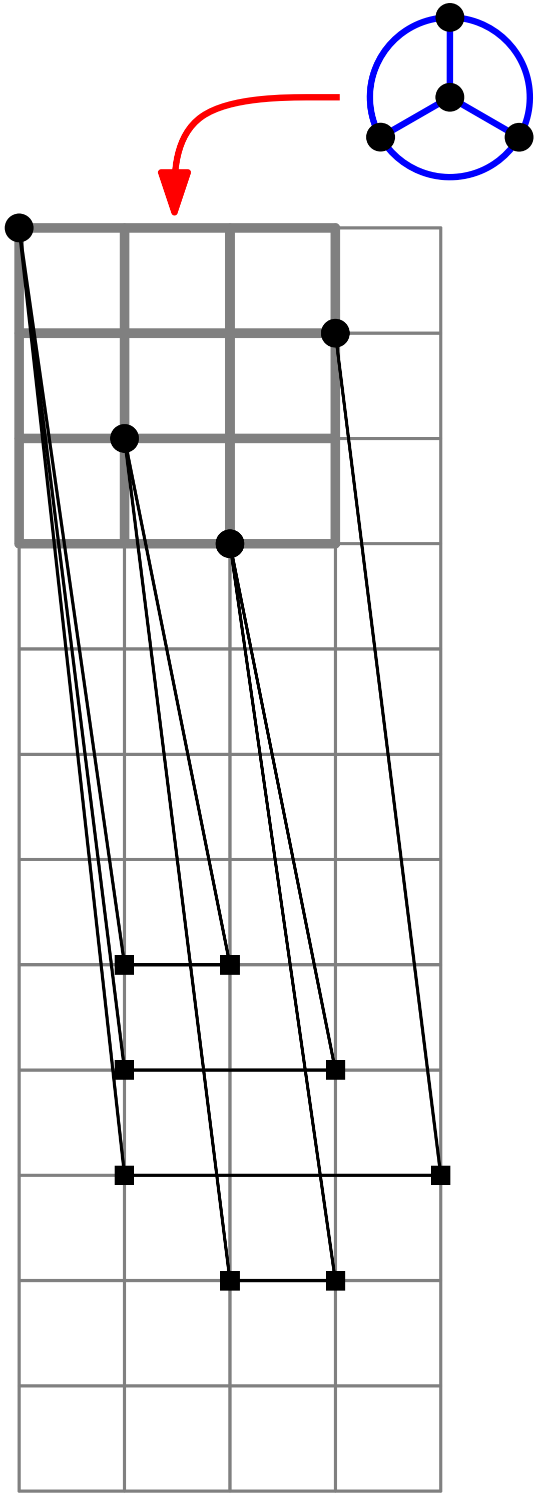

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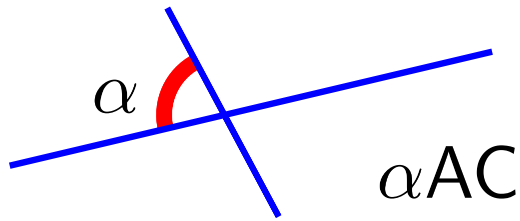

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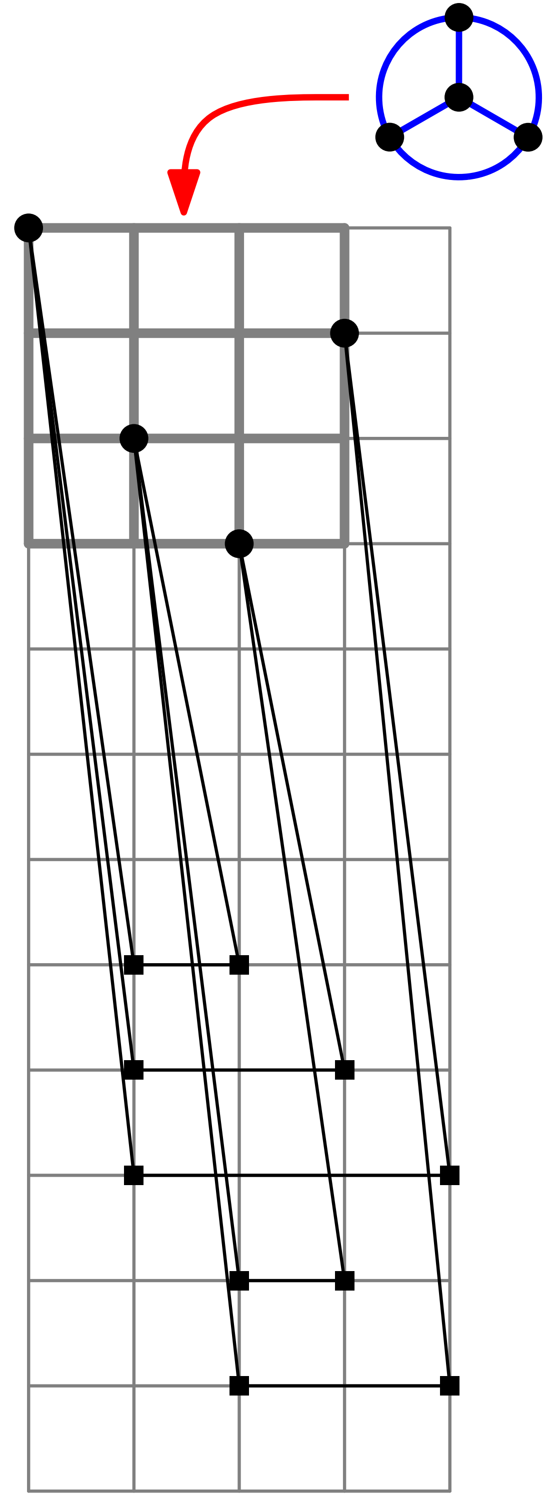

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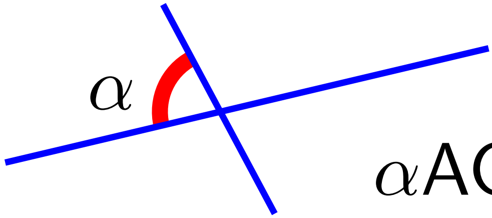

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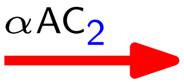
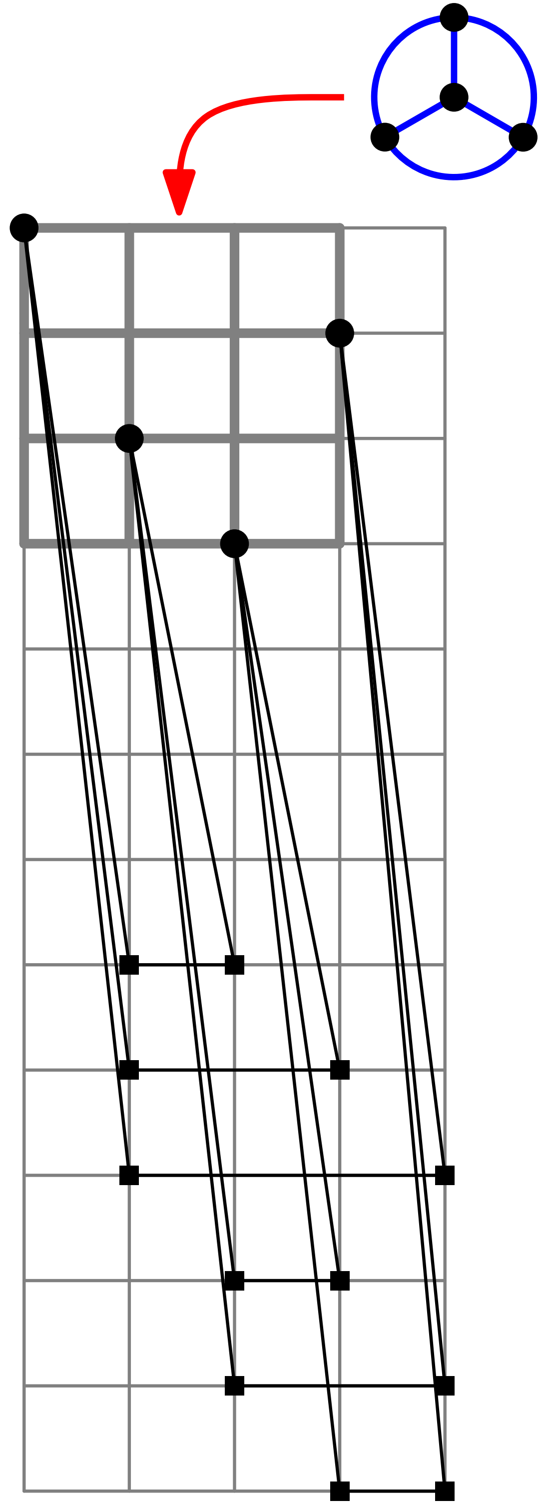


$O(m) \times$ $n + 1$ grid



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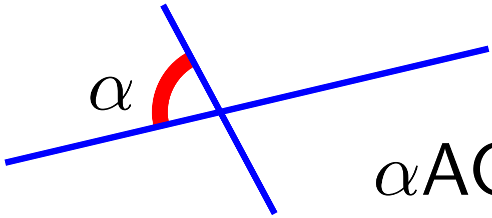
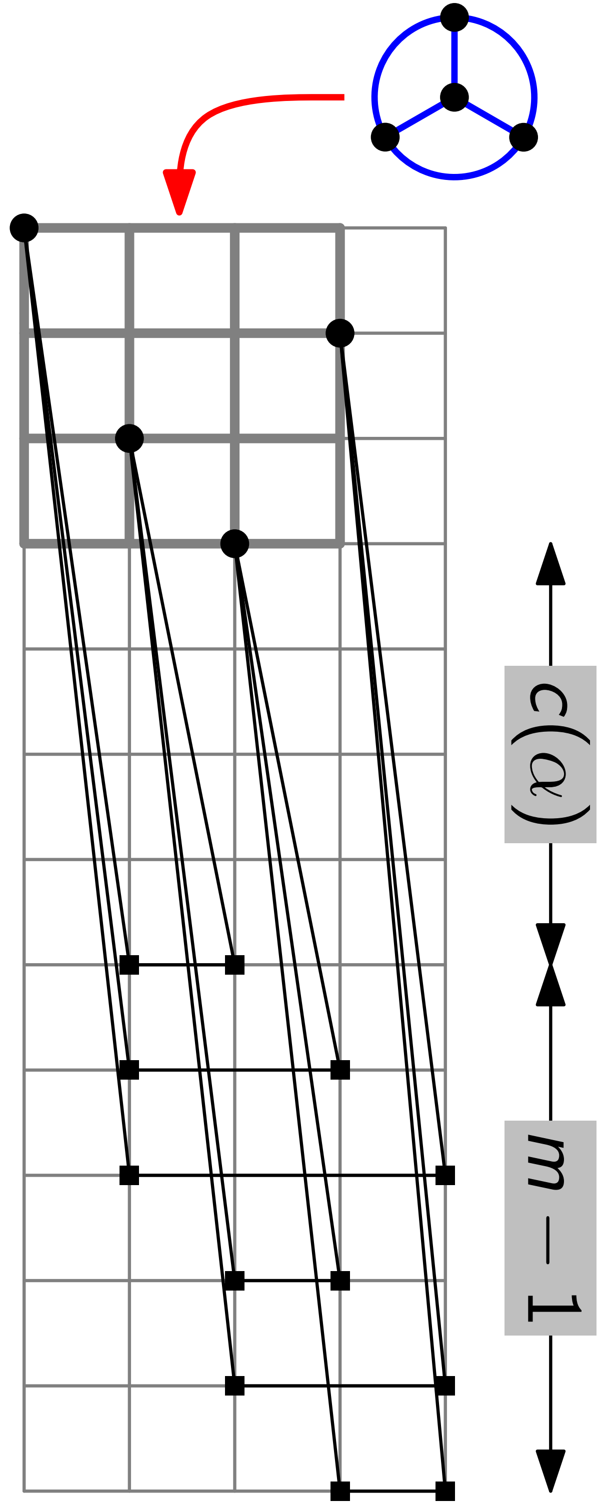


αAC: all crossing angles $\geq \alpha$

any graph with *mapping* $\xrightarrow{\alpha AC_2}$ $O(m) \times n+1$ grid

$c(\alpha)$ depends only on α!



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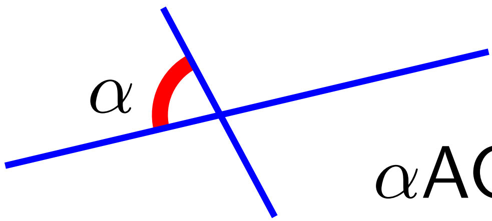

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any graph with *mapping*

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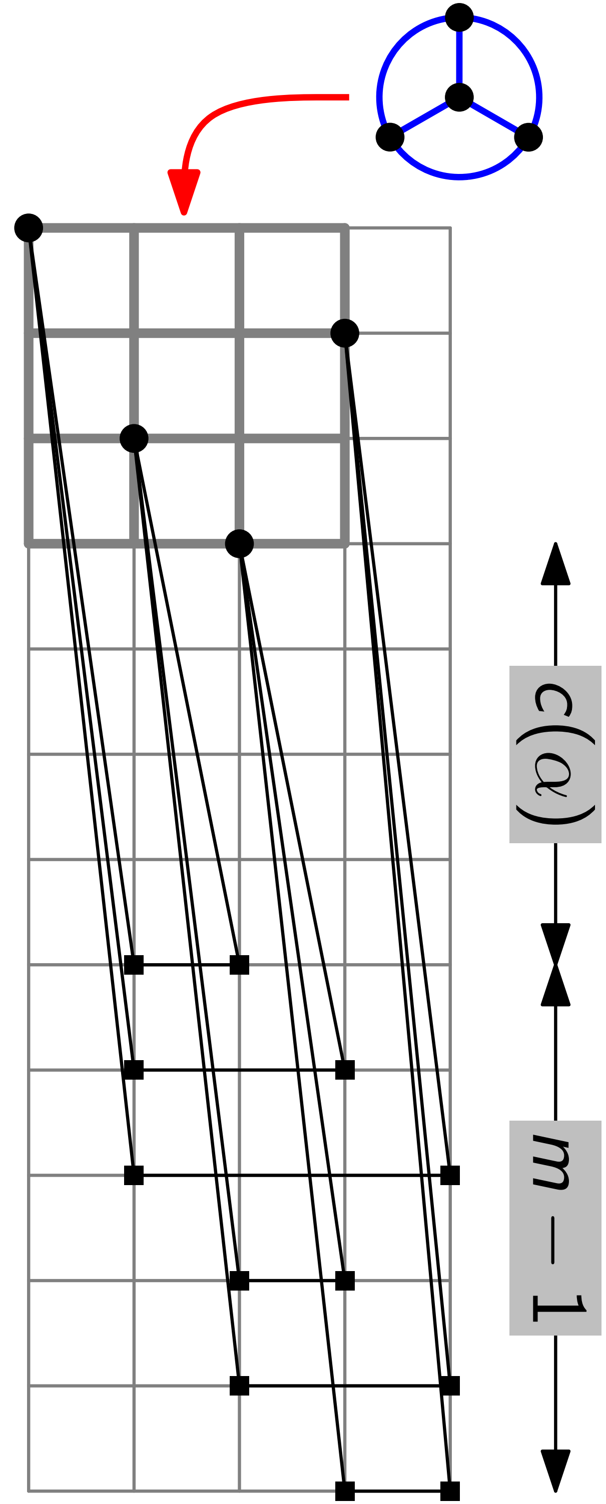
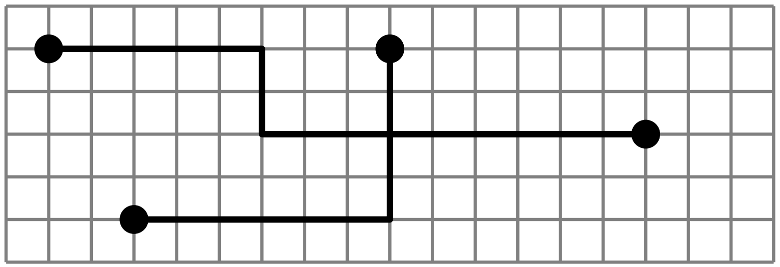
Theorem αAC_0 PSE is NP-hard



