# Metro-Line Crossing Minimization: Hardness, Approximations, and Tractable Cases

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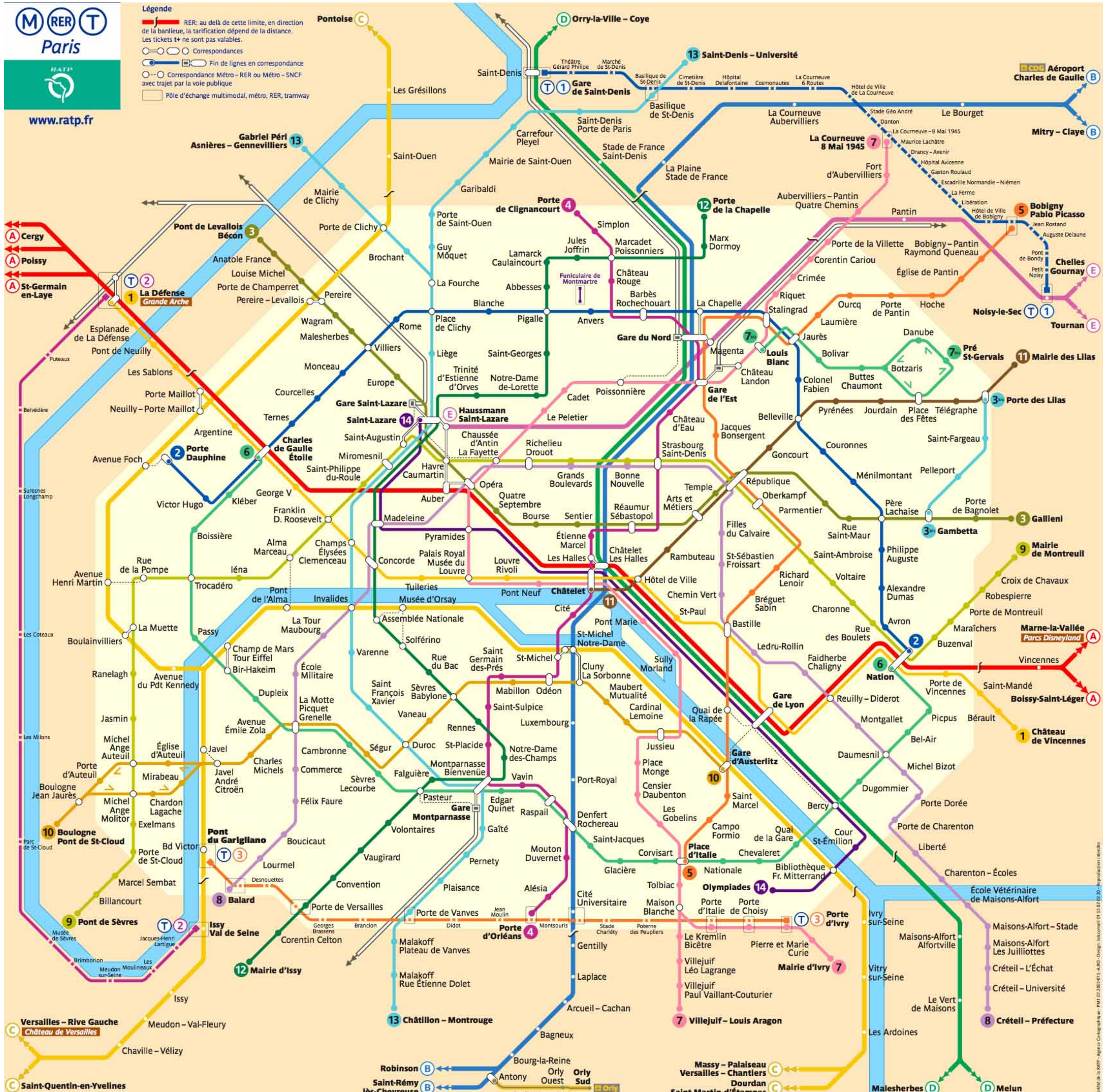
Lehrstuhl für Informatik I
Universität Würzburg

Joint work with Sergey Pupyrev

# Metro Maps – Bordeaux

# Metro Maps – Paris



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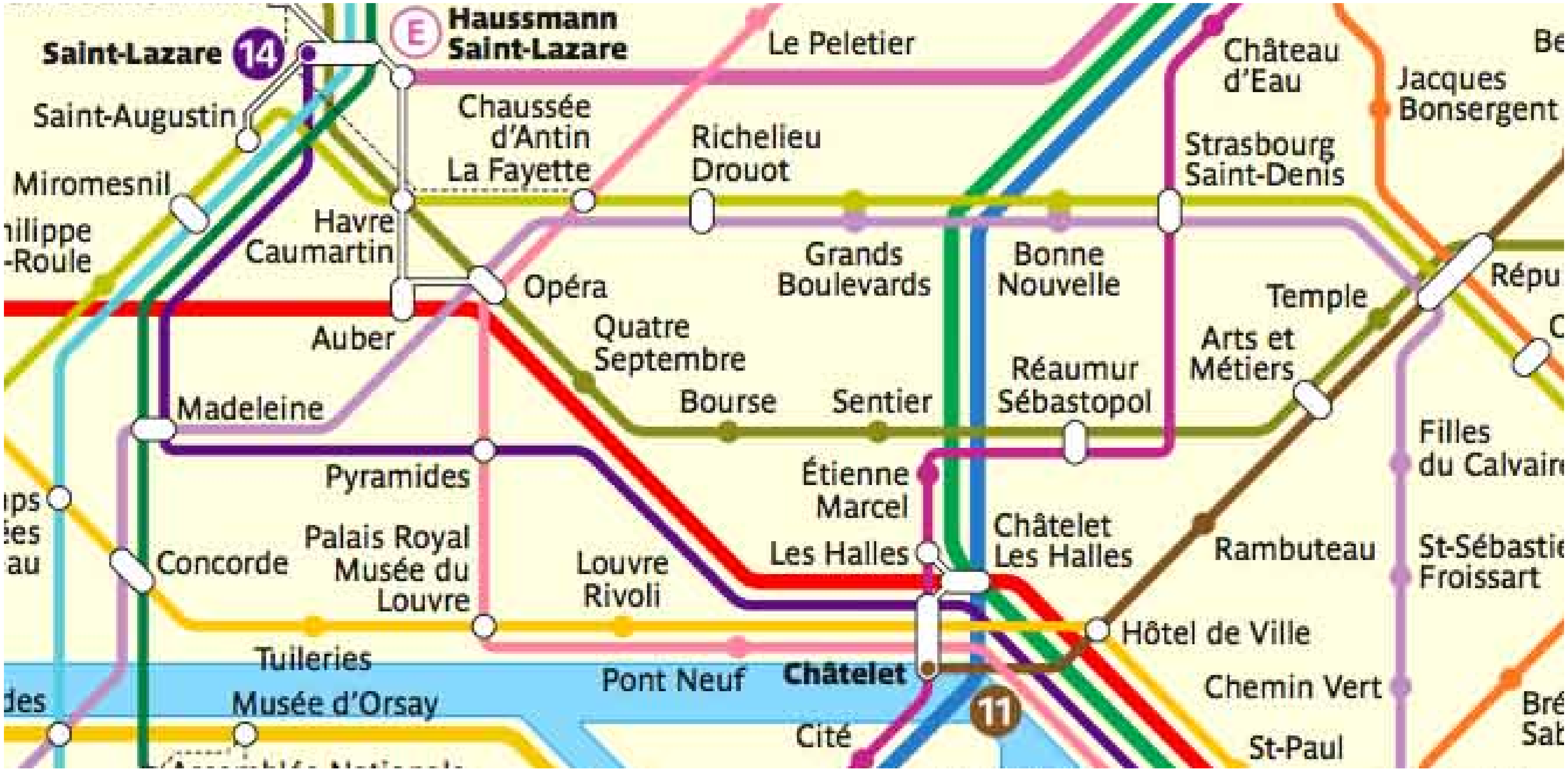

