Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

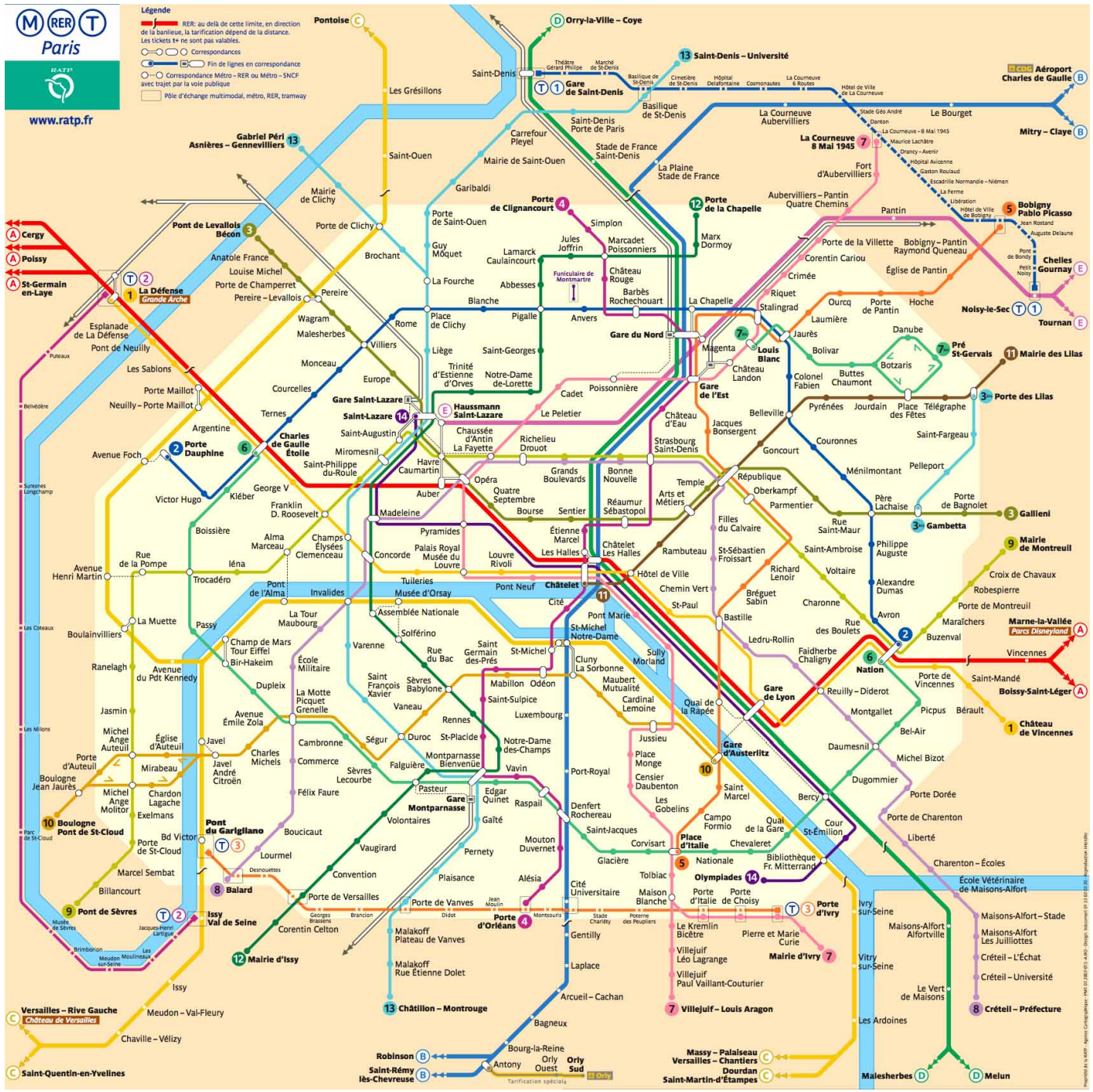
Ordering Metro Lines by Block Crossings

Martin Fink
Lehrstuhl für Informatik I
Universität Würzburg

Joint work with Sergey Pupyrev

Metro Maps – Vienna

Metro Maps – Paris



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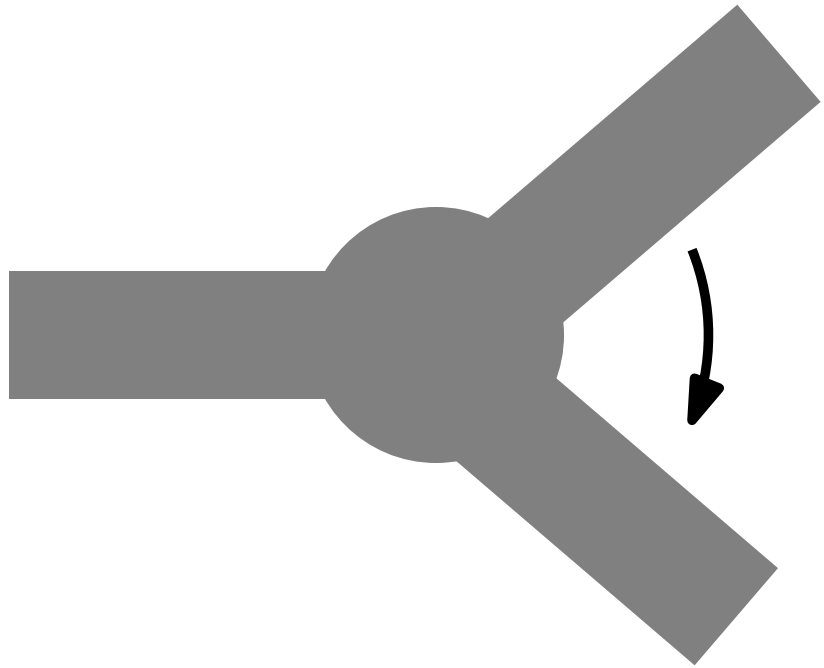

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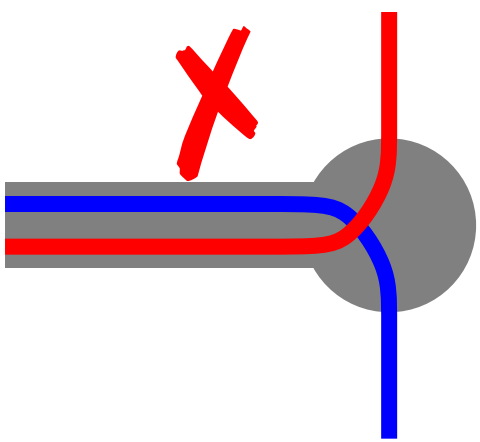

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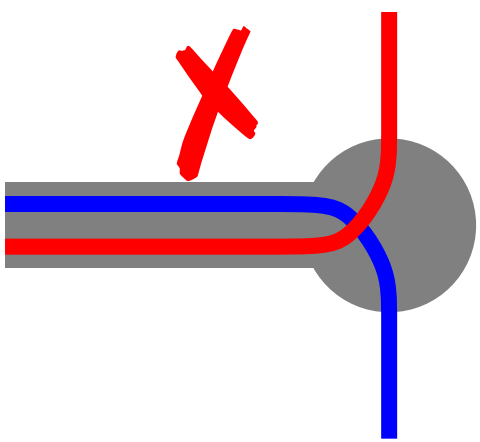

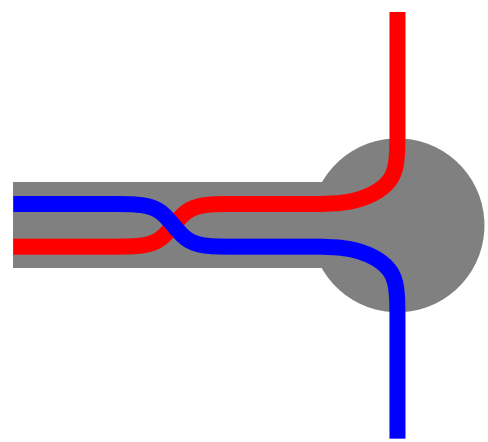

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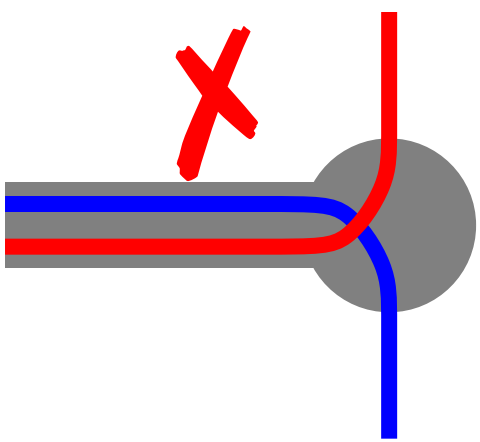

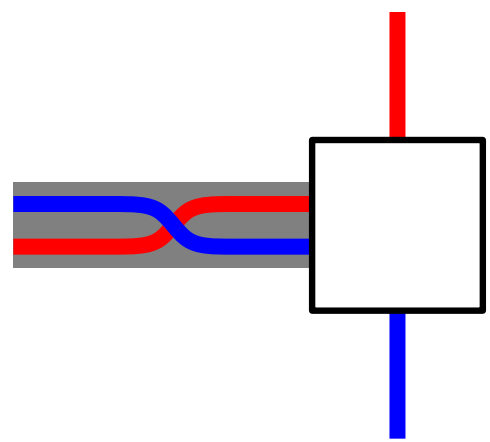

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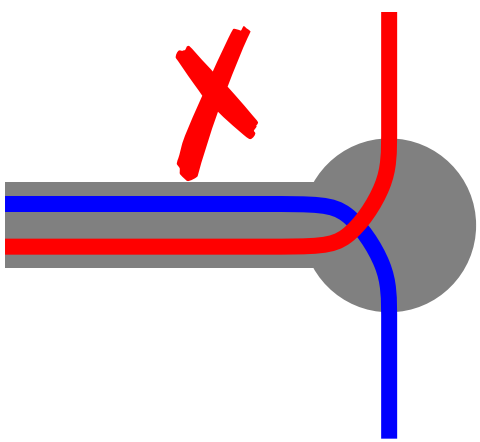

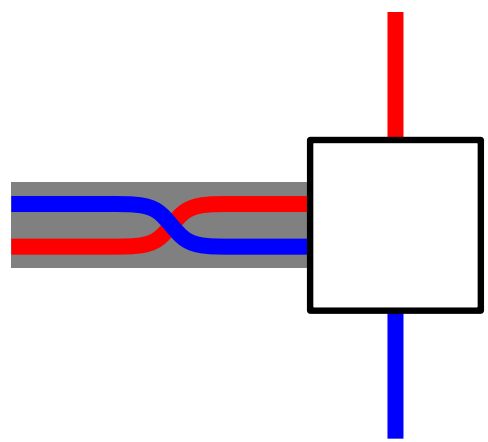

NP-hard [Bekos et al., 2007] & [Fink, Pupyrev, 2013]

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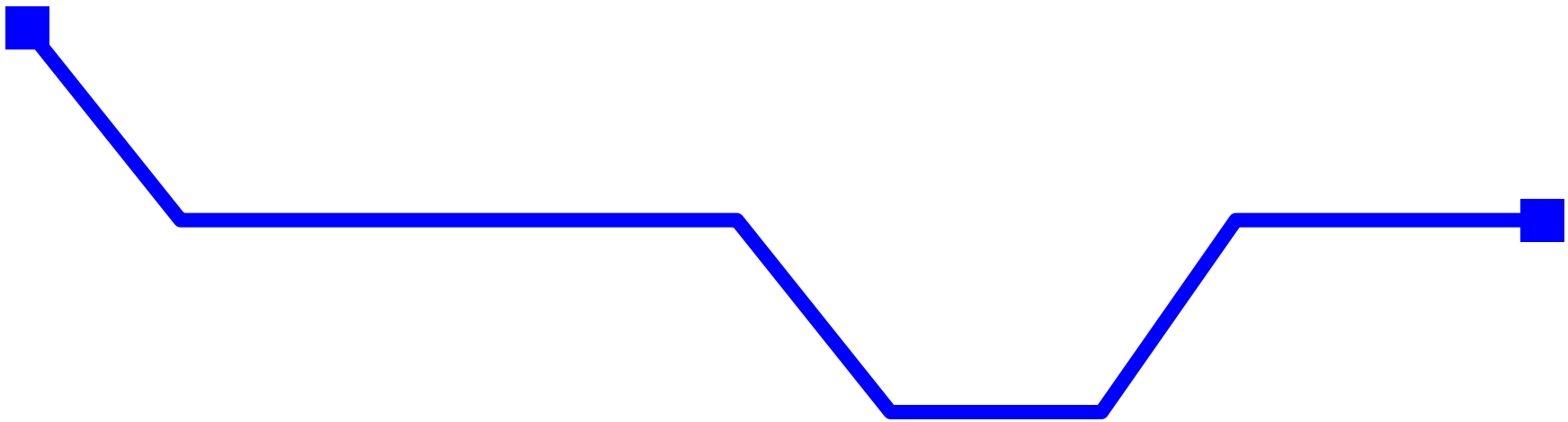

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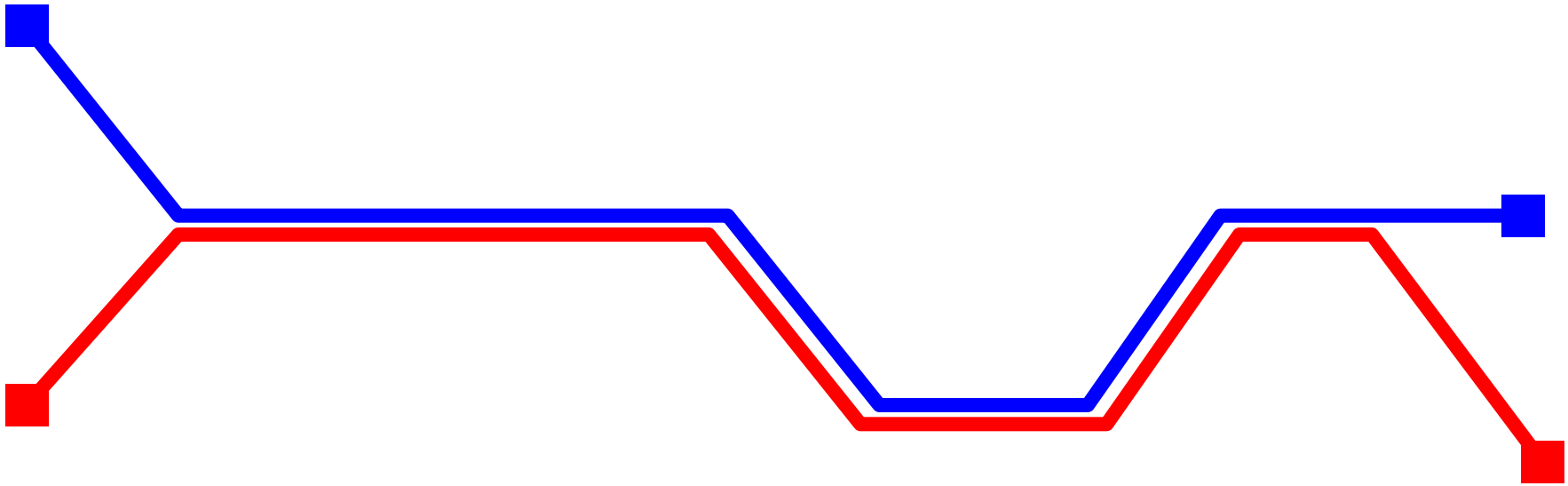

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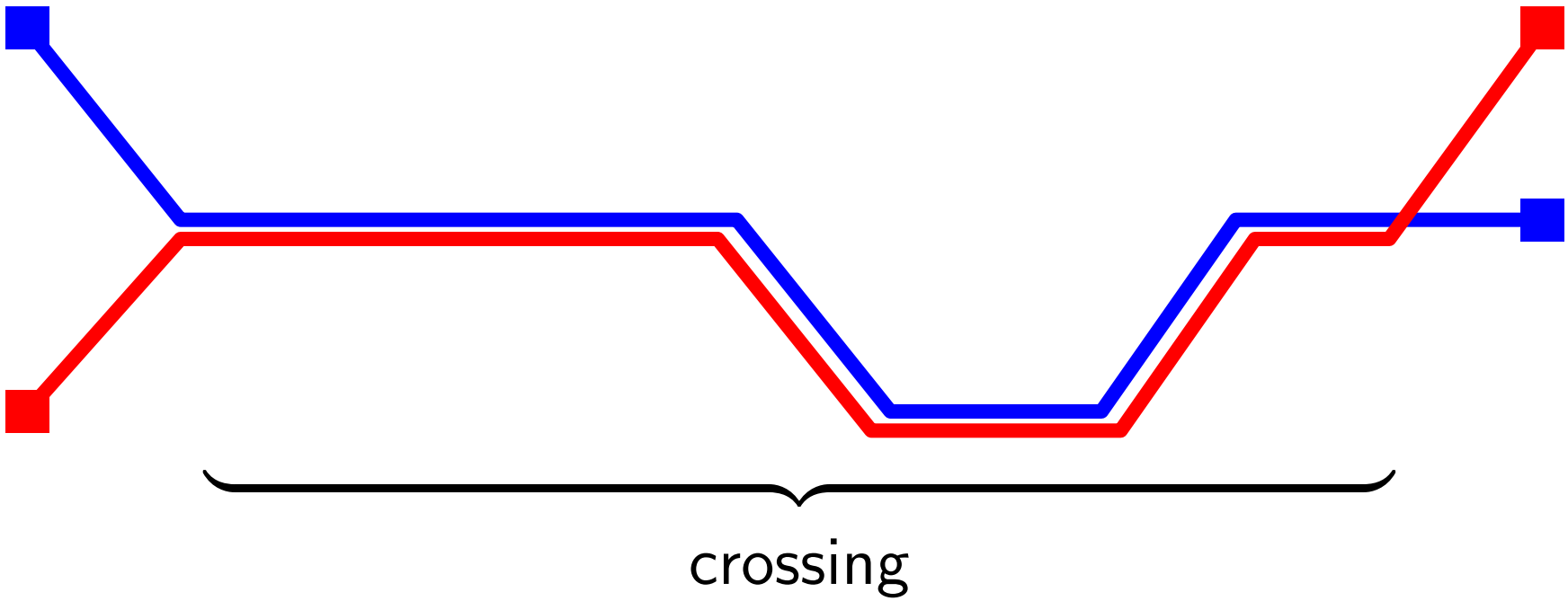

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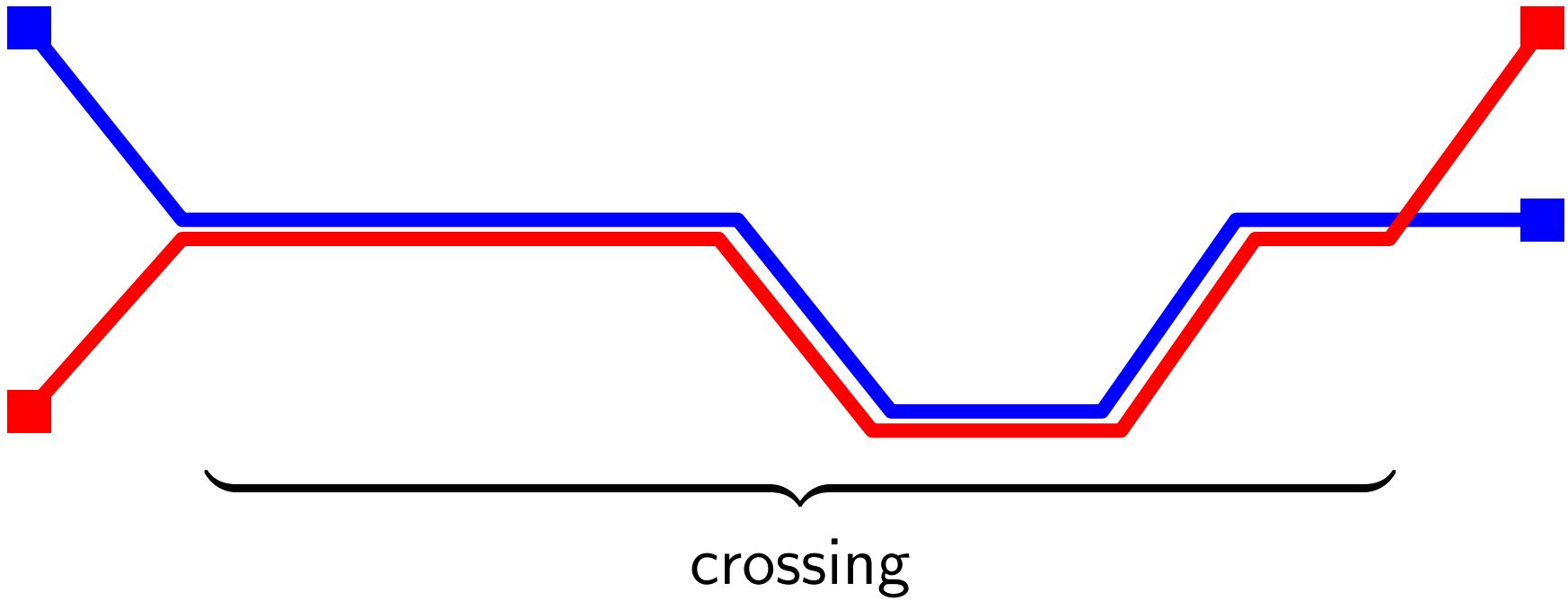

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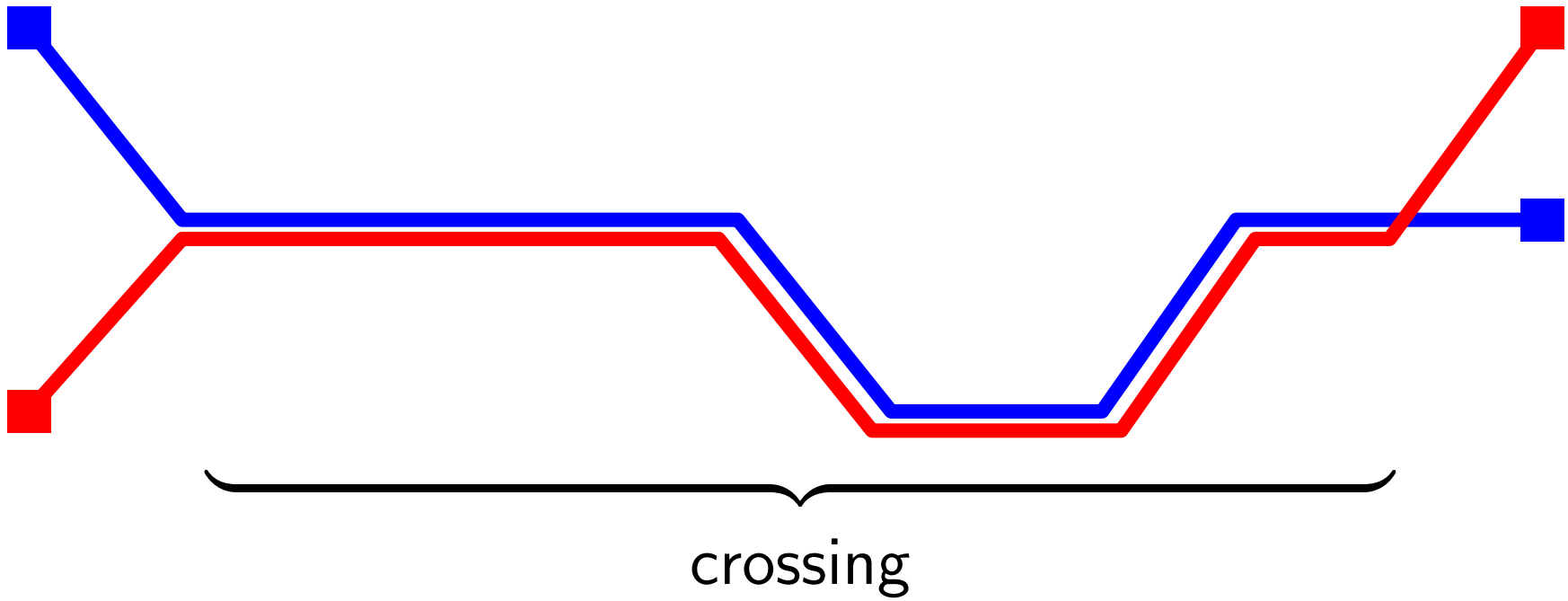
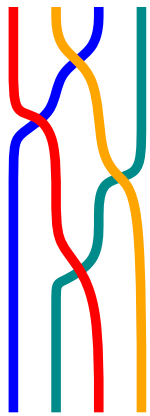


- Crossing Minimization: Solvable in linear time [Pupyrev et al., 2012]
- additional restriction: two lines intersect in a path

New Model: Block Crossings



4 single crossings

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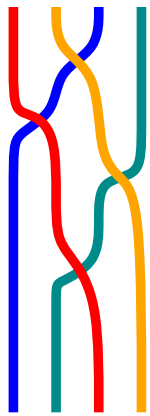

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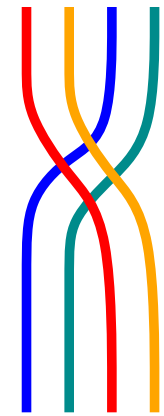

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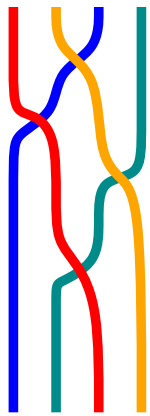

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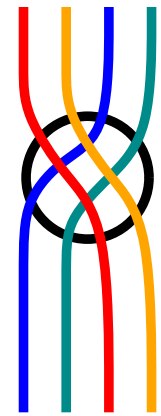

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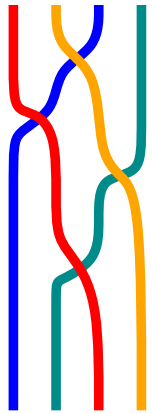

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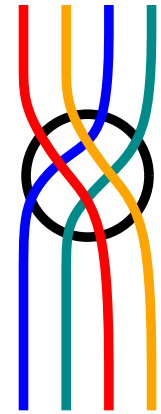

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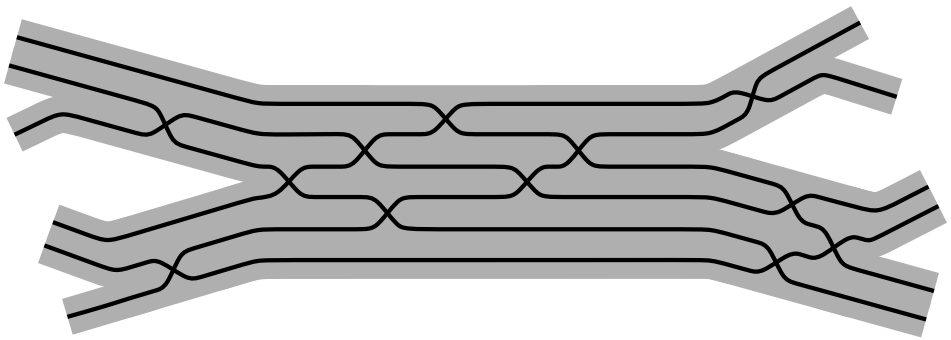

12 single crossings

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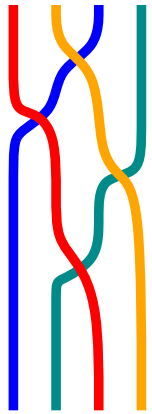

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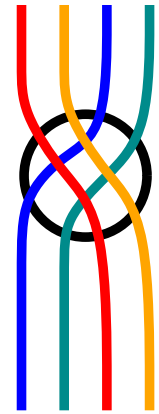

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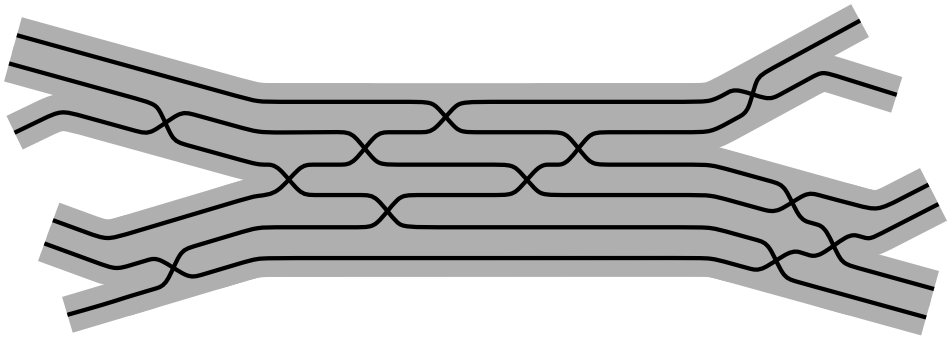

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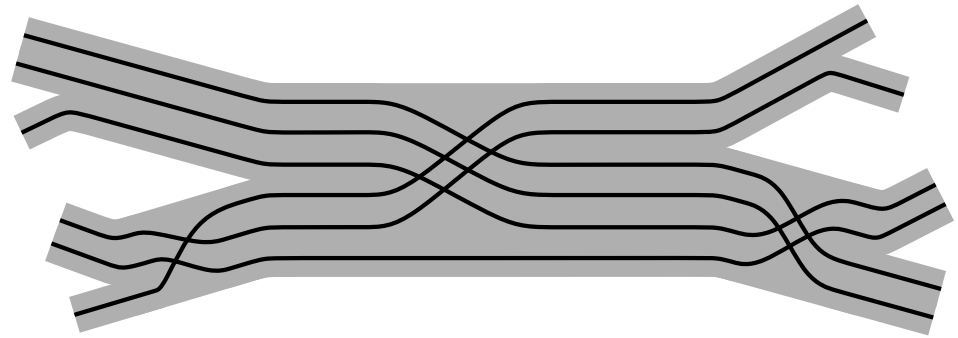

3 block crossings

Block Crossing Minimization

New problem variants:

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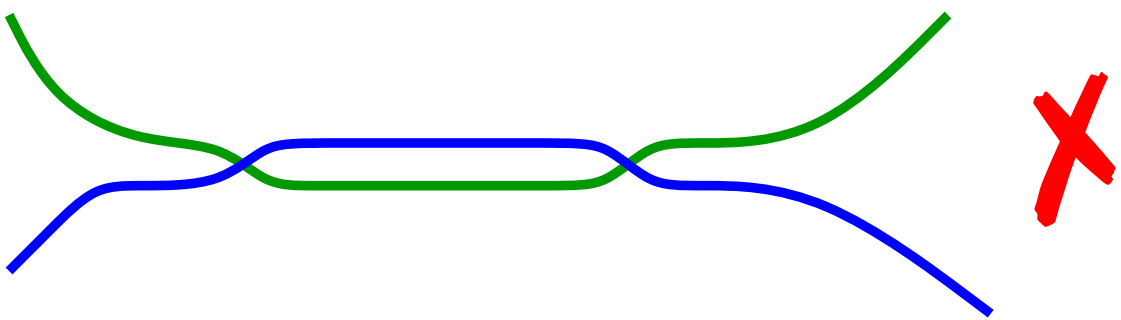
no double crossings!