Overview: Restricted RAC PSE

Additional restriction: *keep edges on grid lines!*



Overview: Restricted RAC PSE

Additional restriction: *keep edges on grid lines!*

1. cycle C_n $\xrightarrow{\text{RAC}_1}$ $n \times n$ grid point set

Overview: Restricted RAC PSE

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1. cycle C_n $\xrightarrow{\text{RAC}_1}$ $n \times n$ grid point set

2. graph with *mapping* $\xrightarrow{\text{RAC}_1\ ?}$ $n \times n$ grid point set

   check in $O(n)$ time

Overview: Restricted RAC PSE

Additional restriction: *keep edges on grid lines!*

1. cycle C_n $\xrightarrow{\text{RAC}_1}$ $n \times n$ grid point set

2. graph with *mapping* $\xrightarrow{\text{RAC}_1\ ?}$ $n \times n$ grid point set

   check in $O(n)$ time

3. *any* binary tree $\xrightarrow{\text{RAC}_1}$ $n \times n$ grid point set

Overview: Restricted RAC PSE

Additional restriction: *keep edges on grid lines!*

1. cycle C_n $\xrightarrow{\text{RAC}_1}$ $n \times n$ grid point set

2. graph with *mapping* $\xrightarrow{\text{RAC}_1\ ?}$ $n \times n$ grid point set

   check in $O(n)$ time

3. *any* binary tree $\xrightarrow{\text{RAC}_1}$ $n \times n$ grid point set

4. *any* maxdeg-3 graph $\xrightarrow{\text{RAC}_2}$ $O(n) \times O(n)$ grid

Overview: Restricted RAC PSE

Additional restriction: *keep edges on grid lines!*

1. cycle C_n $\xrightarrow{\text{RAC}_1}$ $n \times n$ grid point set with mapping

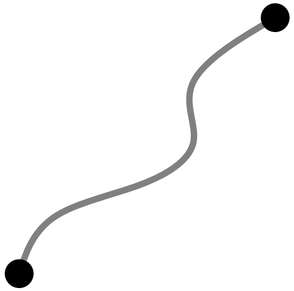
2. graph with *mapping* $\xrightarrow{\text{RAC}_1 \; ?}$ $n \times n$ grid point set

   check in $O(n)$ time

3. *any* binary tree $\xrightarrow{\text{RAC}_1}$ $n \times n$ grid point set

4. *any* maxdeg-3 graph $\xrightarrow{\text{RAC}_2}$ $O(n) \times O(n)$ grid with mapping

1. RAC_1 PSE of cycles



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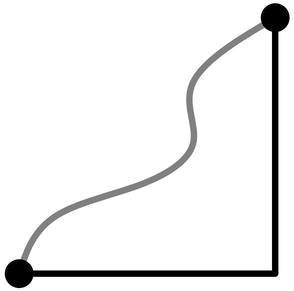
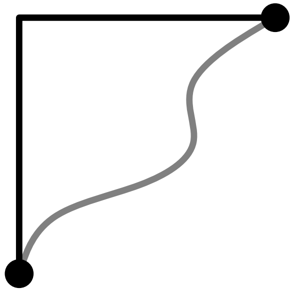
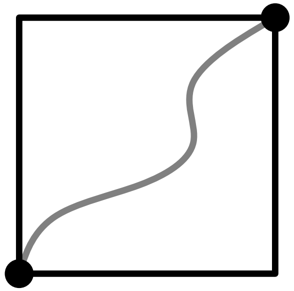


1. RAC_1 PSE of cycles

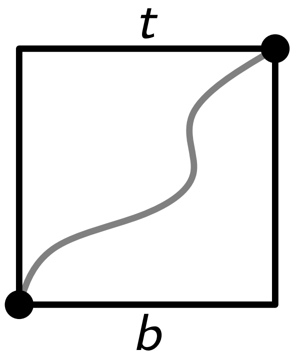


1. RAC_1 PSE of cycles



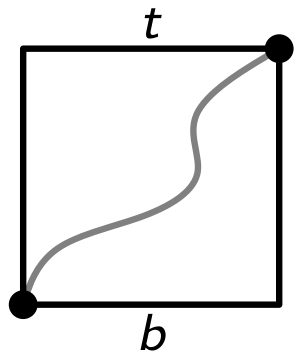
two possibilities per edge

1. RAC$_1$ PSE of cycles

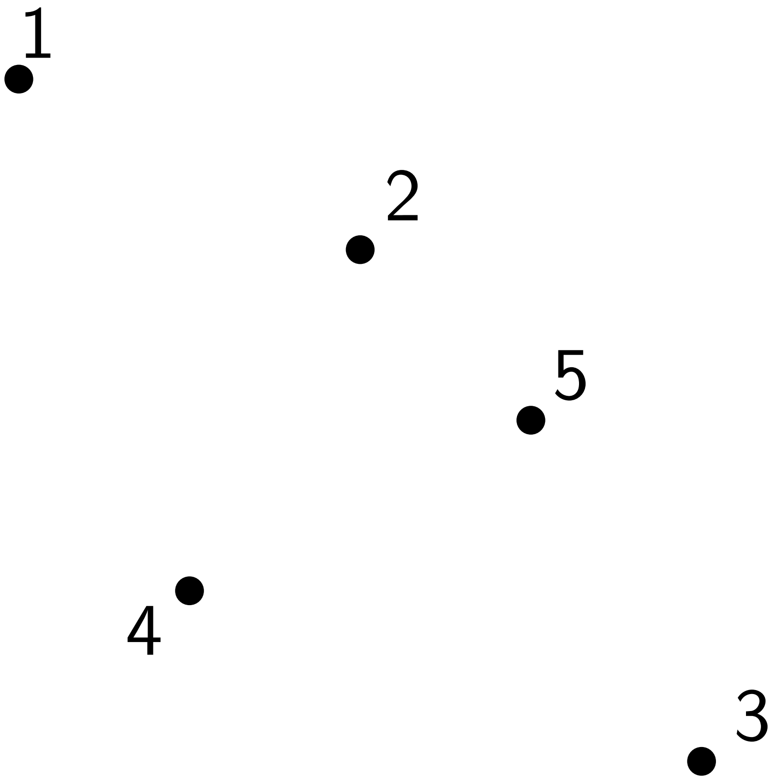


two possibilities per edge

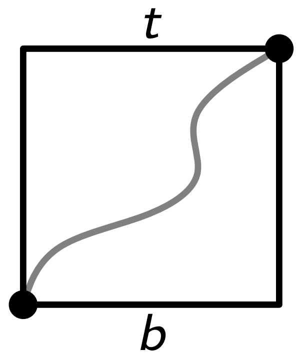
1. RAC_1 PSE of cycles



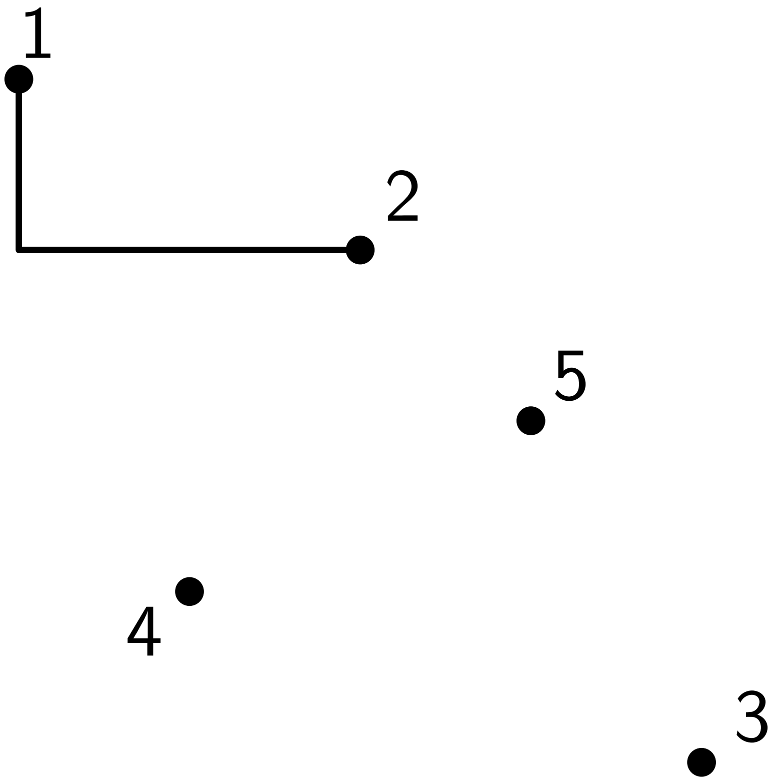
two possibilities per edge



1. RAC_1 PSE of cycles



two possibilities per edge



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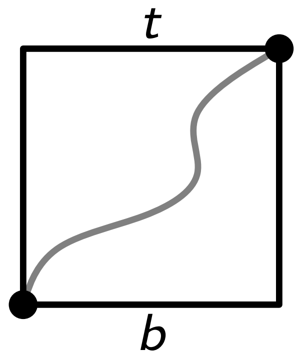
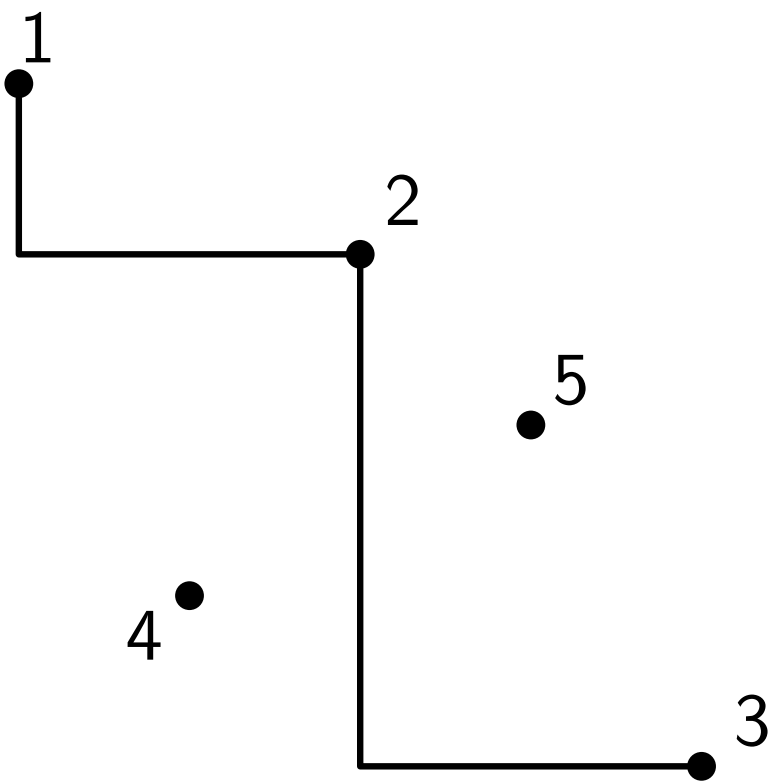


two possibilities per edge



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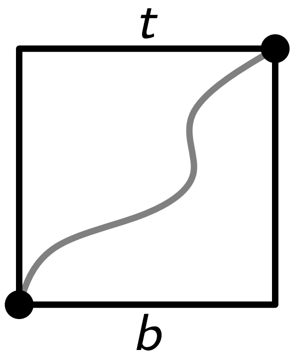
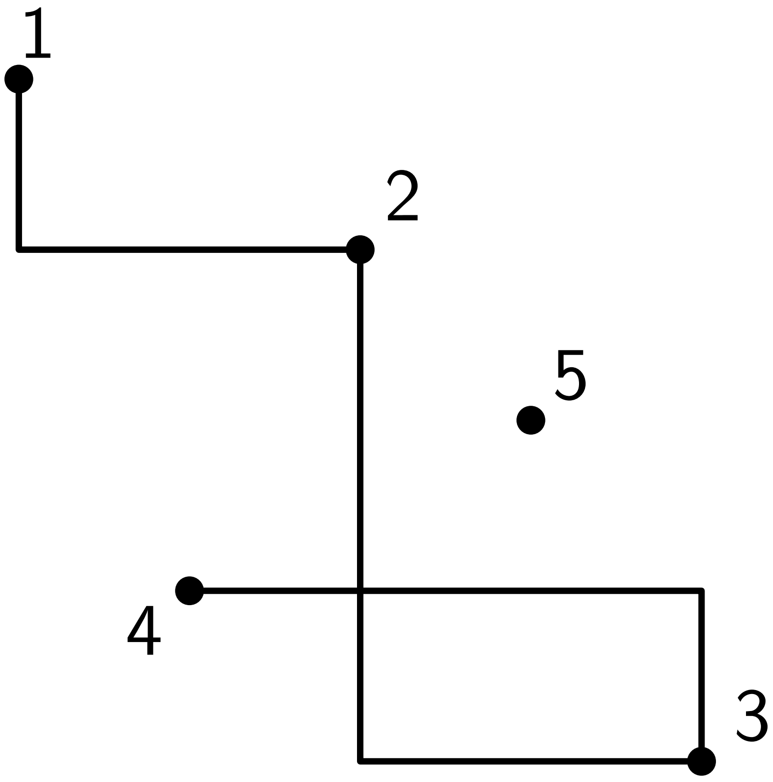


two possibilities per edge



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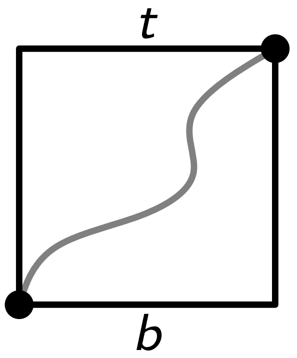
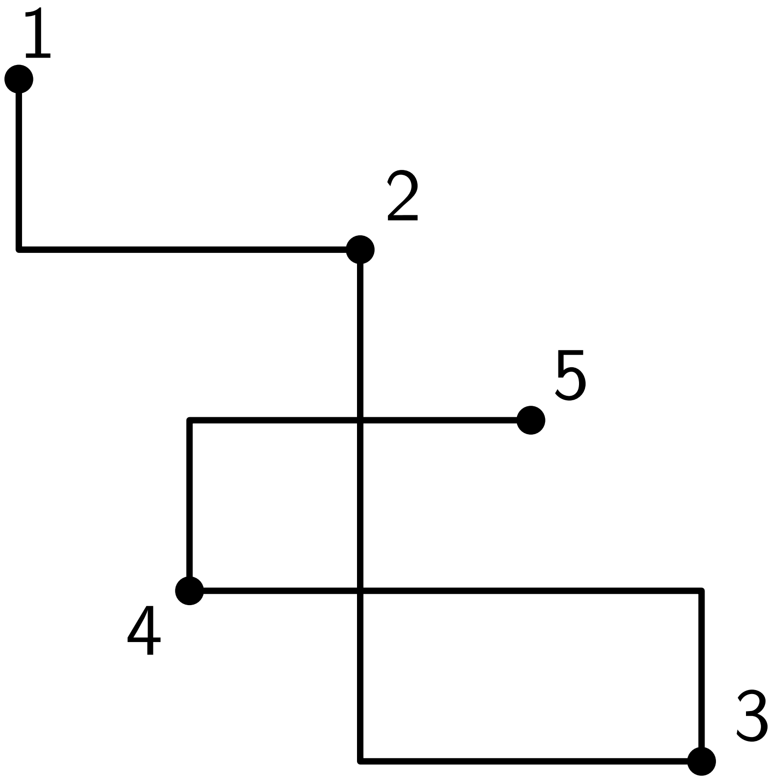
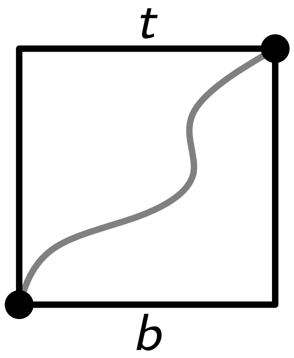