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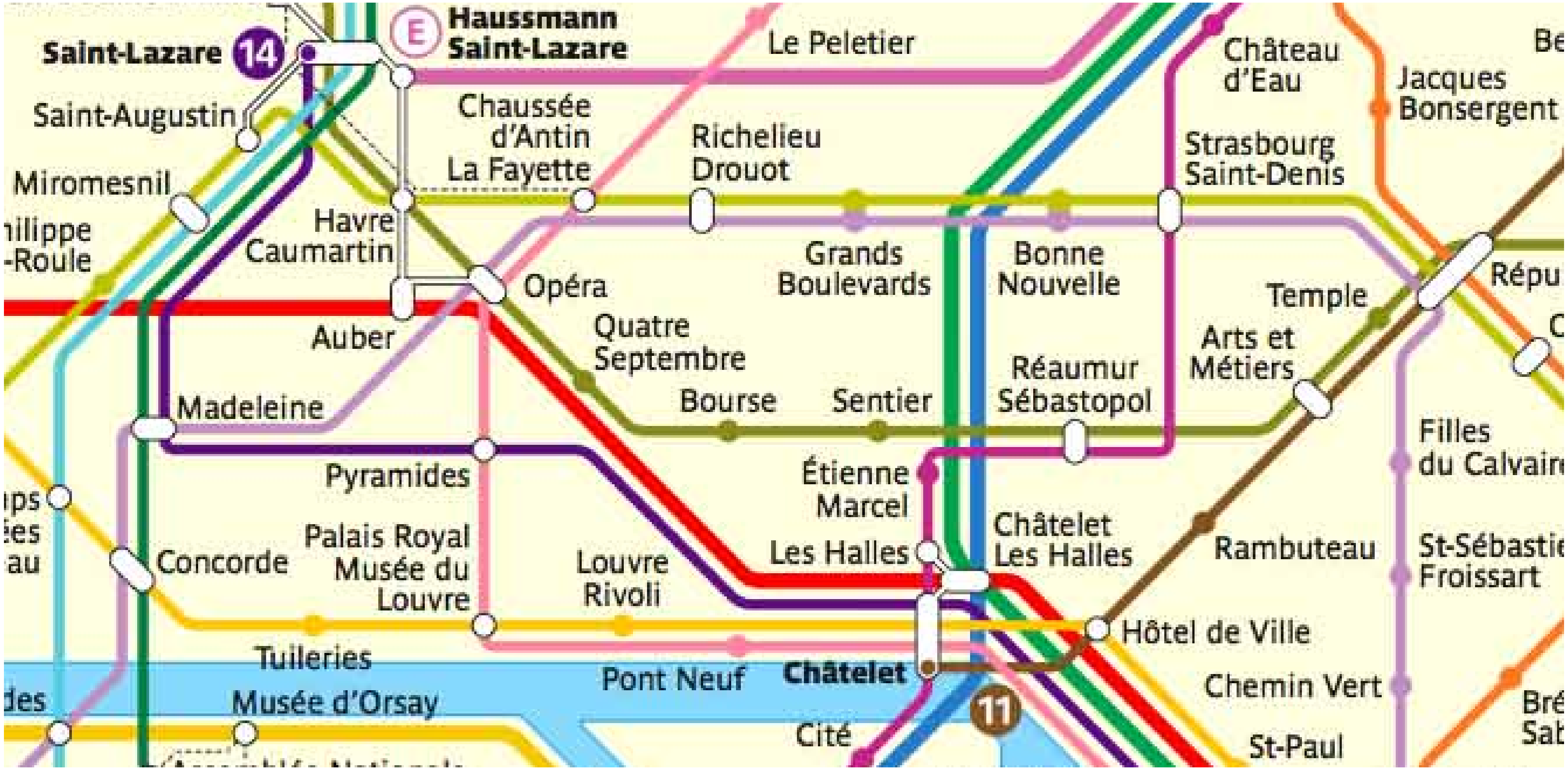


Previous work, e.g. [Nöllenburg and Wolff, 2011]
with focus on drawing underlying graph

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Insert all lines $L$ into *embedded graph* $G = (V, E)$ such that ...

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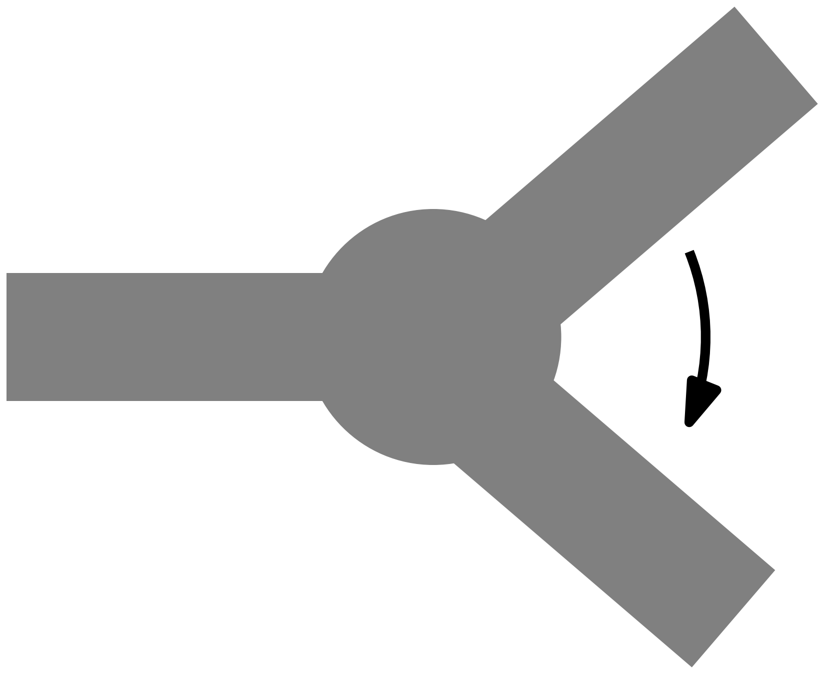

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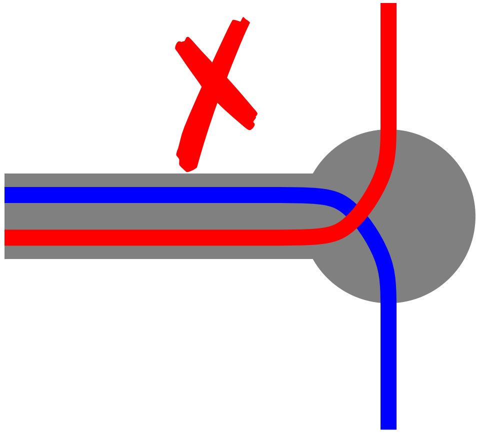

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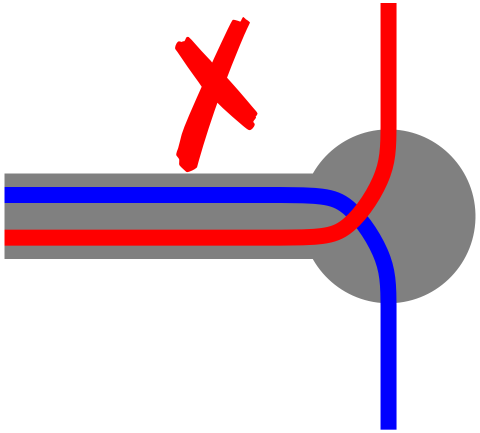

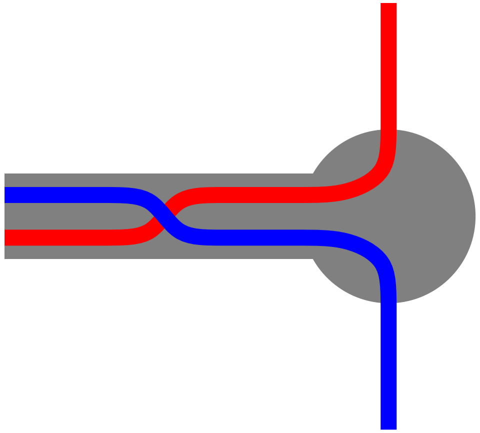

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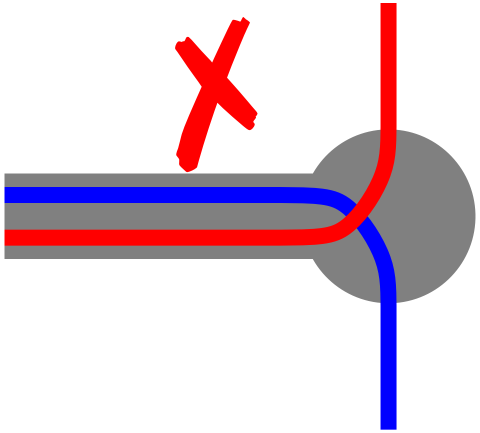

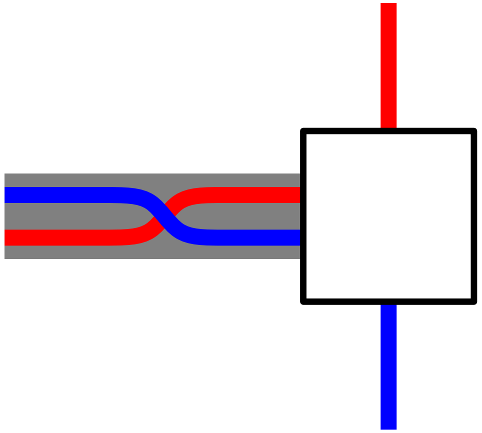

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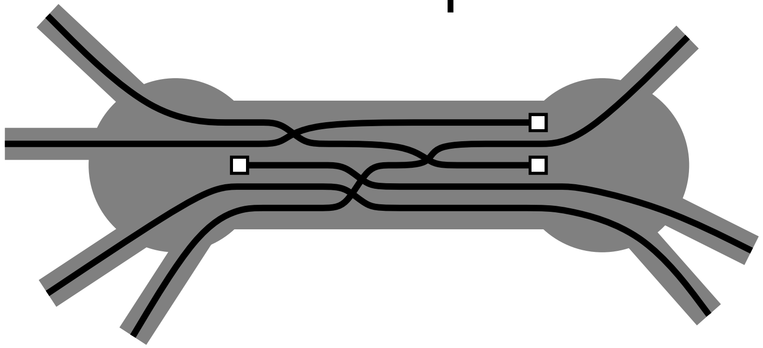
We assume that . . .

- lines are simple paths.
- lines intersect in paths.