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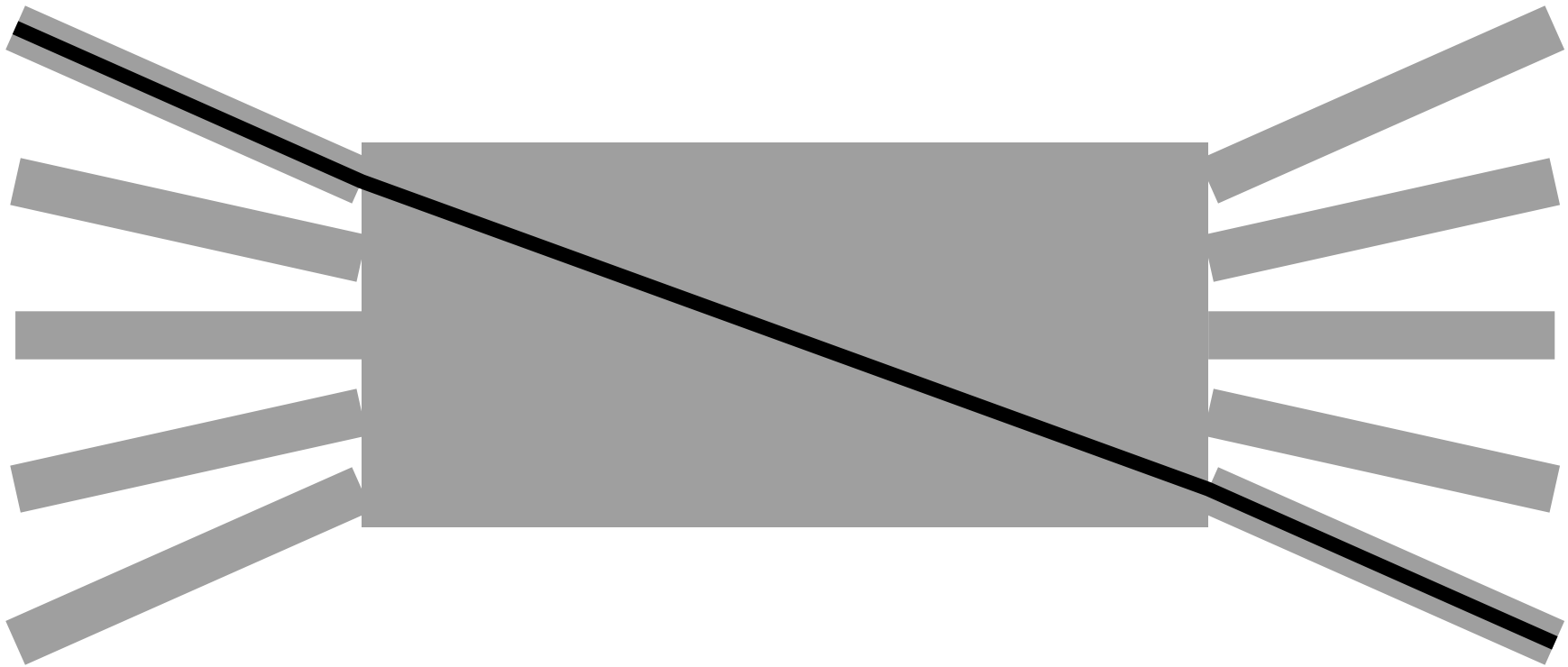

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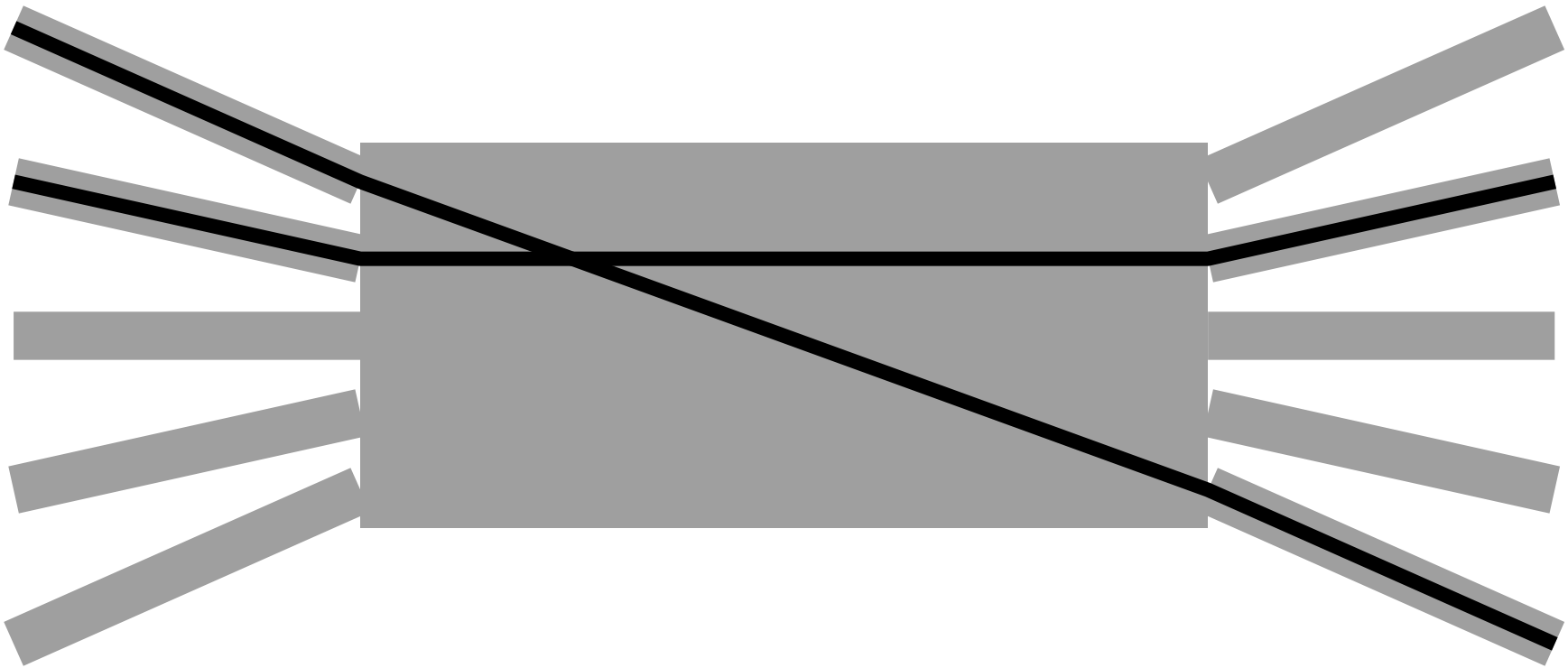

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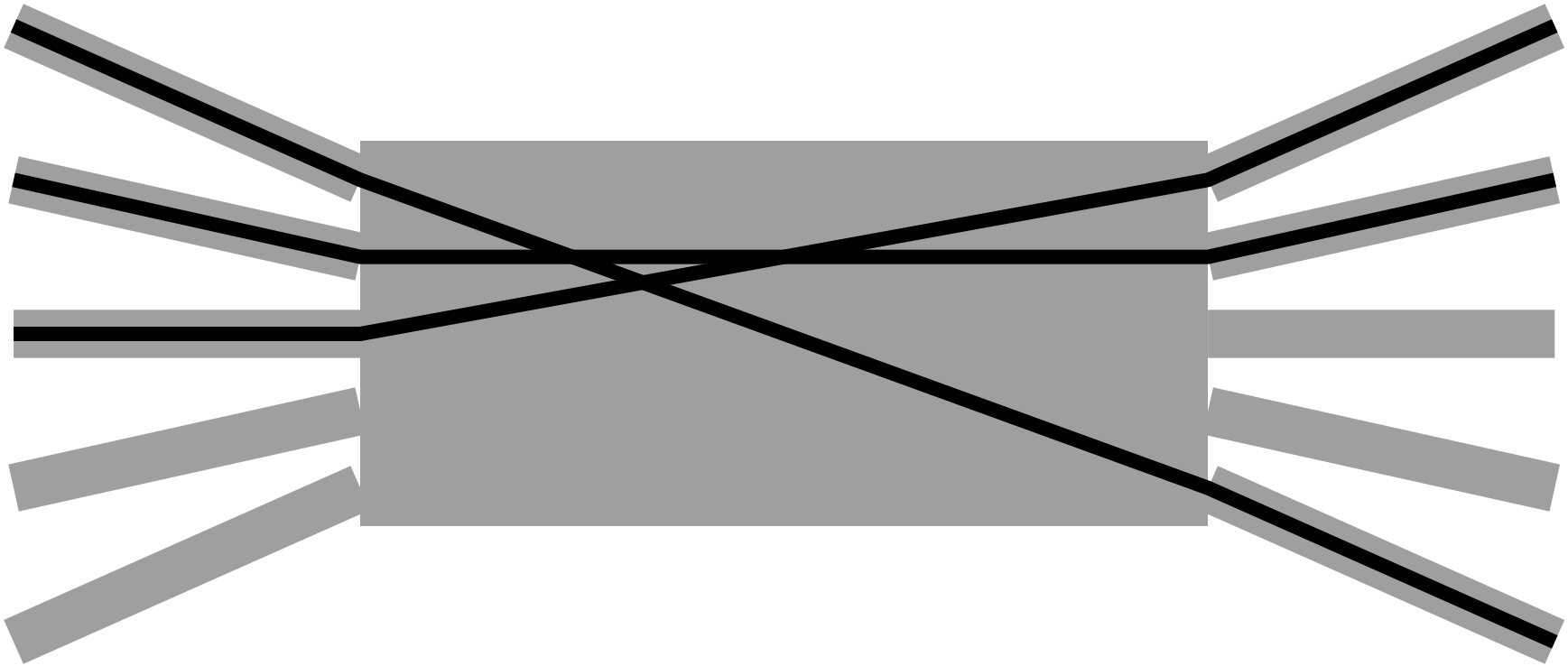

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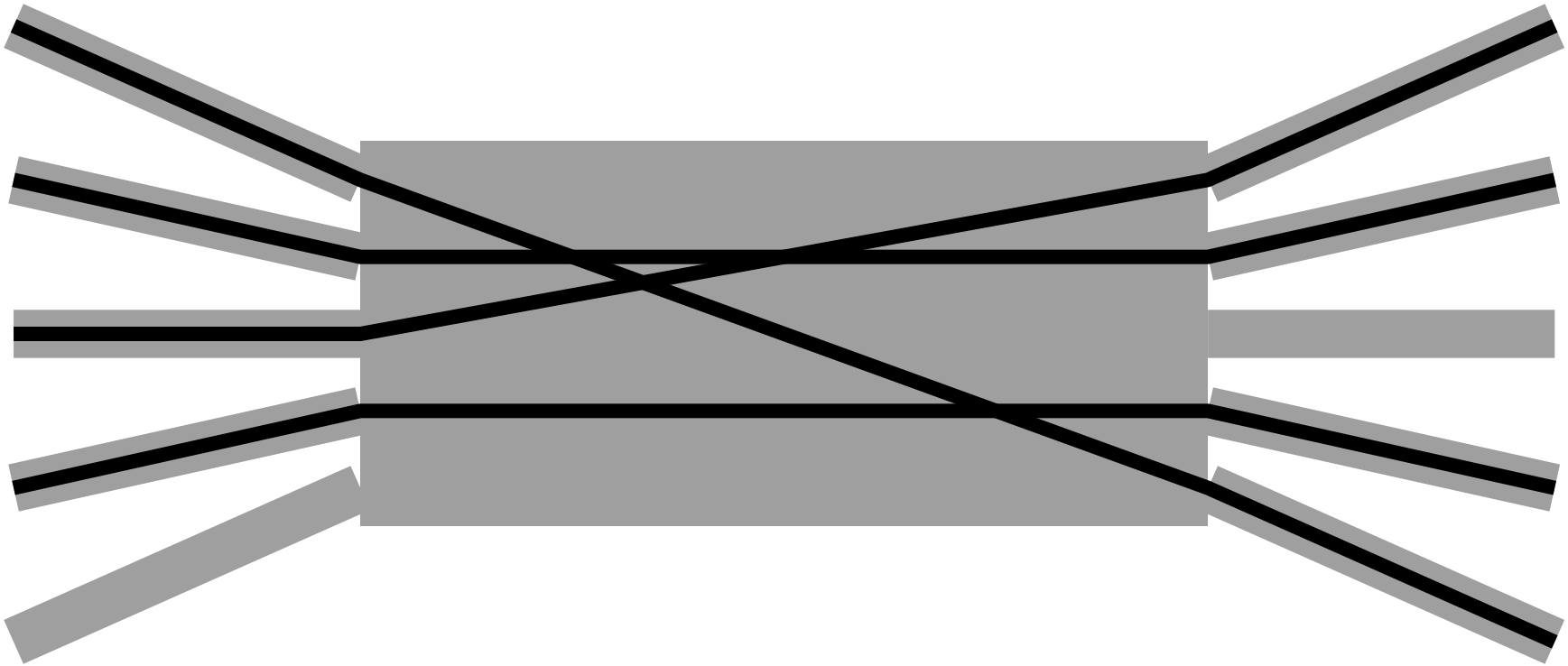

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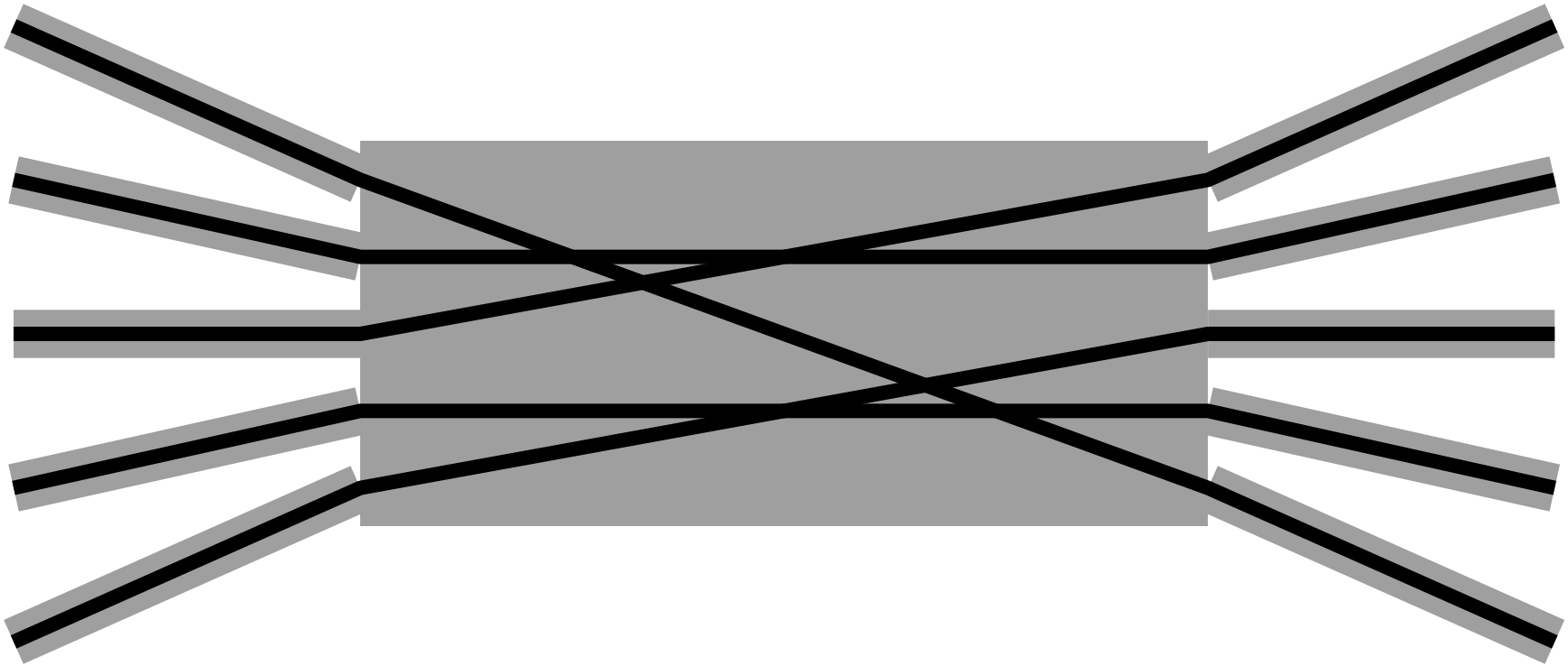

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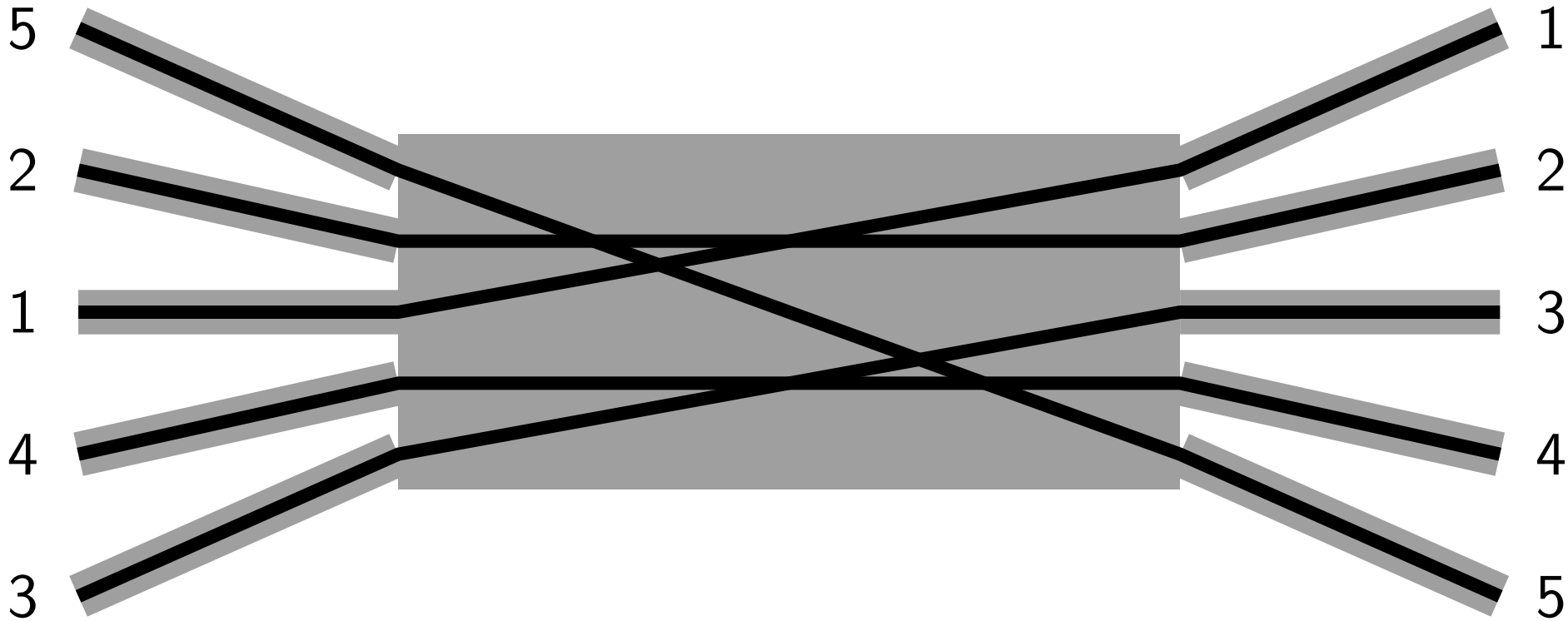

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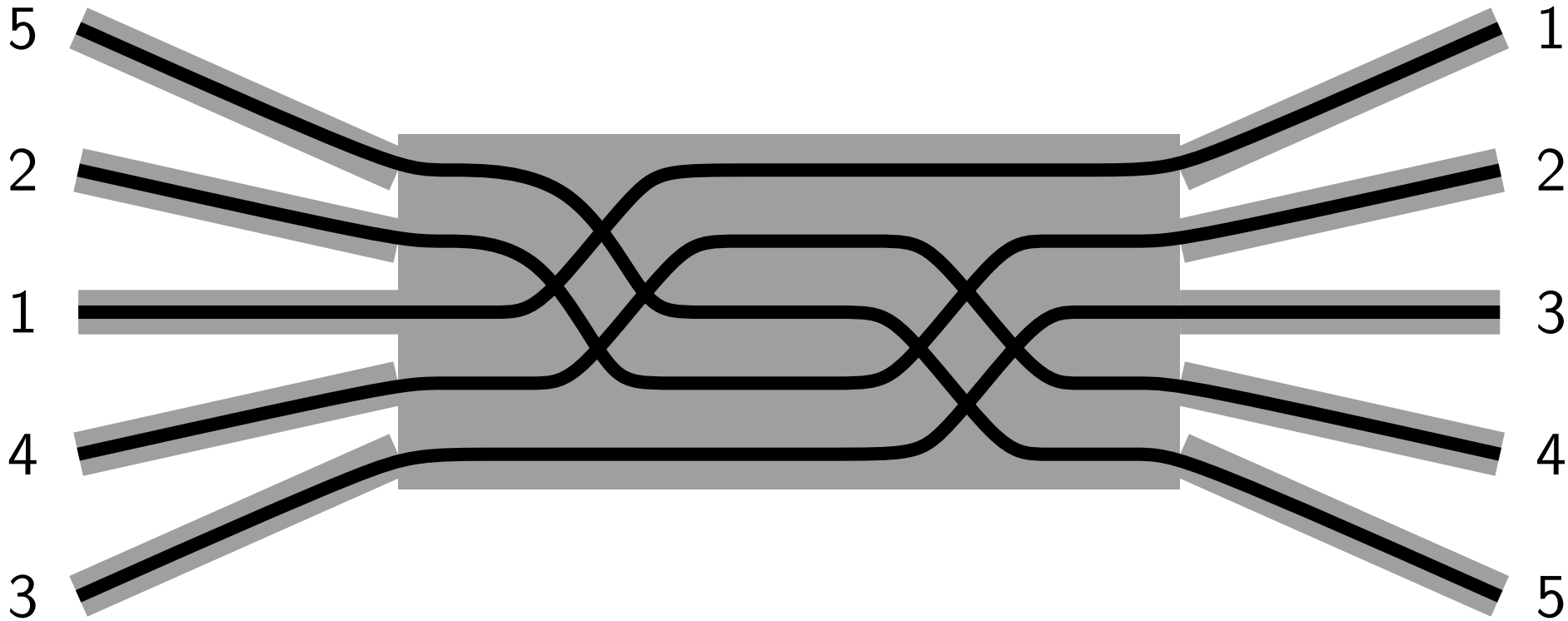

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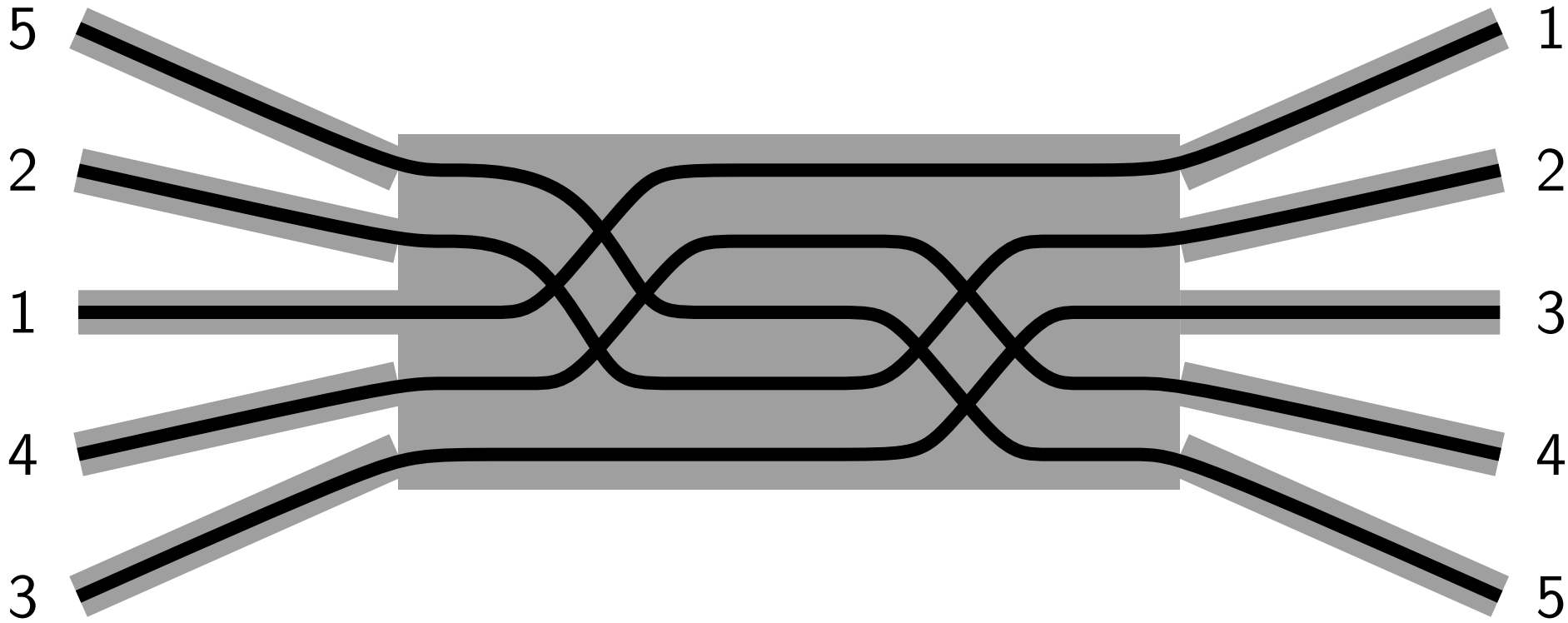

- Equivalent to *Sorting by Transpositions* for permutations

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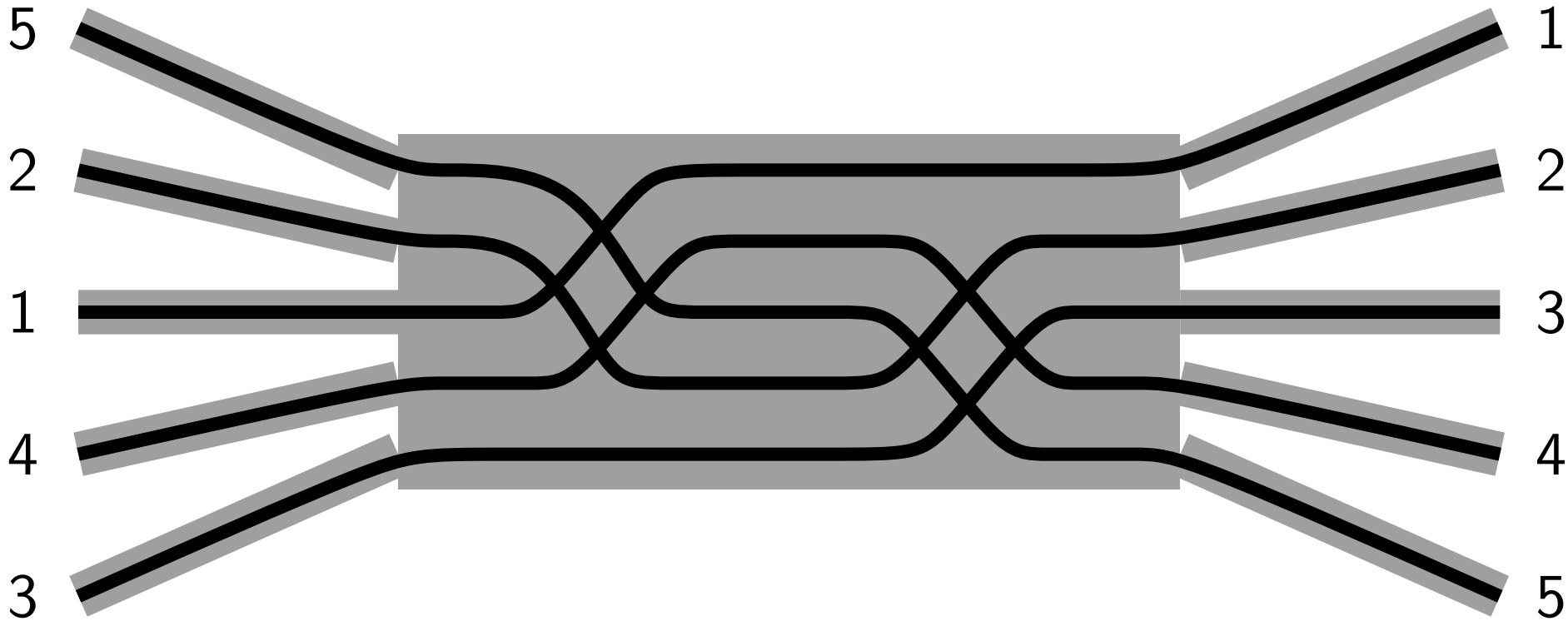

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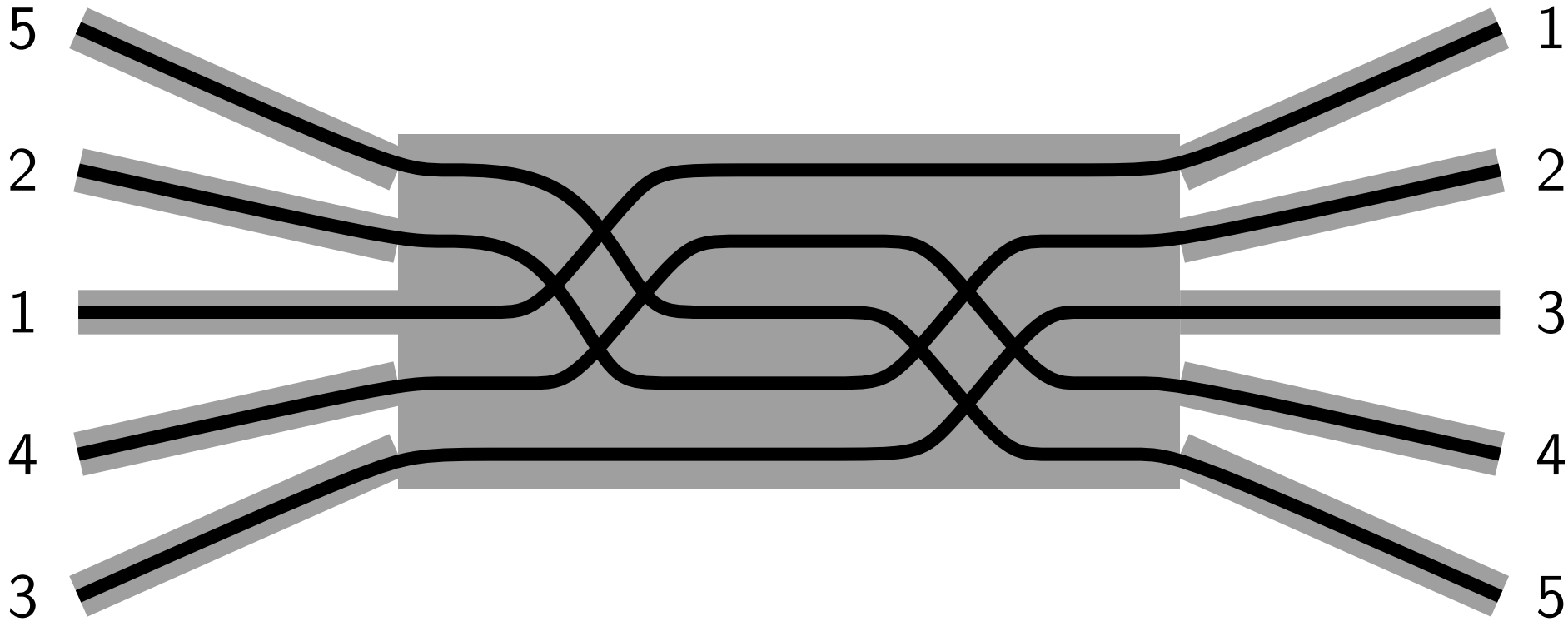