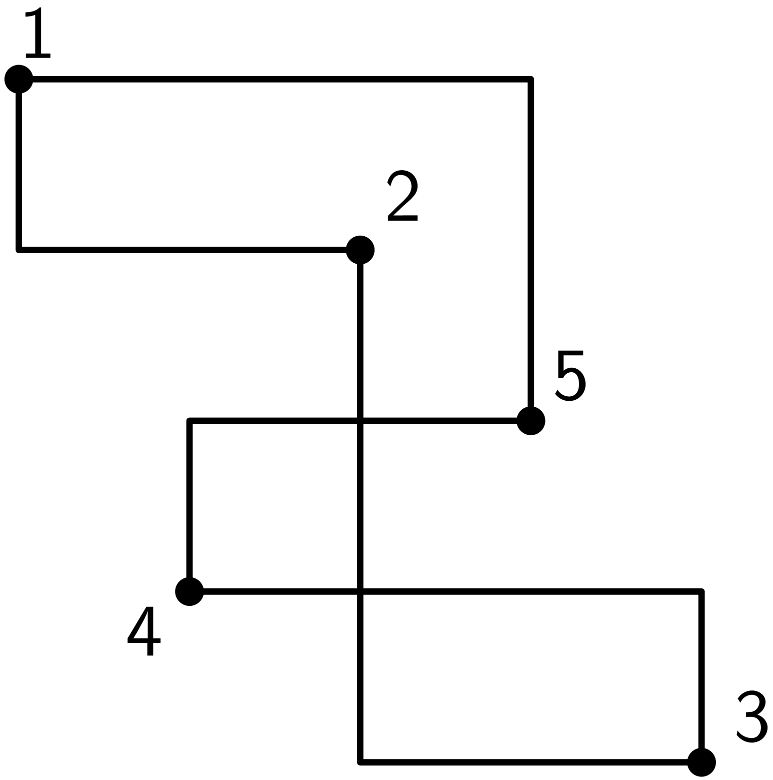
two possibilities per edge



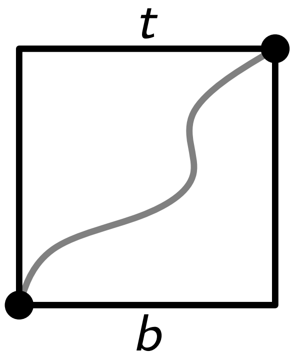
1. RAC_1 PSE of cycles



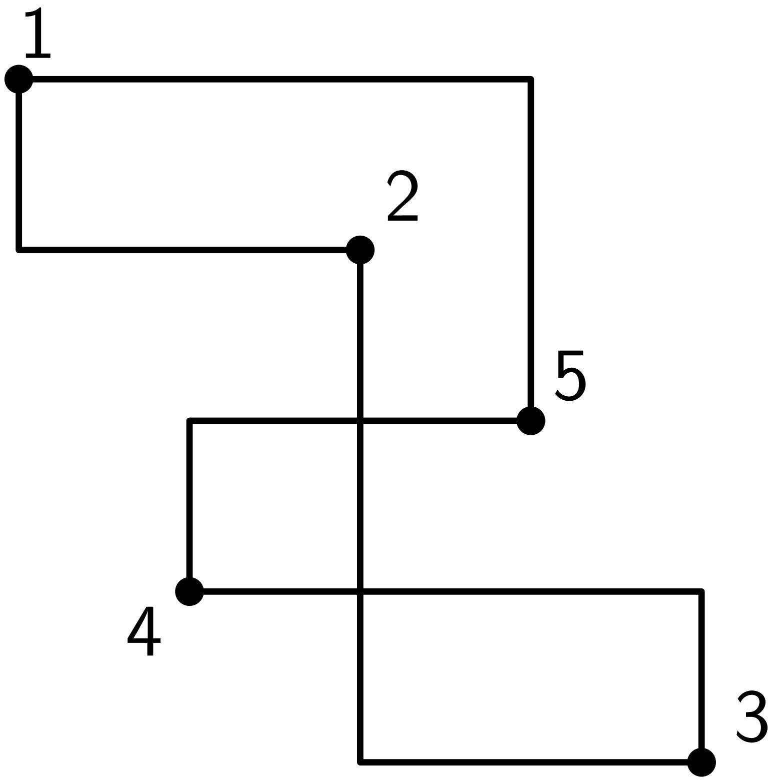
two possibilities per edge



1. RAC$_1$ PSE of cycles

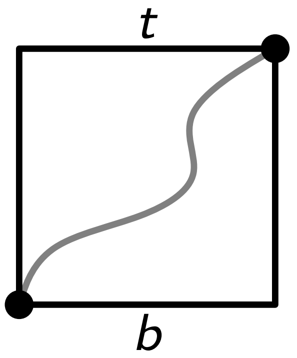


two possibilities per edge

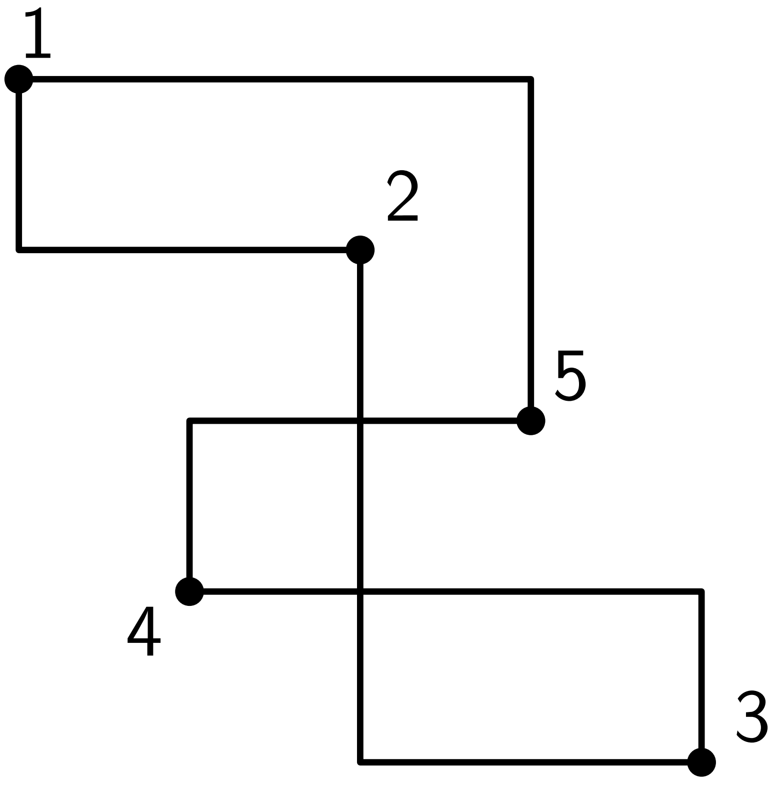


– leave vertices vertically

1. RAC_1 PSE of cycles



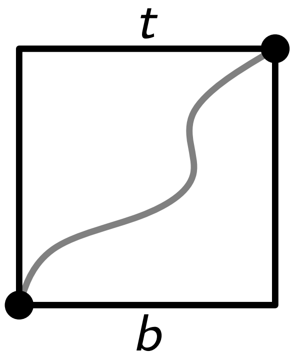
two possibilities per edge



– leave vertices vertically

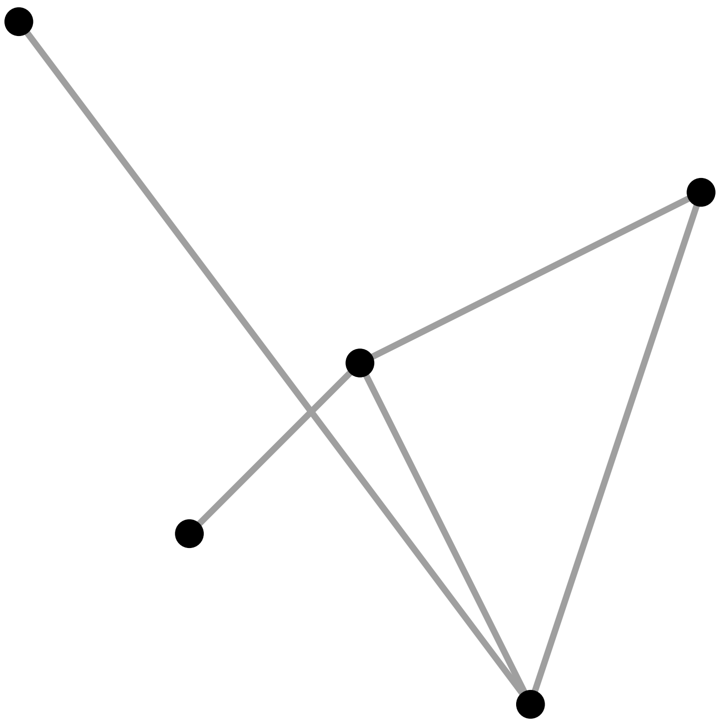
– enter vertices horizontally

2. RAC_1 PSE with mapping

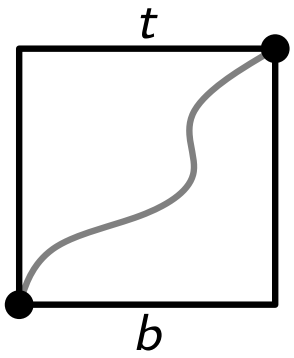


two possibilities per edge

Embeddability testing with mapping

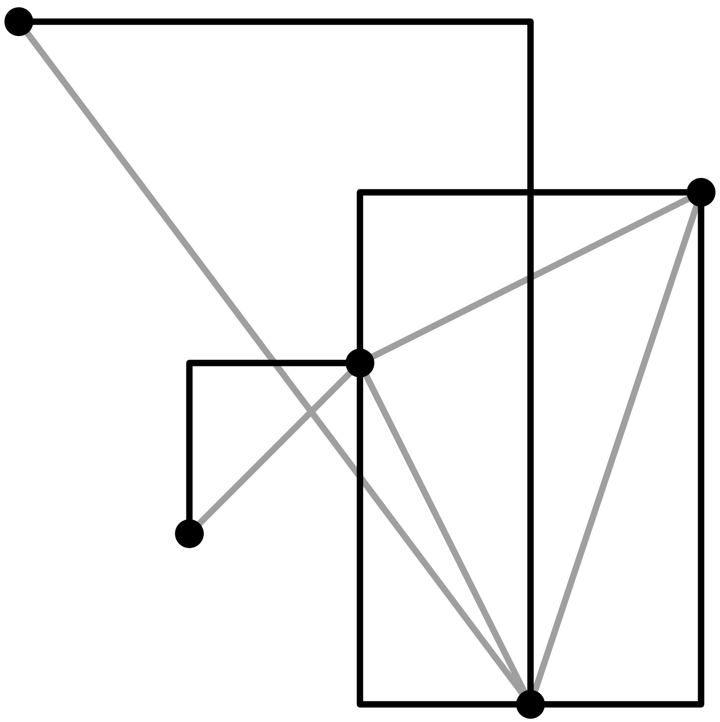


2. RAC_1 PSE with mapping

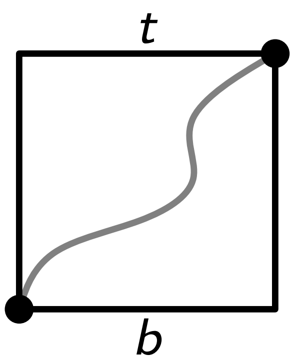


two possibilities per edge

Embeddability testing with mapping

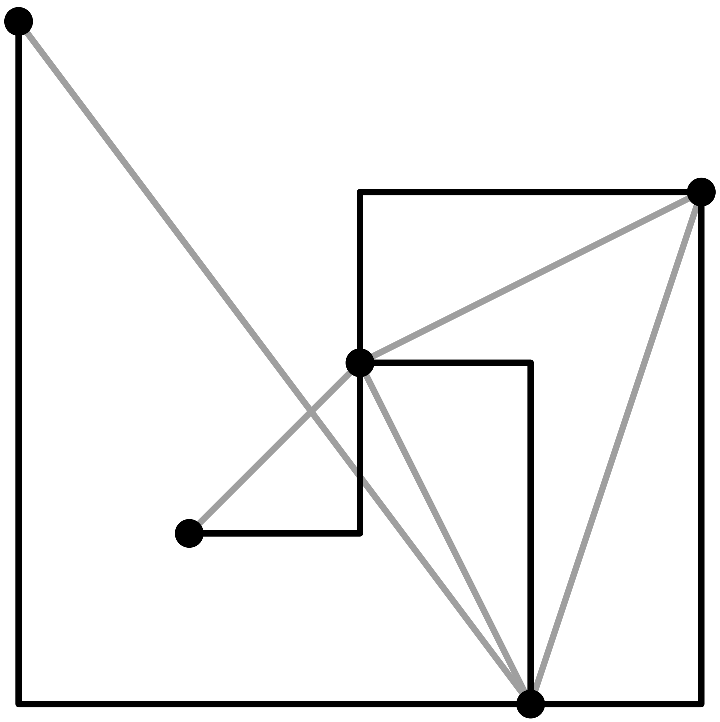


2. RAC$_1$ PSE with mapping

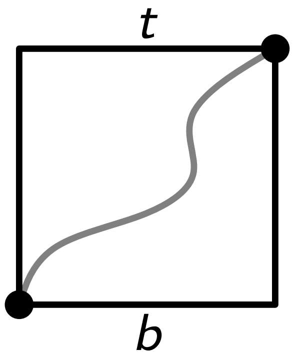


two possibilities per edge

Embeddability testing with mapping



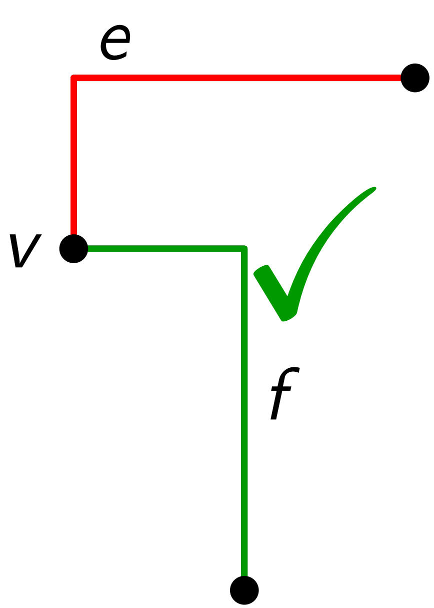
2. RAC_1 PSE with mapping



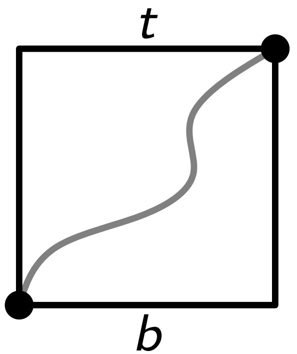
two possibilities per edge

Embeddability testing with mapping

$e_t \wedge f_t$



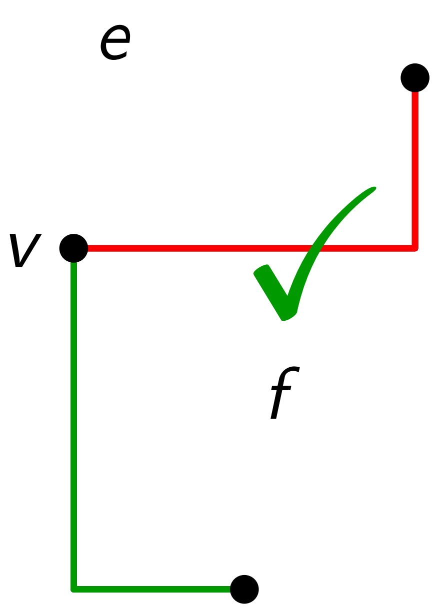
2. RAC_1 PSE with mapping



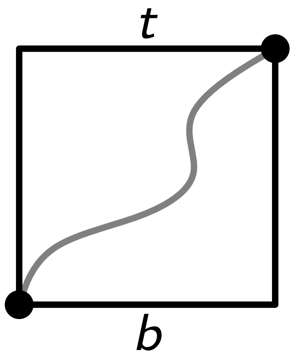
two possibilities per edge

Embeddability testing with mapping

$e_b \wedge f_b$



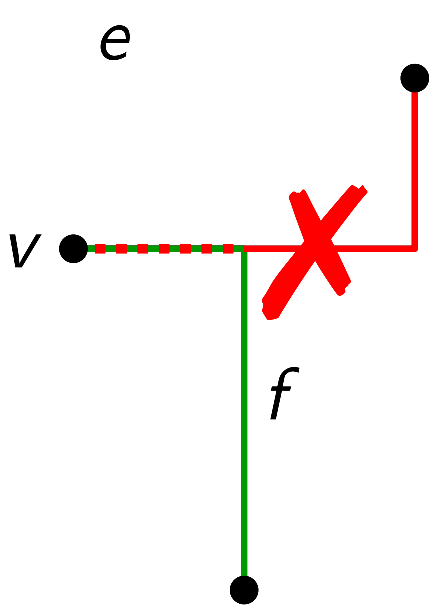
2. RAC_1 PSE with mapping



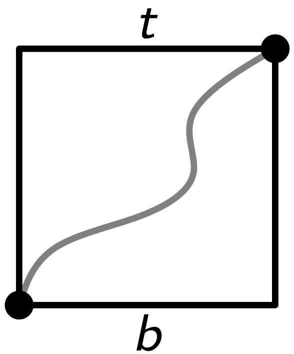
two possibilities per edge