# Problem Variants

- **MLCM**: line ends placed freely



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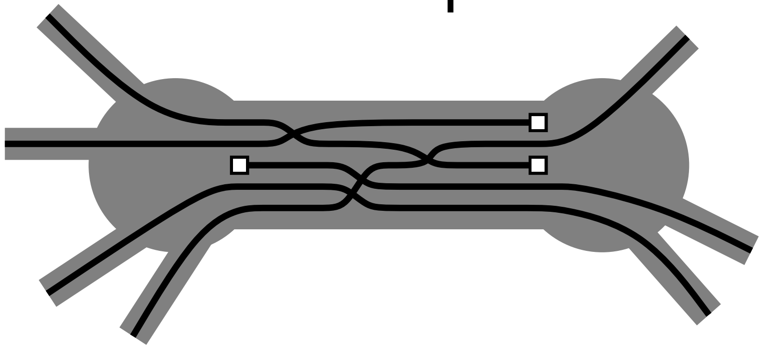
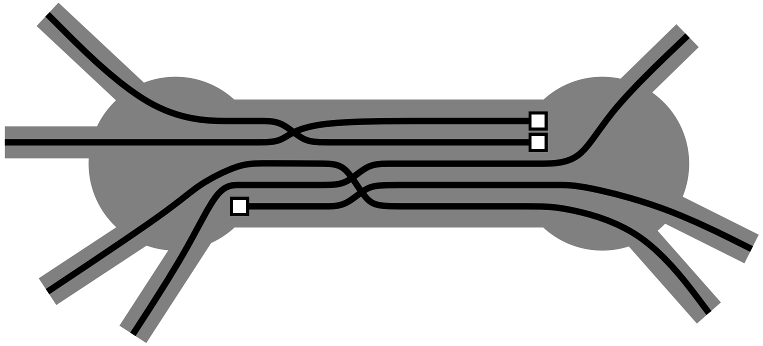


- **MLCM-P**: line ends placed outermost



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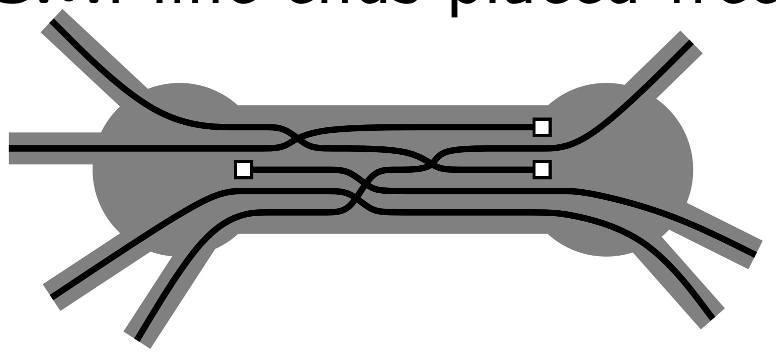

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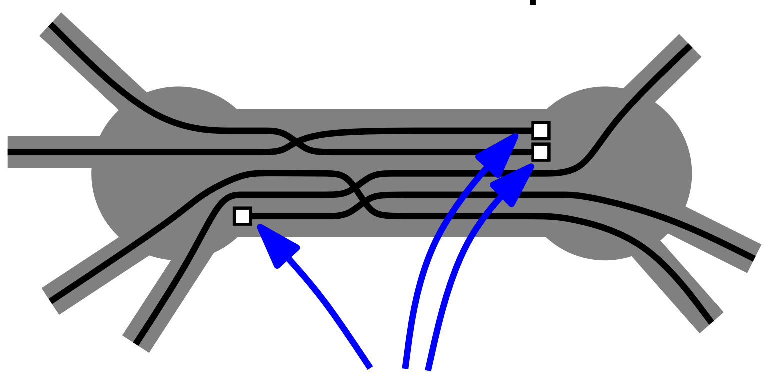
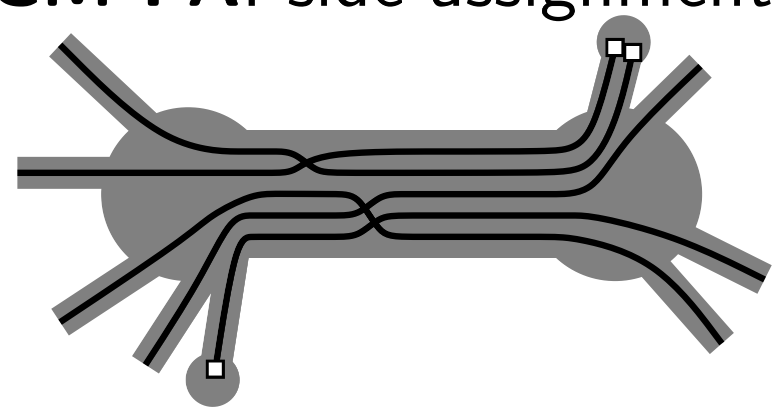


- **MLCM-PA**: side assignment for line ends



equivalent: line ends in degree-1 vertices

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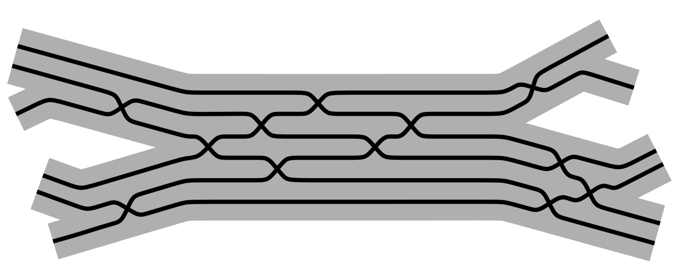
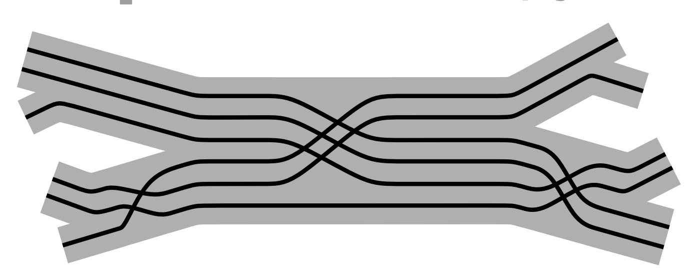

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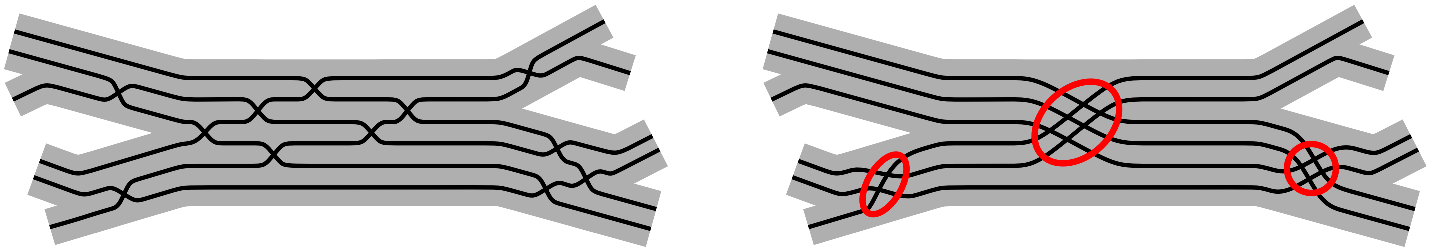

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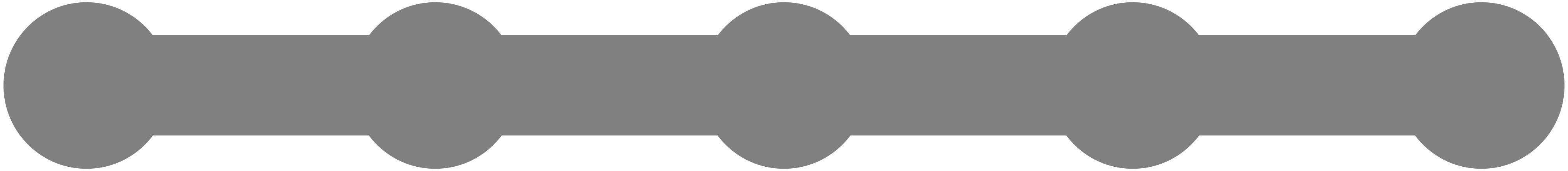

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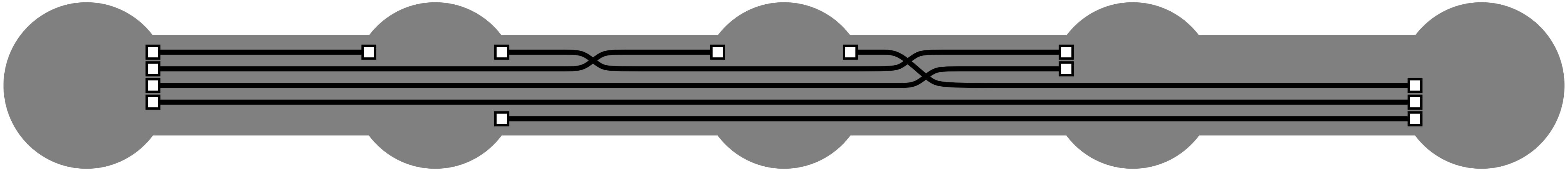

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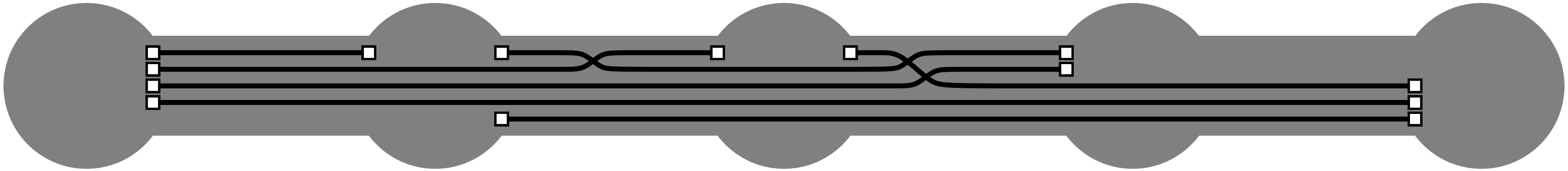