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- simple 3-approximation [Bafna, Pevzner, 1998]

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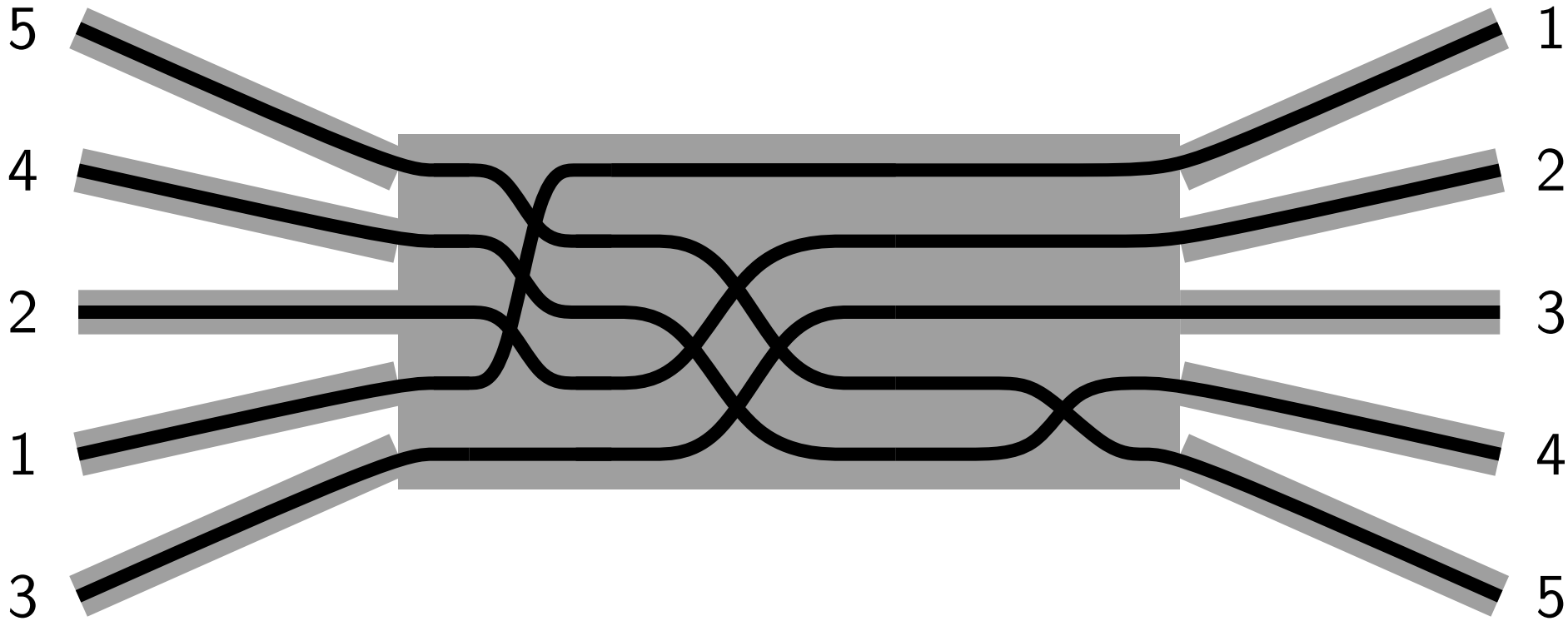

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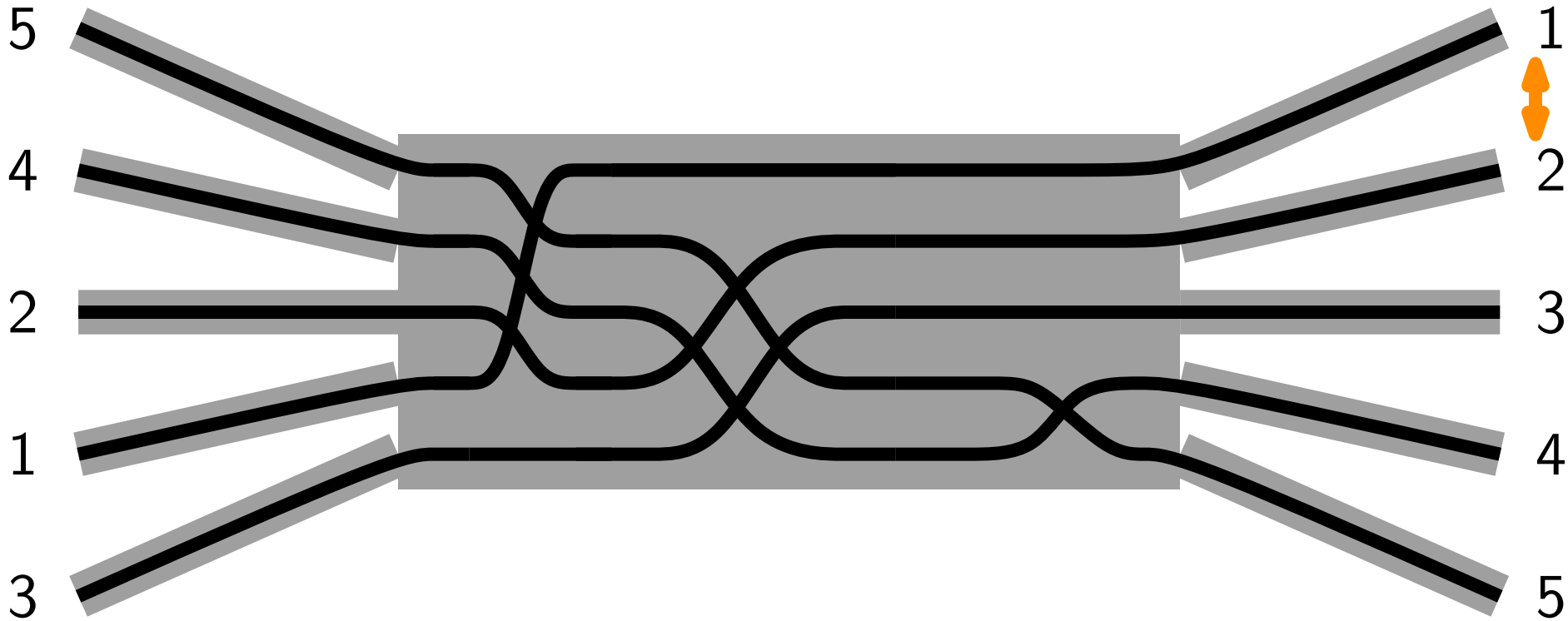

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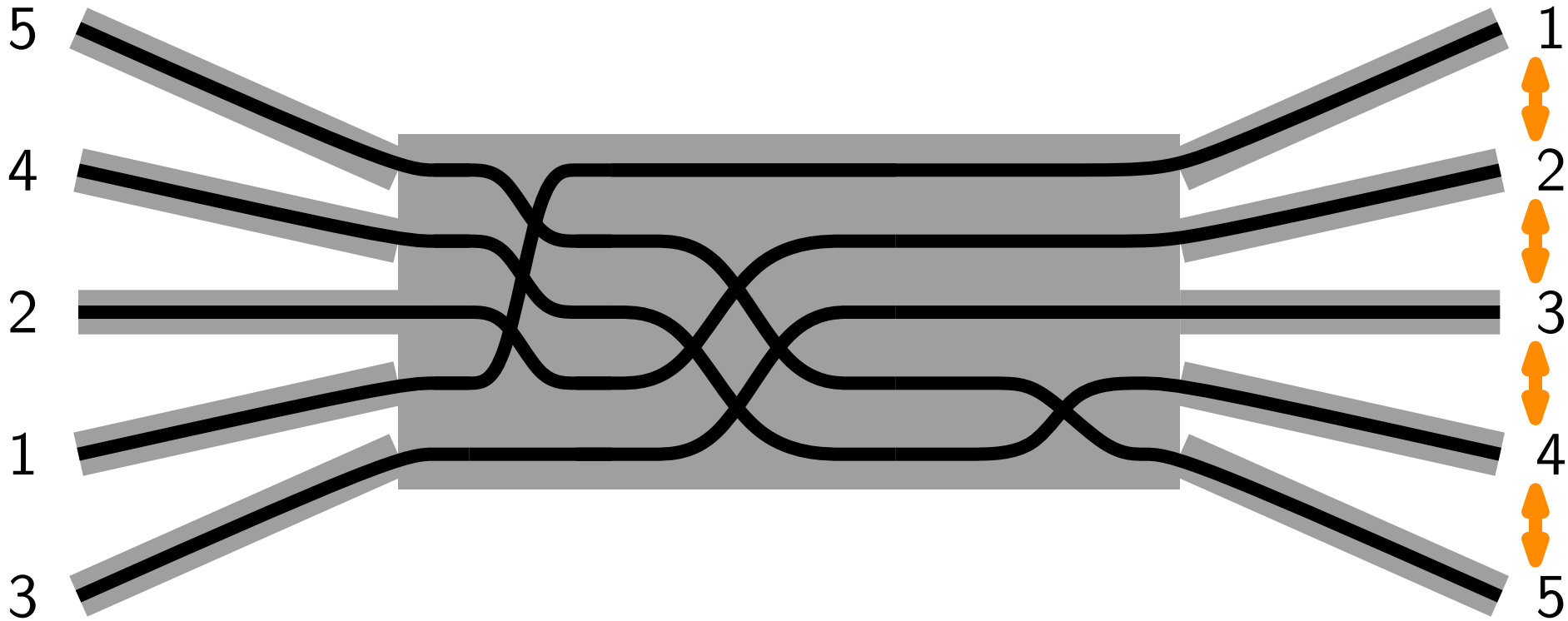

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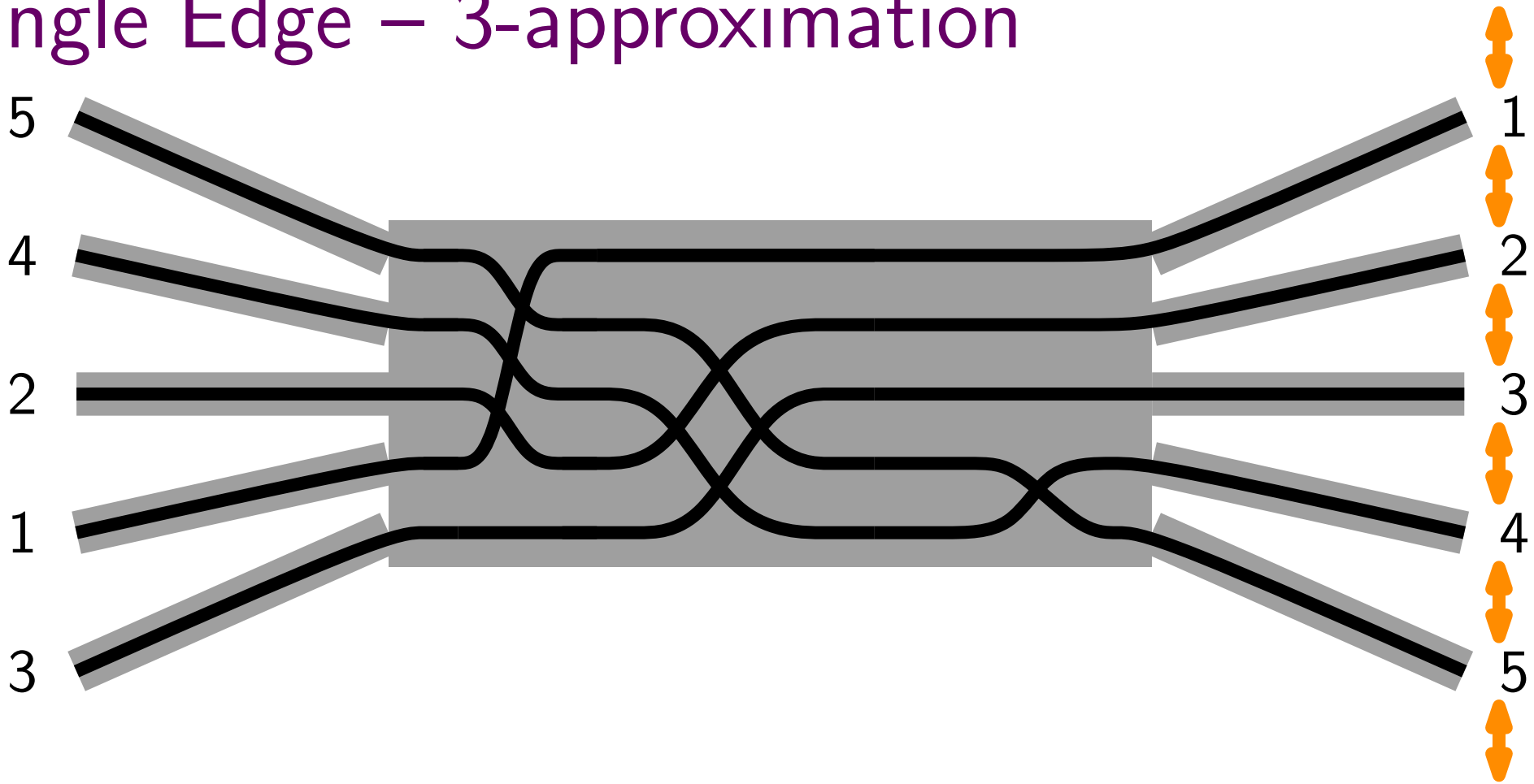

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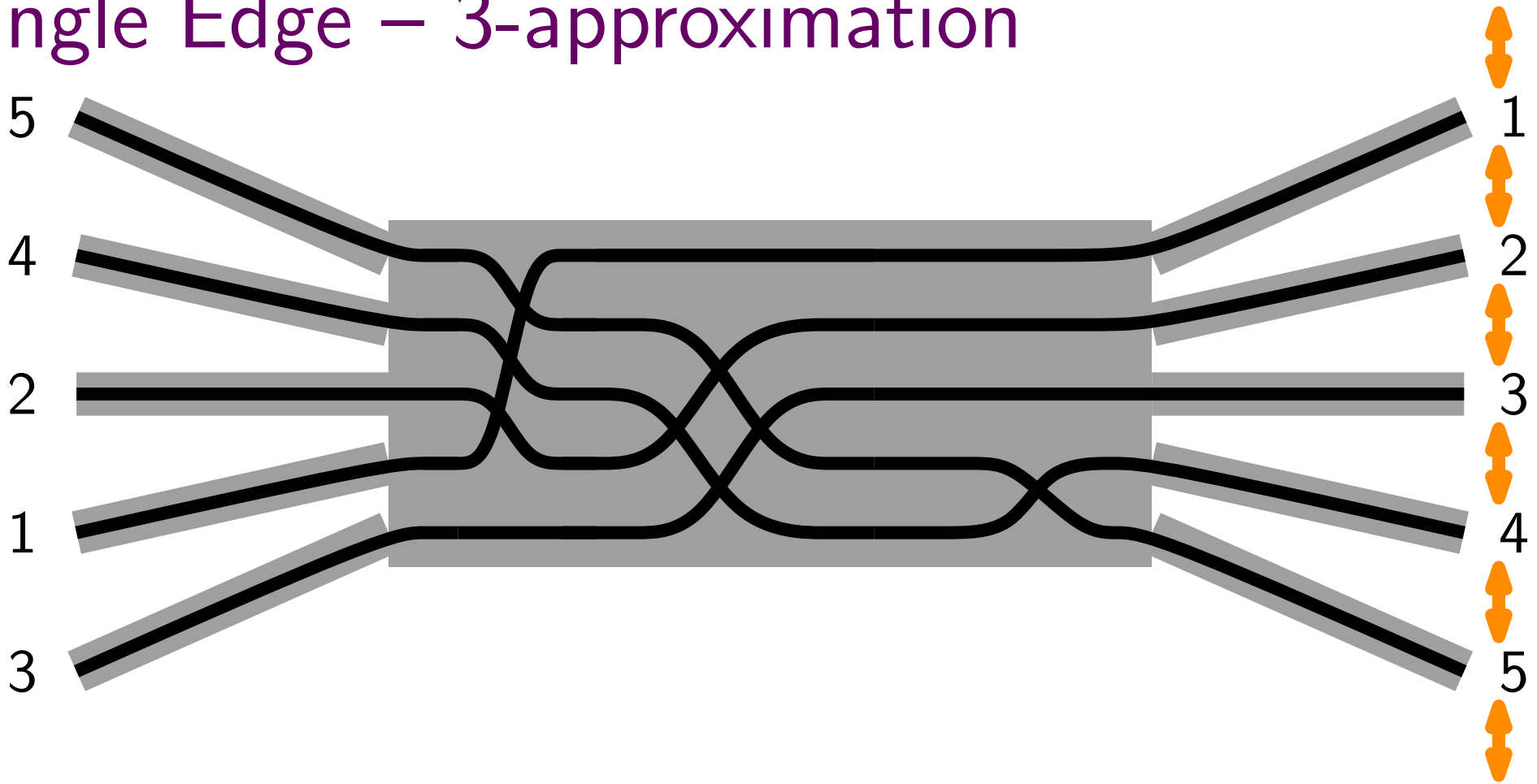

$|L| + 1$ *good pairs*

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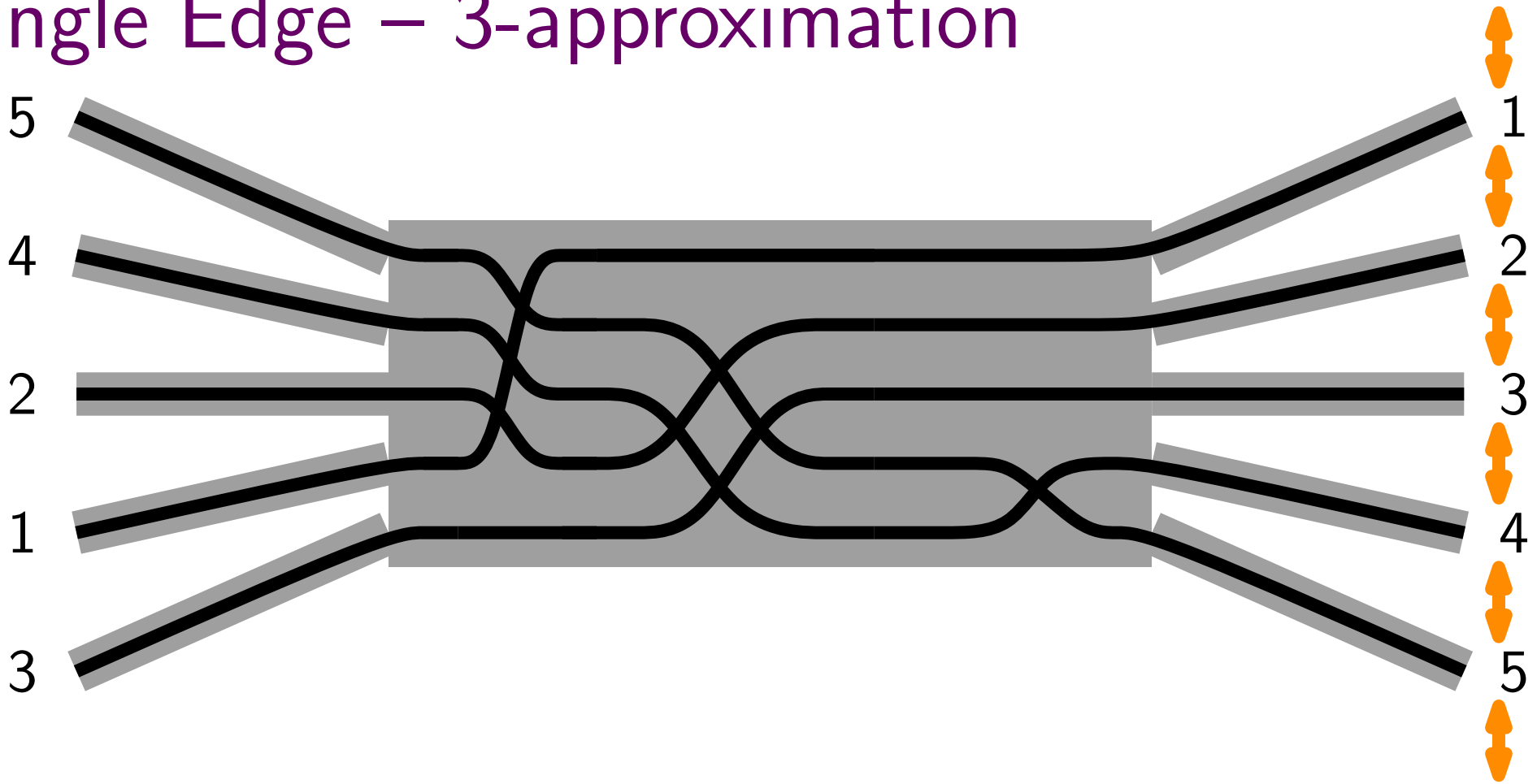

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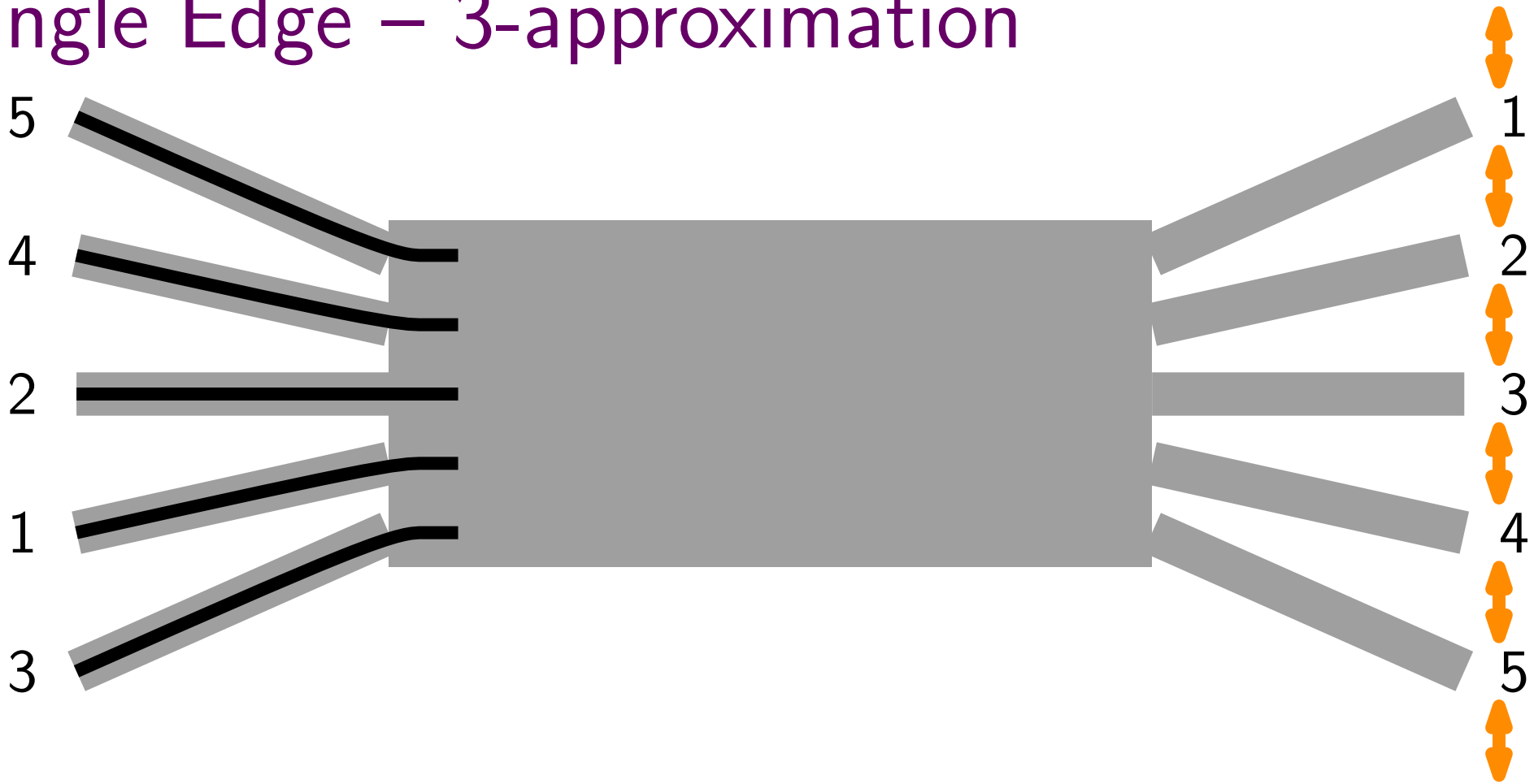

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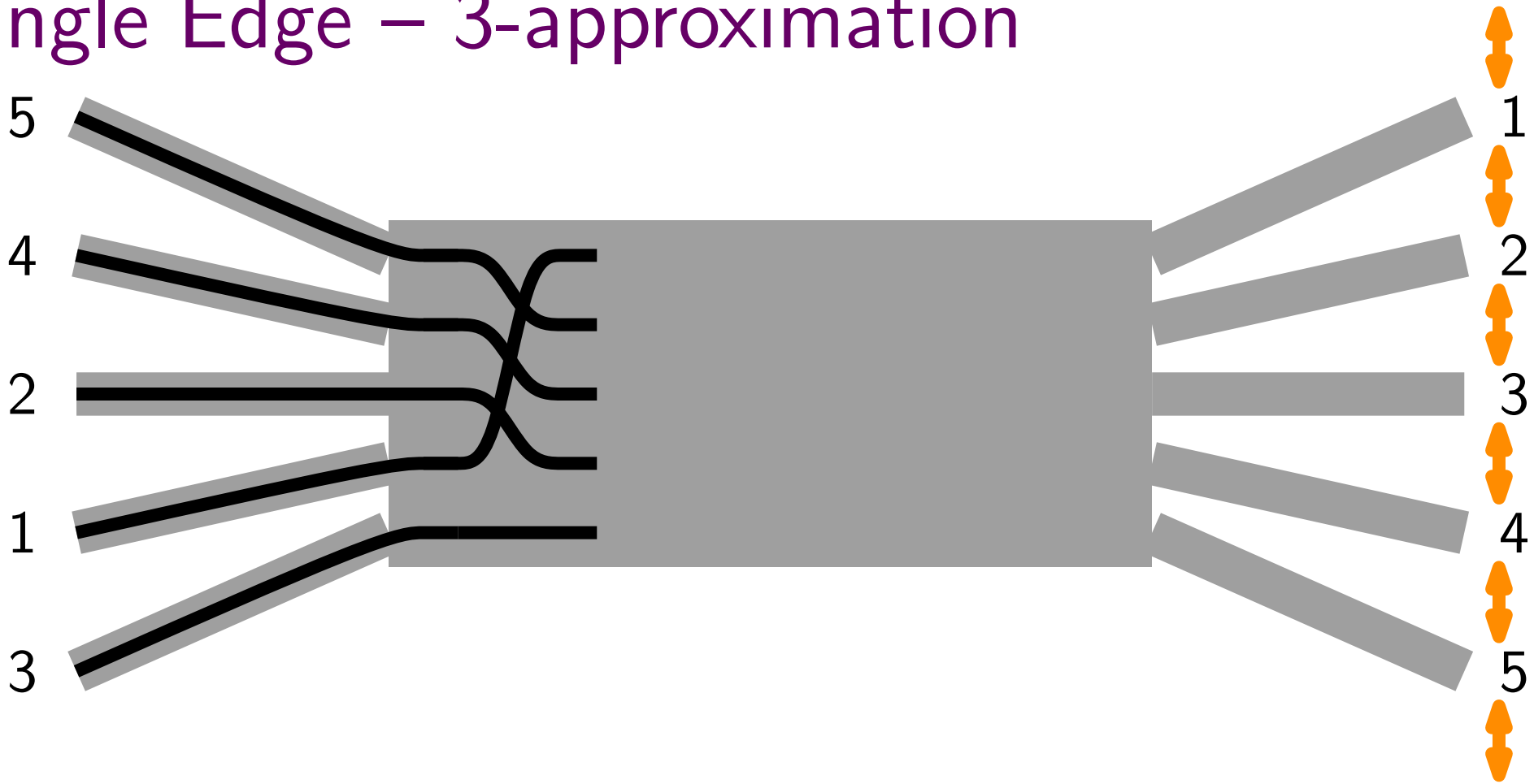

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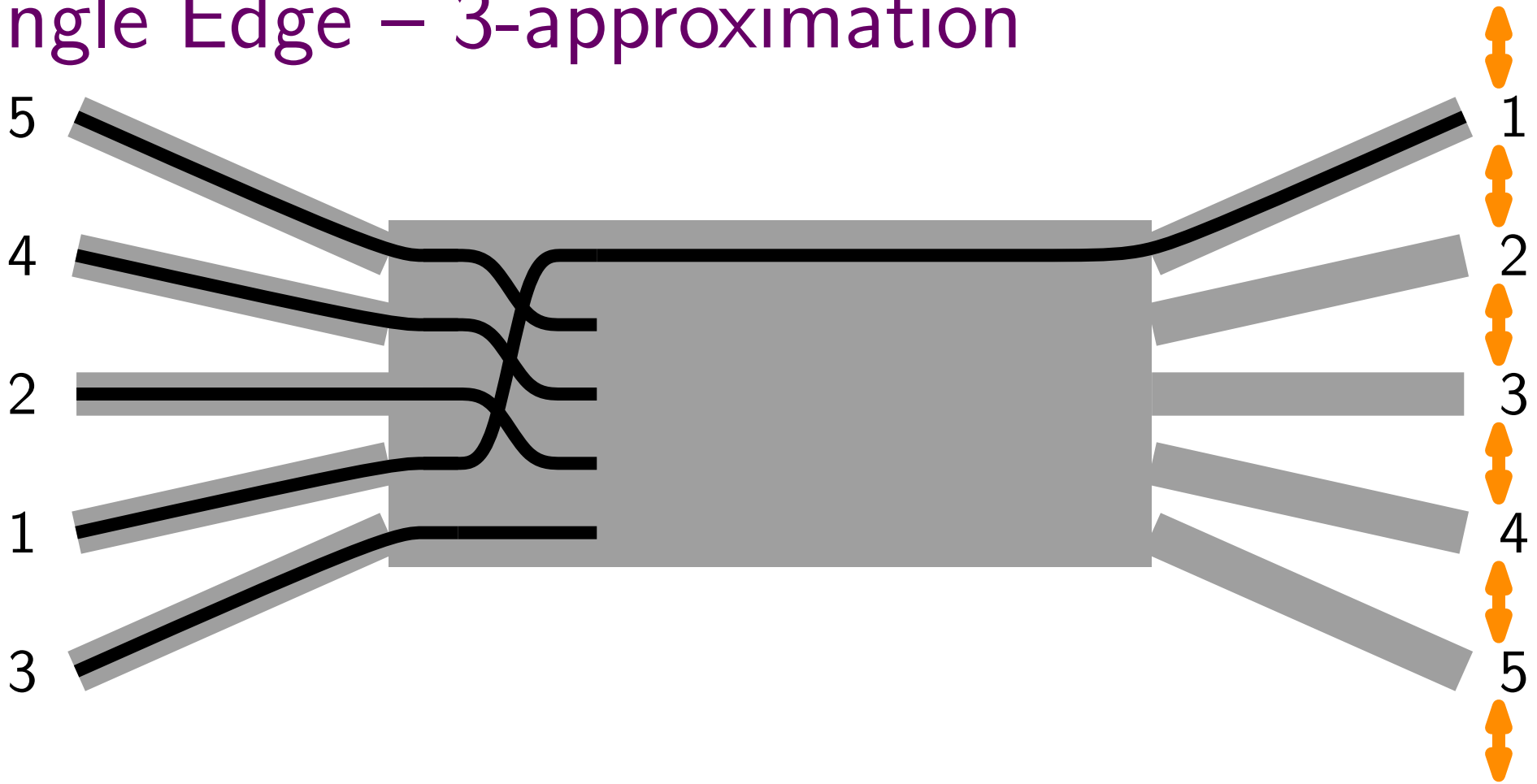