Embeddability testing with mapping

$e_b \wedge f_t$

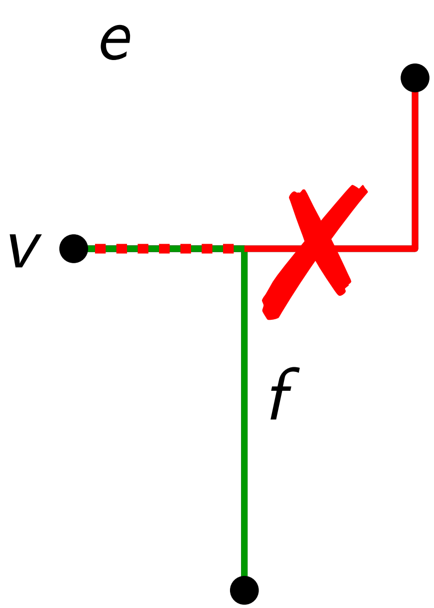


2. RAC_1 PSE with mapping



two possibilities per edge

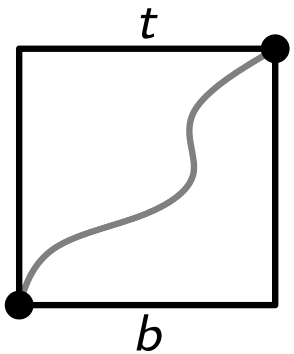
Embeddability testing with mapping



$$\neg(e_b \wedge f_t)$$

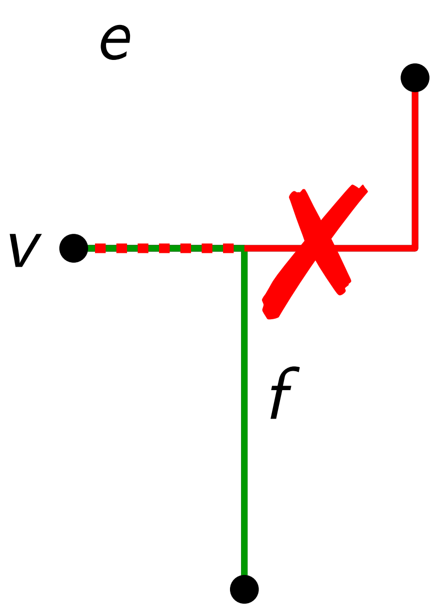
$$\equiv (\neg e_b \vee \neg f_t)$$

2. RAC_1 PSE with mapping



two possibilities per edge

Embeddability testing with mapping



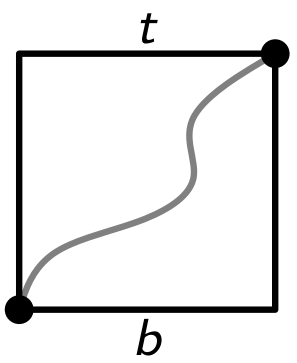
$$\neg(e_b \wedge f_t)$$

$$\equiv (\neg e_b \vee \neg f_t)$$

$$\equiv (e_t \vee \neg f_t)$$

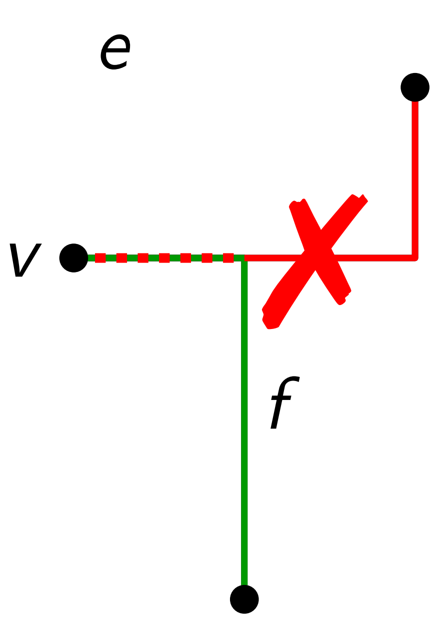
$$e_t \Leftrightarrow \neg e_b$$

2. RAC_1 PSE with mapping



two possibilities per edge

Embeddability testing with mapping



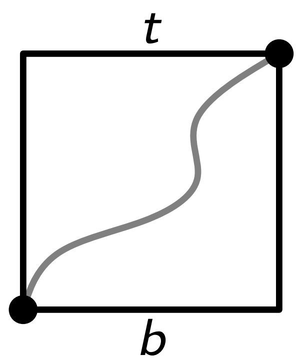
$\neg(e_b \wedge f_t)$

$\equiv (\neg e_b \vee \neg f_t)$

$\equiv (e_t \vee \neg f_t)$

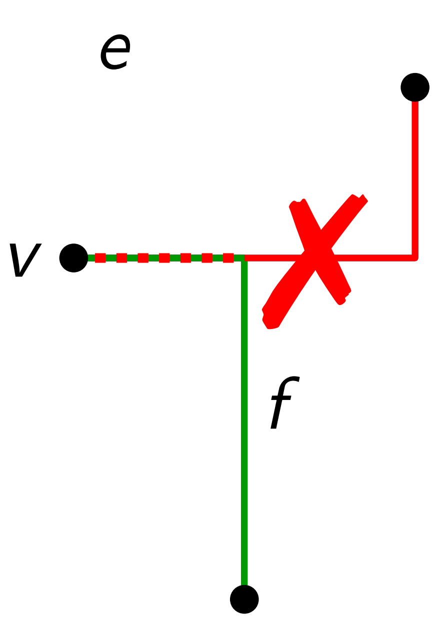
2-SAT clause

2. RAC_1 PSE with mapping



two possibilities per edge

Embeddability testing with mapping



$\neg(e_b \wedge f_t)$

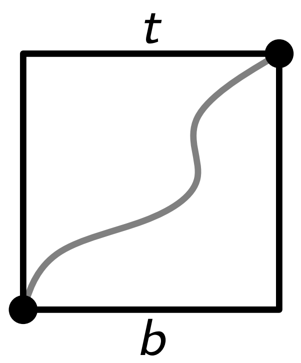
$\equiv (\neg e_b \vee \neg f_t)$

$\equiv (e_t \vee \neg f_t)$

2-SAT clause

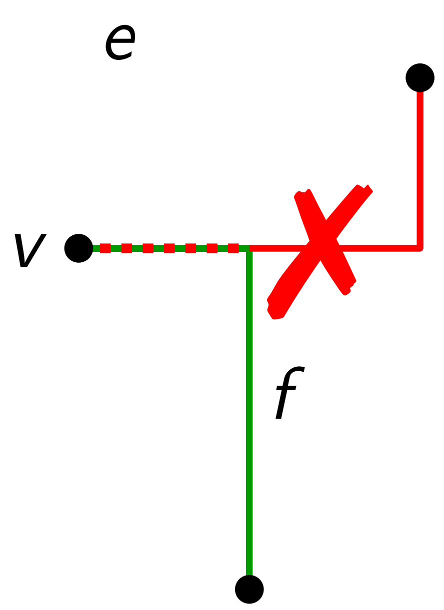
local conditions are sufficient!

2. RAC_1 PSE with mapping



two possibilities per edge

Embeddability testing with mapping in linear time



$\neg(e_b \wedge f_t)$

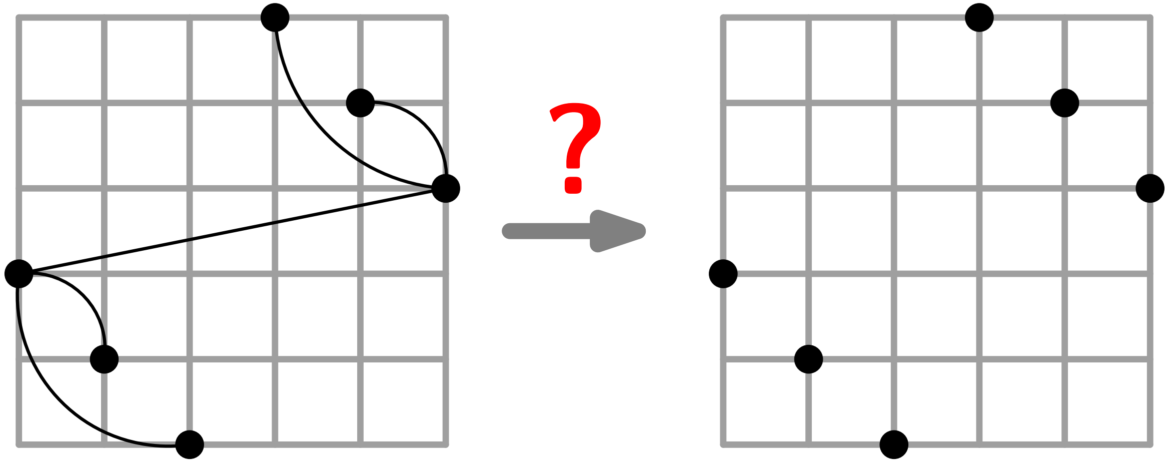
$\equiv (\neg e_b \vee \neg f_t)$

$\equiv (e_t \vee \neg f_t)$

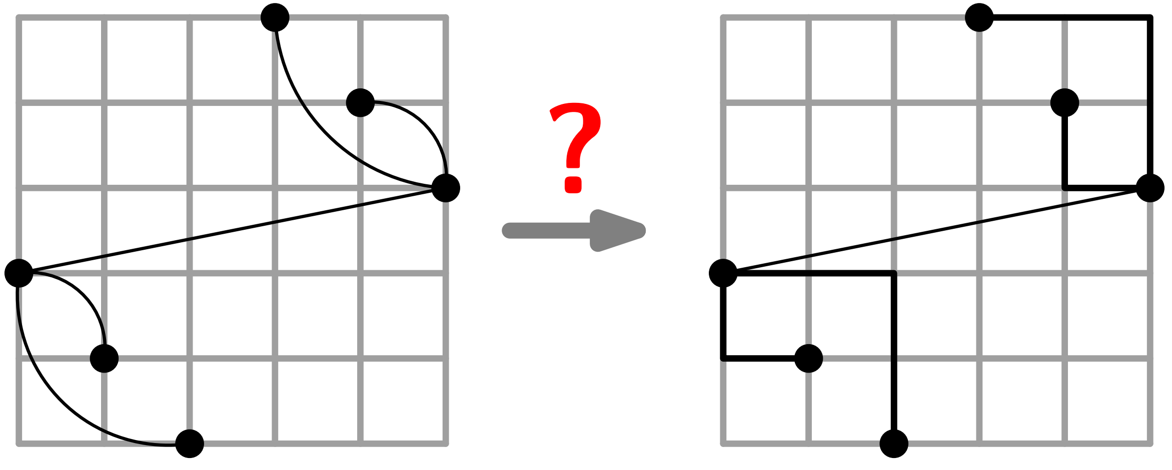
2-SAT clause

In total $\leq n \cdot \binom{4}{2} \cdot 2 = O(n)$ clauses.