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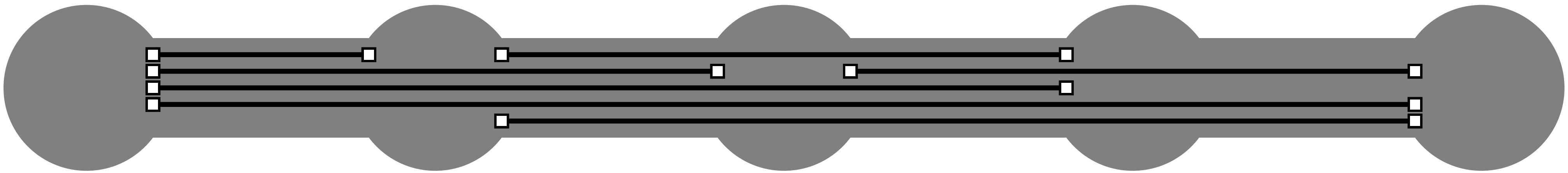

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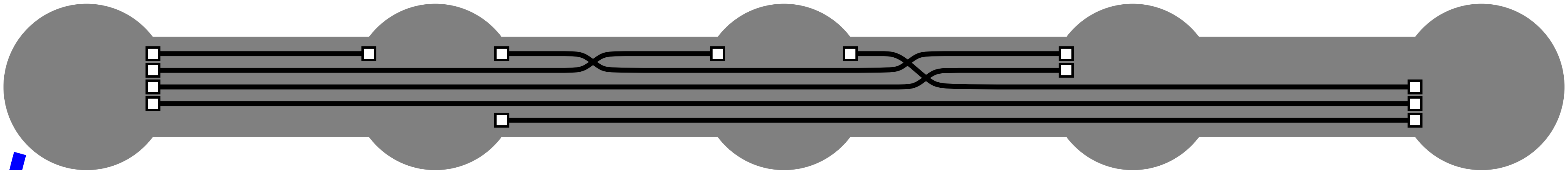

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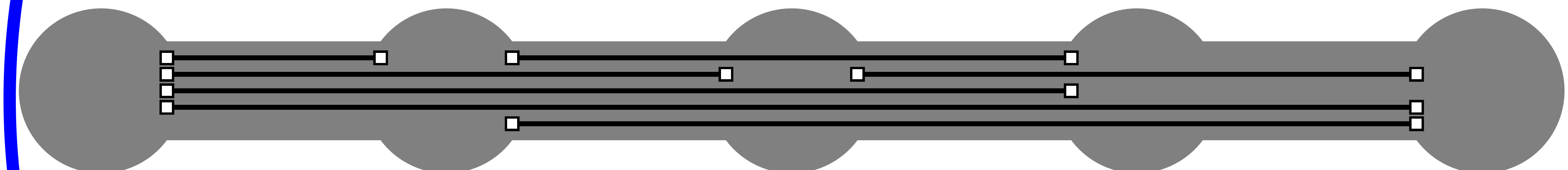

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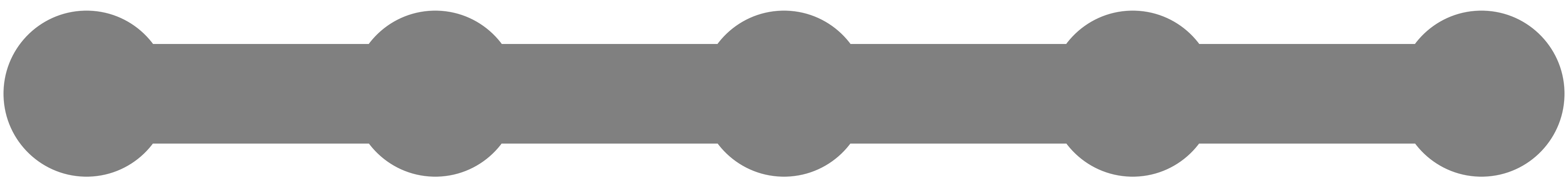

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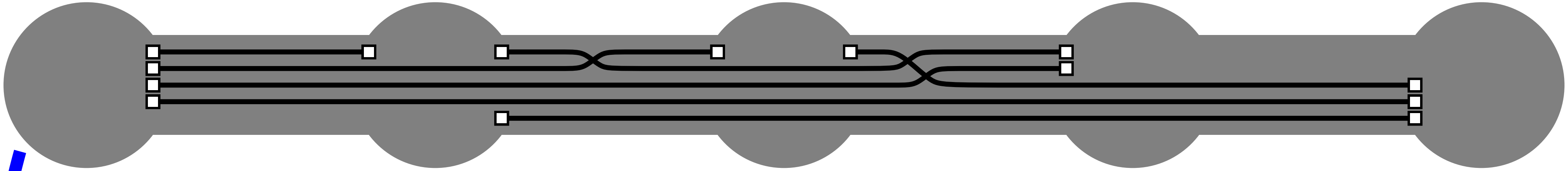

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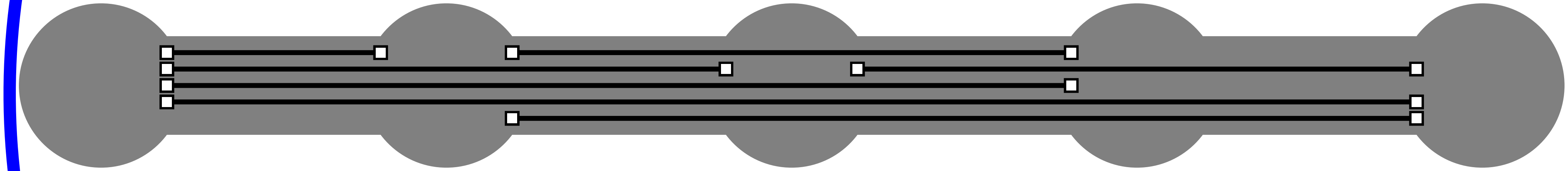

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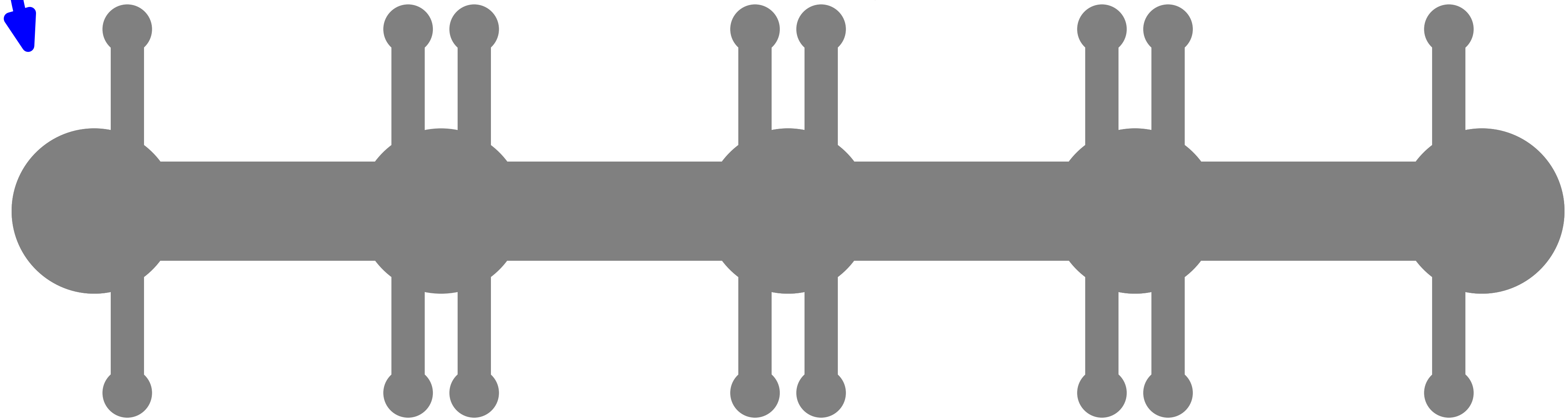

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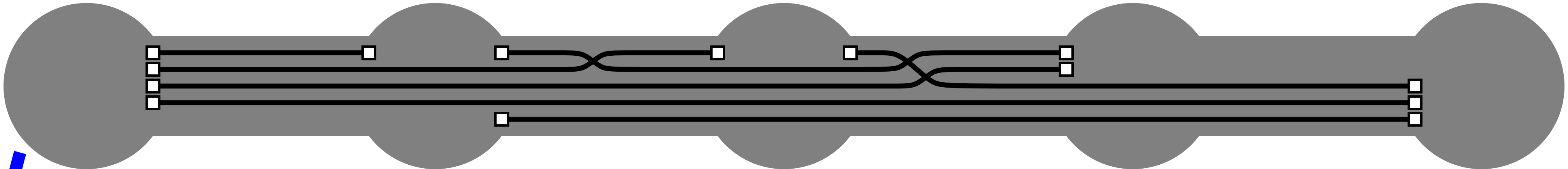

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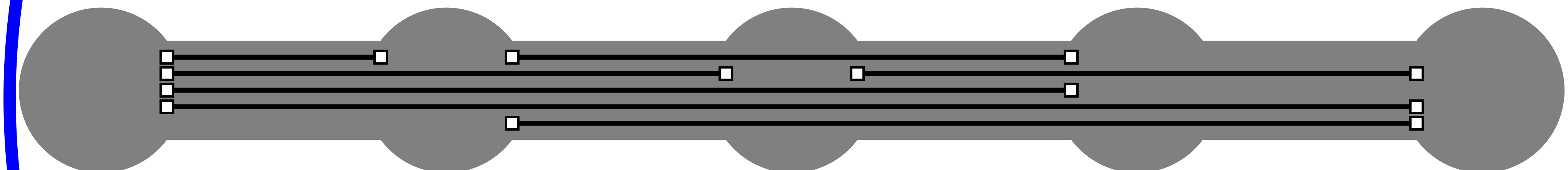