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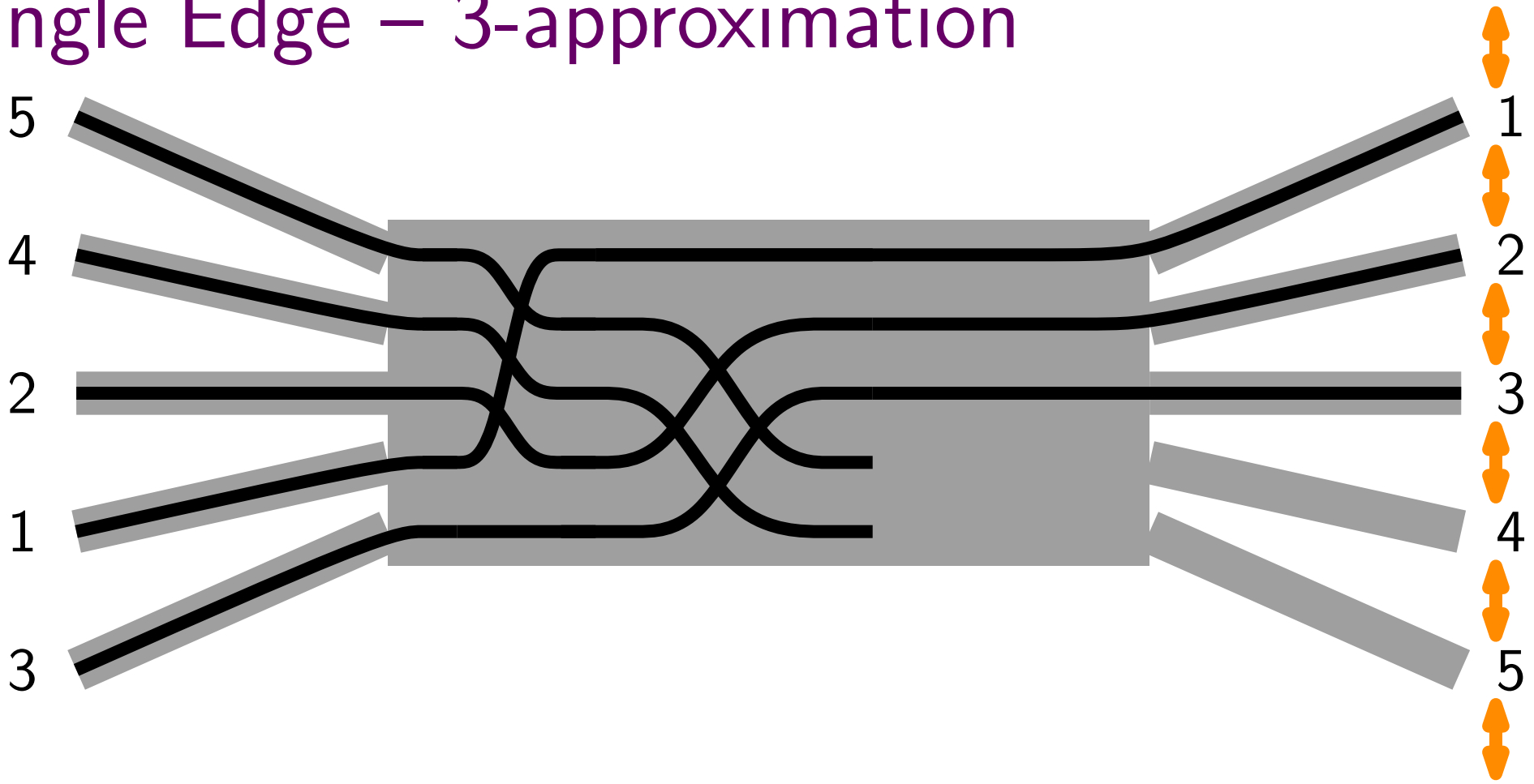

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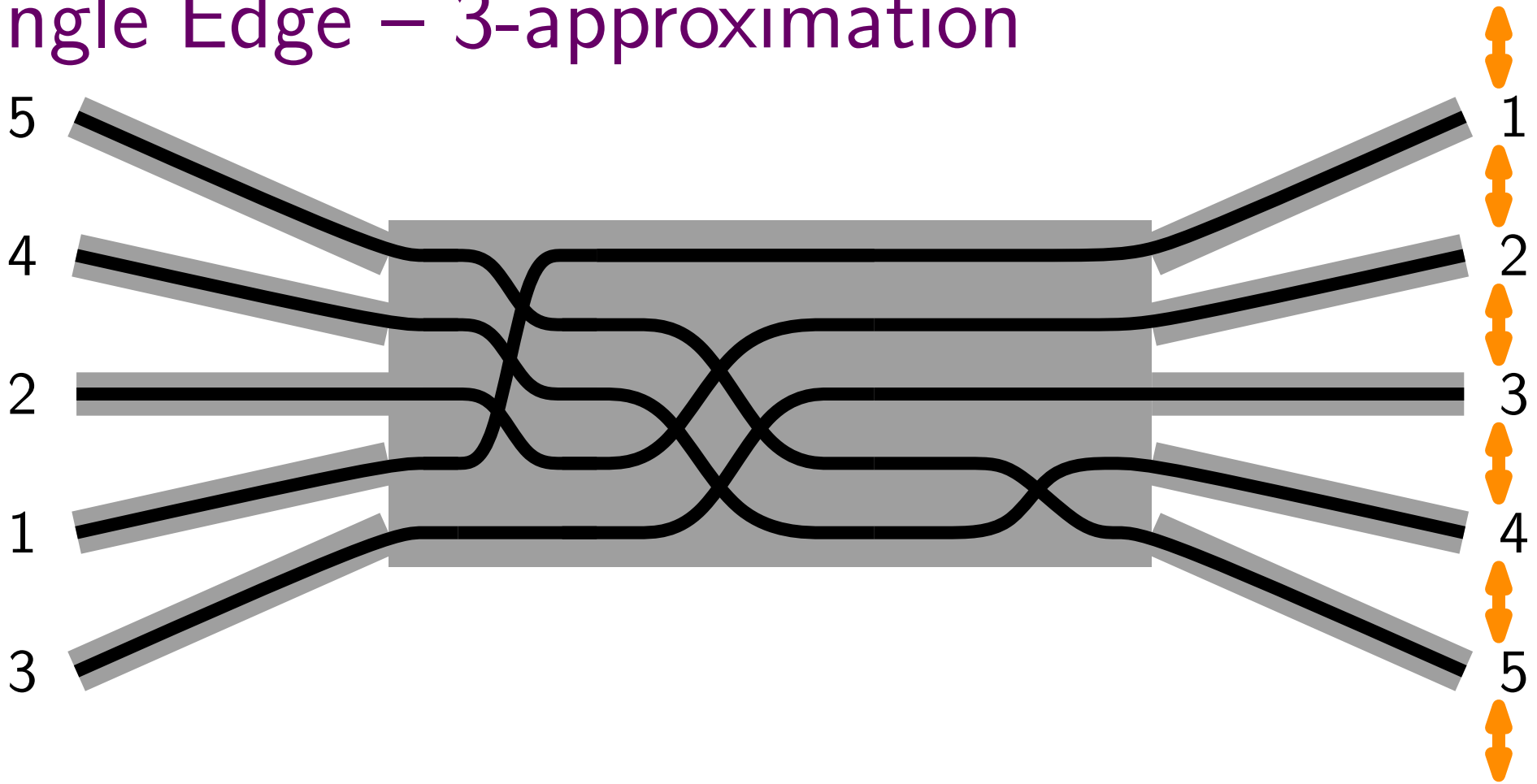

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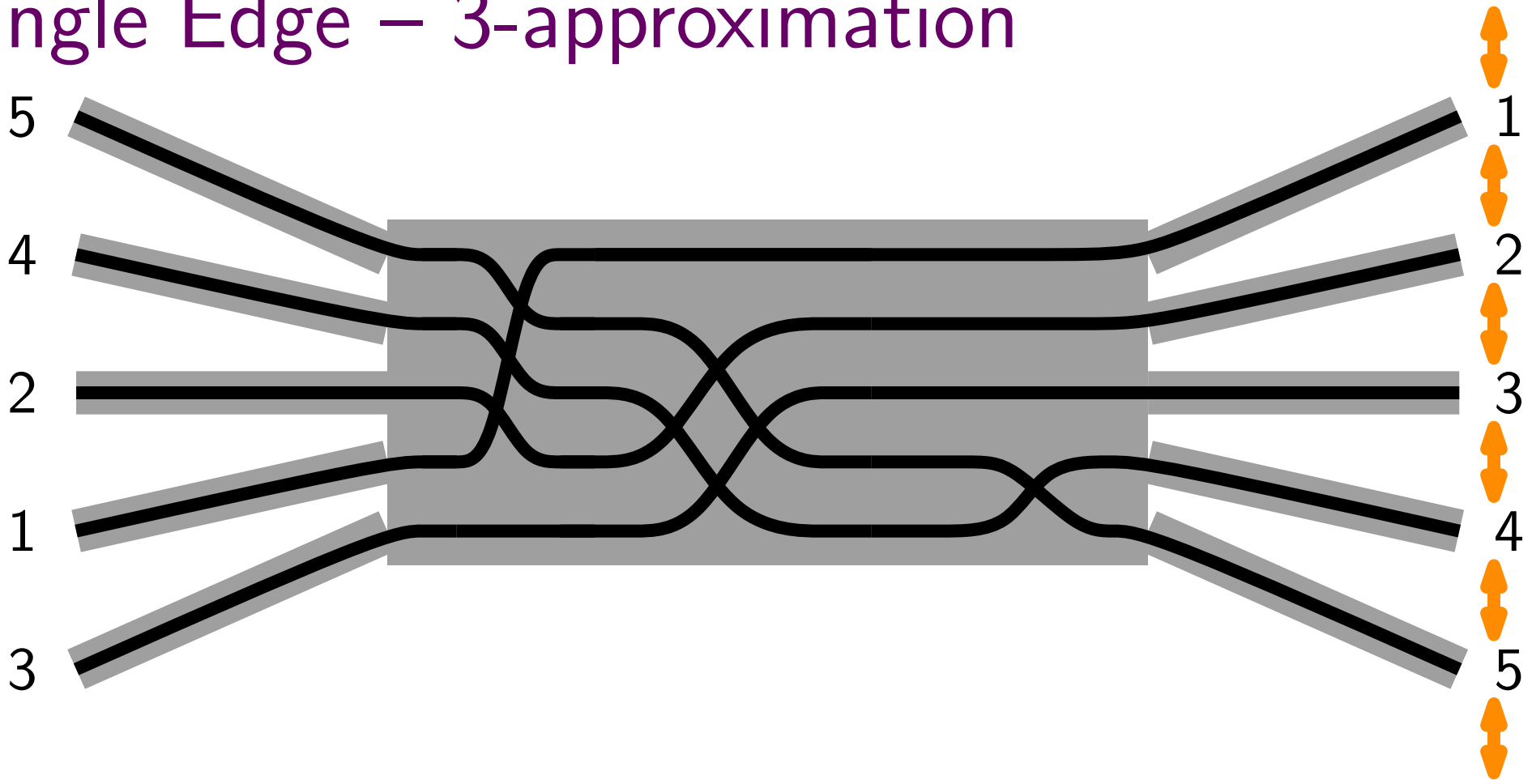

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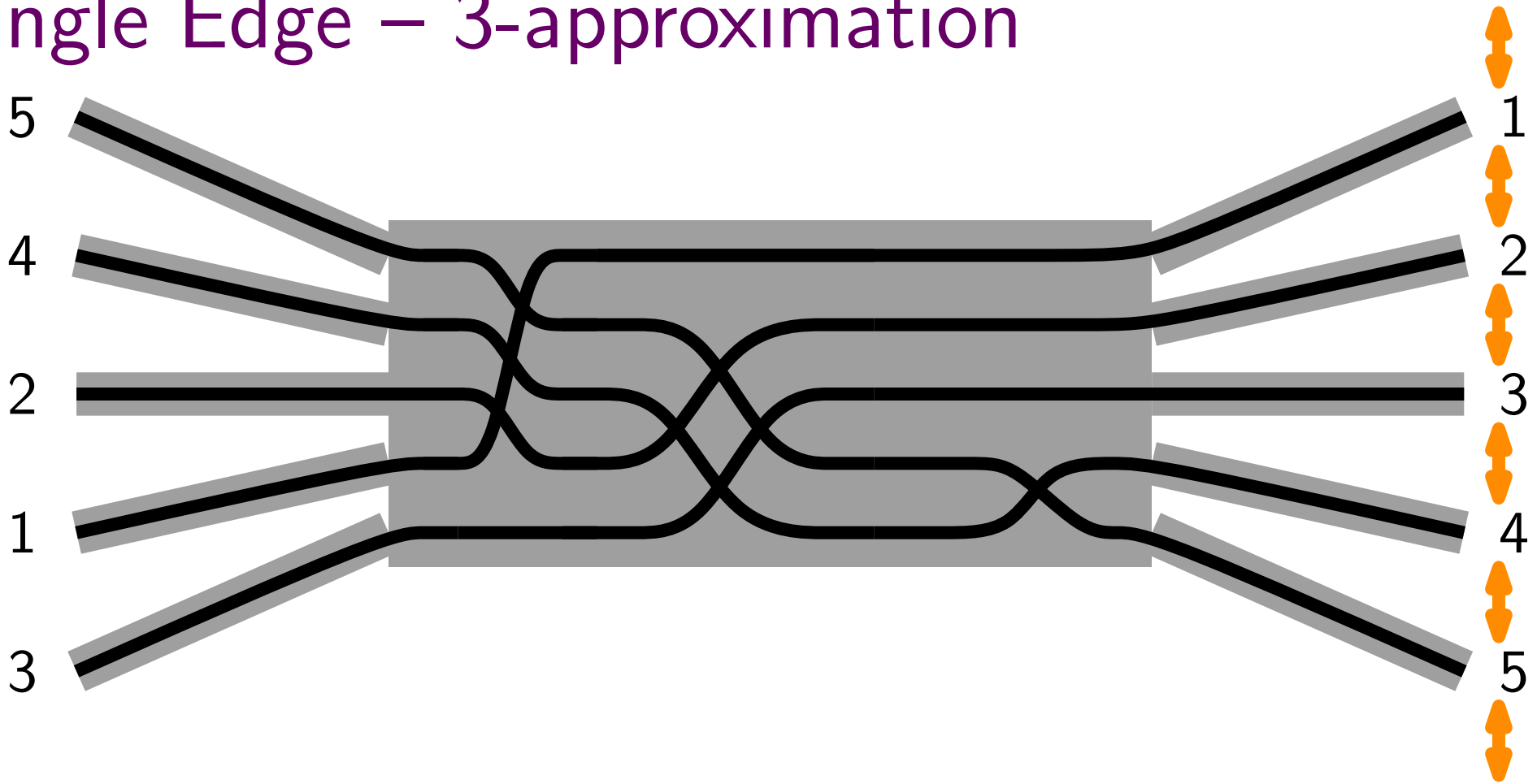

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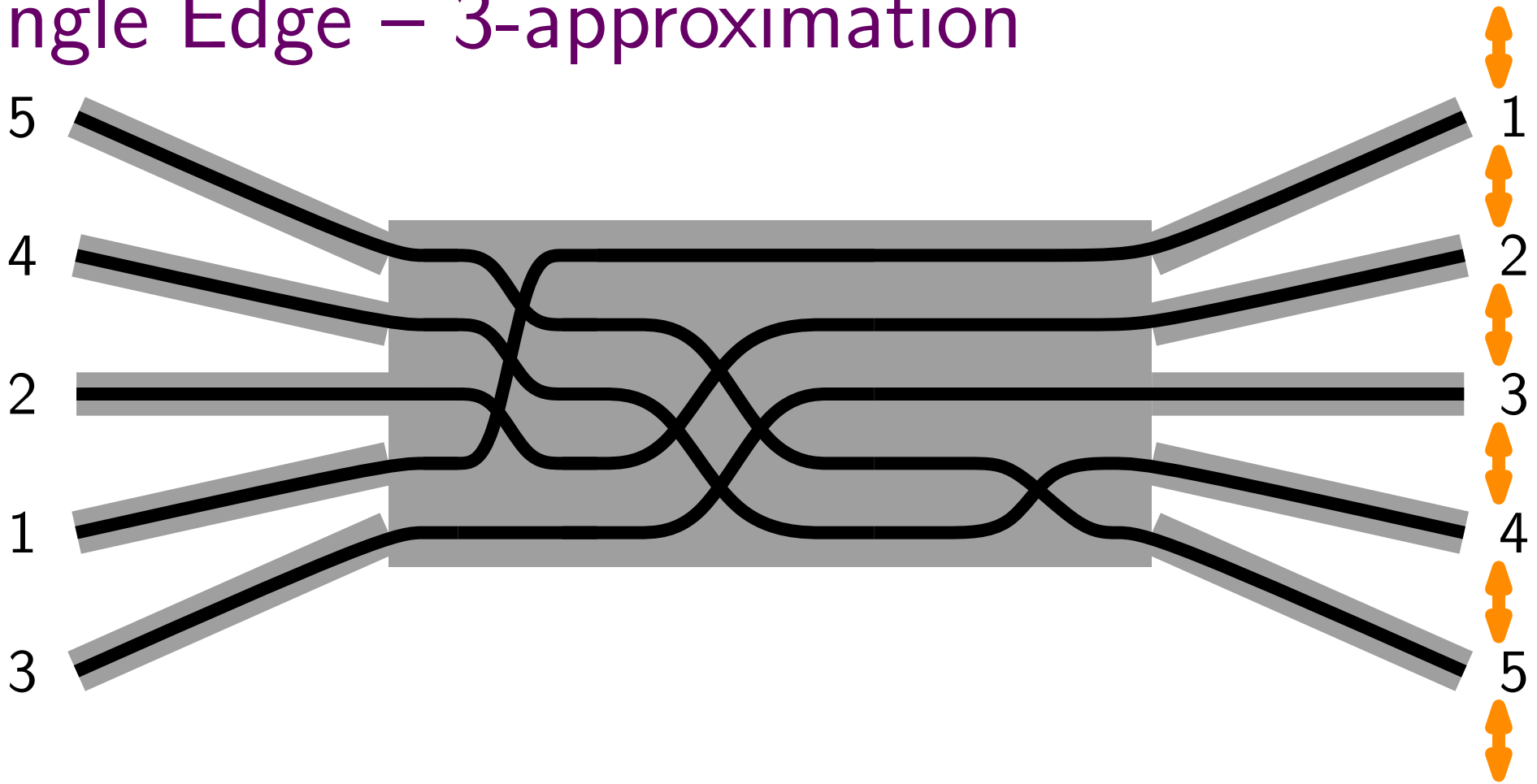

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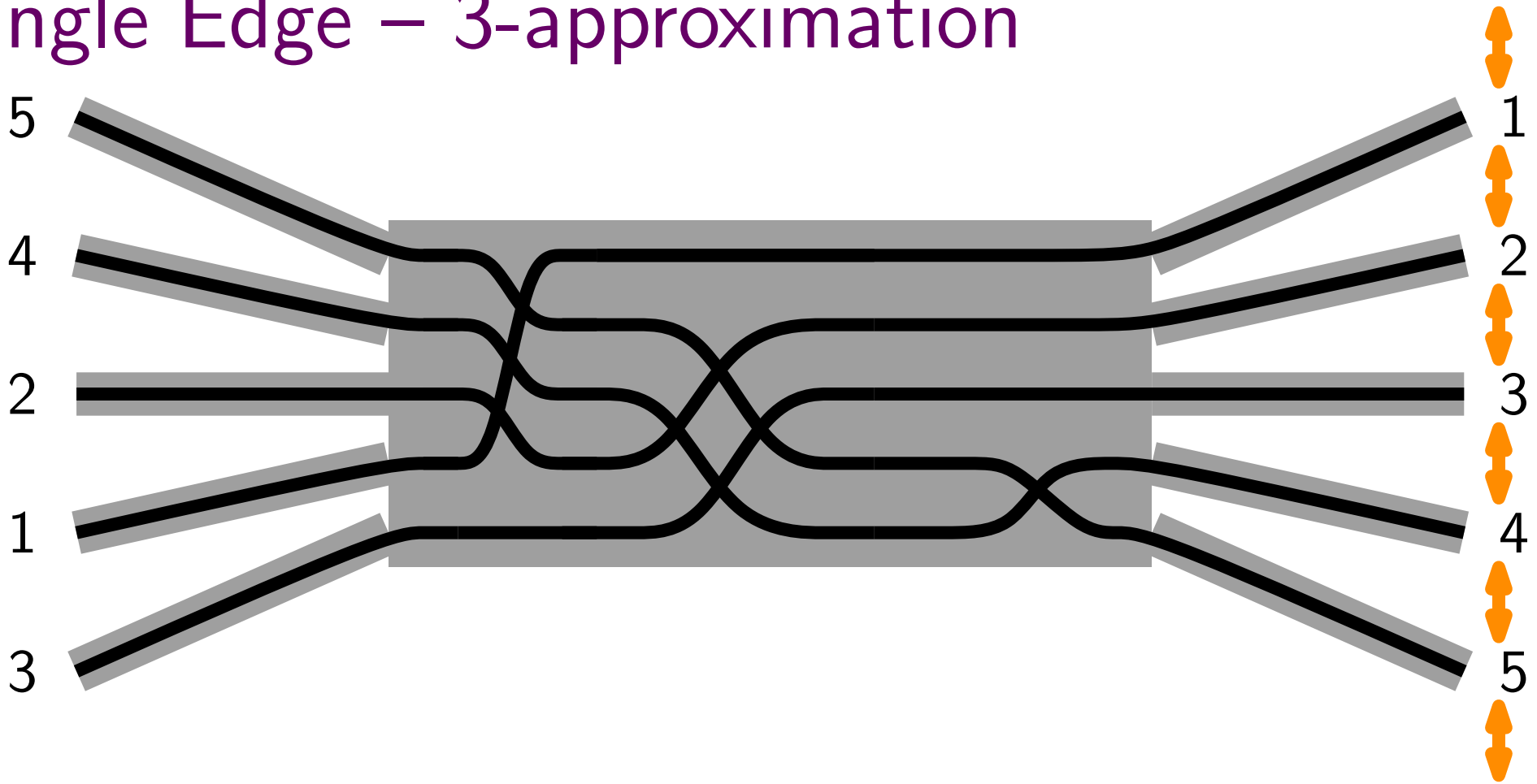


no good pair
for *simple* permutation

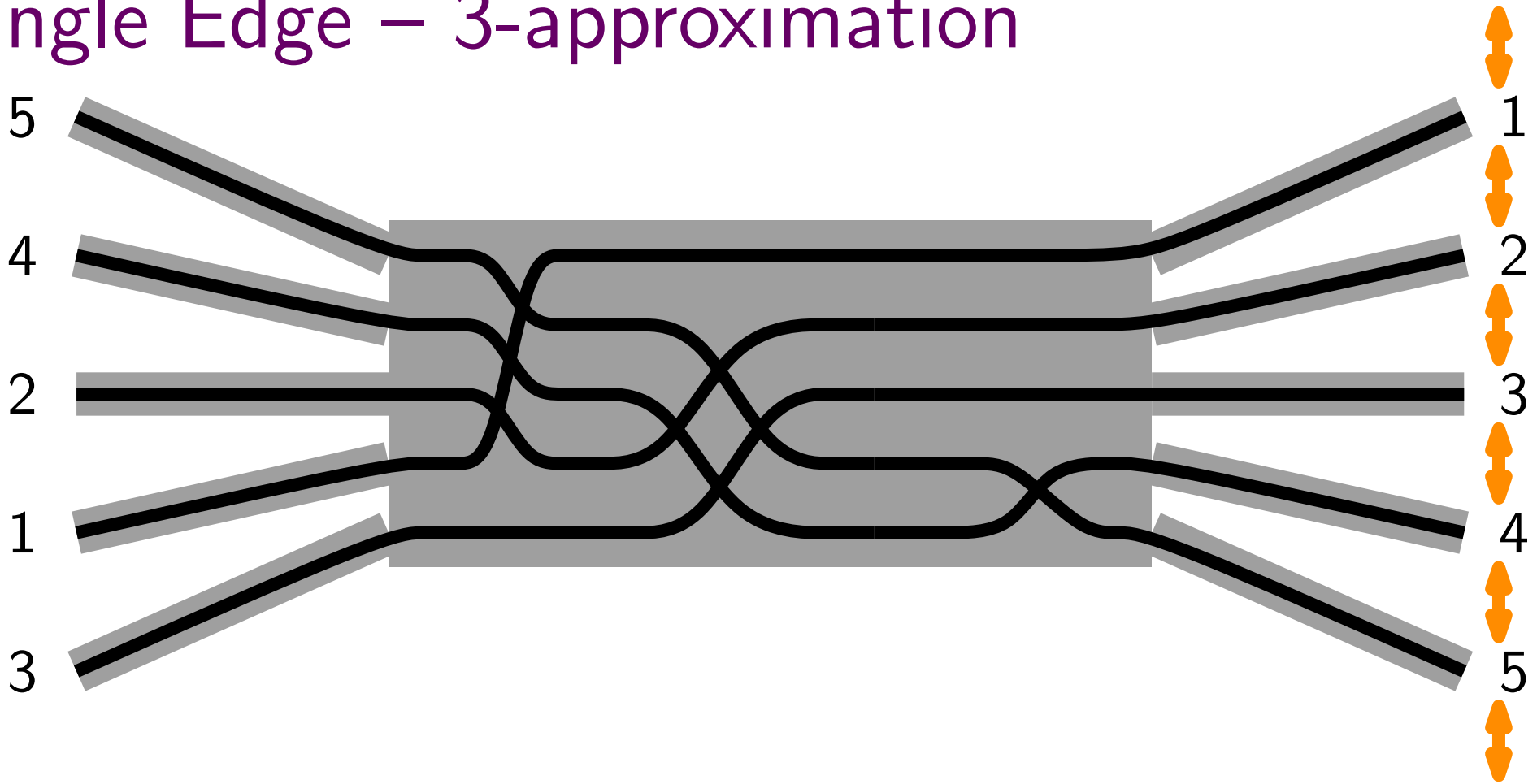
$\leq |L| - 1$ crossings

$|L| + 1$ *good pairs*

≤ 3 good pairs can be created per crossing

3-approx.

Single Edge – 3-approximation



no good pair
for *simple* permutation

$\leq |L| - 1$ crossings

$|L| + 1$ *good pairs*

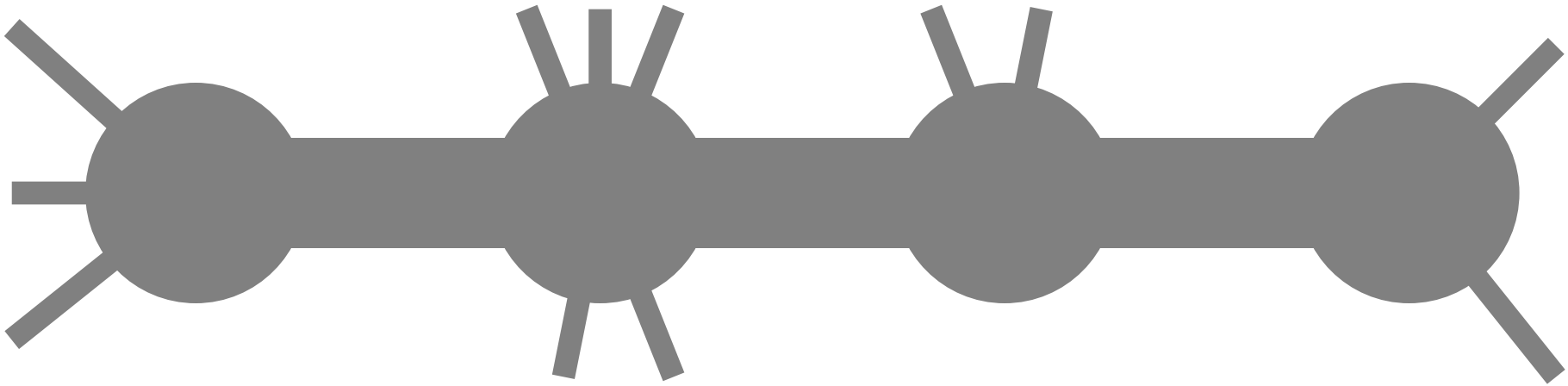
solution uses monotone block crossings!

≤ 3 good pairs can be created per crossing

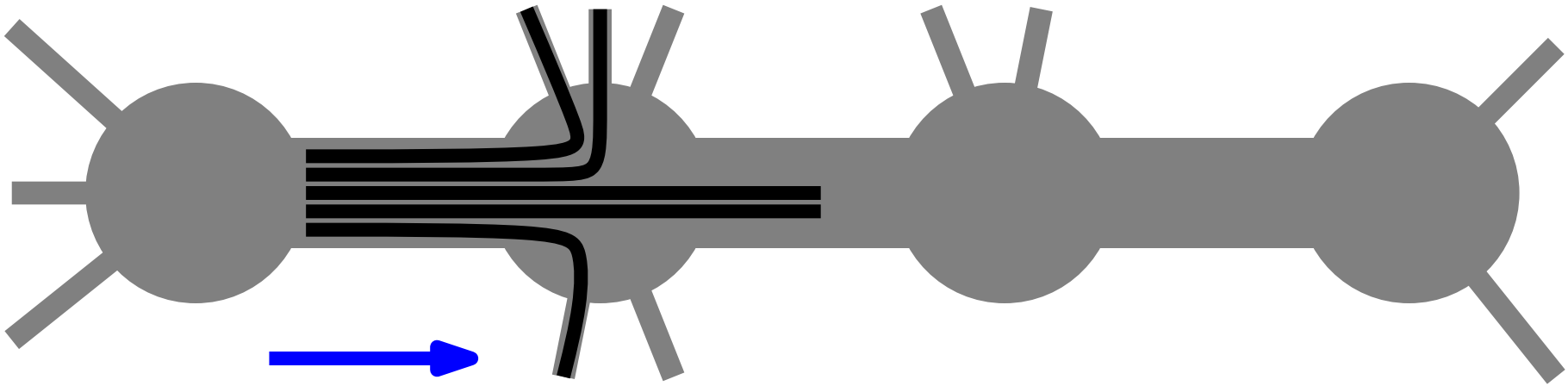
3-approx.