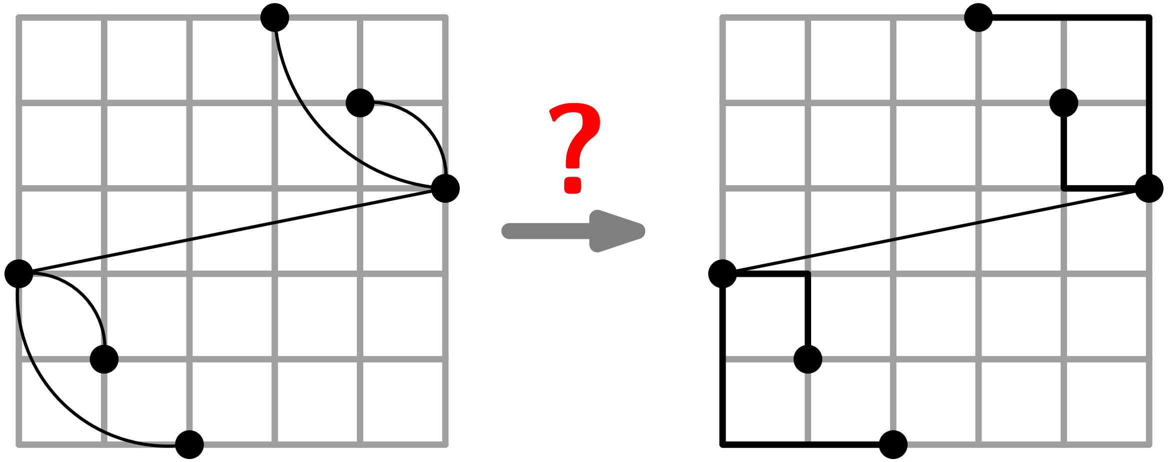
3. RAC_1 PSE of binary trees



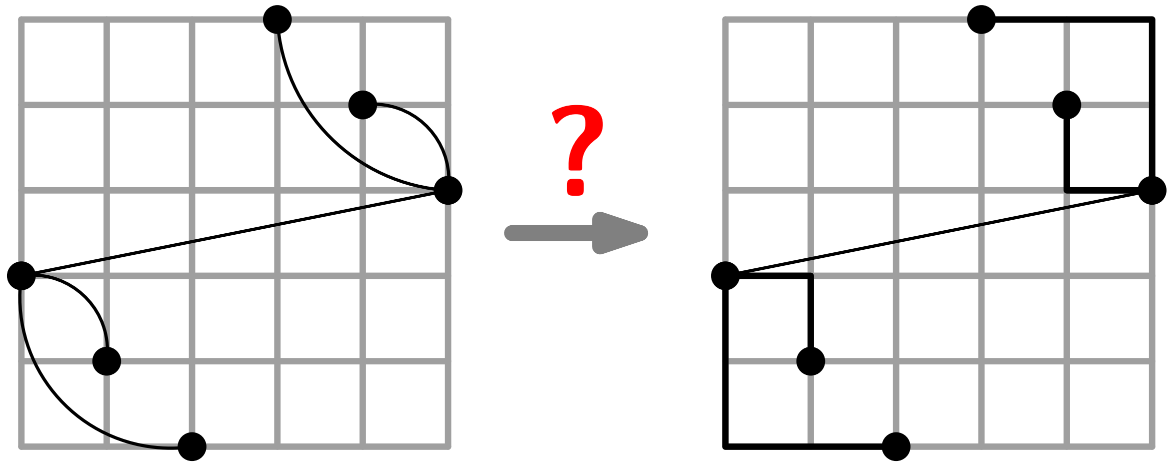
3. RAC_1 PSE of binary trees



3. RAC_1 PSE of binary trees

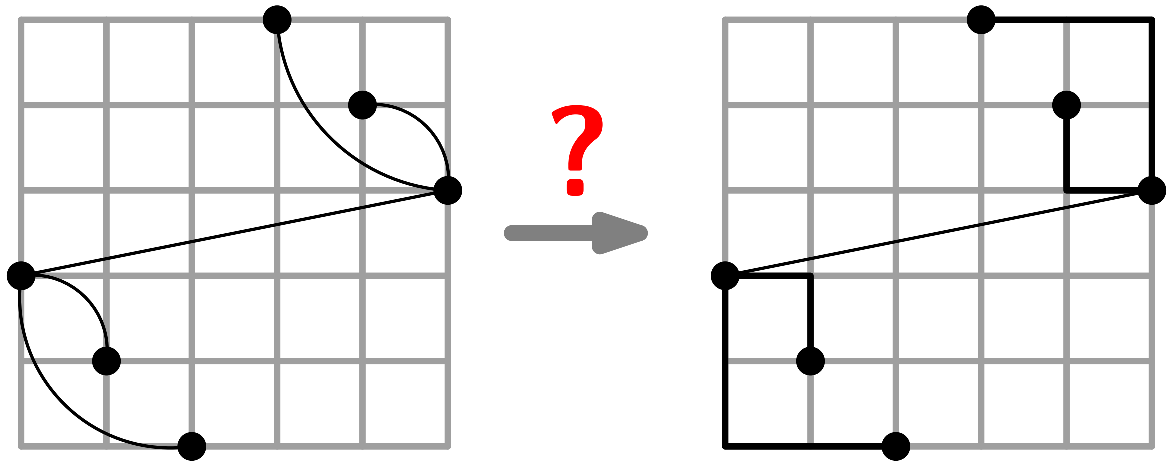


3. RAC_1 PSE of binary trees



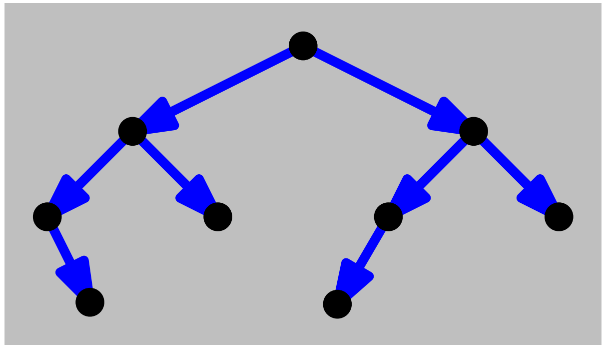
- does not work with every mapping

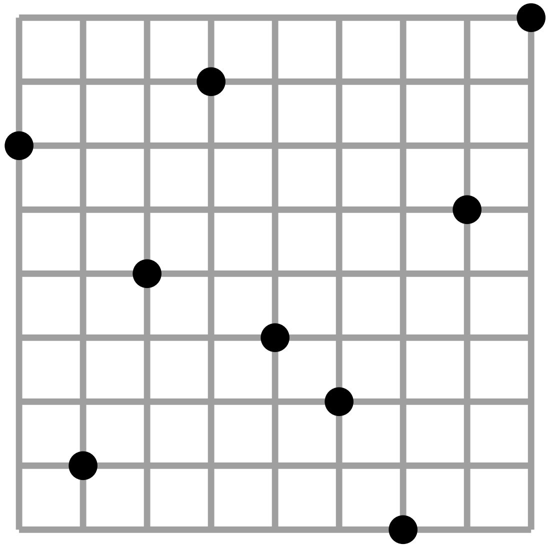
3. RAC_1 PSE of binary trees



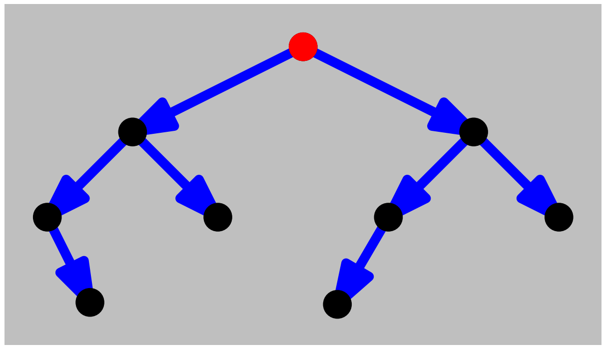
- does not work with every mapping
- *we choose* the mapping

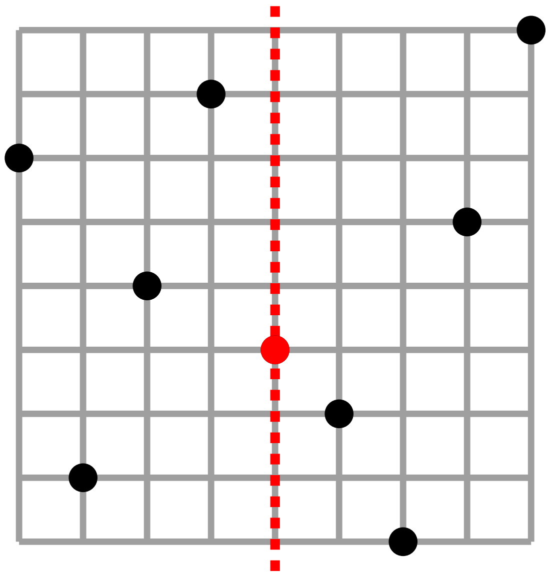
3. RAC$_1$ PSE of binary trees





3. RAC$_1$ PSE of binary trees





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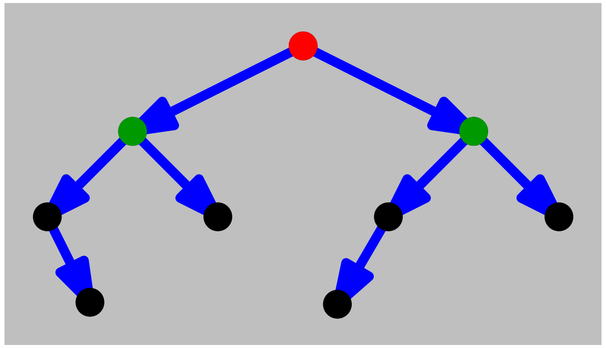

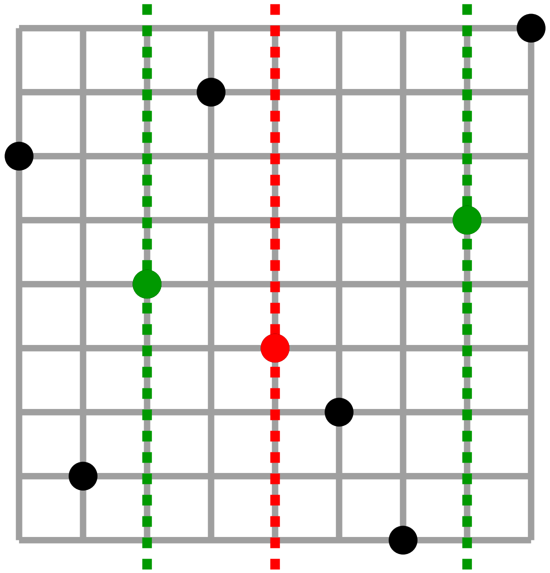

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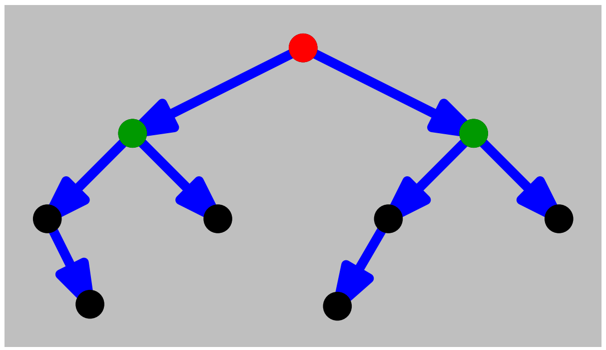
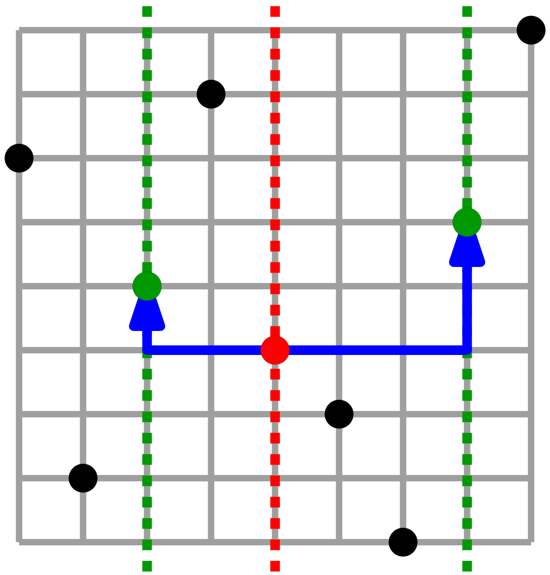
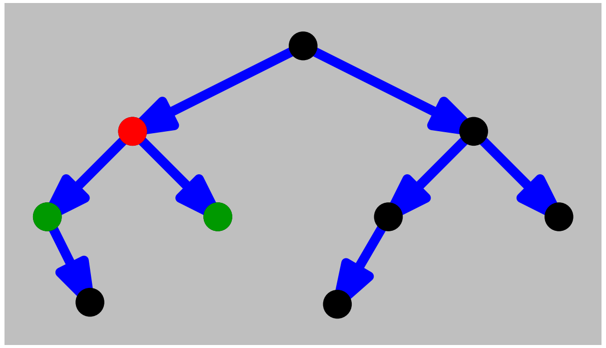
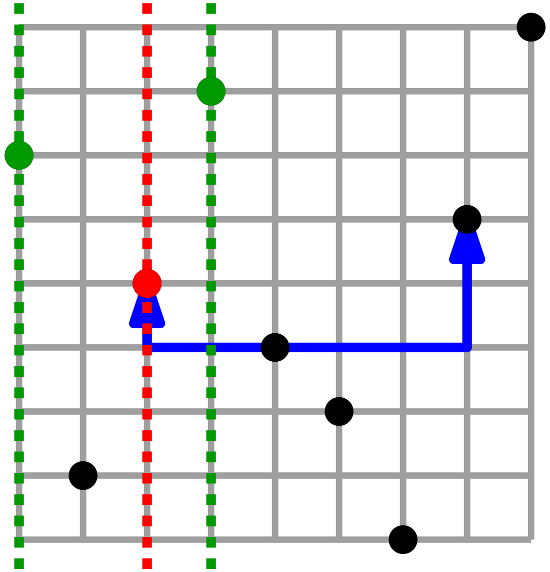