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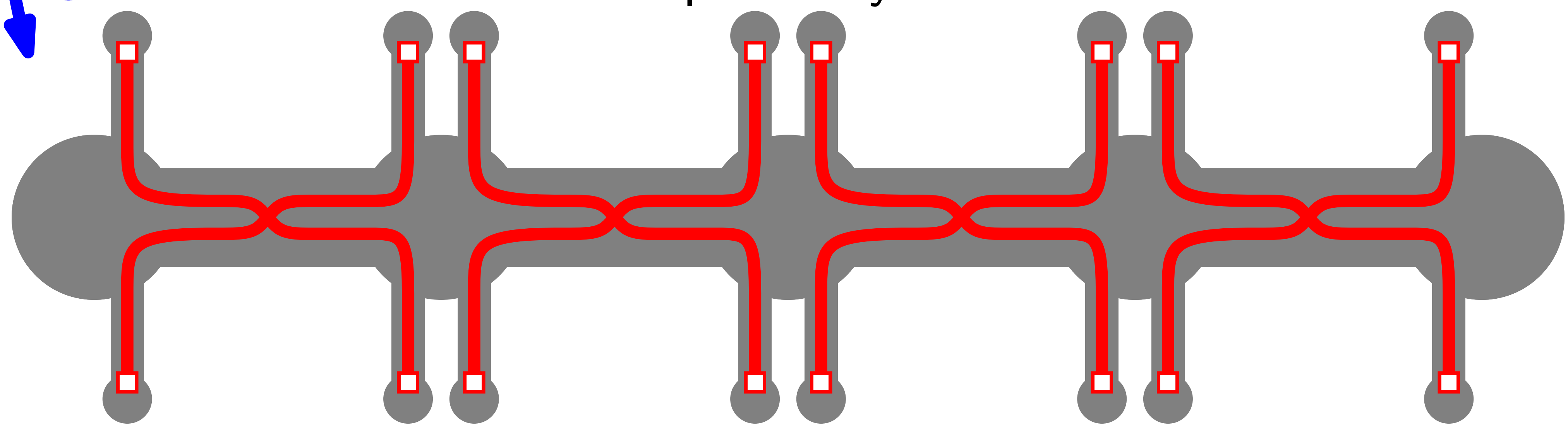

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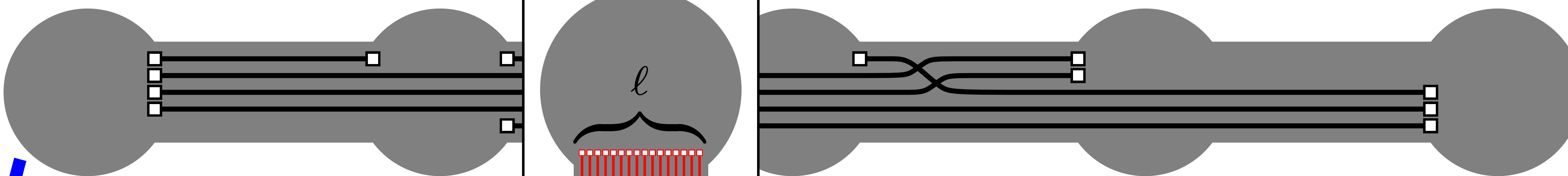

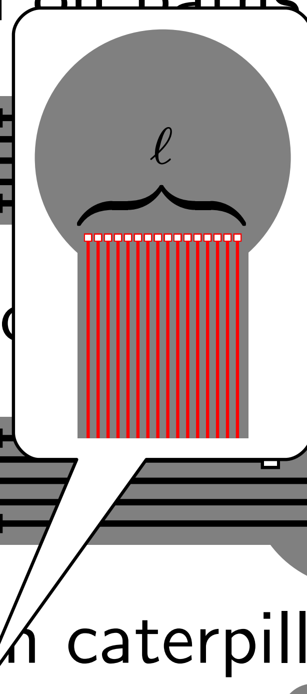


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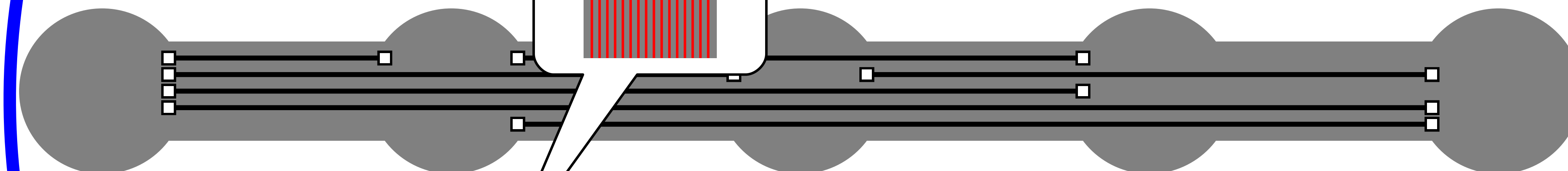

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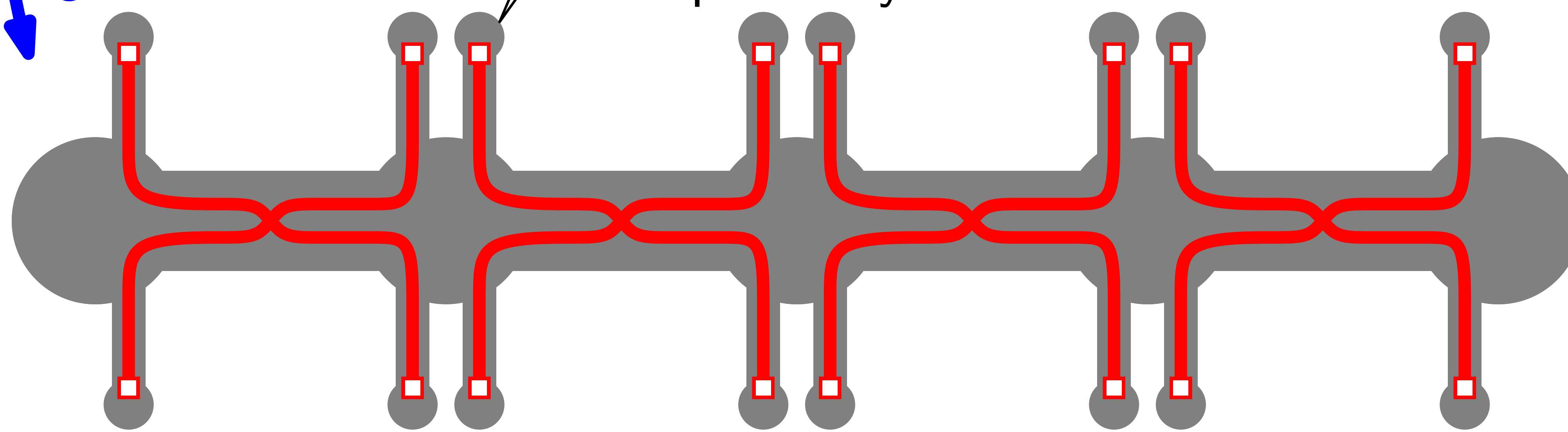

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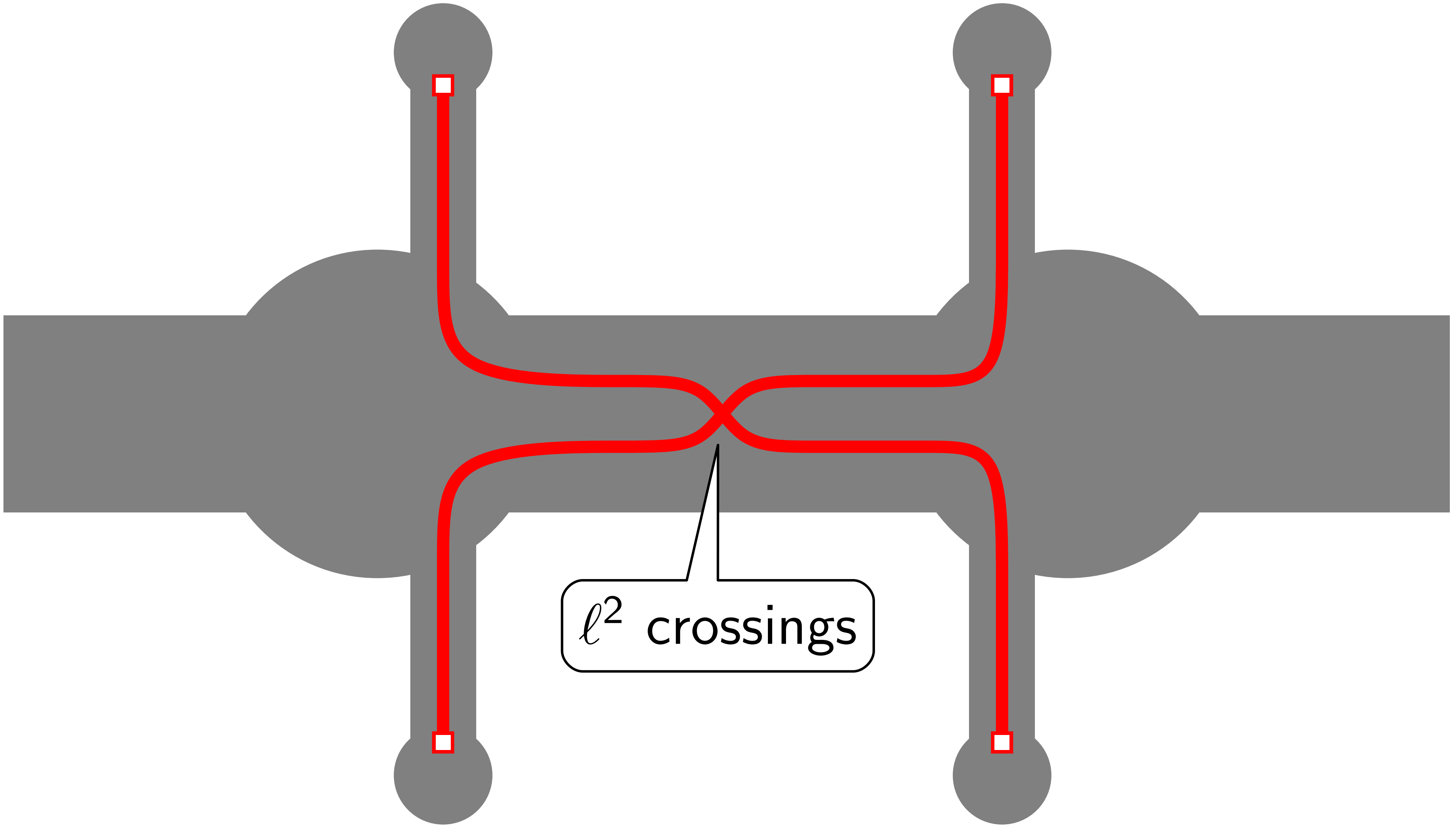