Path

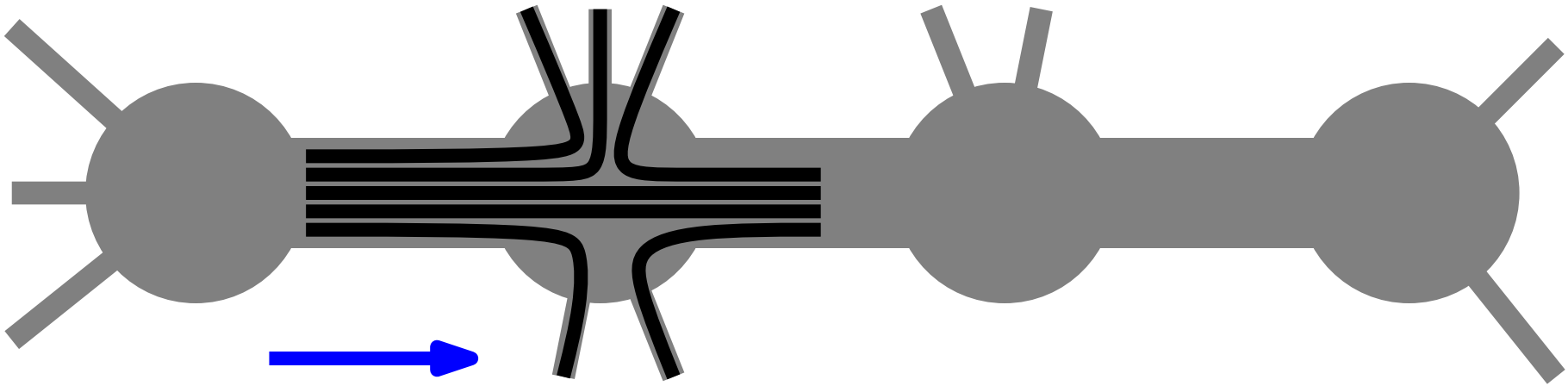
Path



Path



Path



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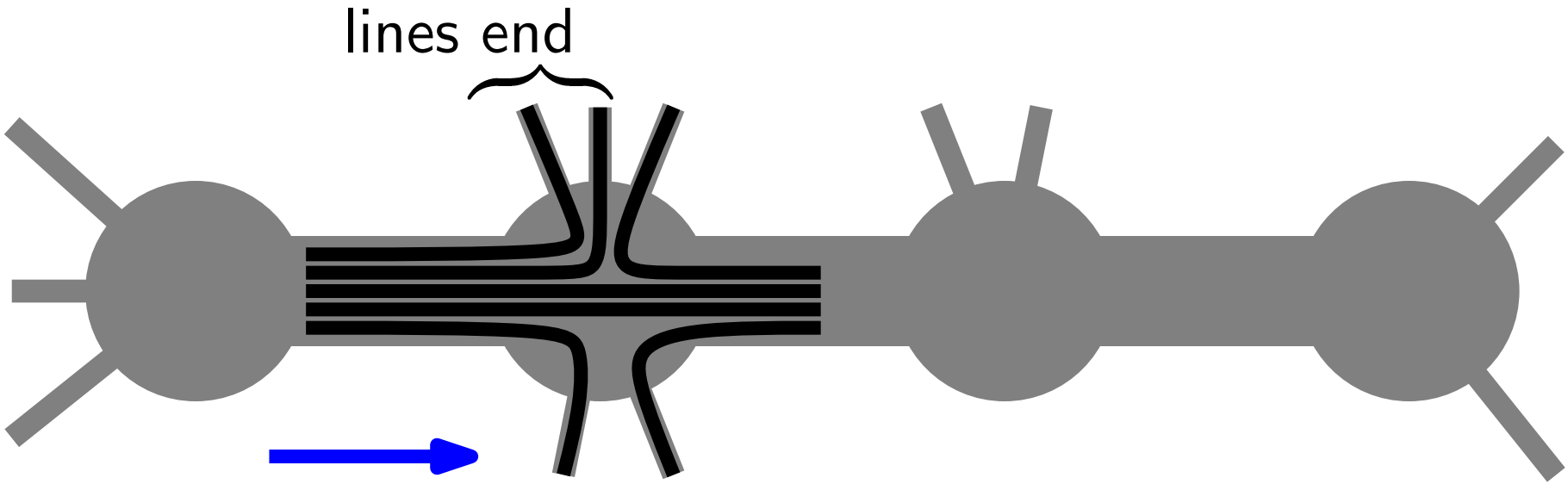

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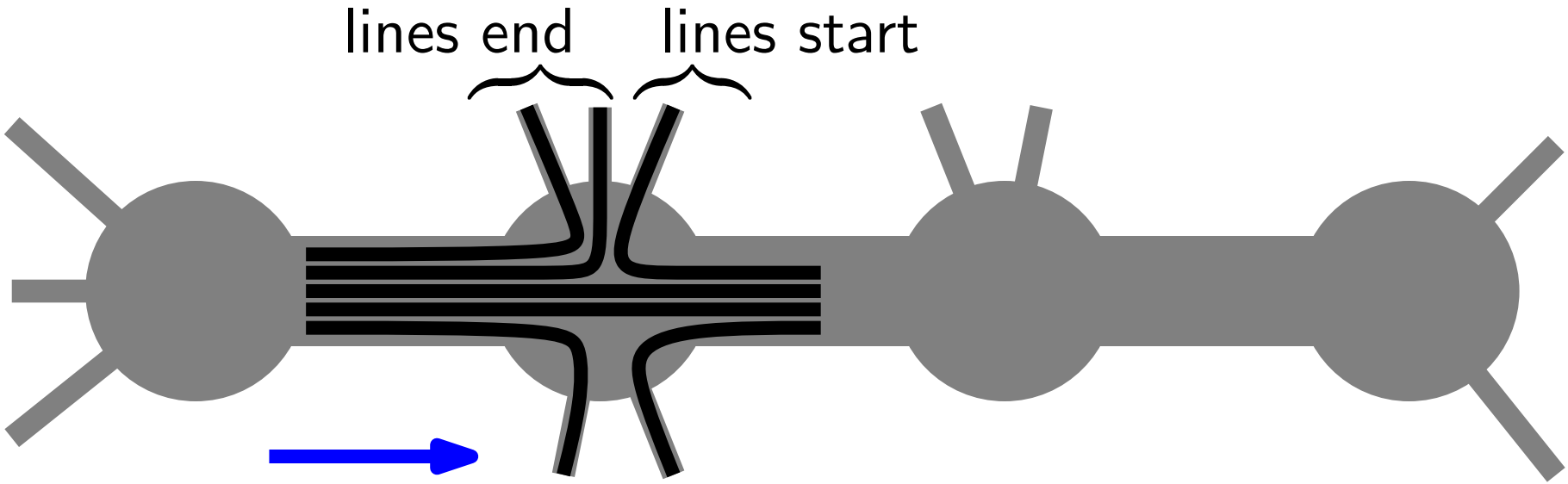
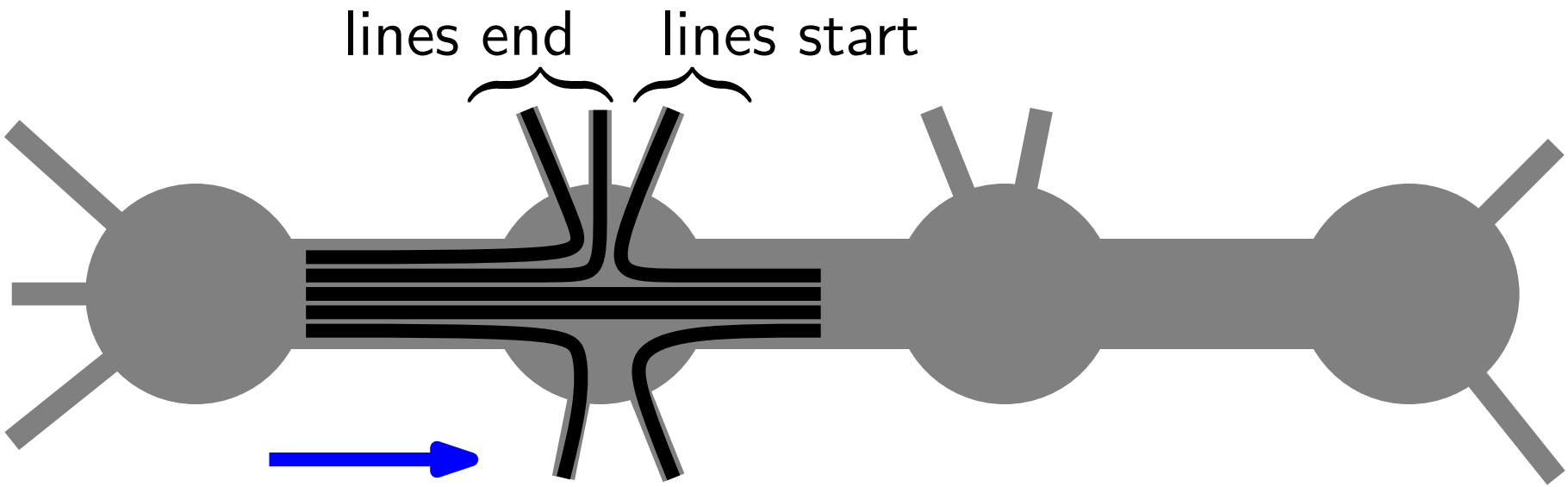
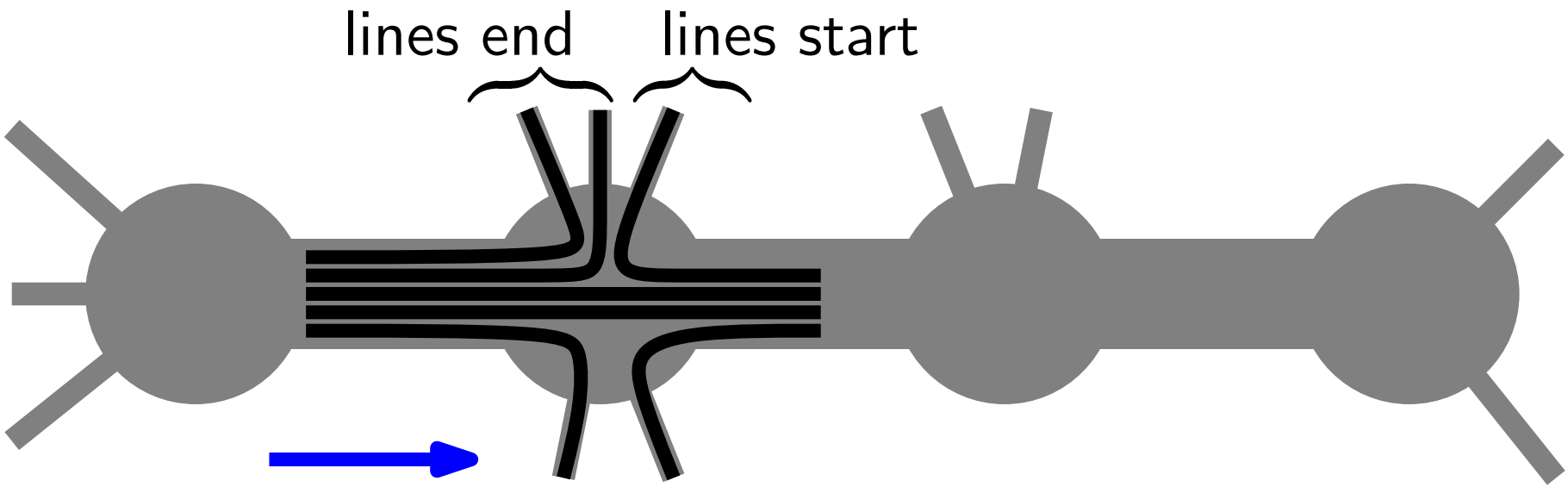


Path

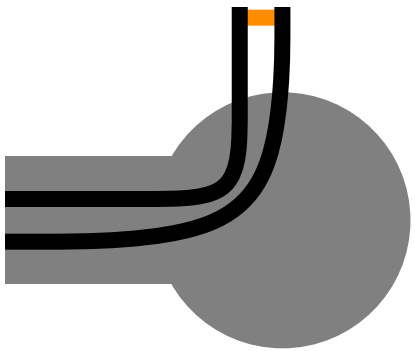


- Redefine good pairs:

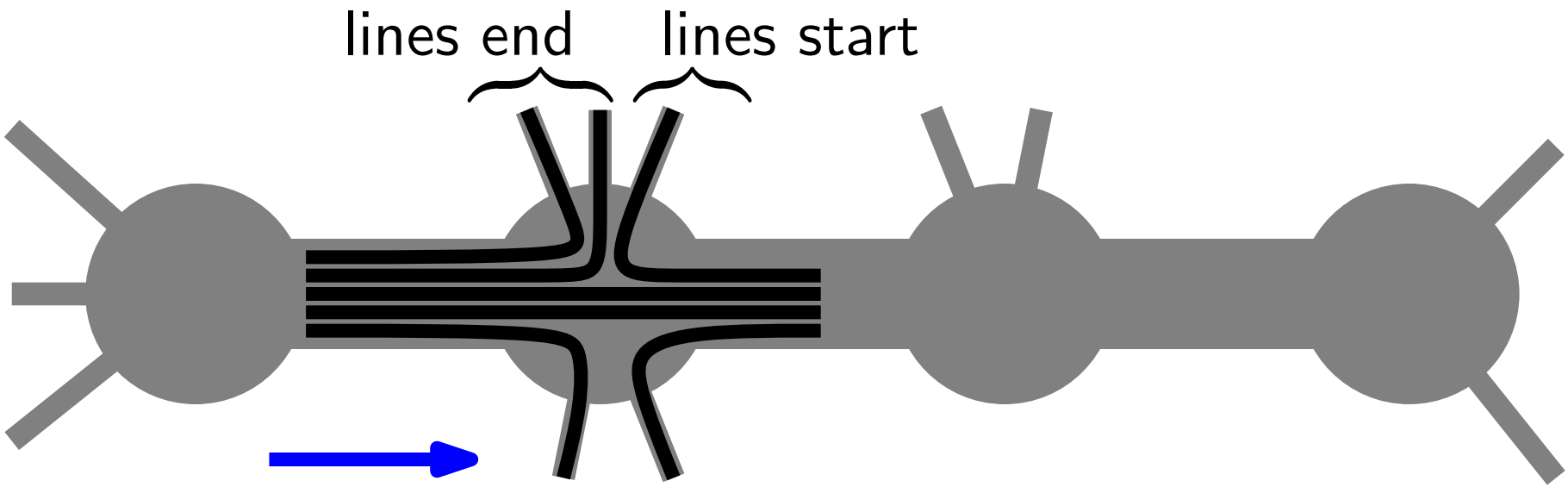
Path



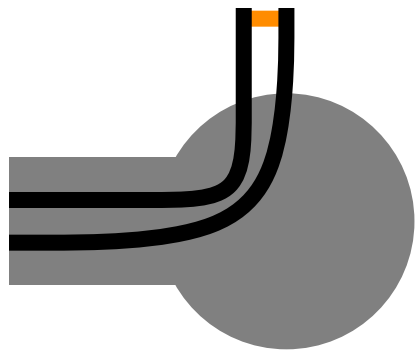
- Redefine good pairs:
  - lines end together

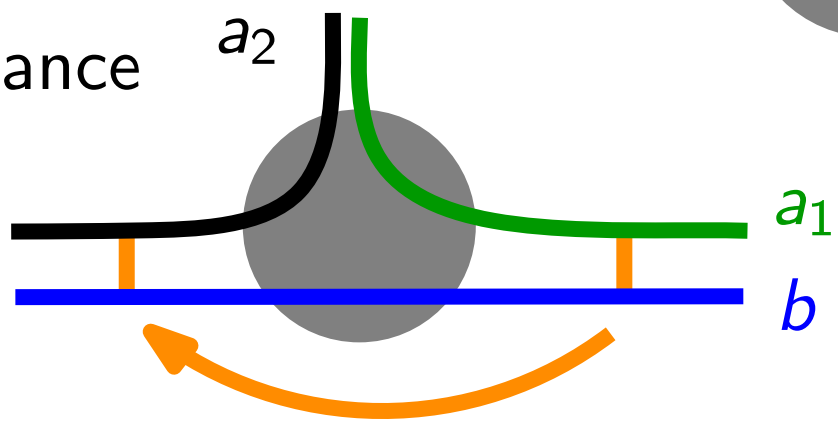


Path

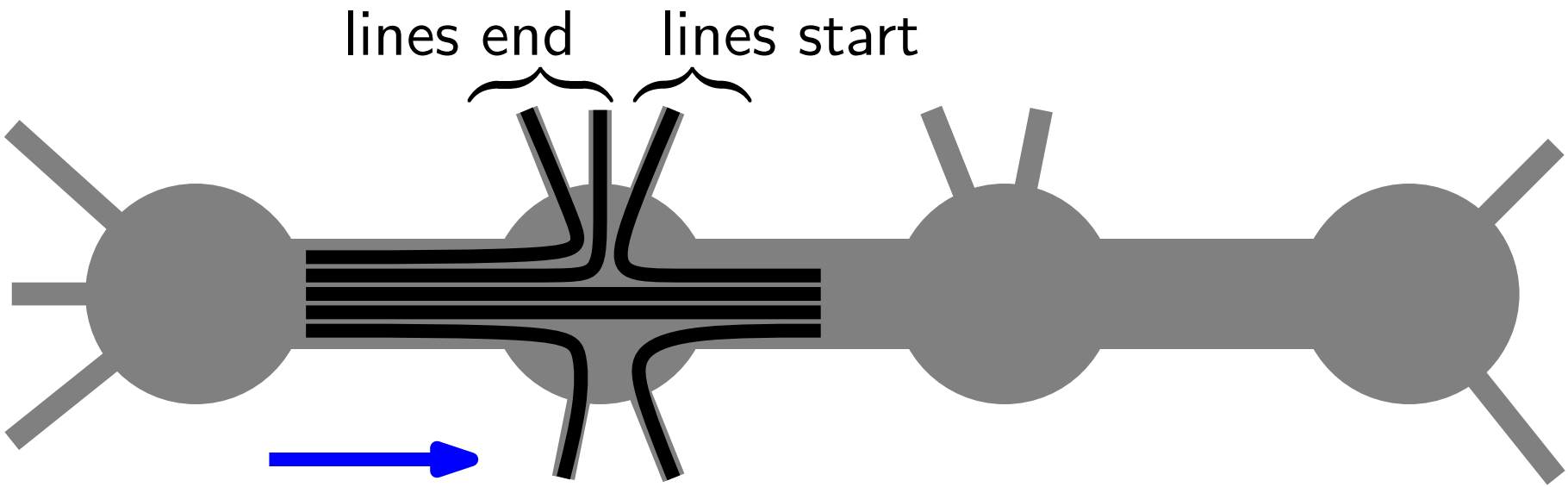


- Redefine good pairs:
  - lines end together
  - inheritance





Path



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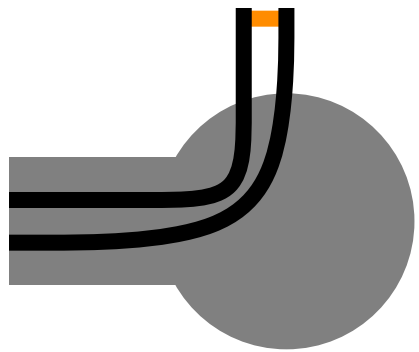
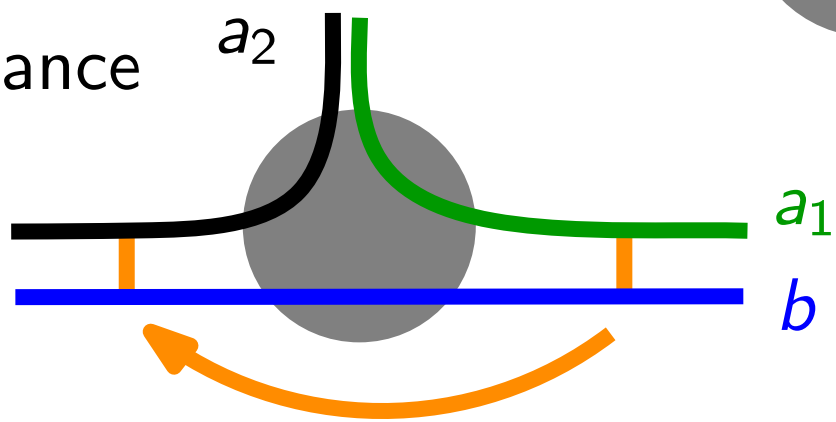
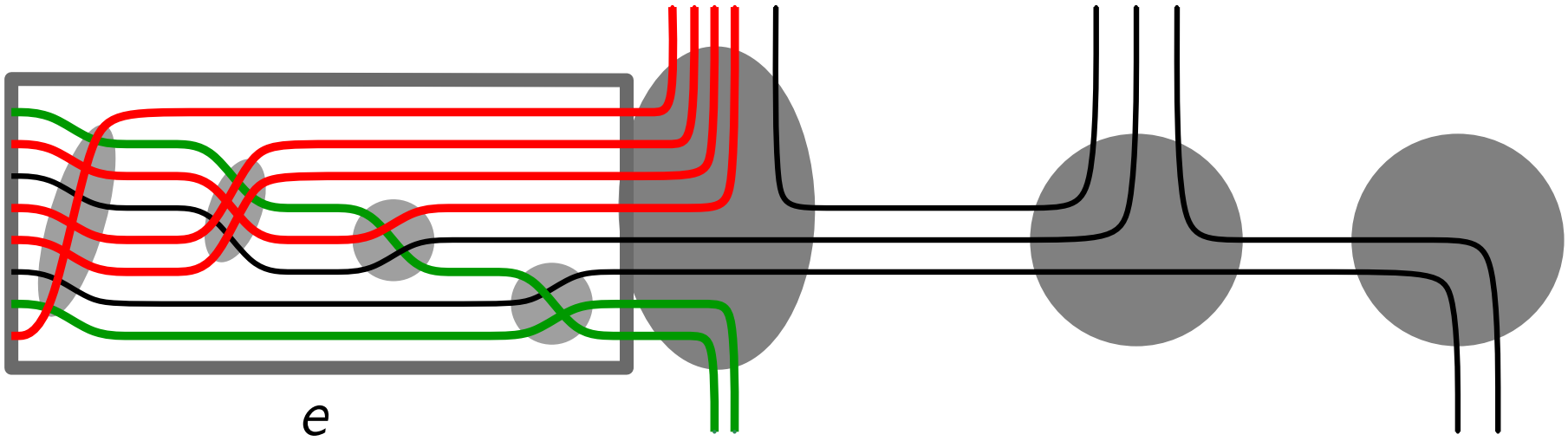




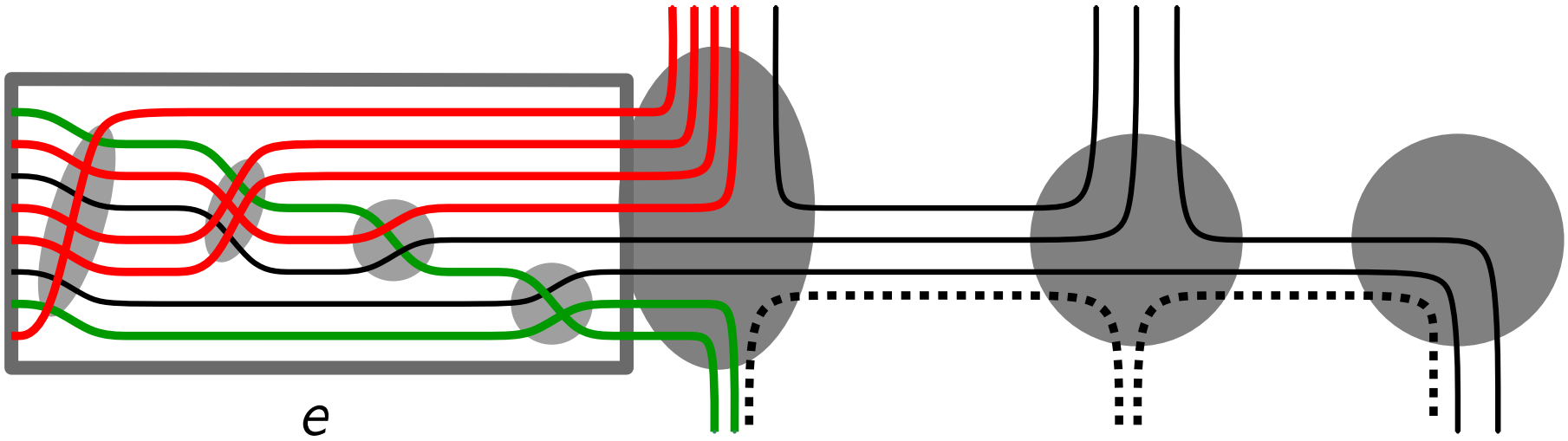
≤ 3 good pairs created per crossing

Path – 3-approximation



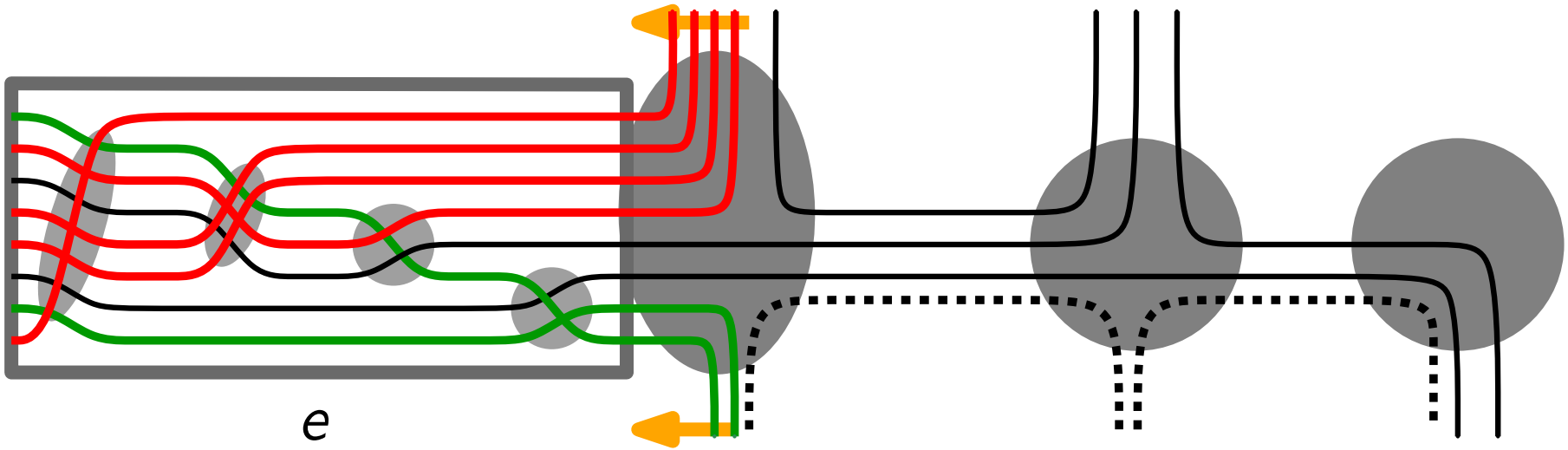
- treat edges from left to right

Path – 3-approximation



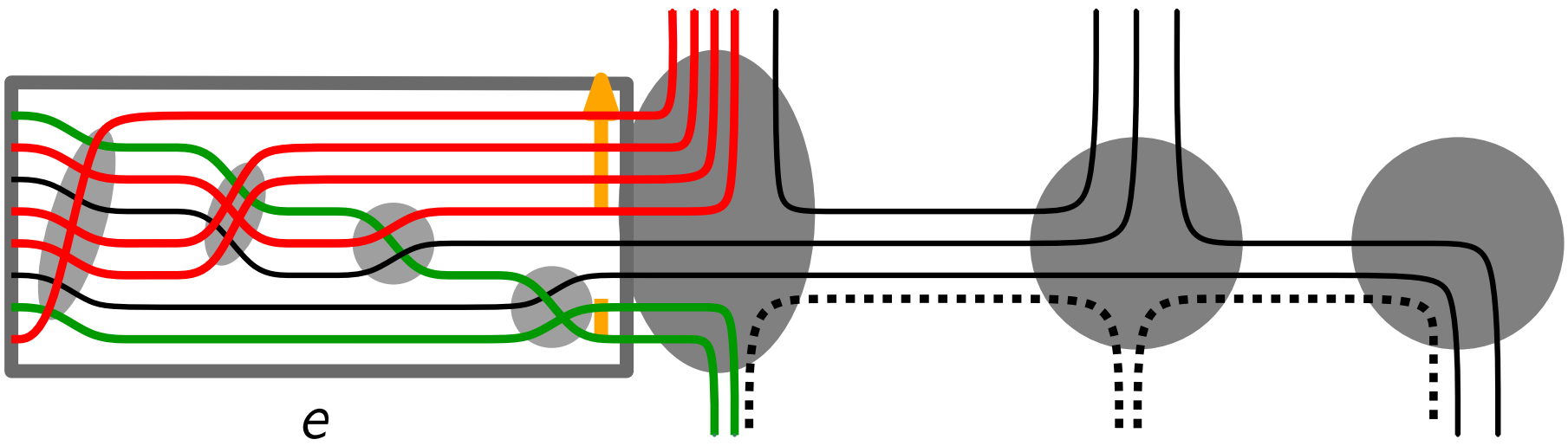
- treat edges from left to right

Path – 3-approximation



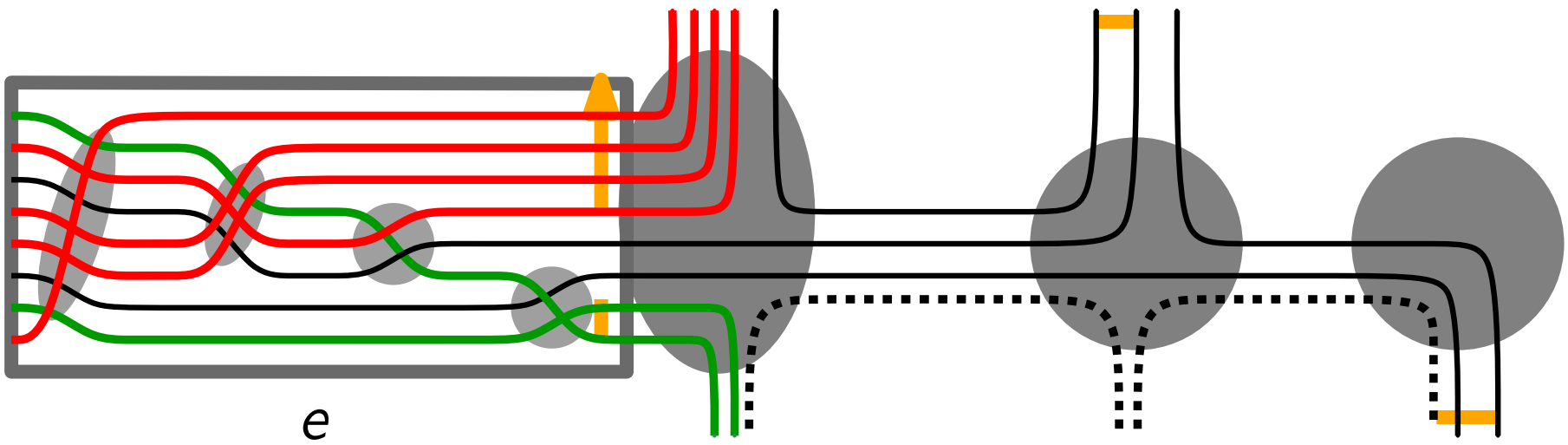
- treat edges from left to right
- identify good pairs