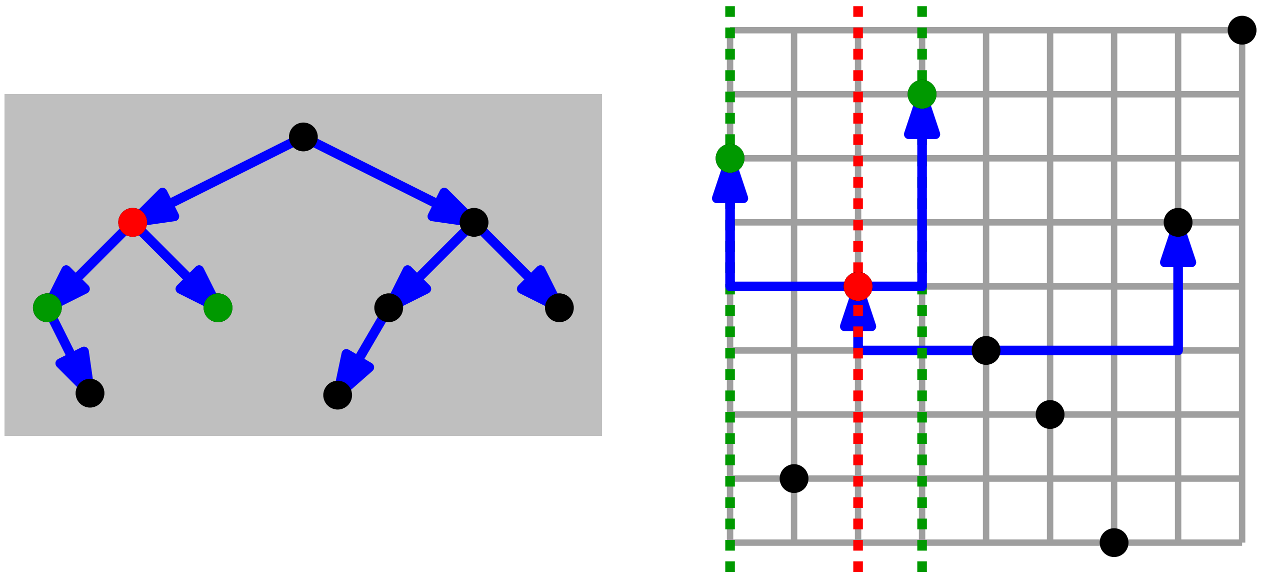


3. RAC_1 PSE of binary trees

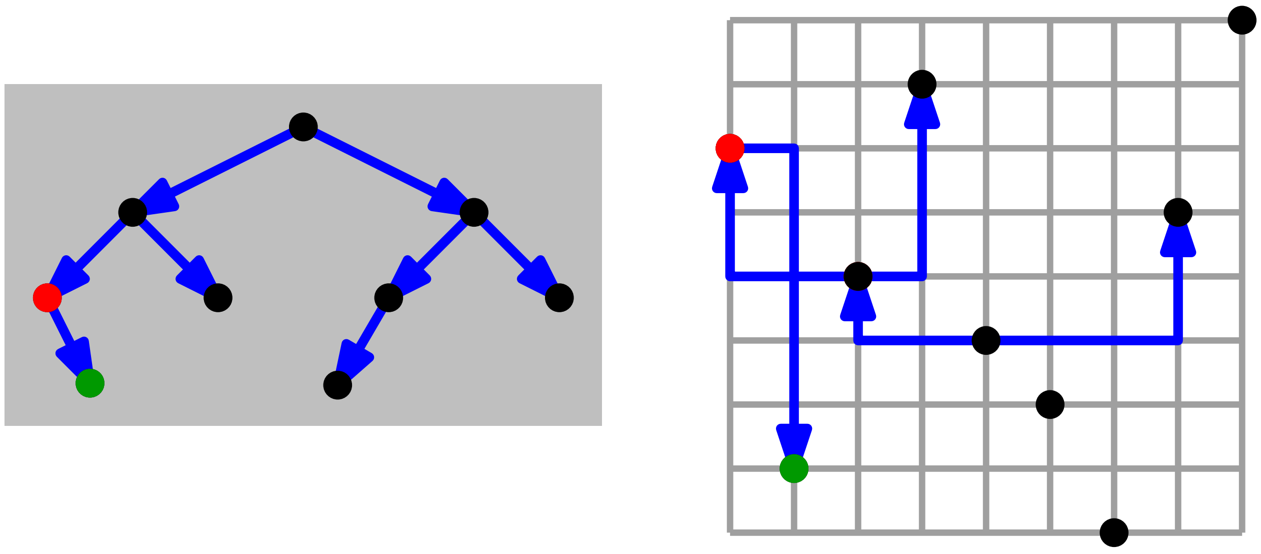




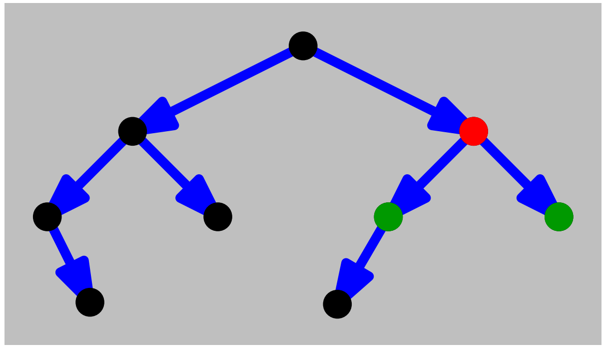
3. RAC_1 PSE of binary trees

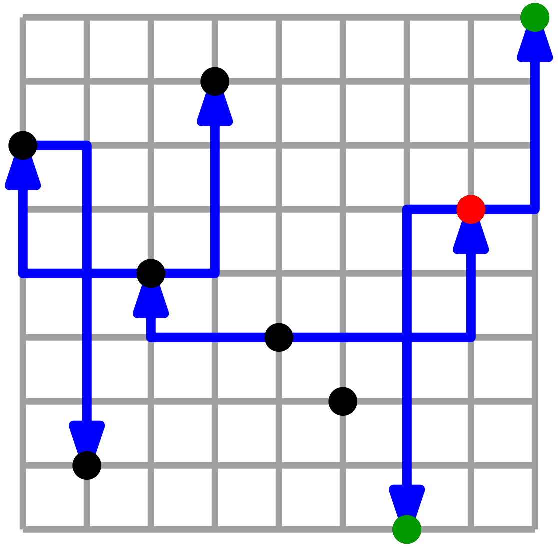


3. RAC_1 PSE of binary trees

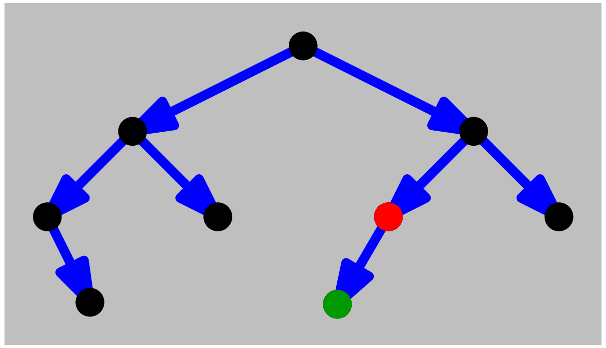


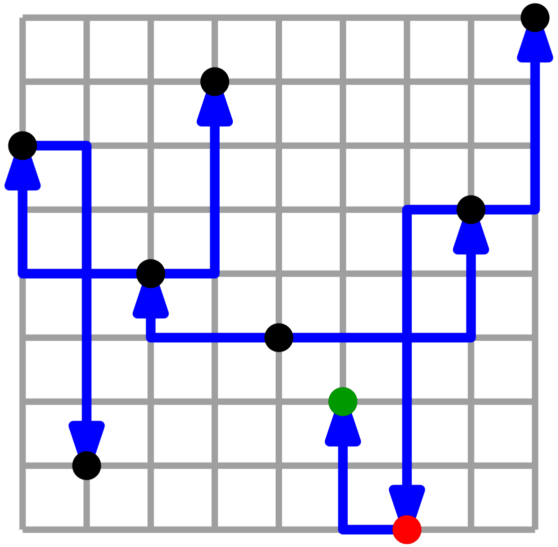
3. RAC_1 PSE of binary trees





3. RAC_1 PSE of binary trees





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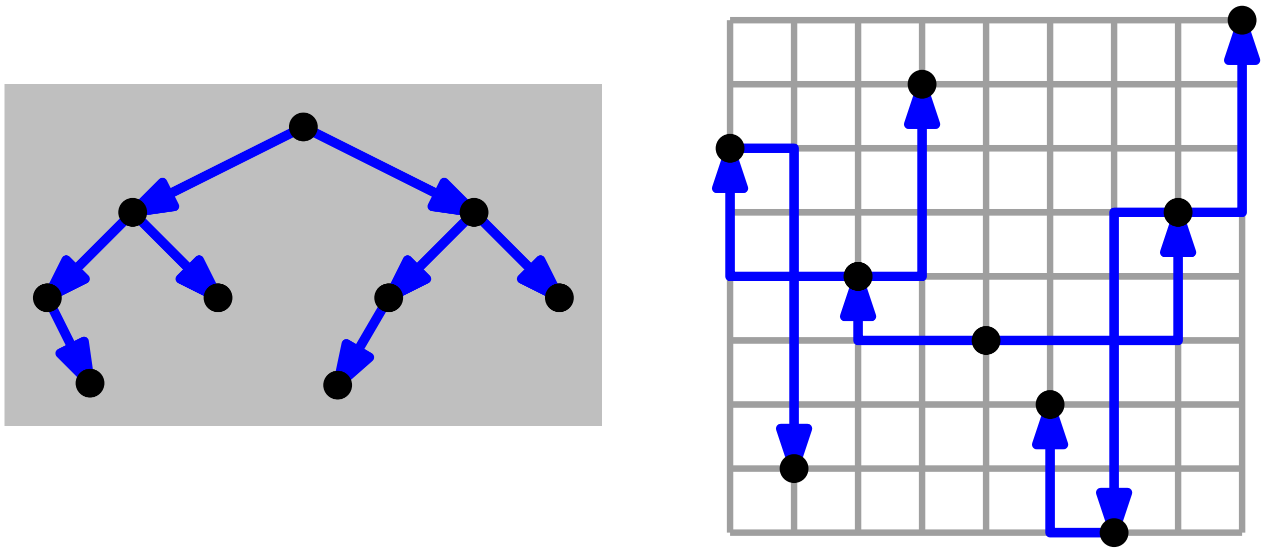
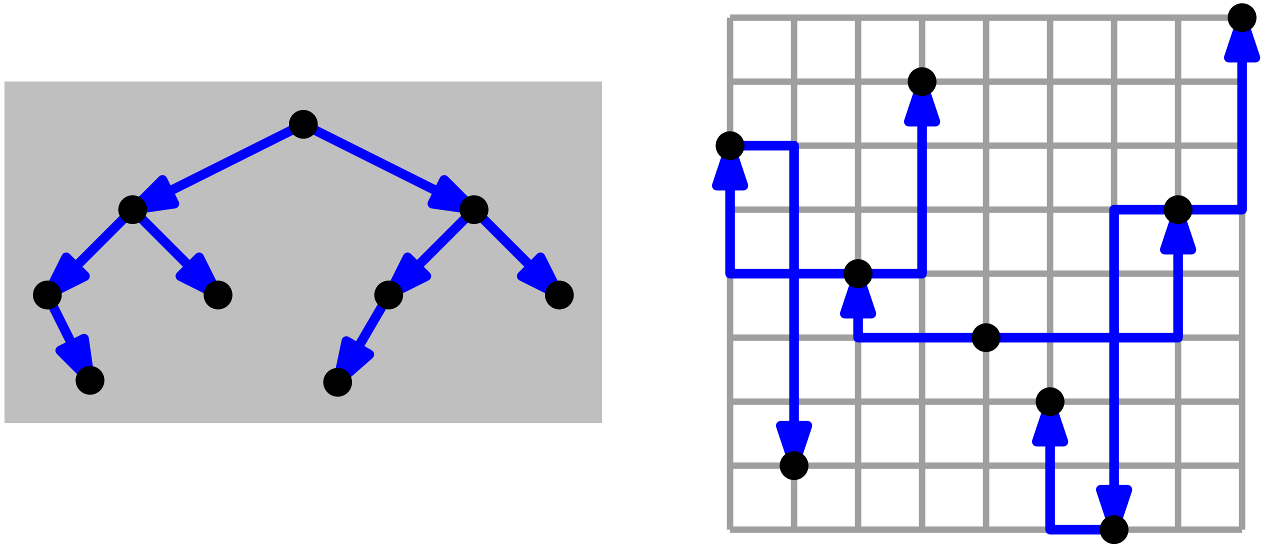


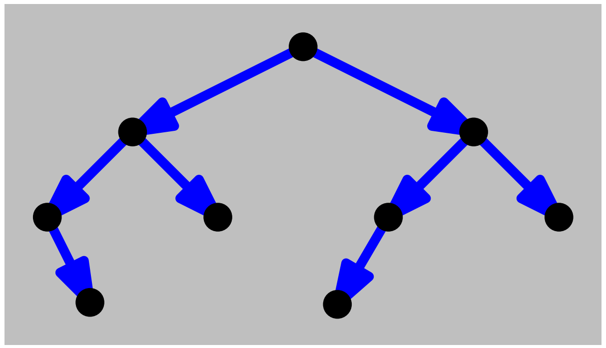
independentely by Di Giacomo et al.

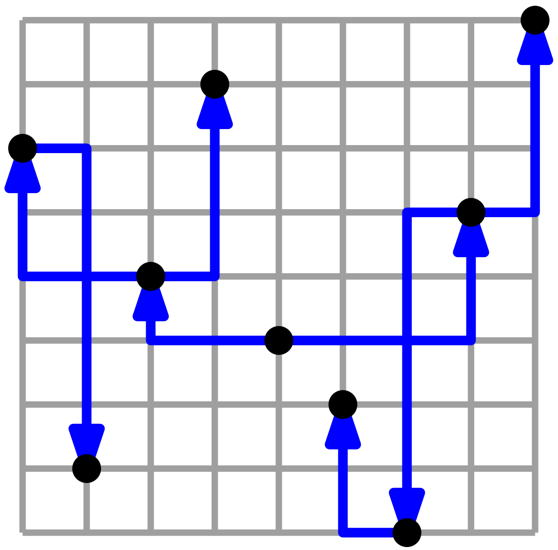
3. RAC_1 PSE of binary trees



– What about larger classes of graphs?

3. RAC_1 PSE of binary trees

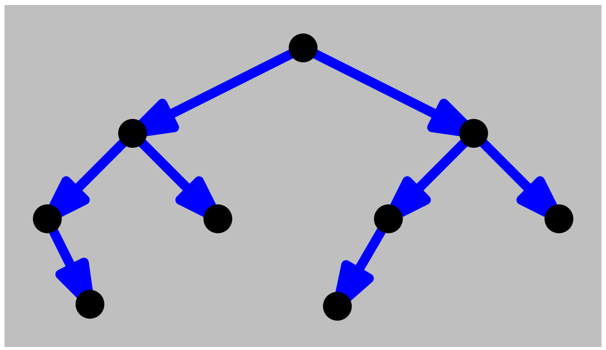


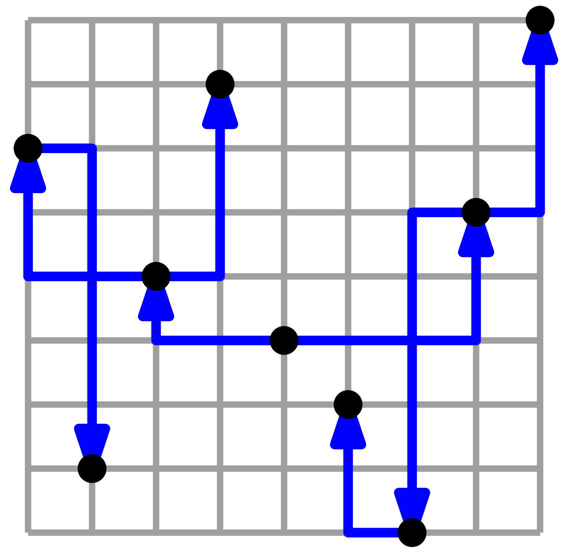


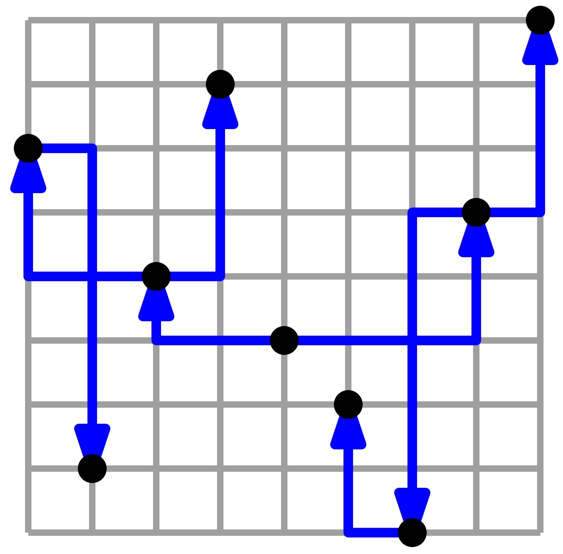
– What about larger classes of graphs?

– What about the *planar* case?

3. RAC_1 PSE of binary trees







– What about larger classes of graphs?

– What about the *planar* case?