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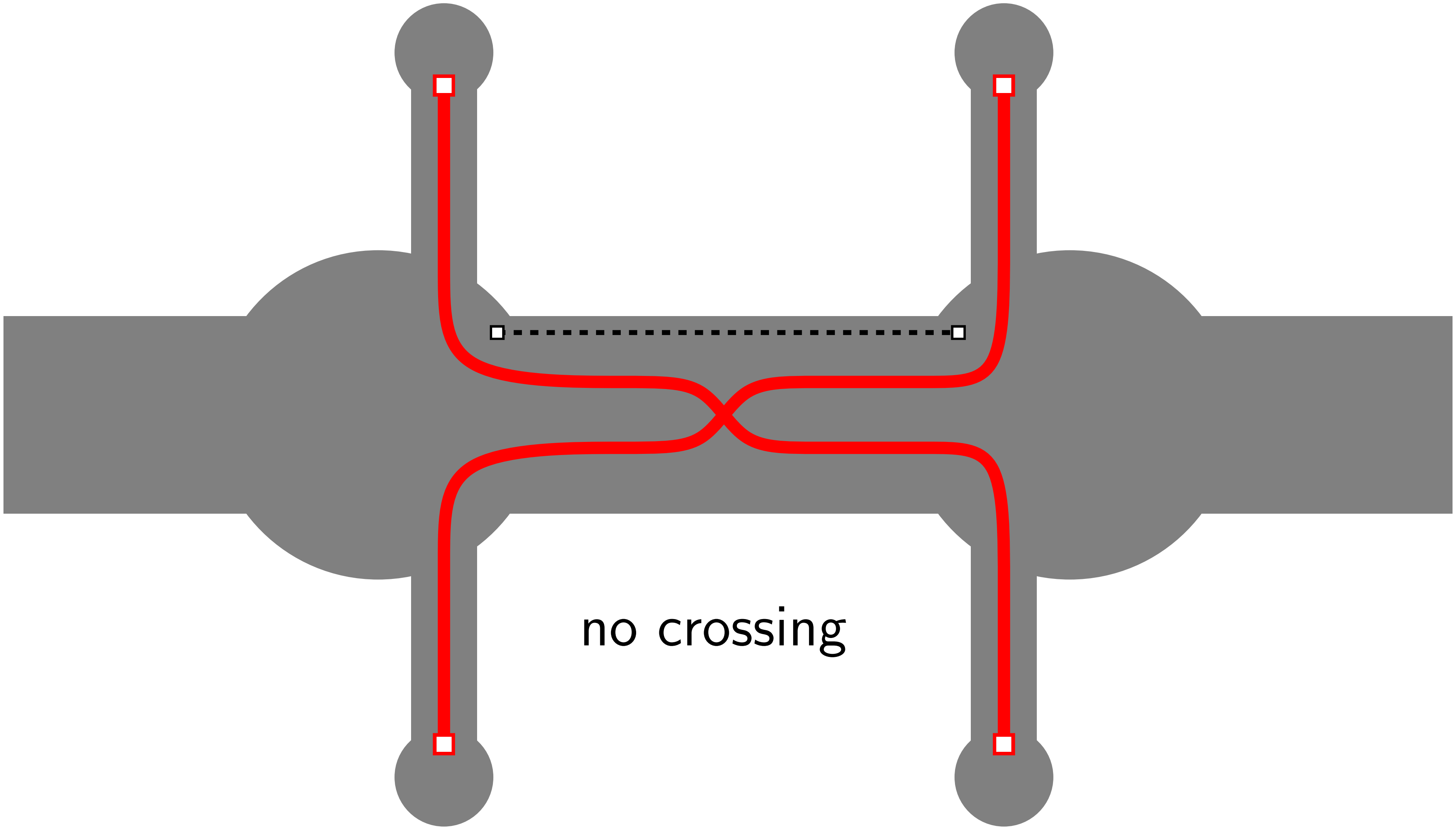

no crossing

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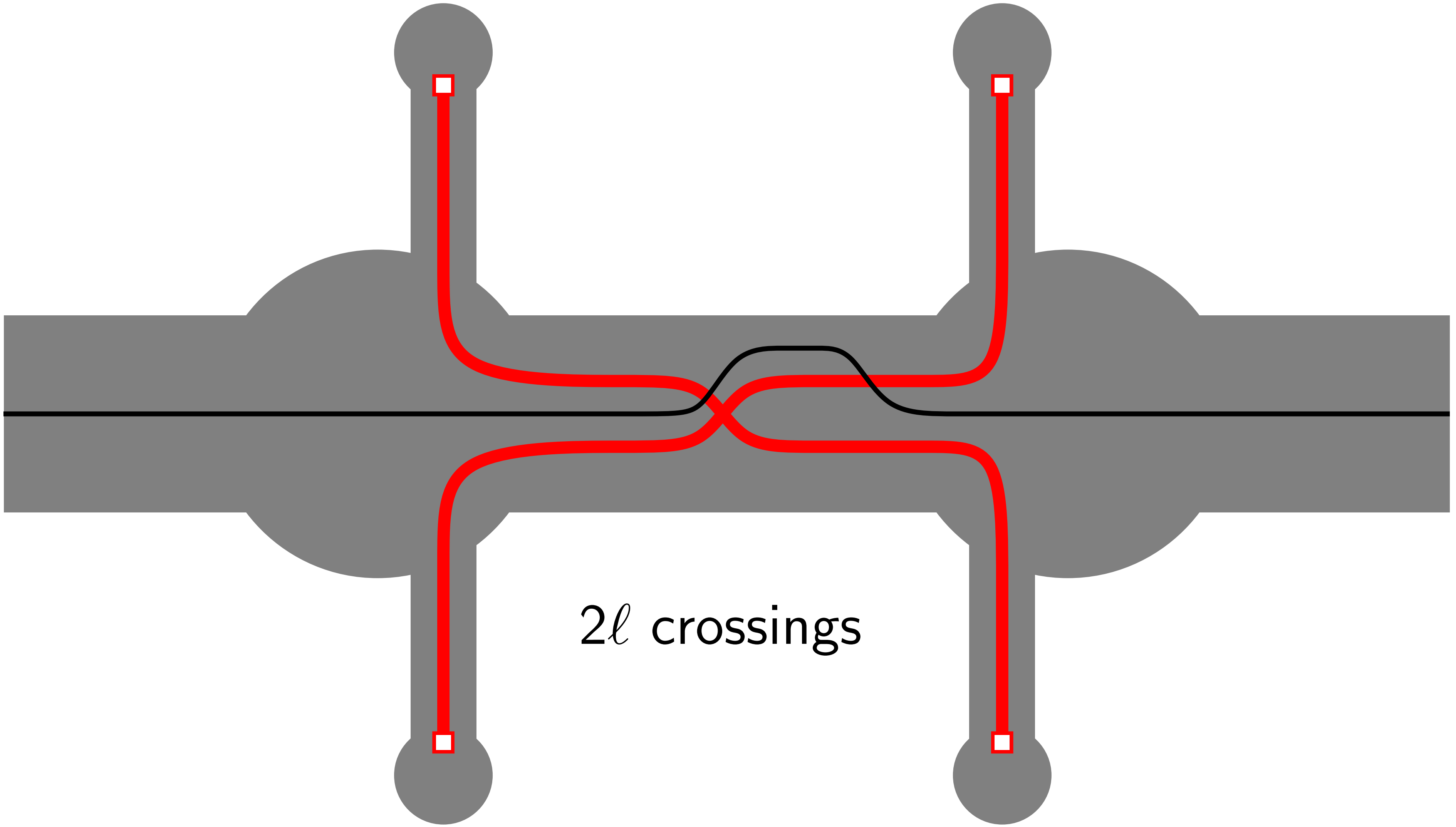