Path – 3-approximation



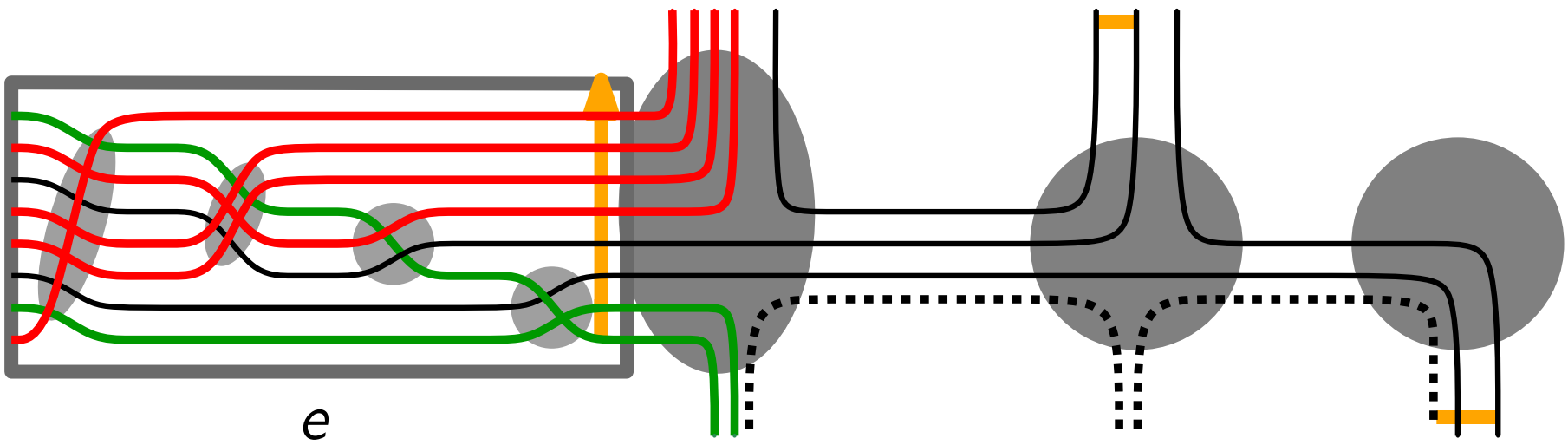
- treat edges from left to right

Path – 3-approximation



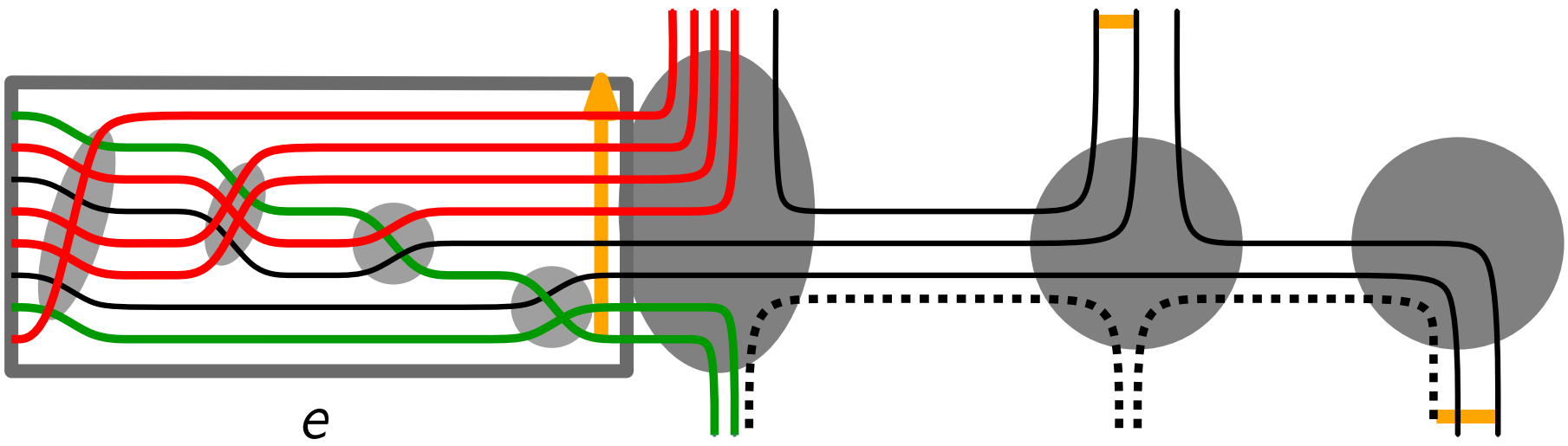
- treat edges from left to right

Path – 3-approximation



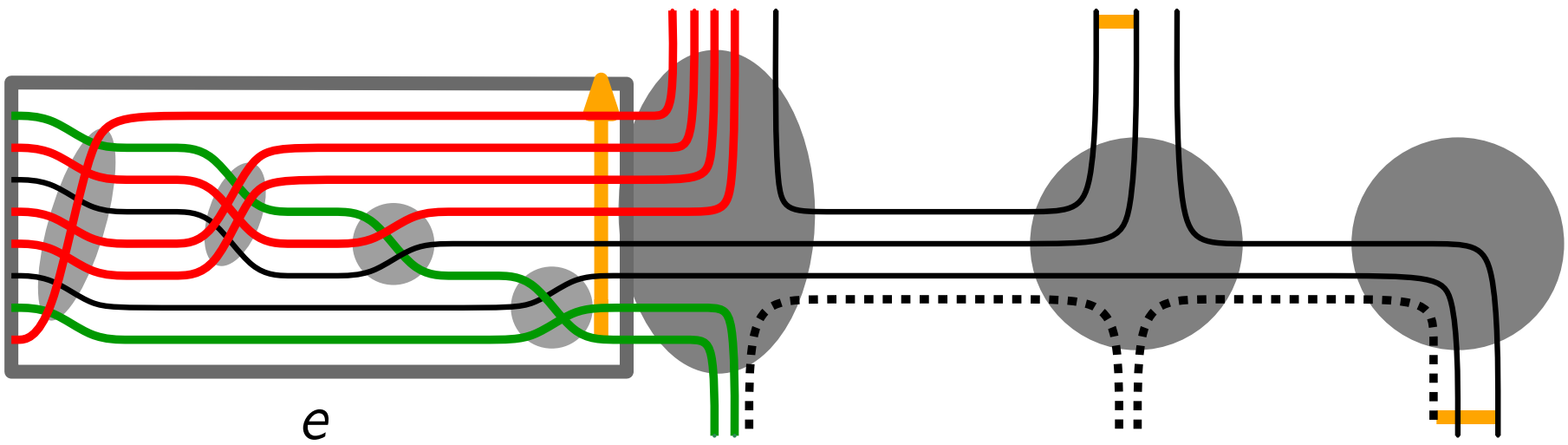
- treat edges from left to right

Path – 3-approximation



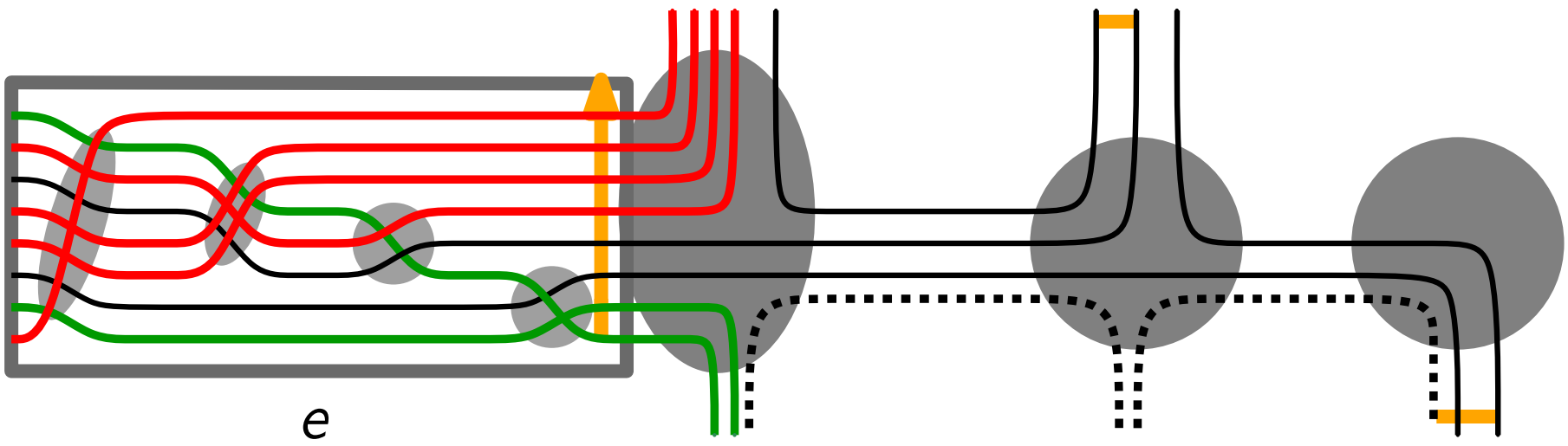
- treat edges from left to right
- bring ending lines to top/bottom keeping good pairs together

Path – 3-approximation



- treat edges from left to right
- bring ending lines to top/bottom keeping good pairs together
- create 1 good pair per block crossing crossing

Path – 3-approximation

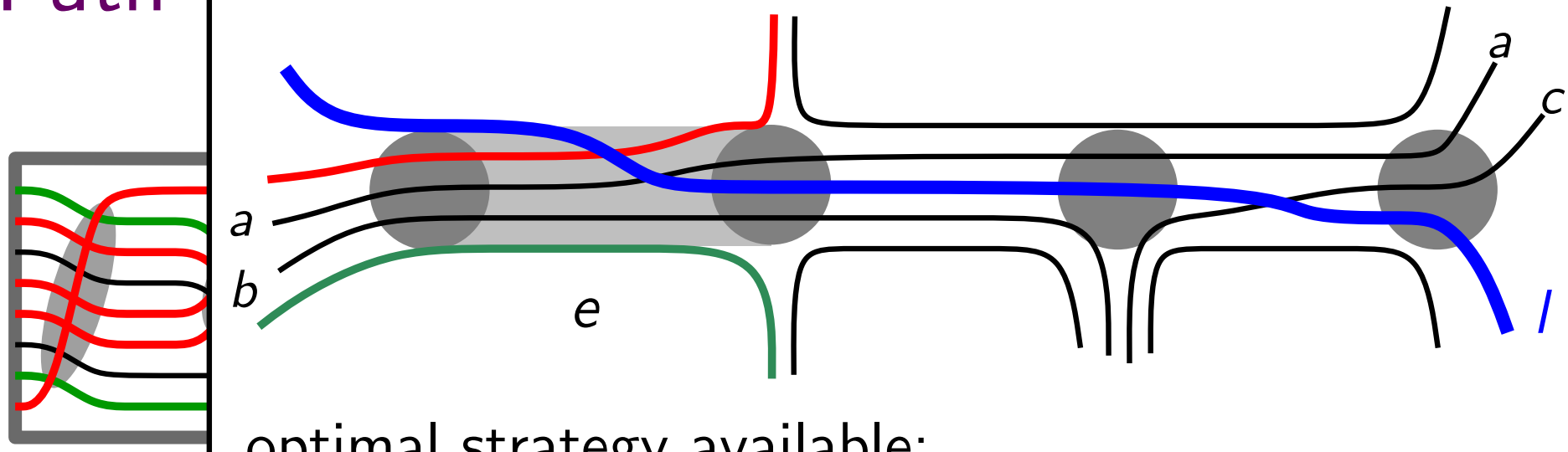


- treat edges from left to right
- bring ending lines to top/bottom keeping good pairs together
- create 1 good pair per block crossing crossing
- optimum creates up to 3

→ 3-approximation

Path -

Special Case: Destroying good pairs



optimal strategy available:

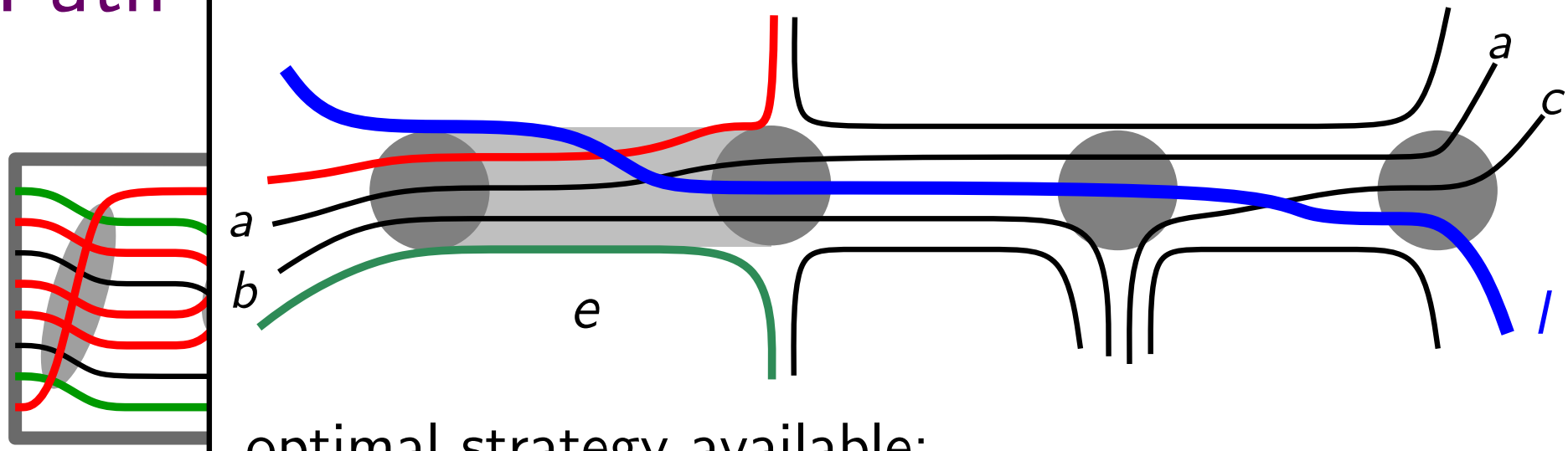
- treat
  right
- bring ending lines to top/bottom keeping good pairs together

- optimum creates up to 3

→ 3-approximation

Path -

Special Case: Destroying good pairs



optimal strategy available:

– preferably destroy non-initial good pair

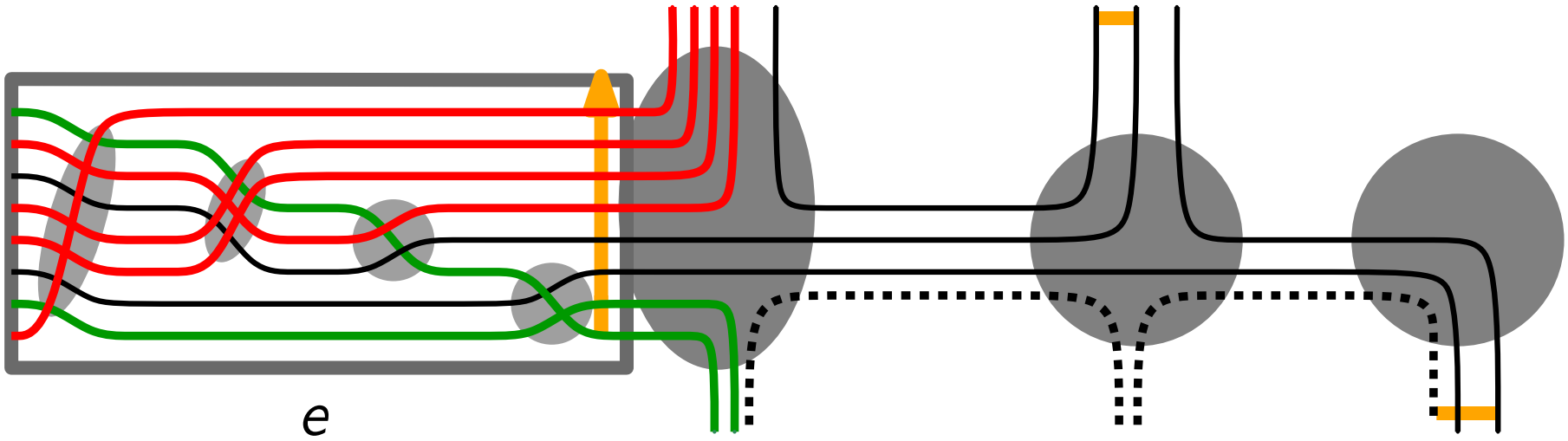
– if none: destroy longest-living good pair

- treat
  right
- bring ending lines to top/bottom keeping good pairs together

- optimum creates up to 3

→ 3-approximation

Path – 3-approximation



- treat edges from left to right
- bring ending lines to top/bottom keeping good pairs together

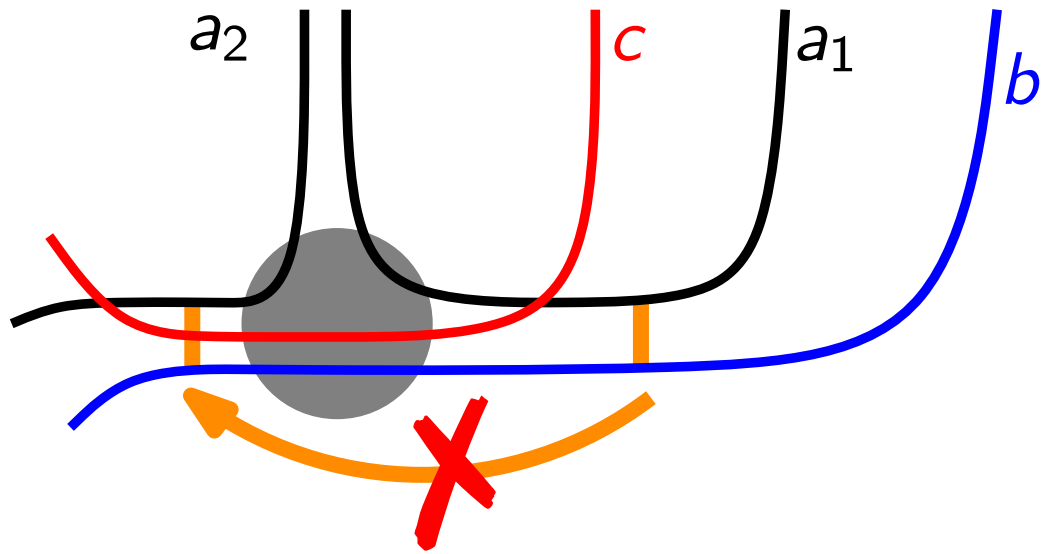
- create 1 good pair per block crossing crossing
- optimum creates up to 3

→ 3-approximation

- algorithm can be adjusted for monotone block crossings

Path – 3-approximation

- adjust inheritance of good pairs:



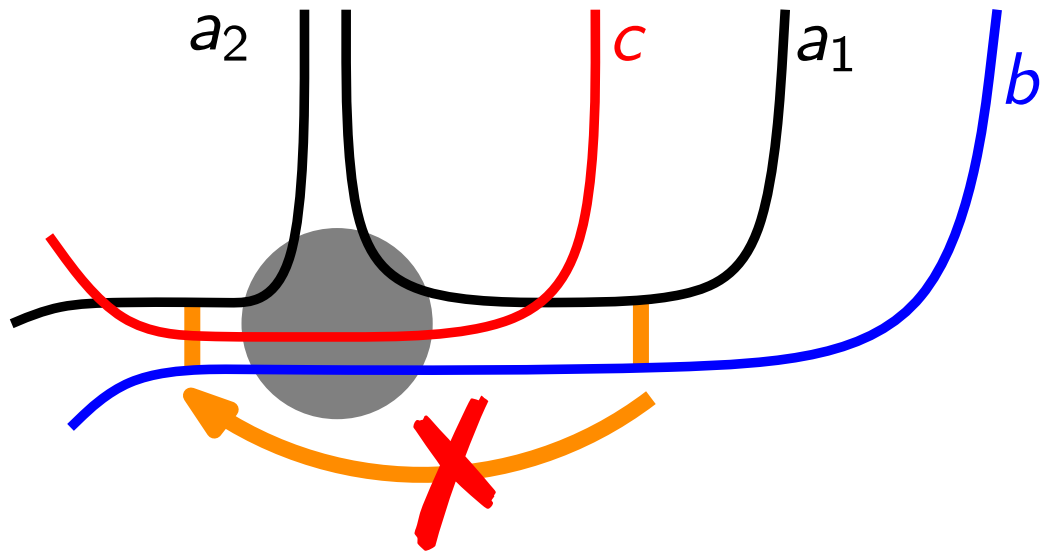
- t… air per
  r… rossing
- b… s up to 3
  top/bottom keeping good pairs together

→ 3-approximation

- algorithm can be adjusted for monotone block crossings

Path – 3-approximation

- adjust inheritance of good pairs:



- new strategy when destroying good pairs: ensure monotonicity

→ 3-approximation

- algorithm can be adjusted for monotone block crossings

- t… air per ri… rossing
- b… s up to 3 top/bottom keeping good pairs together

Trees - an upper bound

- root at some leave

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- root at some leave
- after treating edge
  recursively order subtrees

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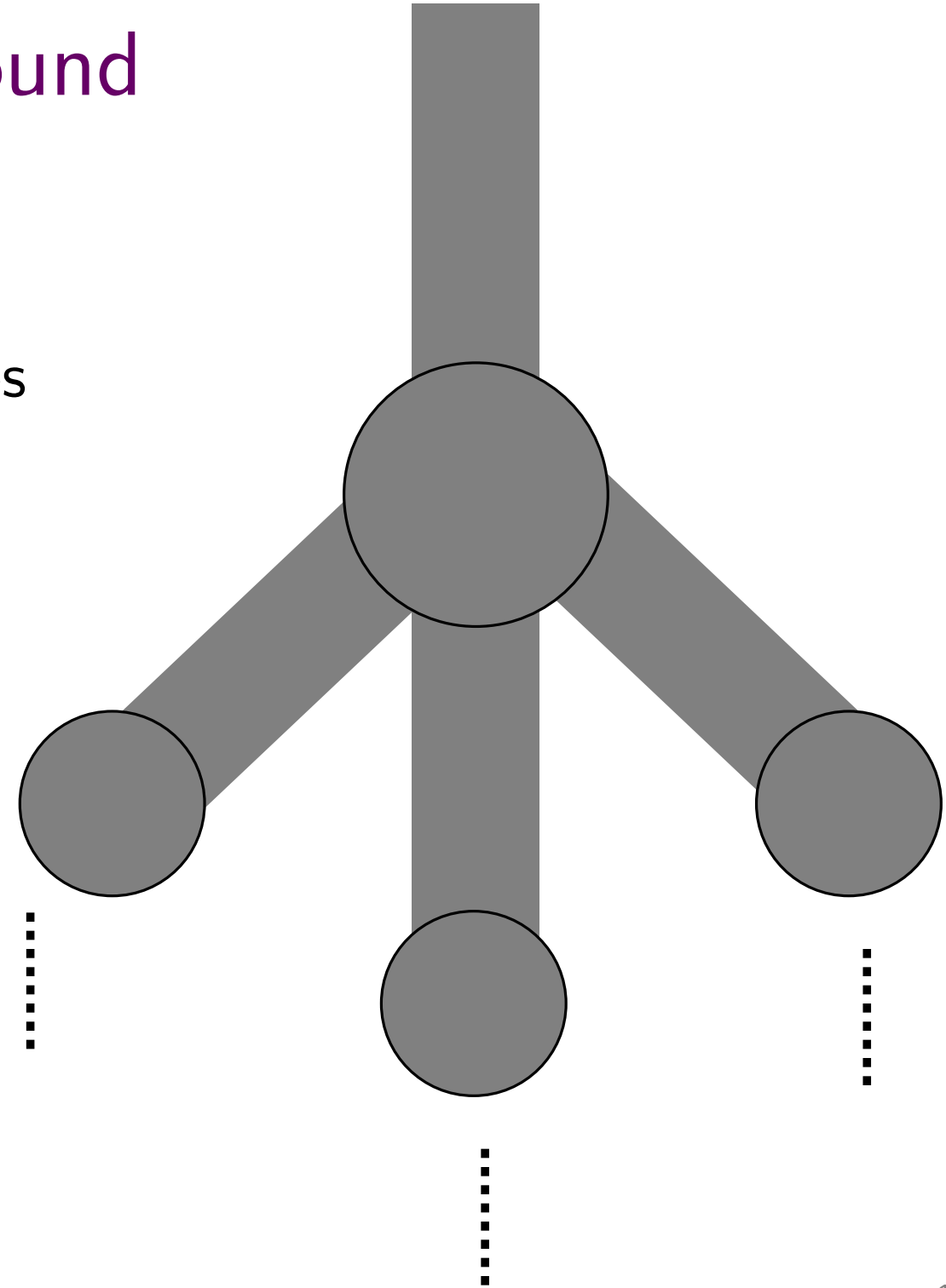

Trees - an upper bound

- root at some leave
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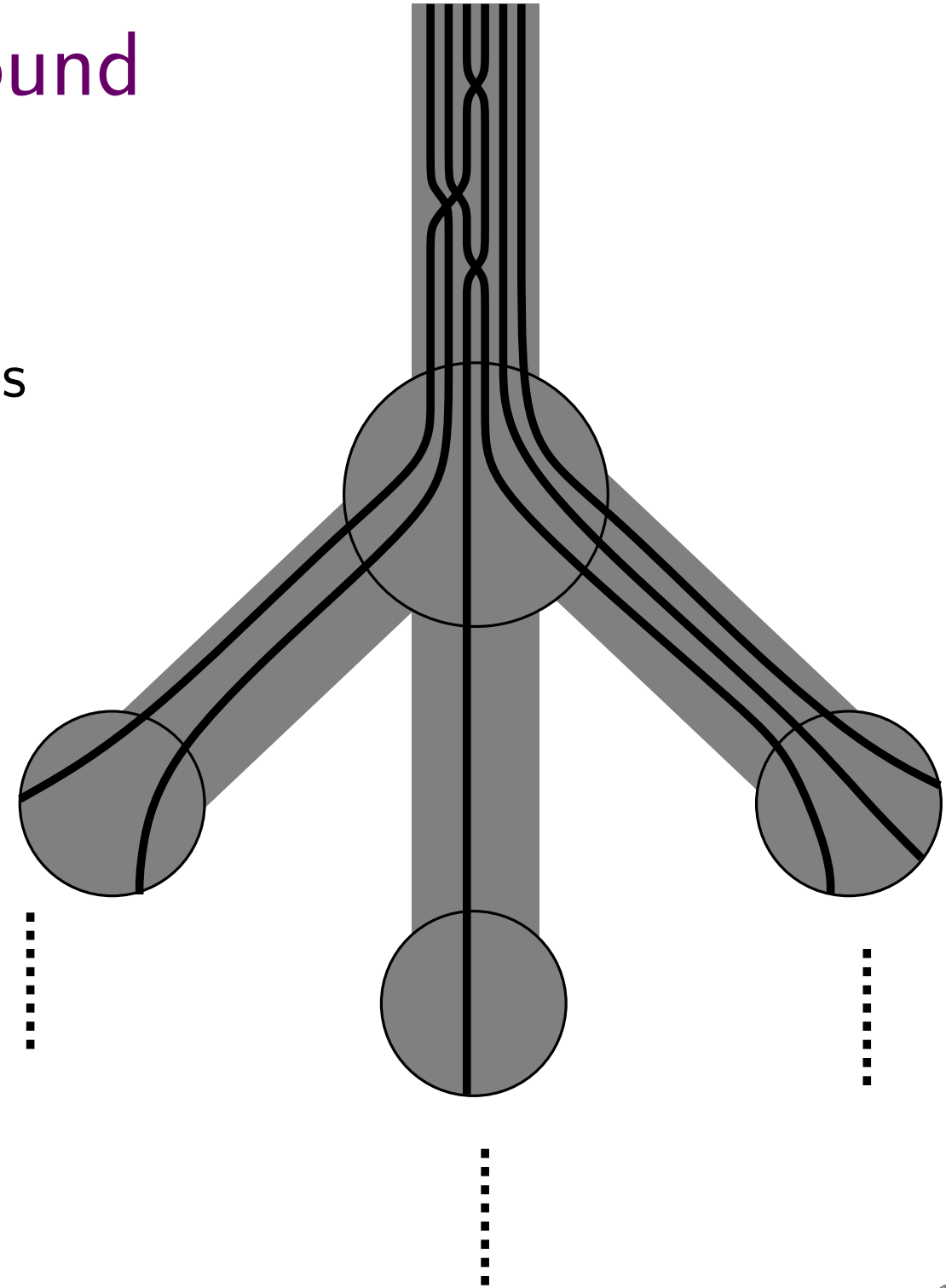
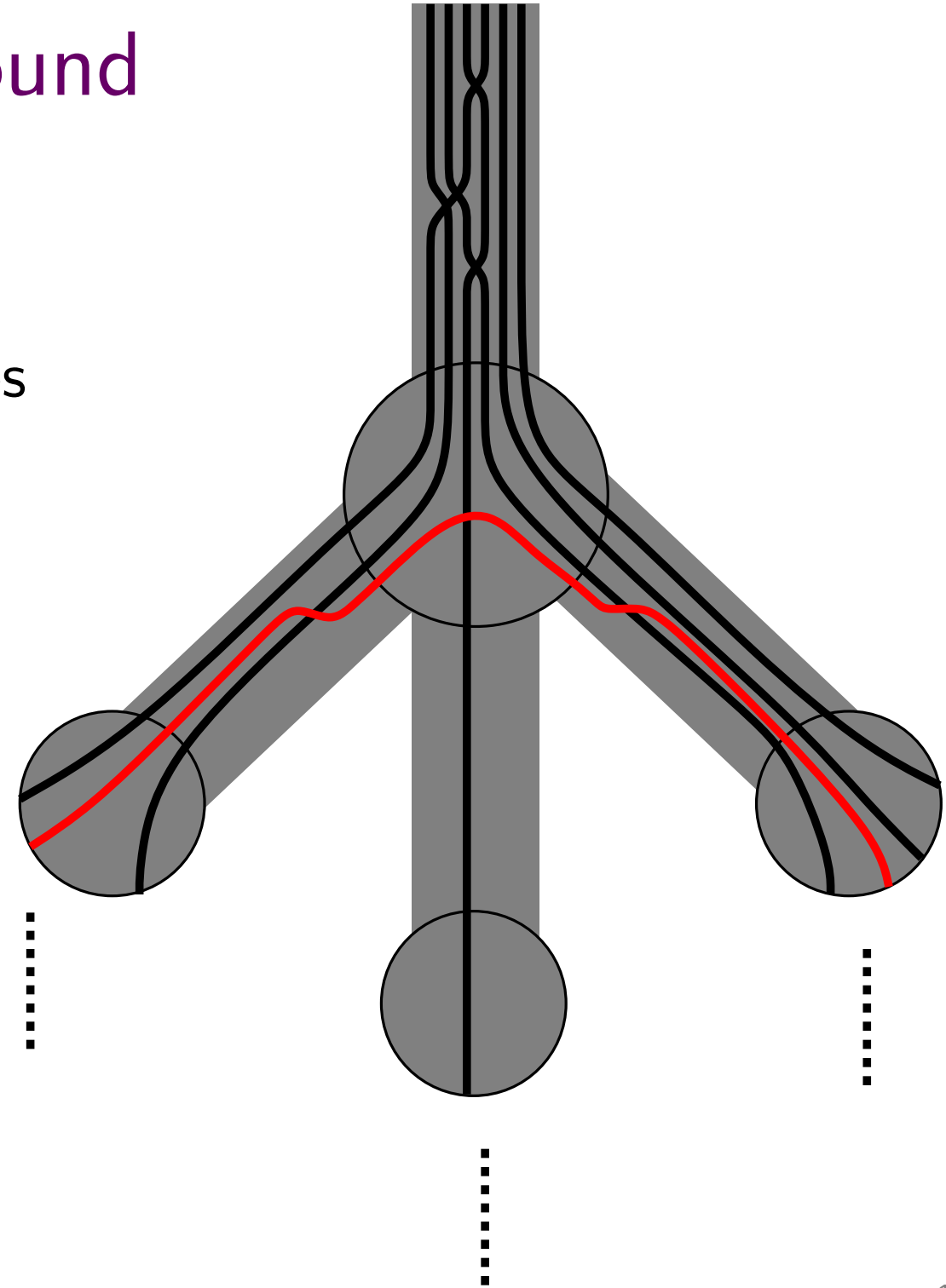