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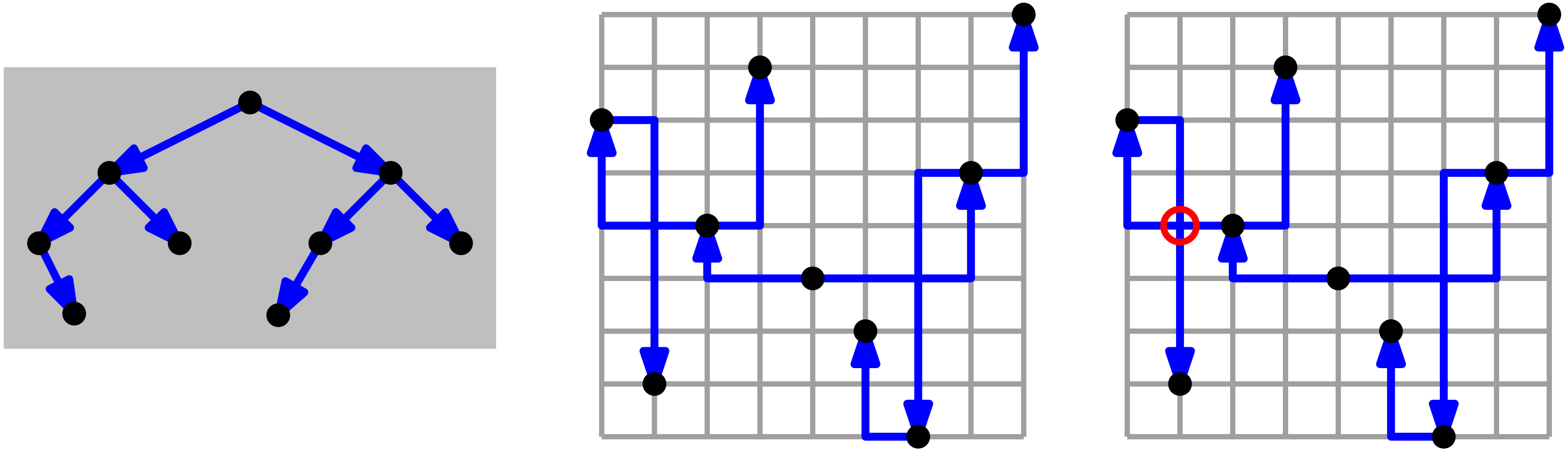

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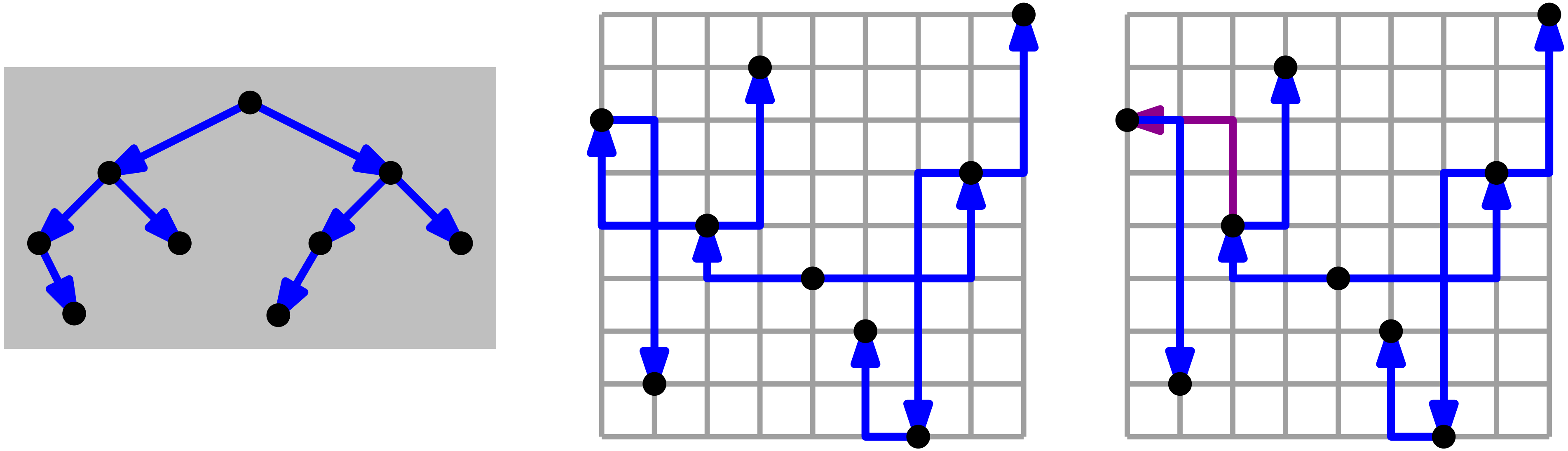

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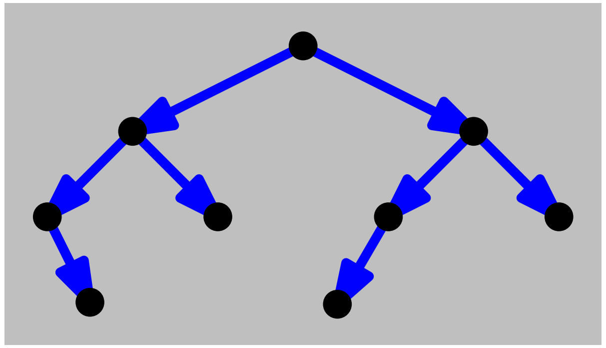

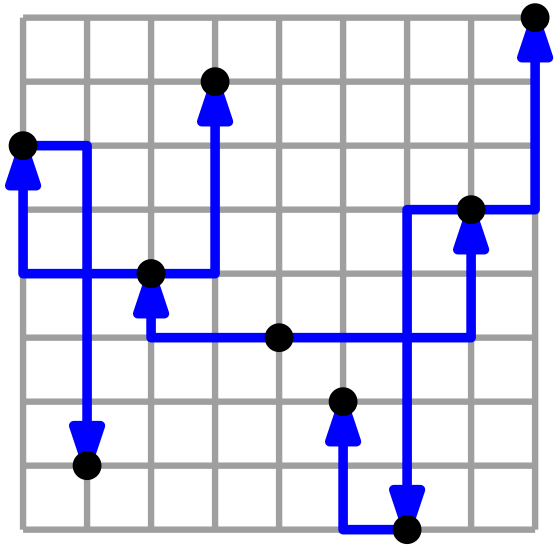

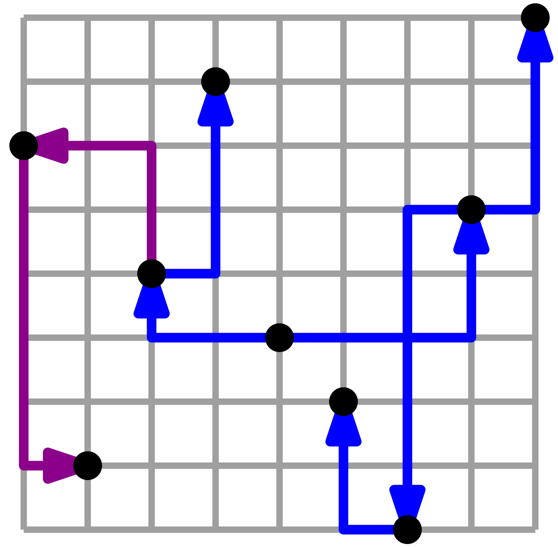

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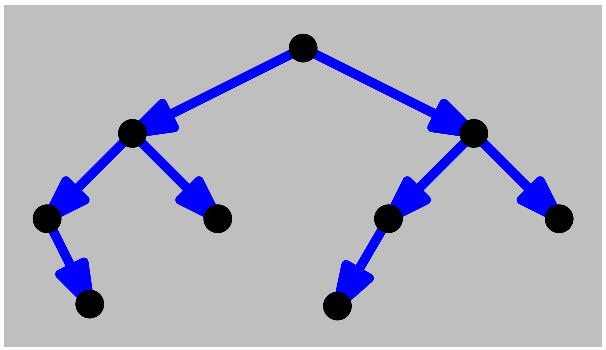
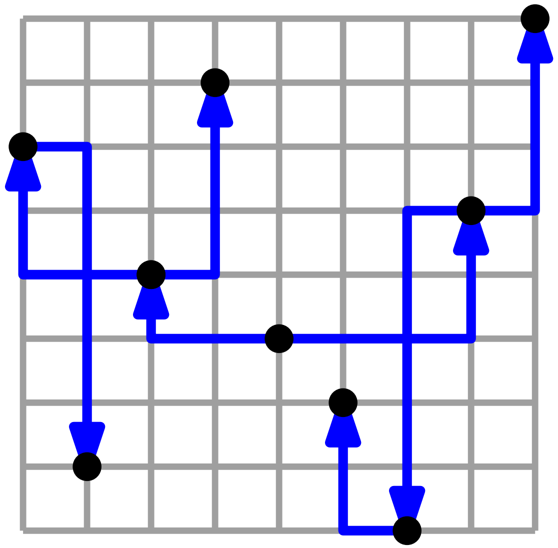
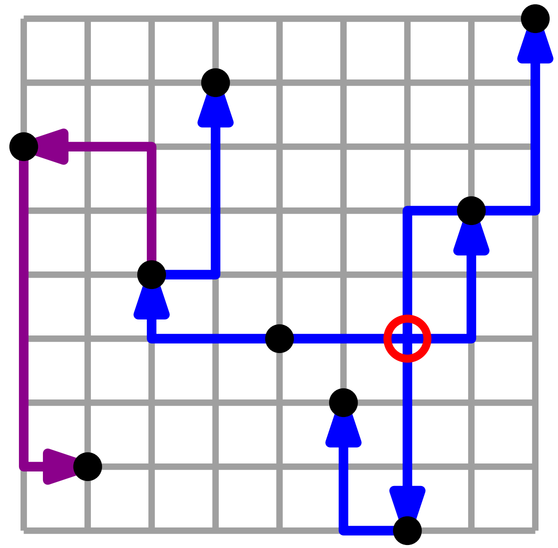

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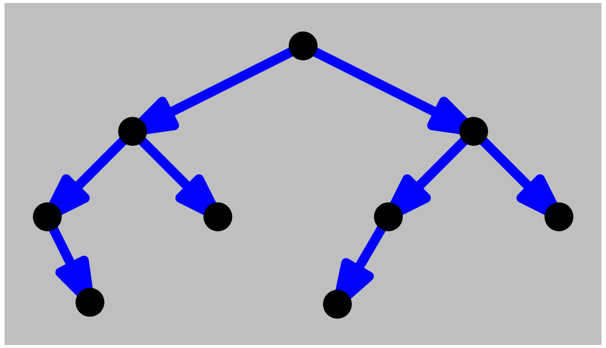
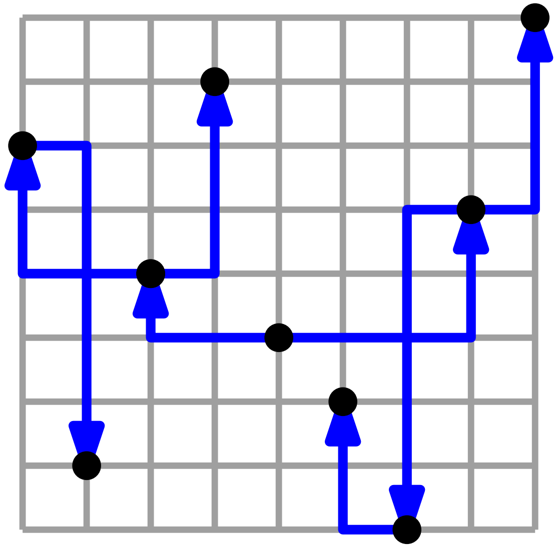
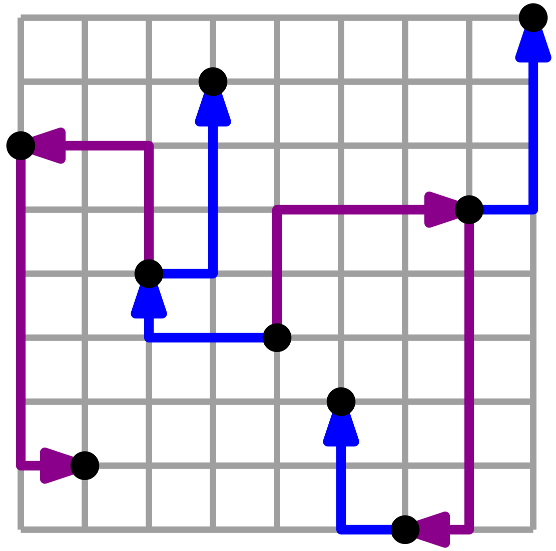




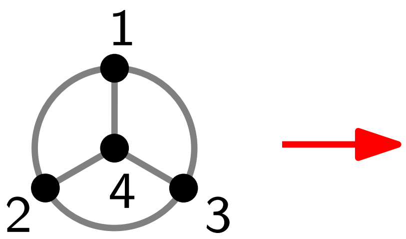


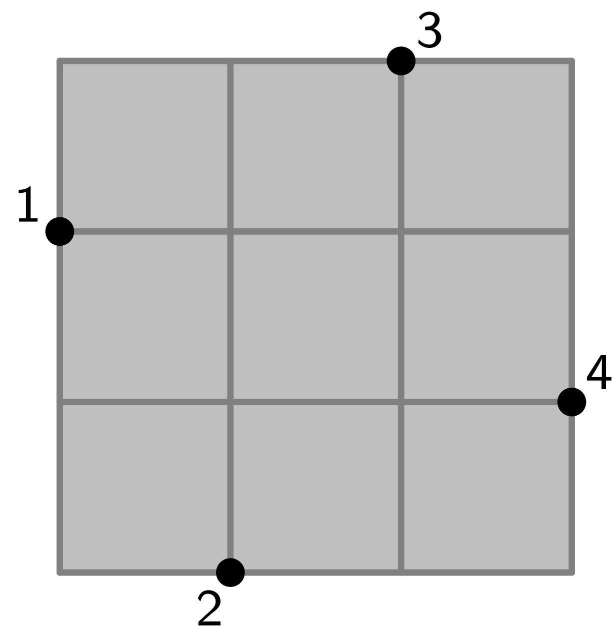
– What about larger classes of graphs?

– What about the *planar* case?

4. RAC_2 PSE of maxdeg-3 graphs

any maxdeg-3 graph $\xrightarrow{RAC_2}$ $O(n) \times O(n)$ grid (with mapping)





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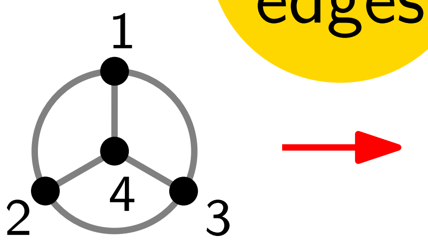

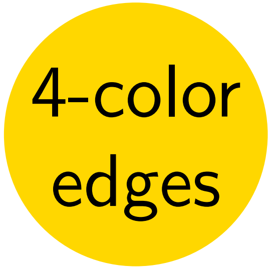

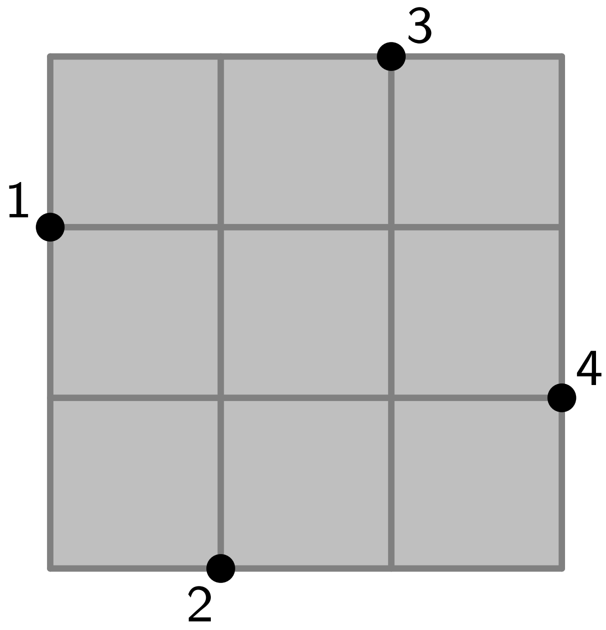

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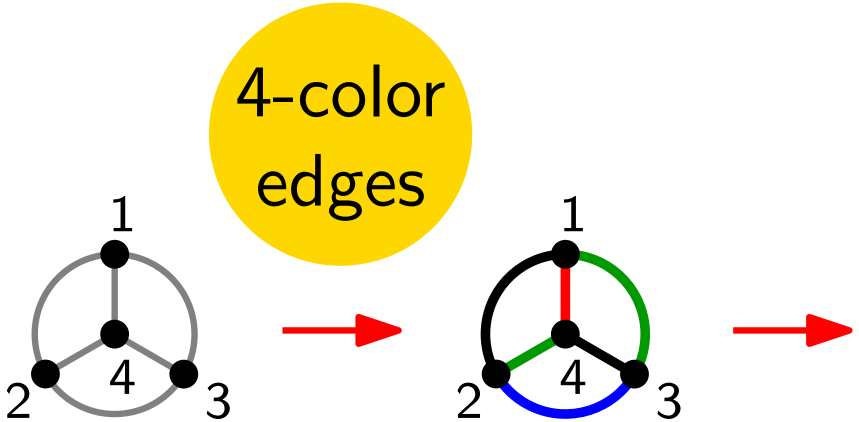