2$\ell$ crossings

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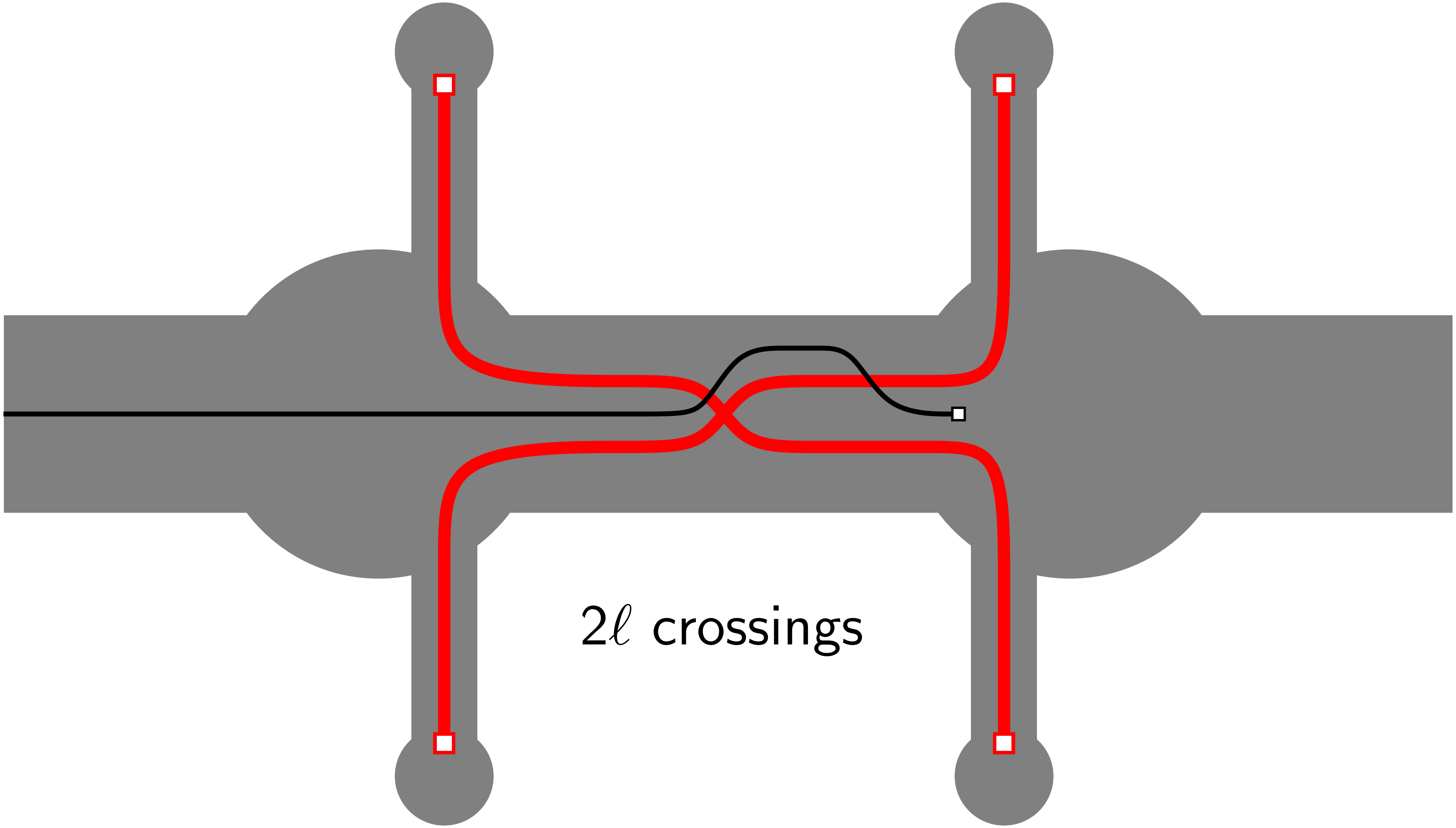

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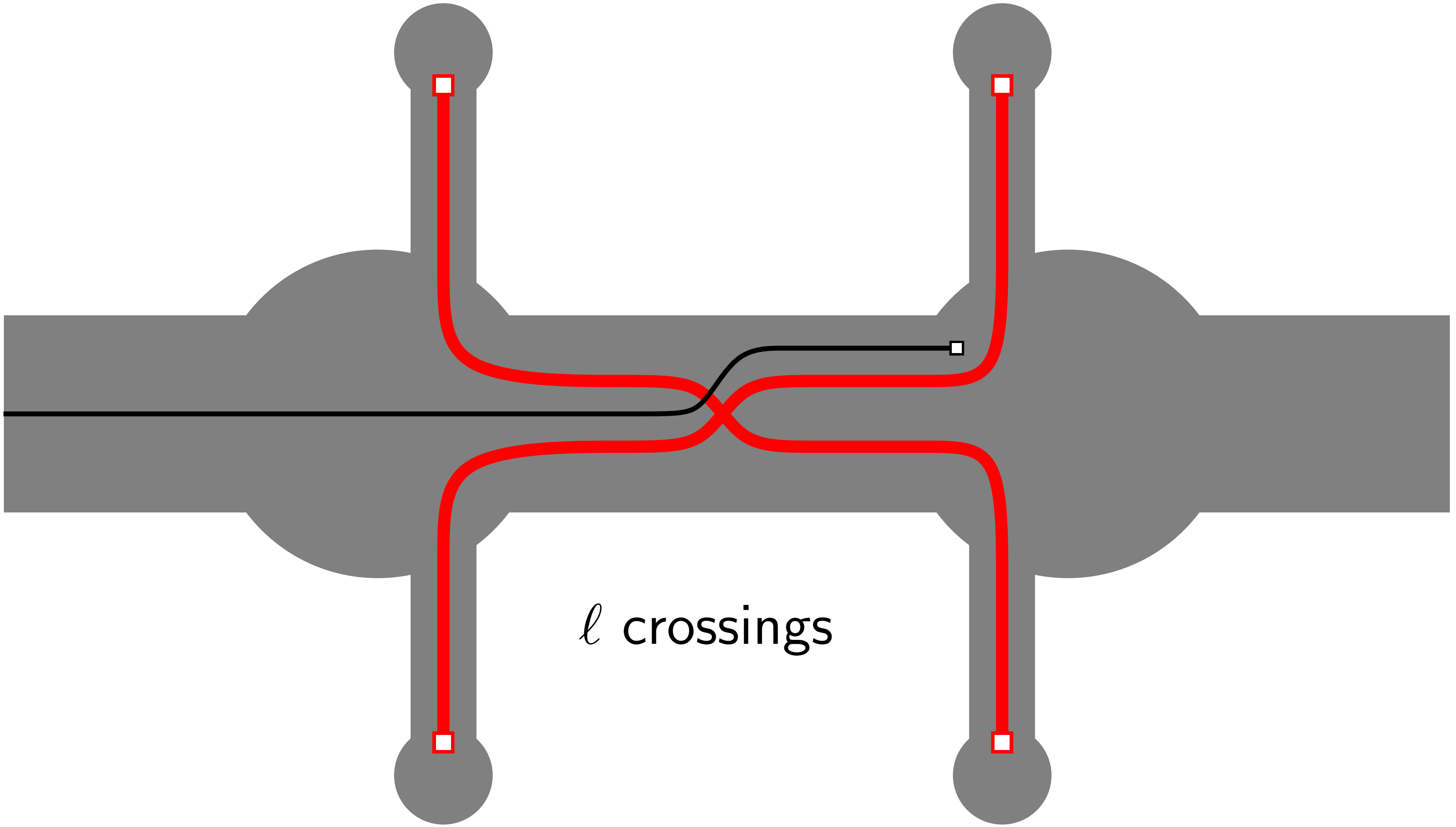

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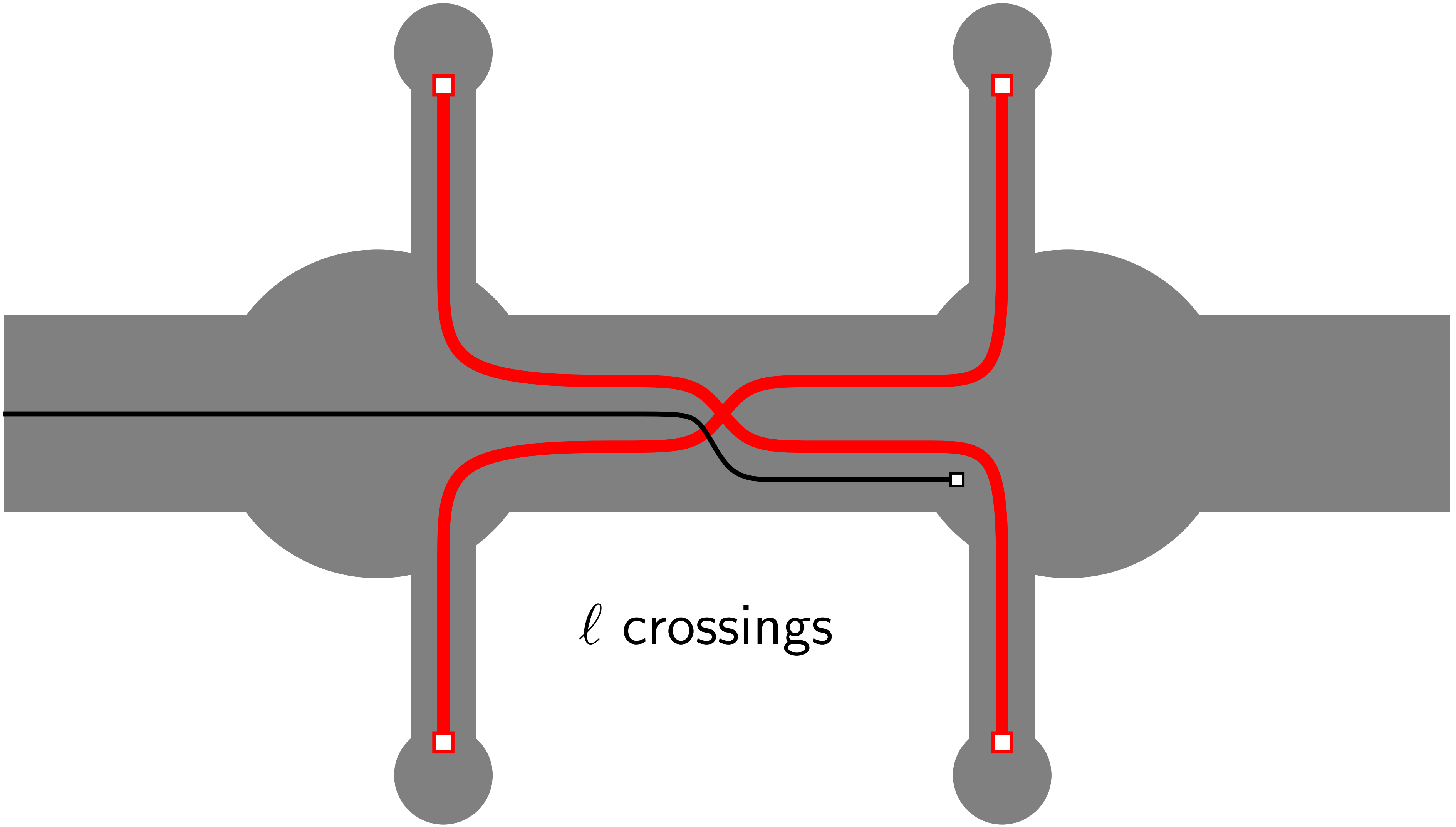