Trees - an upper bound

- root at some leave
- after treating edge
  recursively order subtrees

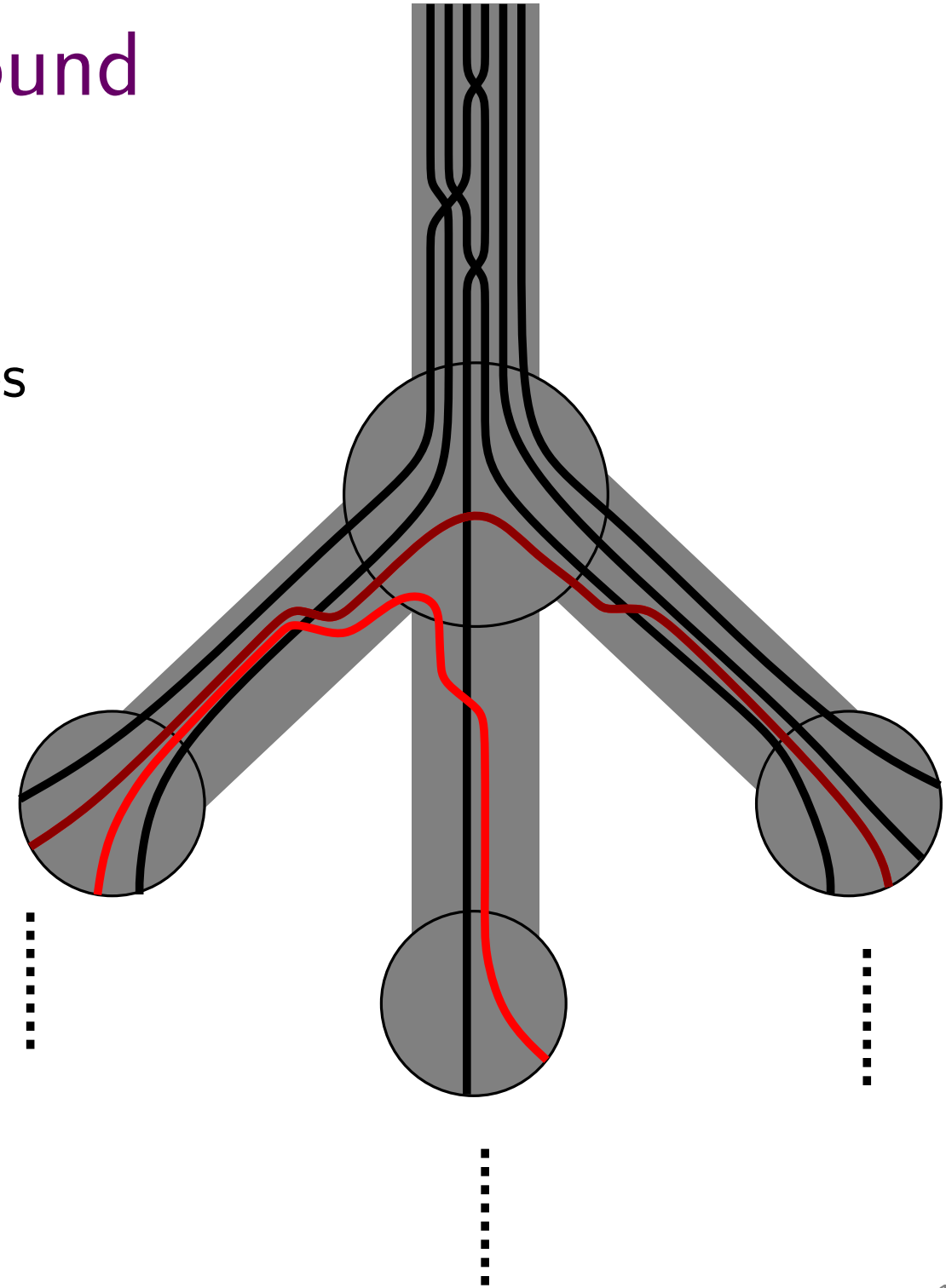
- insert lines between
  subtrees



Trees - an upper bound

- root at some leave
- after treating edge recursively order subtrees

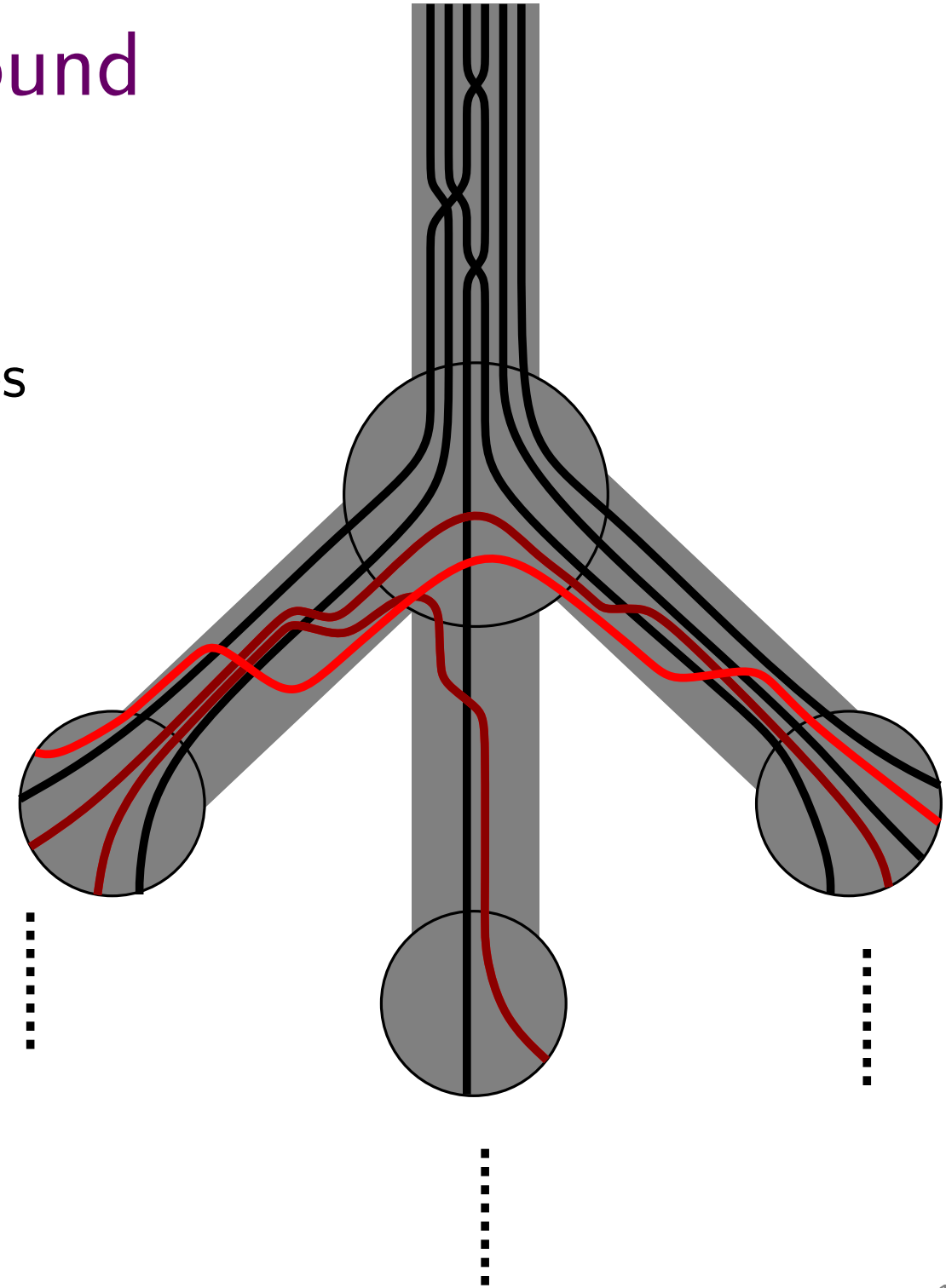
- insert lines between subtrees
- ≤ 2 crossings per line



Trees - an upper bound

- root at some leave
- after treating edge recursively order subtrees

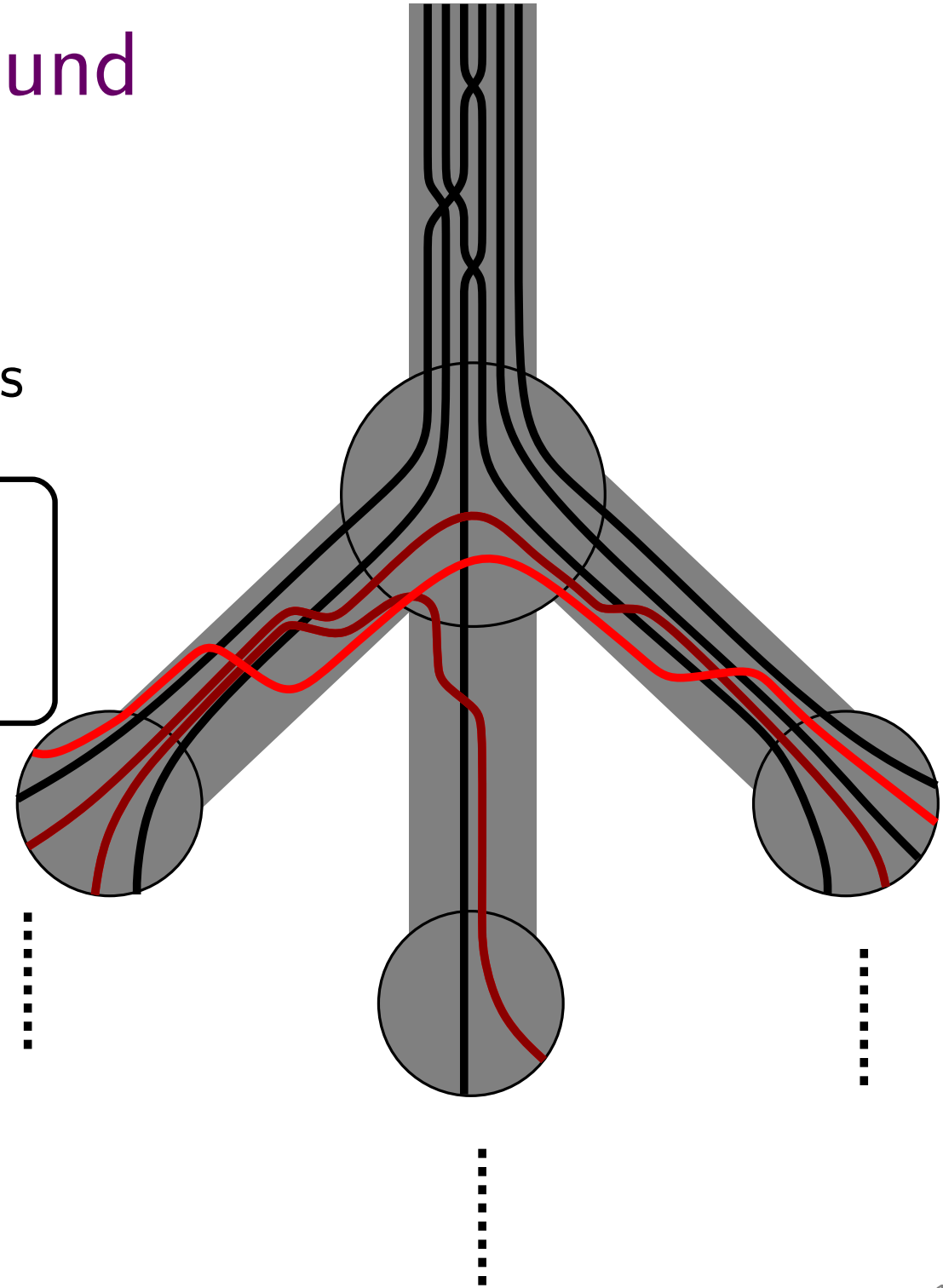
- insert lines between subtrees
- ≤ 2 crossings per line
- right insertion order needed for:
  – avoiding vertex crossings – avoiding double crossings



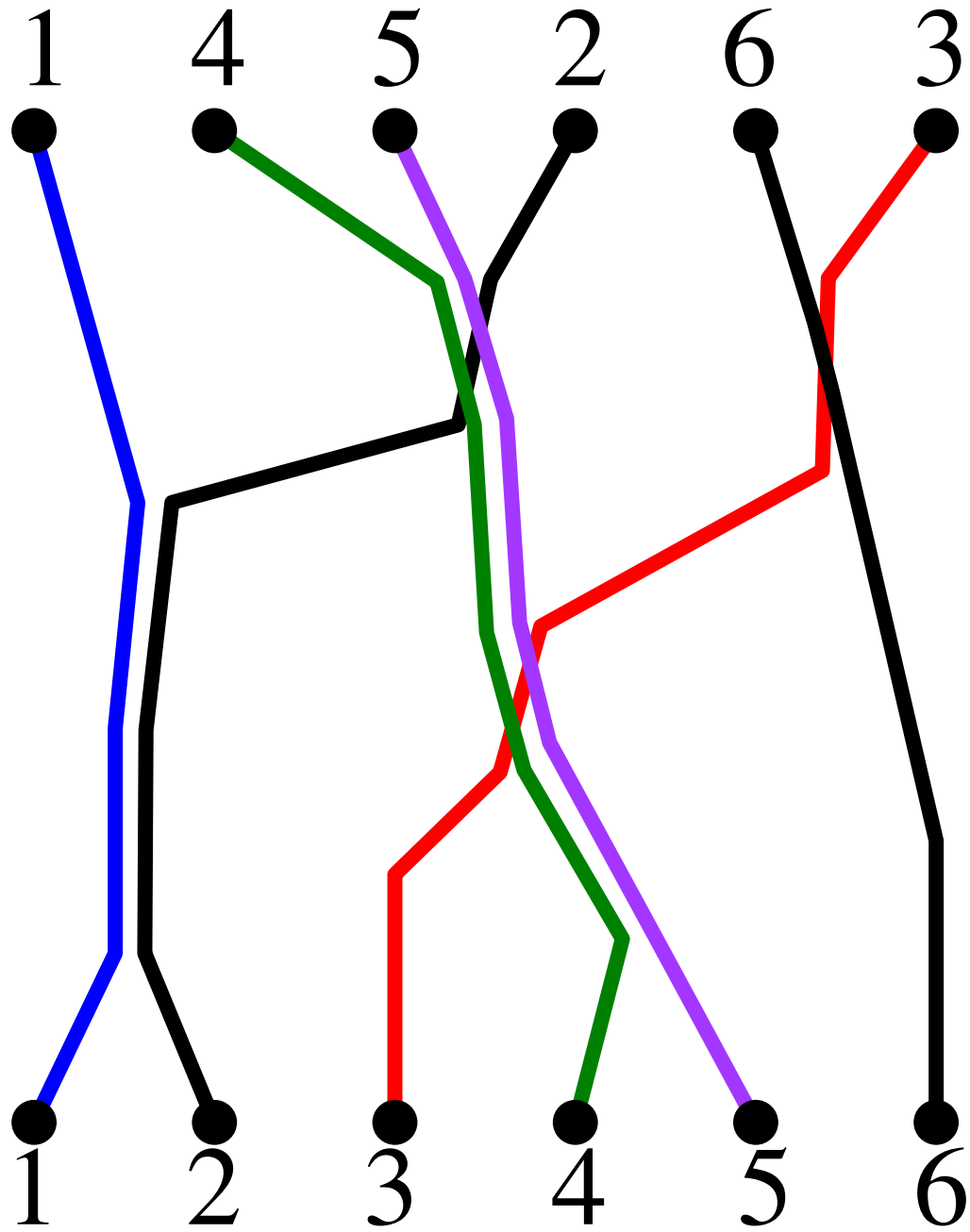
Trees - an upper bound

- root at some leave
- after treating edge recursively order subtrees
- in su
- ≤ 2 crossings per line
- right insertion order needed for:
  – avoiding vertex crossings – avoiding double crossings

worst-case instances:
$2|L| - 3$ crossings necessary

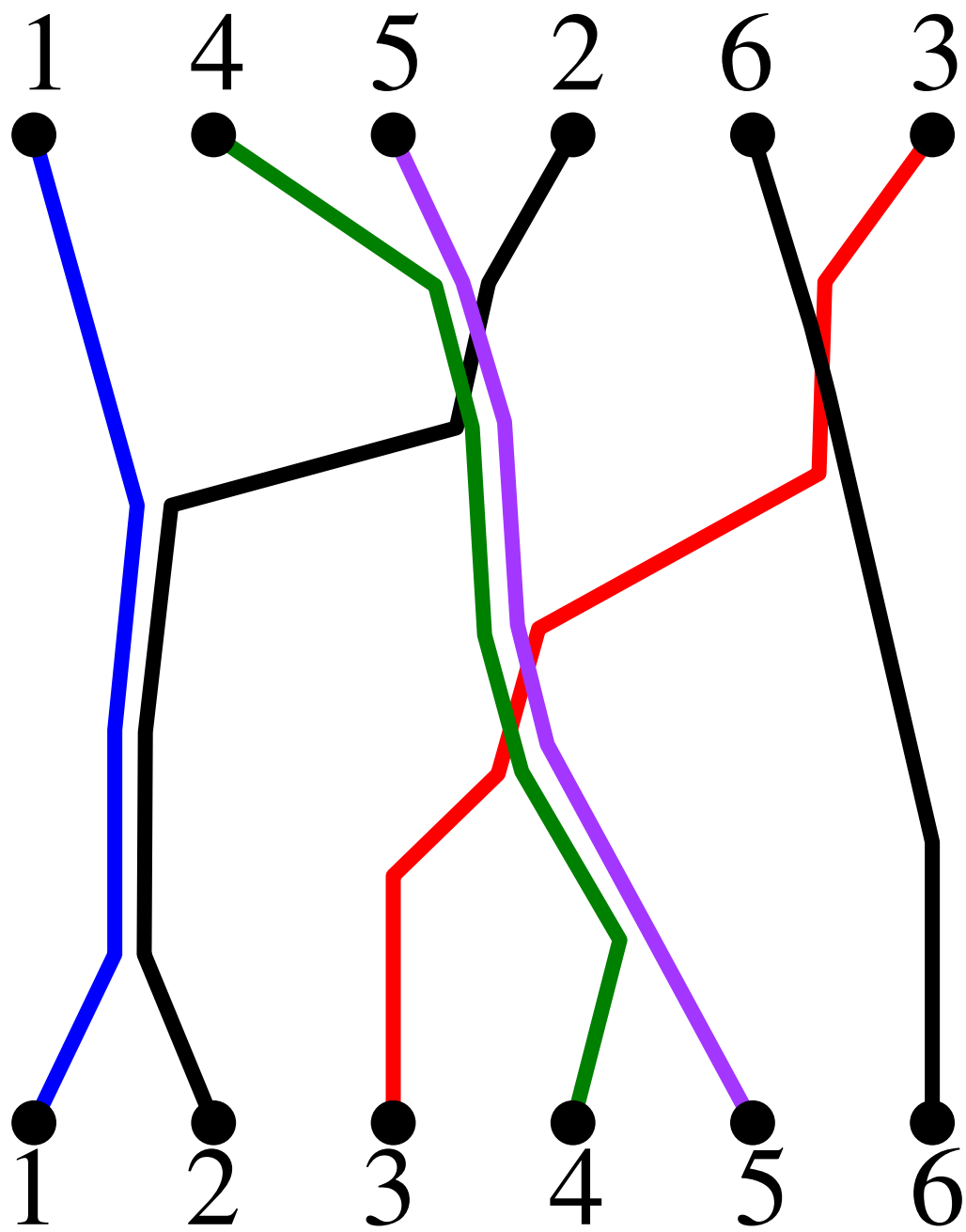


Upward Trees



Upward Trees

- simplification
- use tree algorithm
- 6-approximation for monotone block crossings



General Graphs

- Process edges in arbitrary order

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- Completely sort lines on an edge

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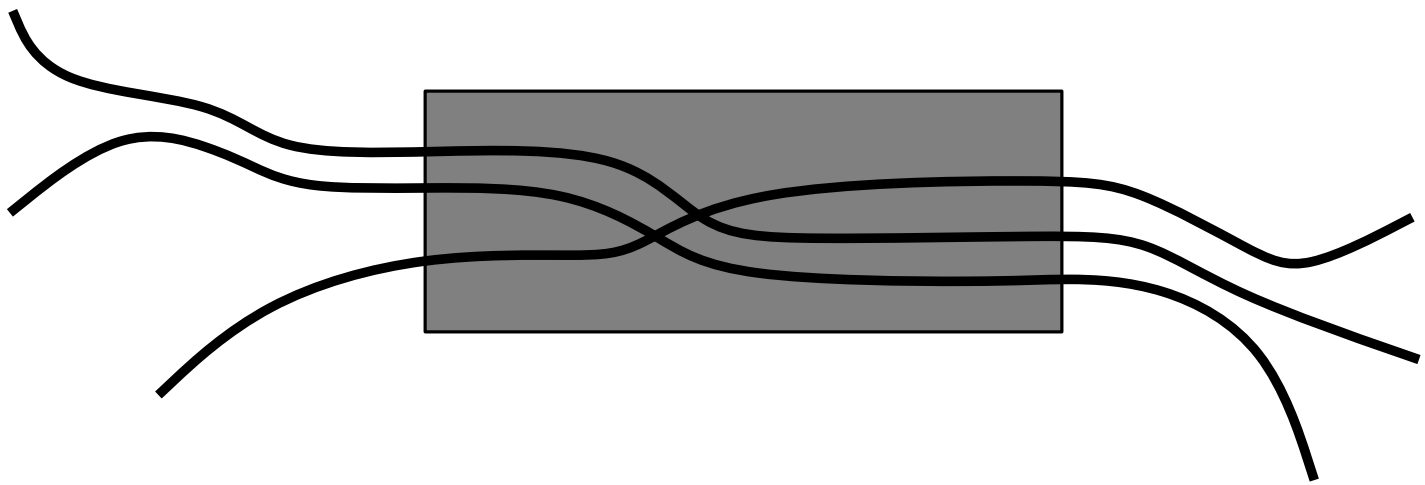

General Graphs

- Process edges in arbitrary order
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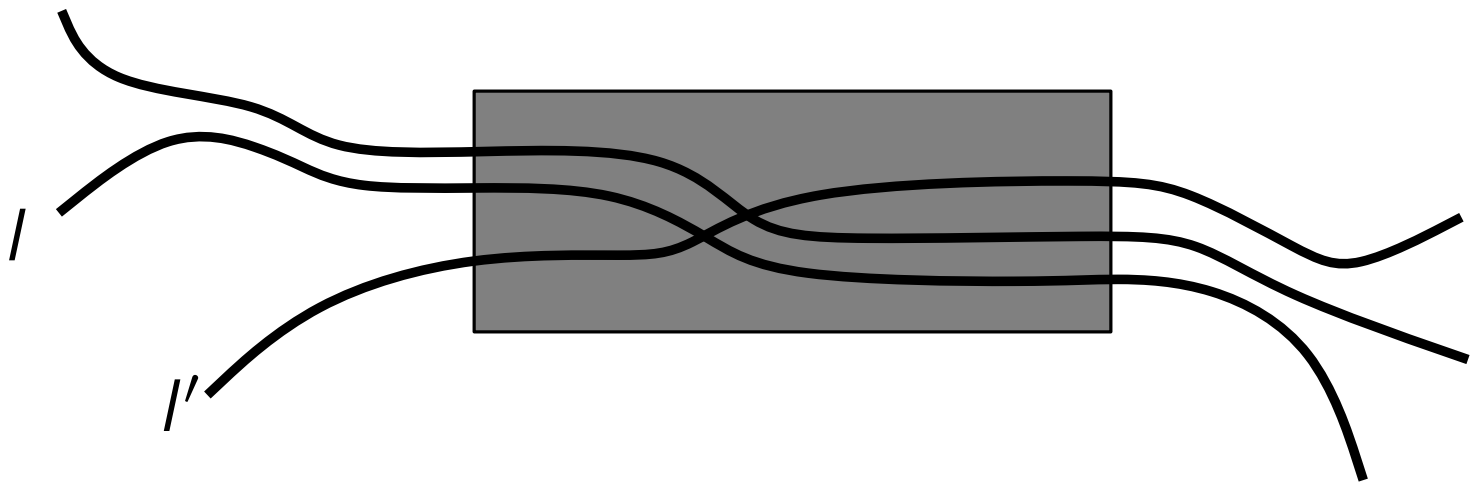
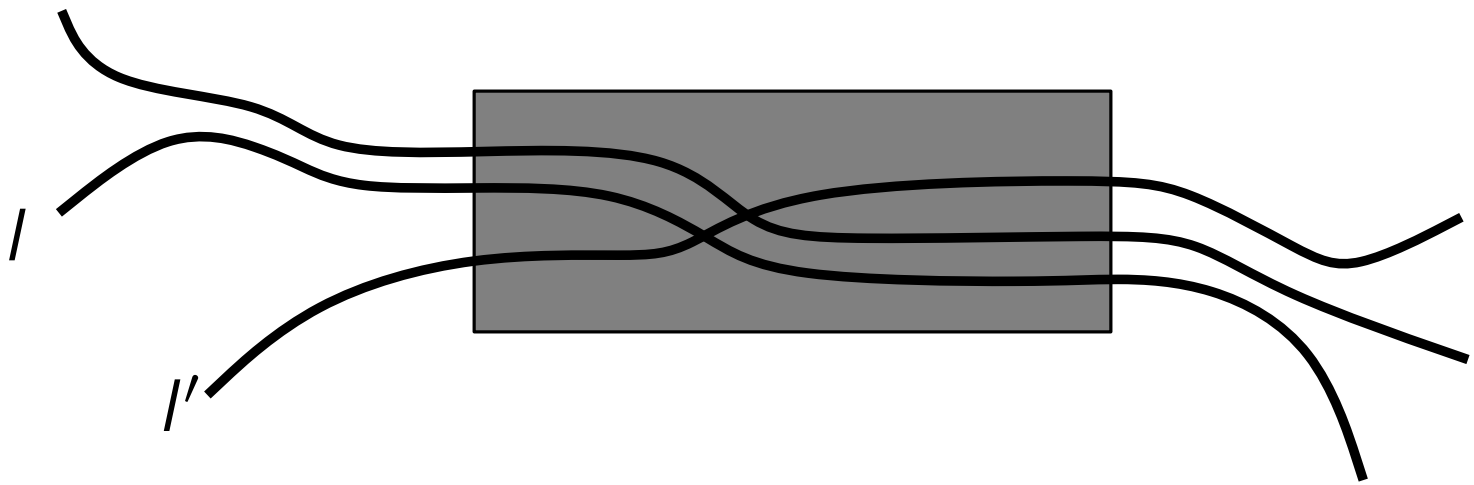


- lines l, l' seen together on edge

  → will never cross (again)

General Graphs

- Process edges in arbitrary order
- Completely sort lines on an edge



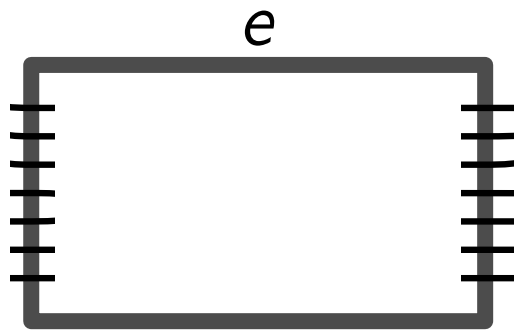
- lines l, l' seen together on edge

  → will never cross (again)

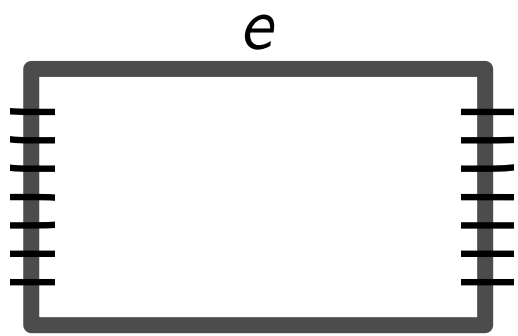
- lines l, l' seen together for the first time