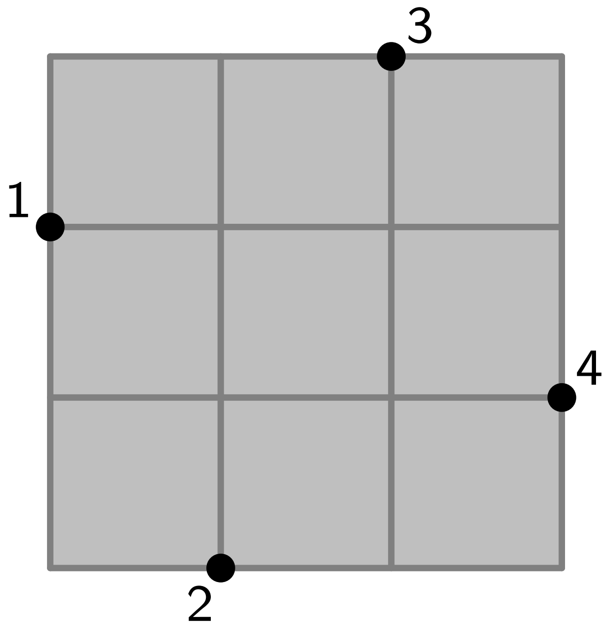

Vizing 1964:

maxdeg $\Delta \Rightarrow (\Delta + 1)$-edge-colorable

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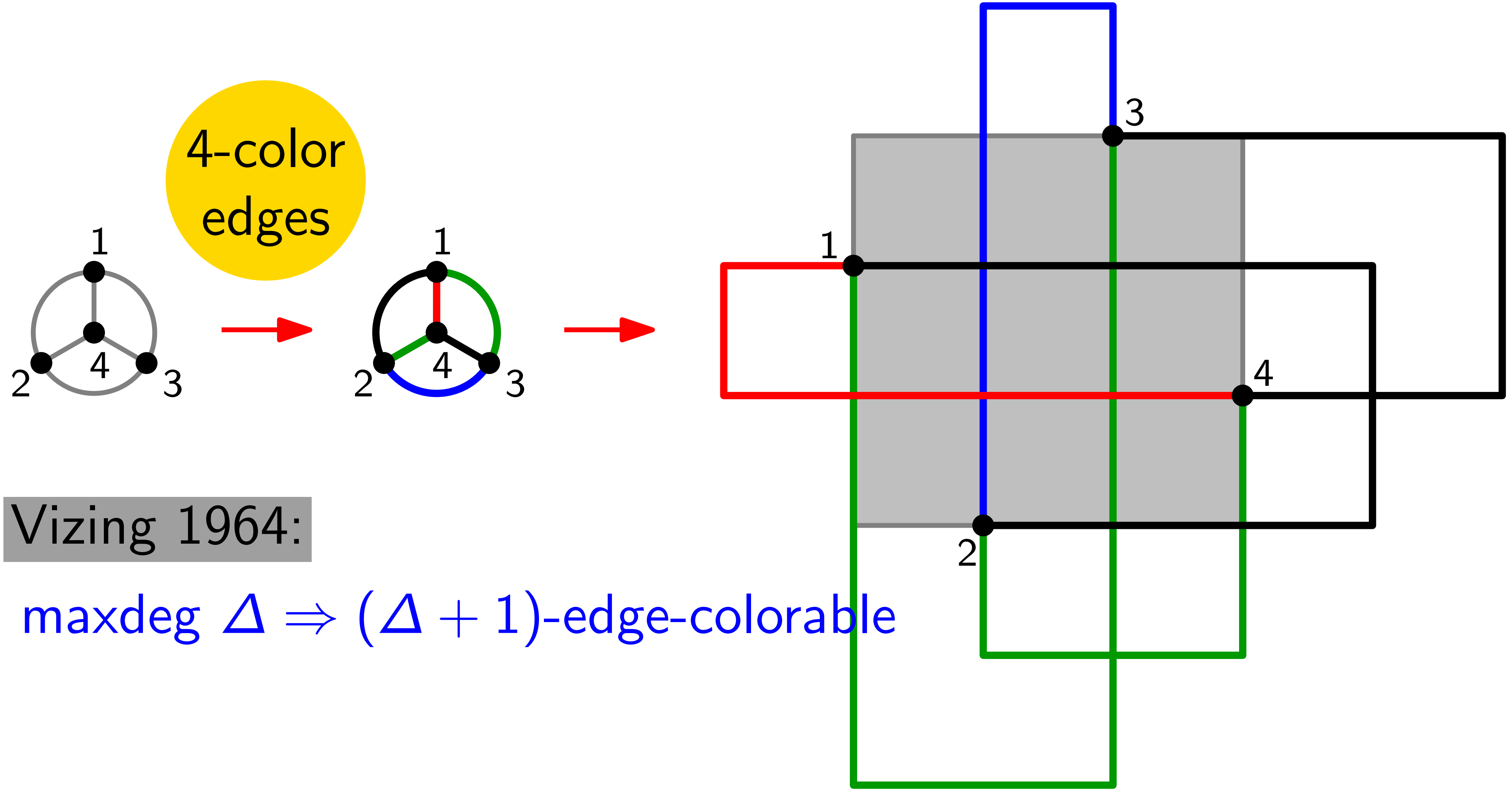

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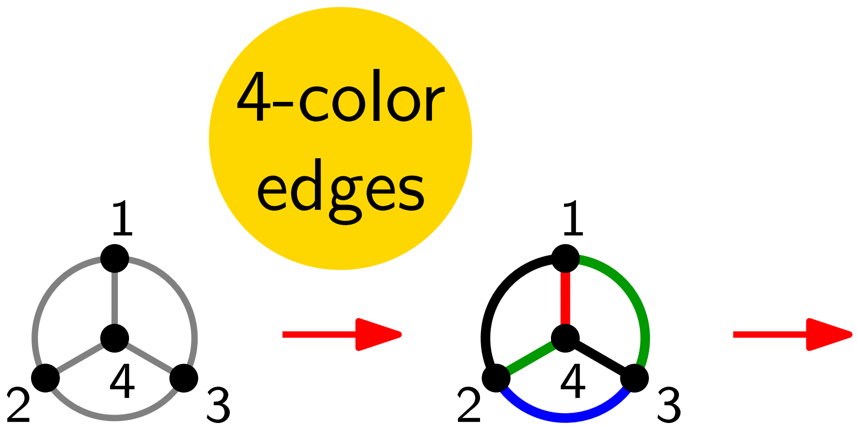

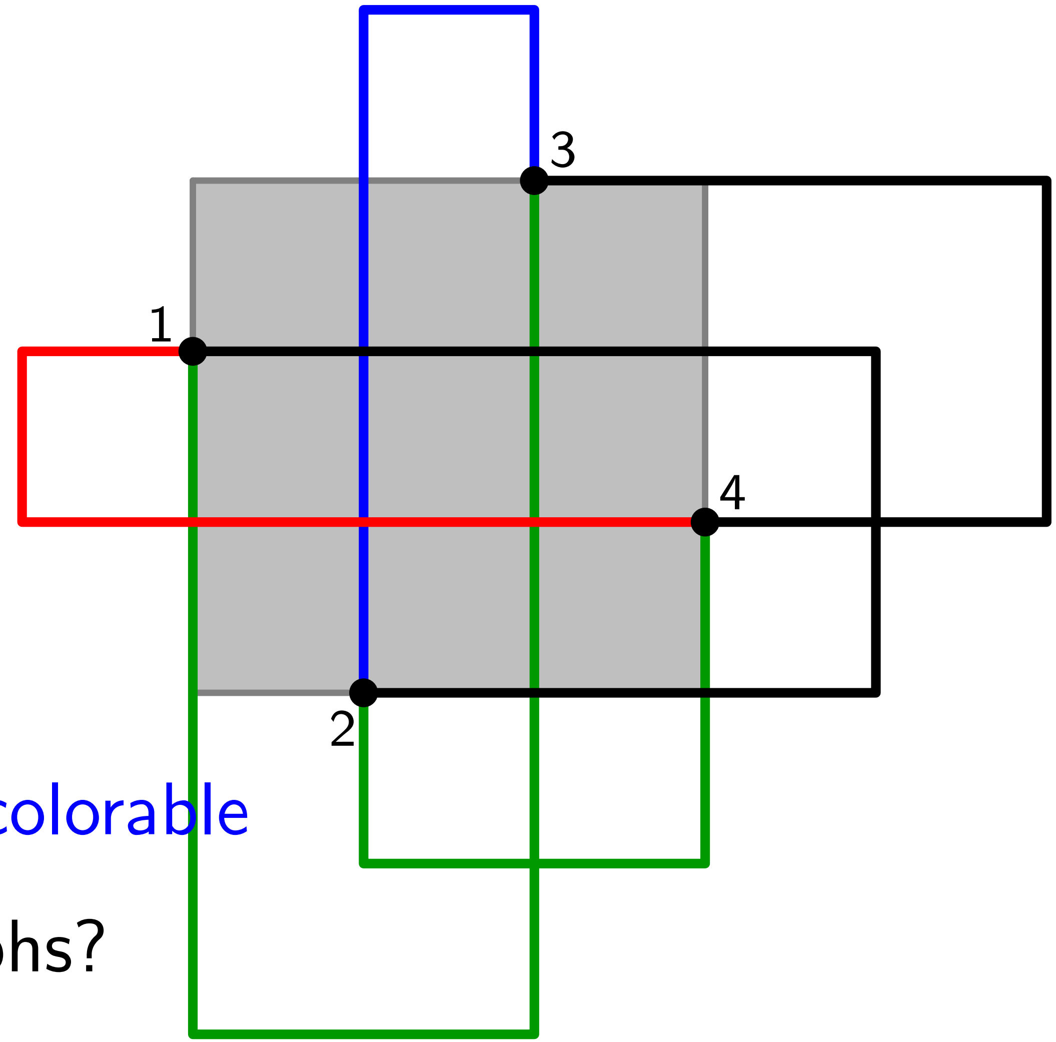

Vizing 1964:

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– What about maxdeg-4 graphs?

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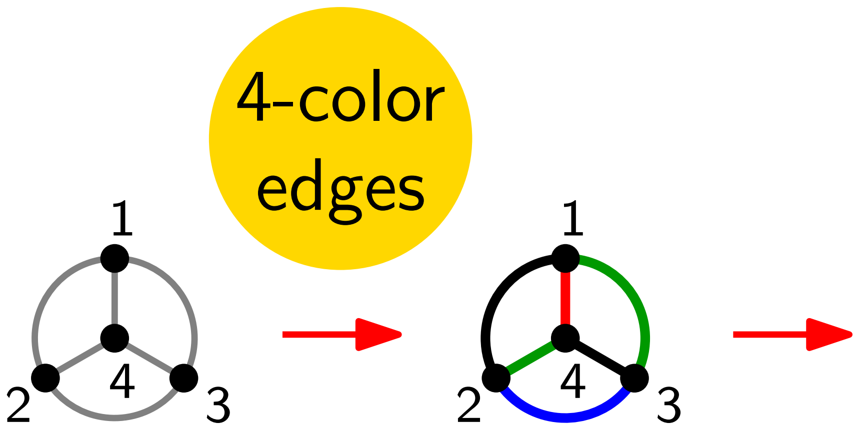

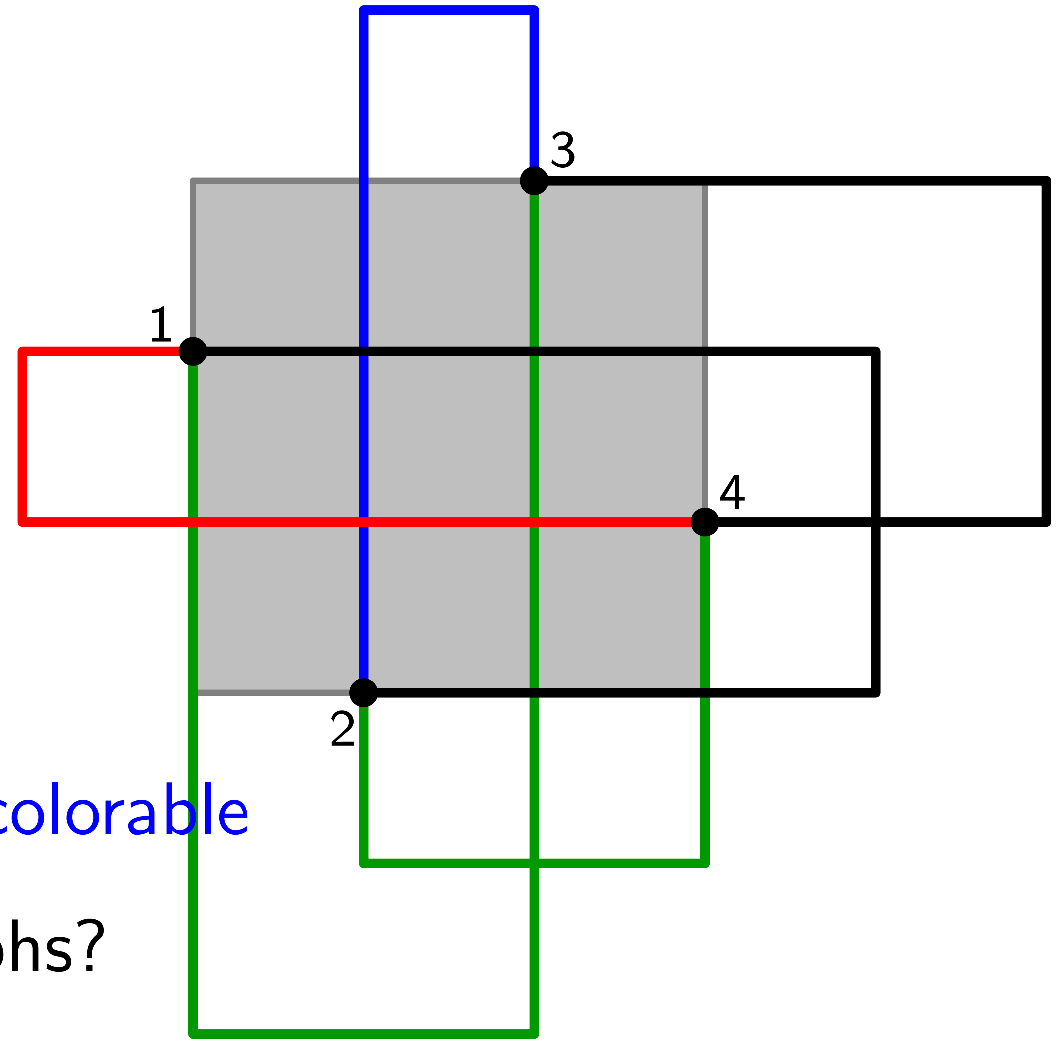

Vizing 1964:

maxdeg $\Delta \Rightarrow (\Delta + 1)$-edge-colorable

– What about maxdeg-4 graphs?

– Can we minimize the area?

Conclusion

- **Unrestricted RAC/αAC PSE:**

any graph $+$
grid point set

with mapping →

- RAC$_3$
- αAC$_2$
- [αAC$_1$]

Conclusion

- **Unrestricted RAC/αAC PSE:**

  any graph + grid point set —with mapping→
  - RAC$_3$
  - αAC$_2$
  - [αAC$_1$]

- **Restricted RAC PSE**

  binary tree → RAC$_1$

  any graph —with mapping→ check RAC$_1$

  any maxdeg-3 graph —with mapping→ RAC$_2$

Conclusion

- **Unrestricted RAC/αAC PSE:**

  any graph + grid point set —with mapping→
  - RAC_3
  - αAC_2
  - [αAC_1]

- **Restricted RAC PSE**

  binary tree → RAC_1

  any graph —with mapping→ check RAC_1

  any maxdeg-3 graph —with mapping→ RAC_2

- *Many open Problems!*

Thank you!