  → information gain

General Graphs – Sorting an Edge

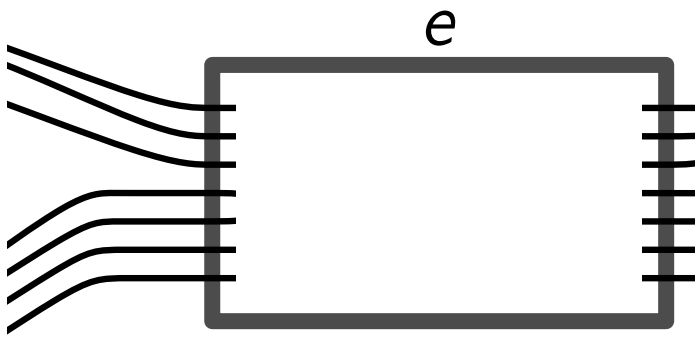


General Graphs – Sorting an Edge



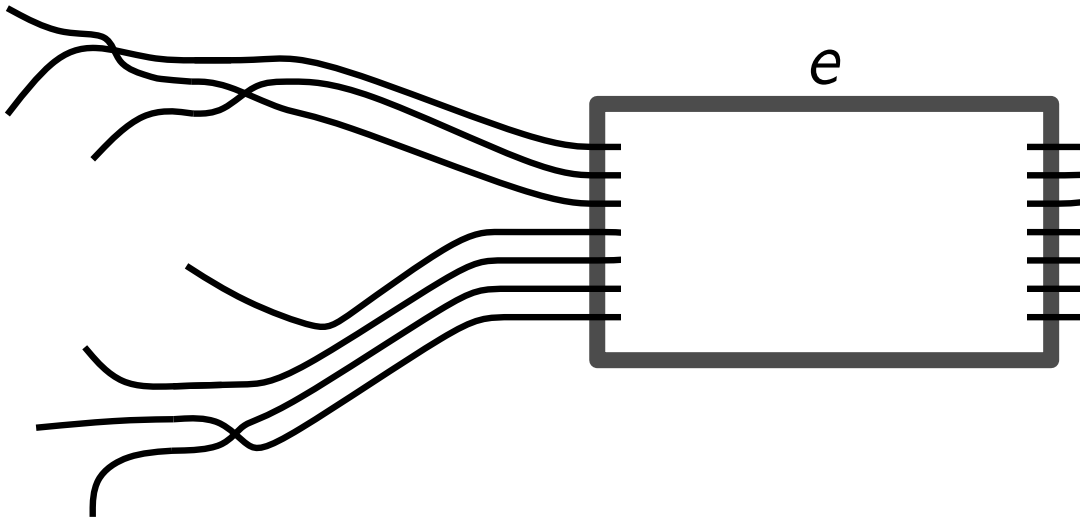
- follow lines

General Graphs – Sorting an Edge



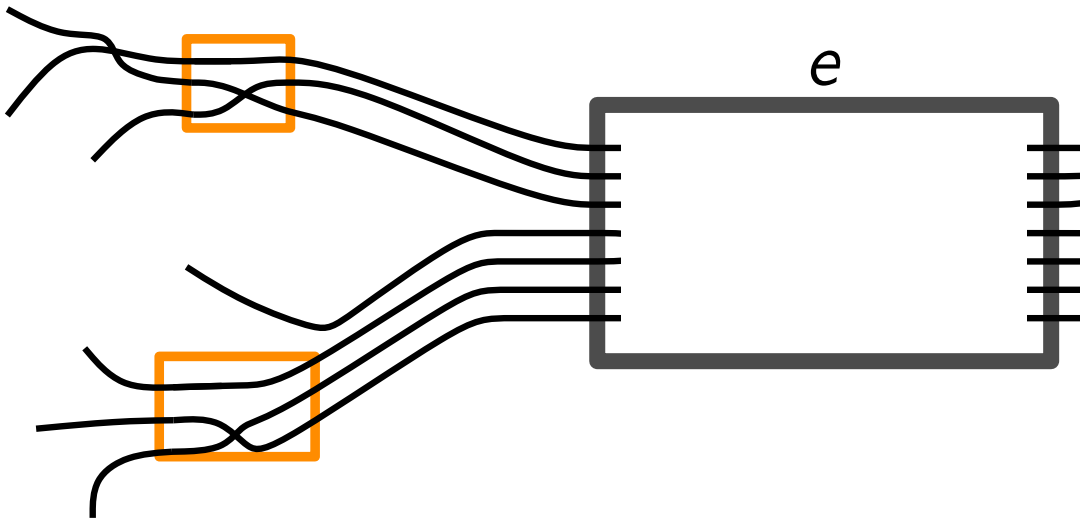
- follow lines

General Graphs – Sorting an Edge



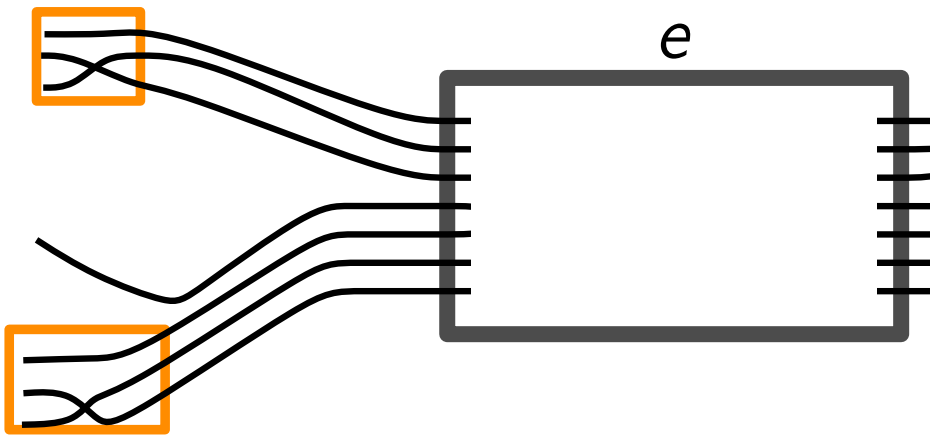
- follow lines

General Graphs – Sorting an Edge



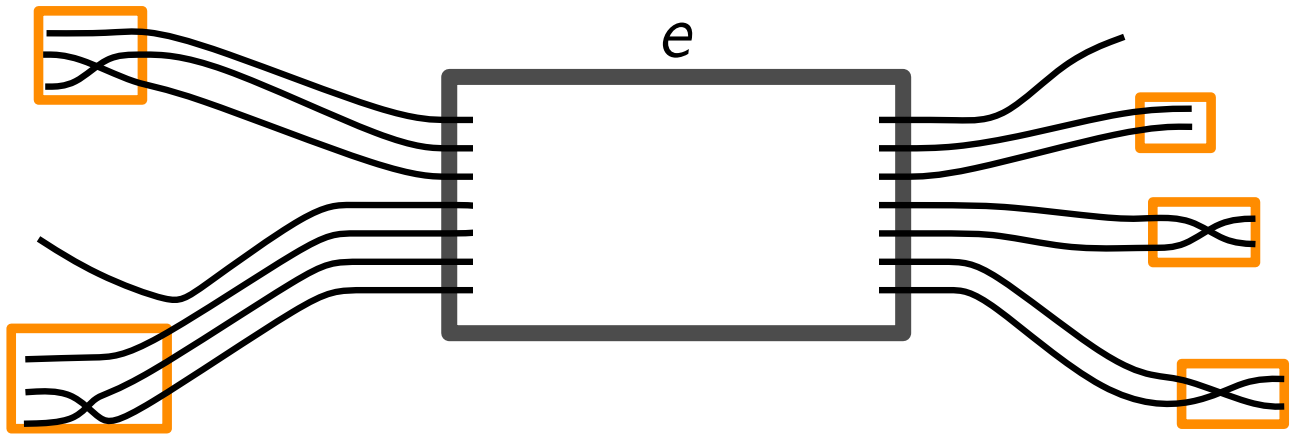
- follow lines
- find cut edges

General Graphs – Sorting an Edge



- follow lines
- find cut edges

General Graphs – Sorting an Edge



- follow lines
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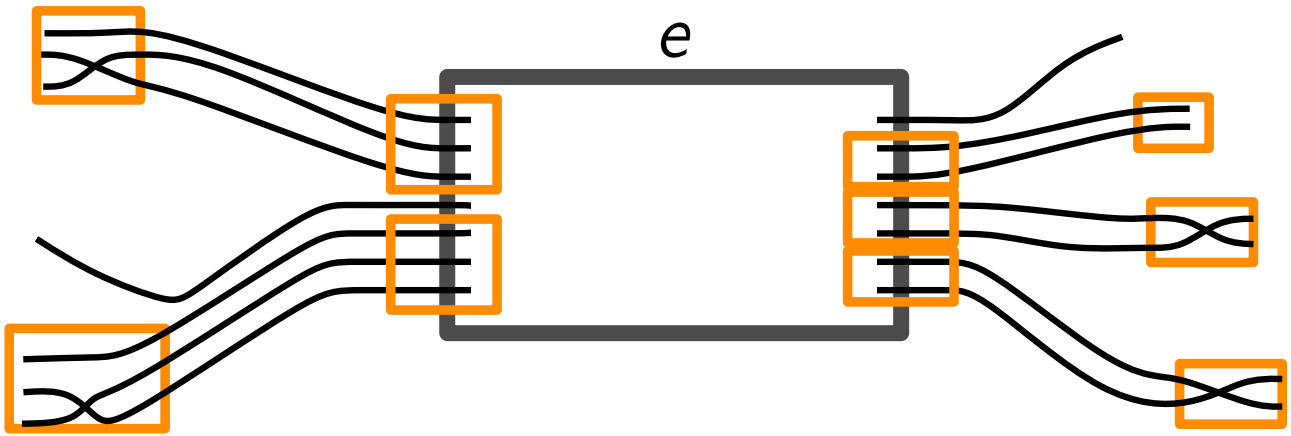
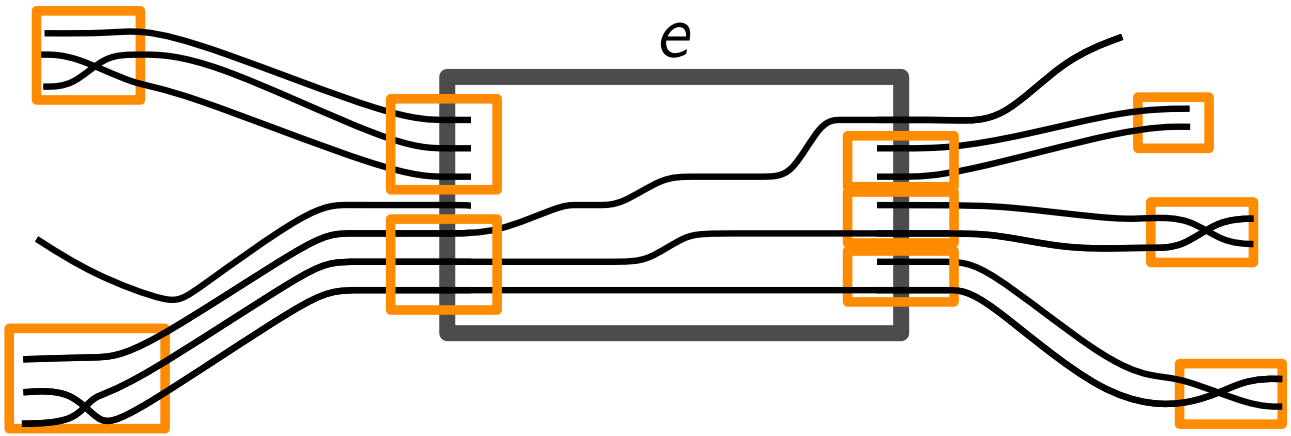
General Graphs – Sorting an Edge



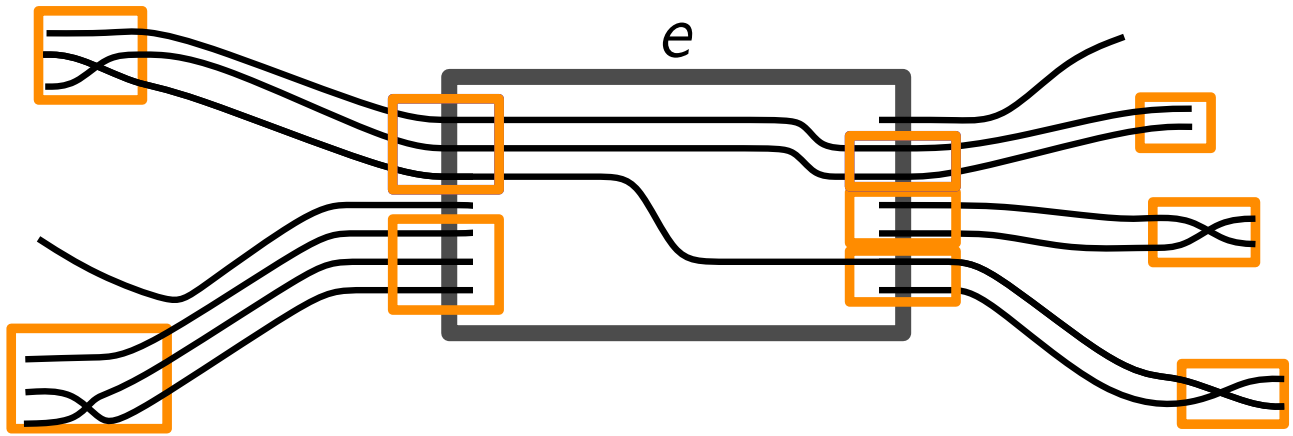
- follow lines
- find cut edges
- identify *groups* of lines

General Graphs – Sorting an Edge



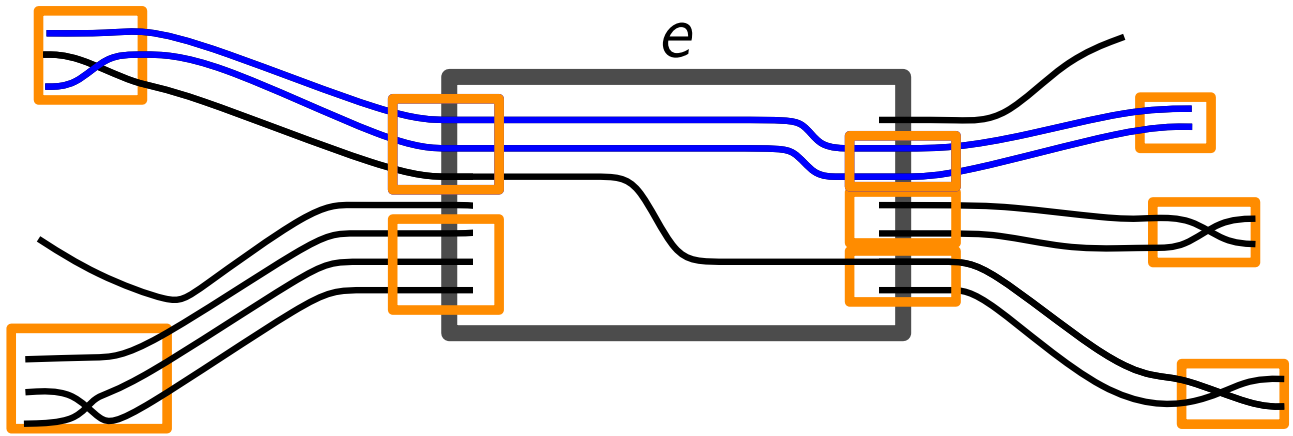
- follow lines
- find cut edges
- identify *groups* of lines
- group stays parallel

General Graphs – Sorting an Edge



- follow lines
- find cut edges
- identify *groups* of lines
- group stays parallel
- consider pairs of groups

General Graphs – Sorting an Edge



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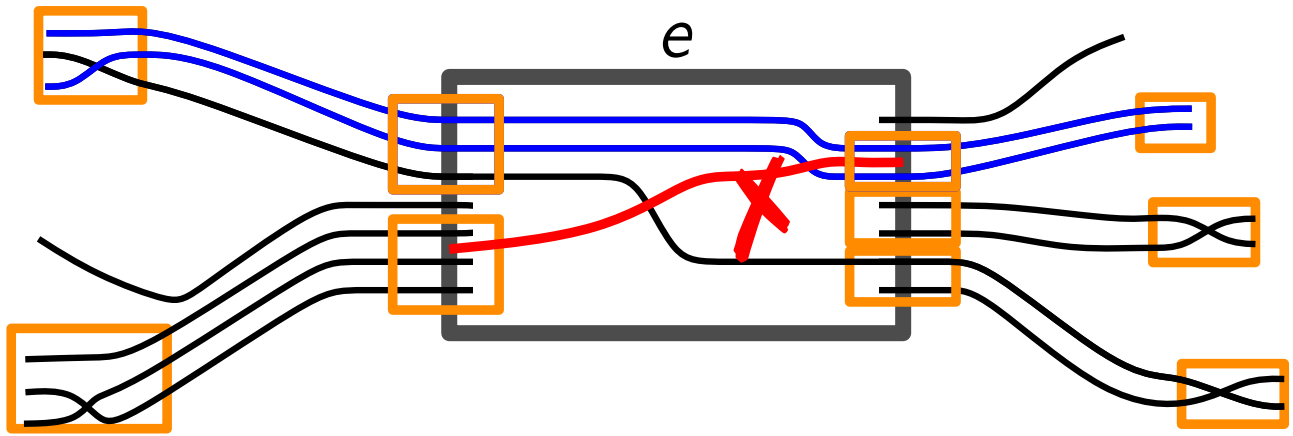

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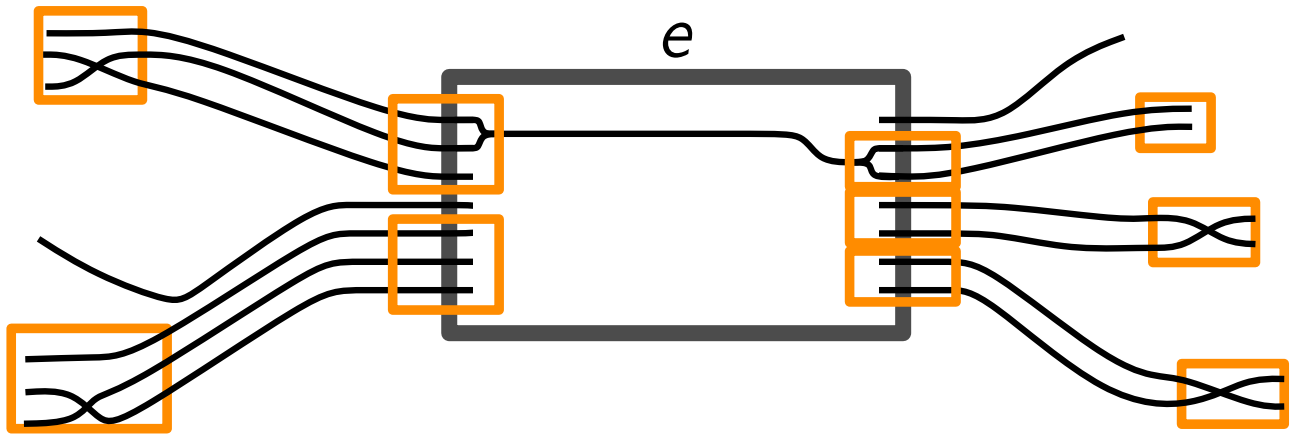

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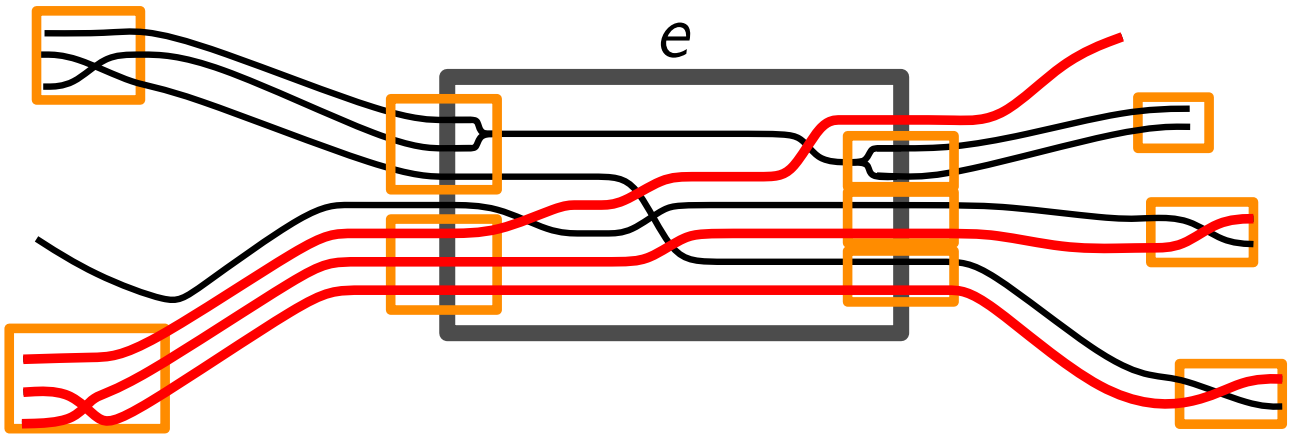

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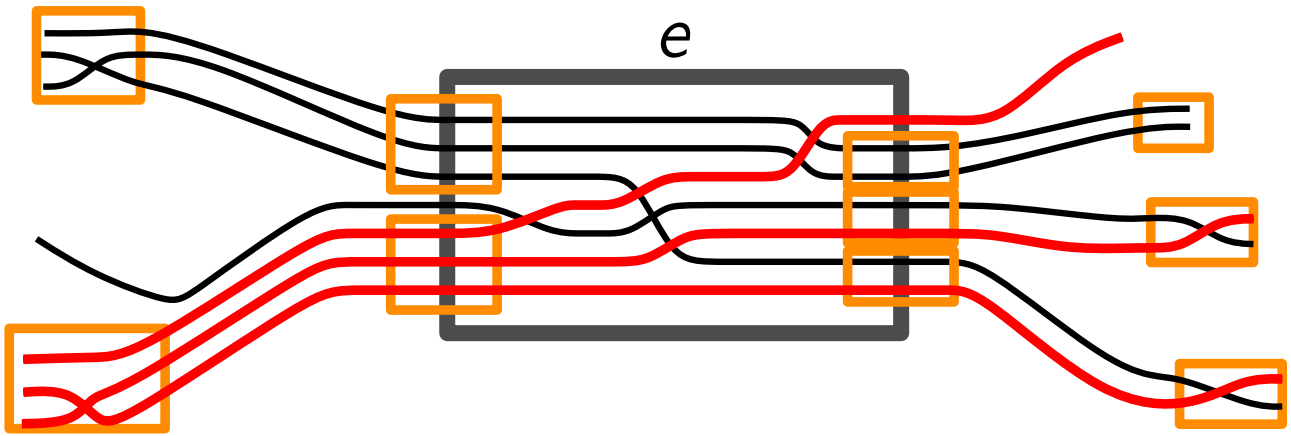

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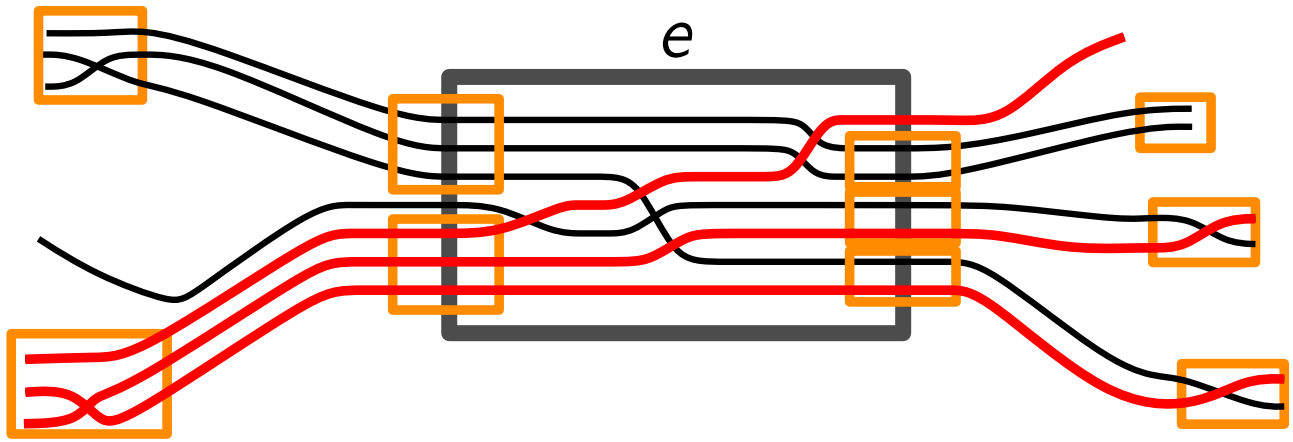
General Graphs – Analysis



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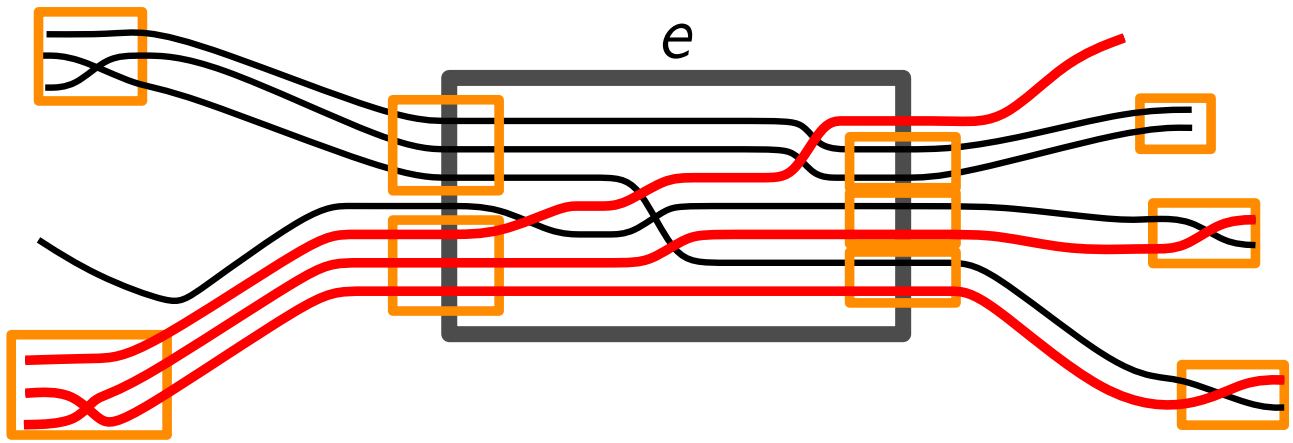

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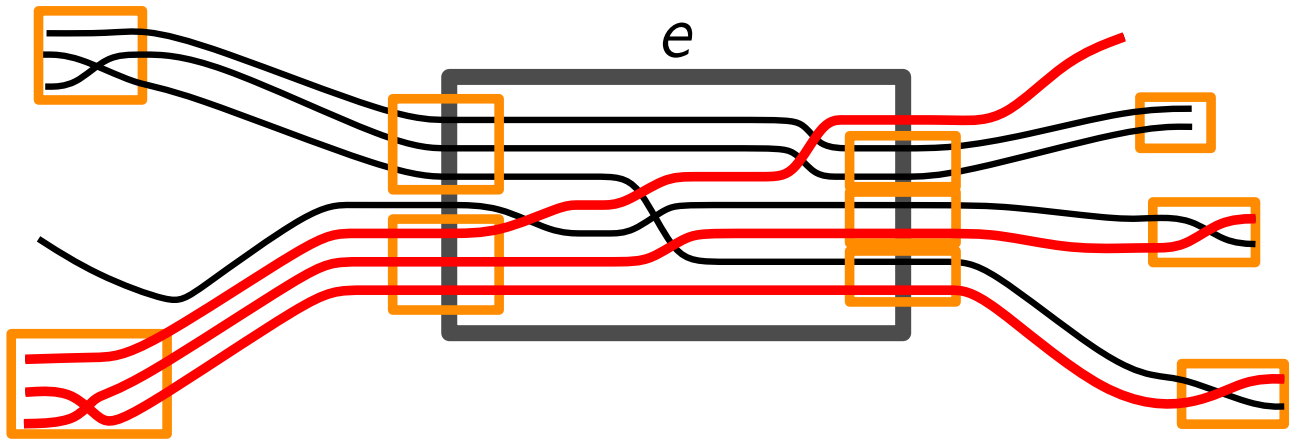

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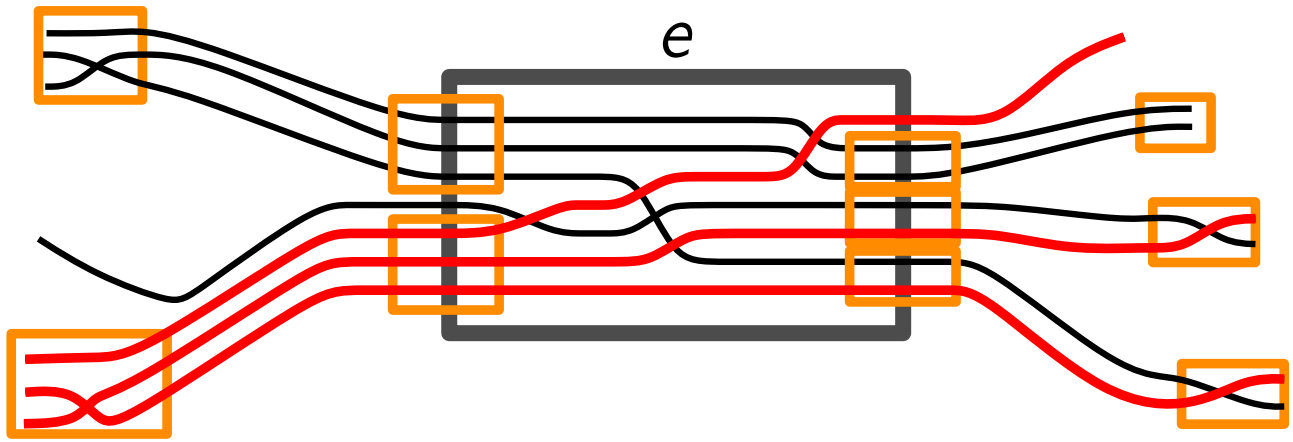

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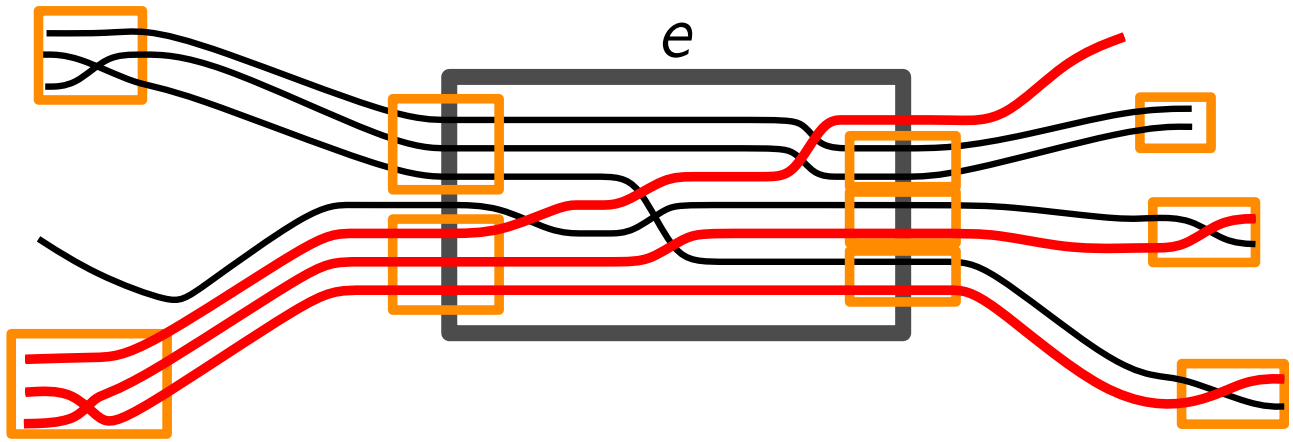

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- $\sum_{e \in E} (\mathrm{bc}(e))^2 \leq |L|^2$
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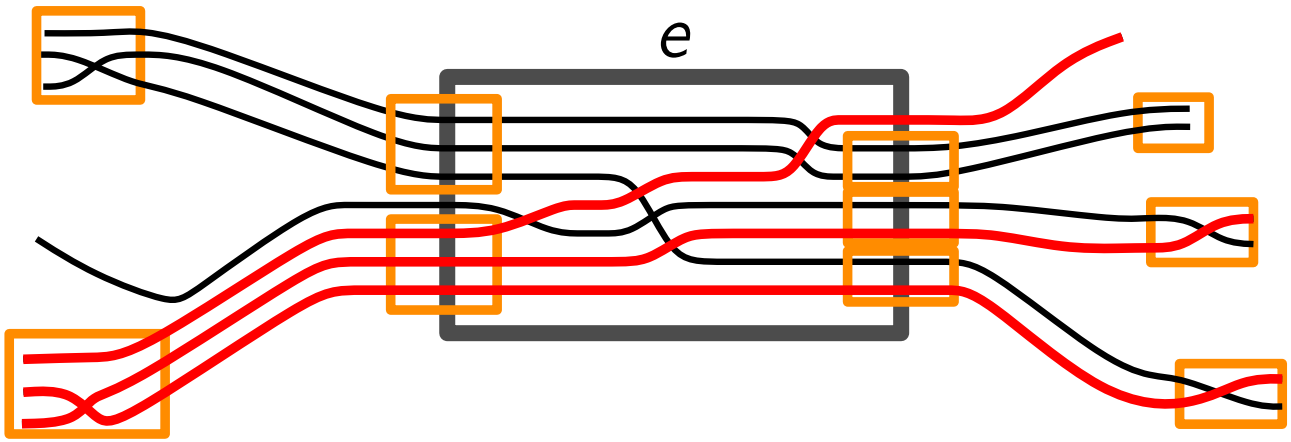

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worst-case instances: $\Omega(|L|\sqrt{|E'|})$ block crossings necessary

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  - information gain
  - block crossings on edge
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Conclusion

- new model for counting crossings
- approximations for paths and upward trees
- tight upper bound for trees
- algorithm for general graphs
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Thank you!