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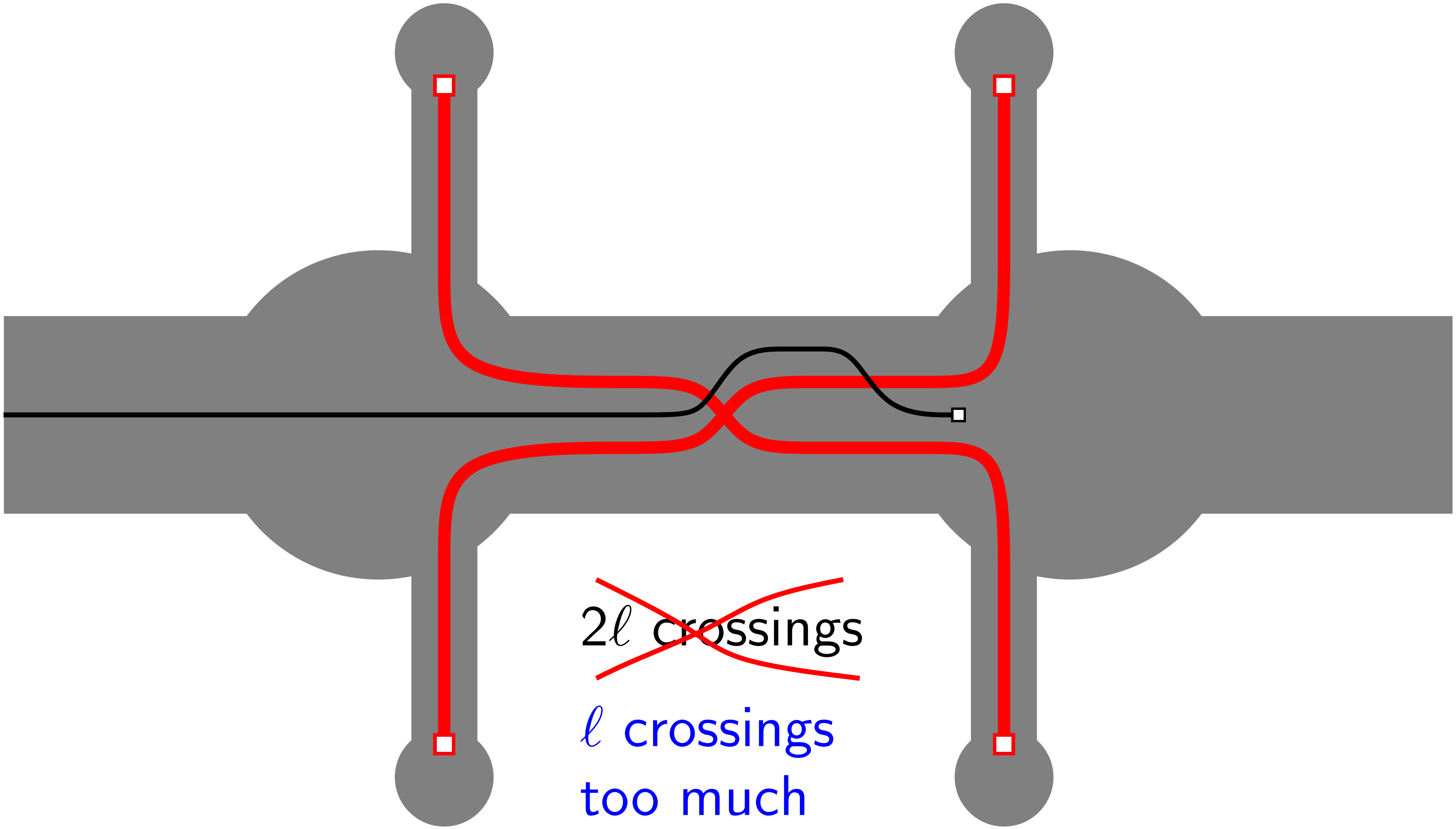

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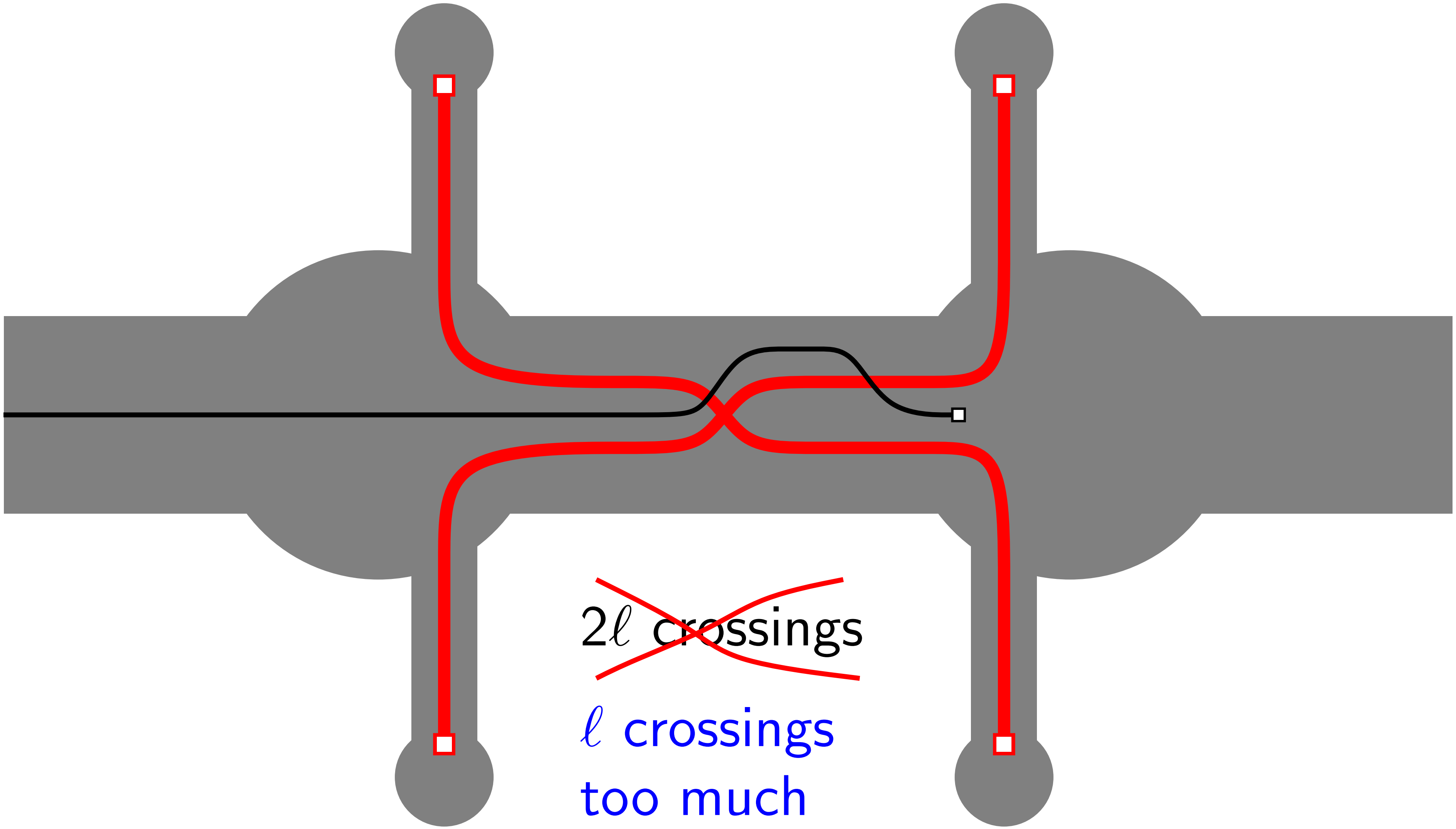


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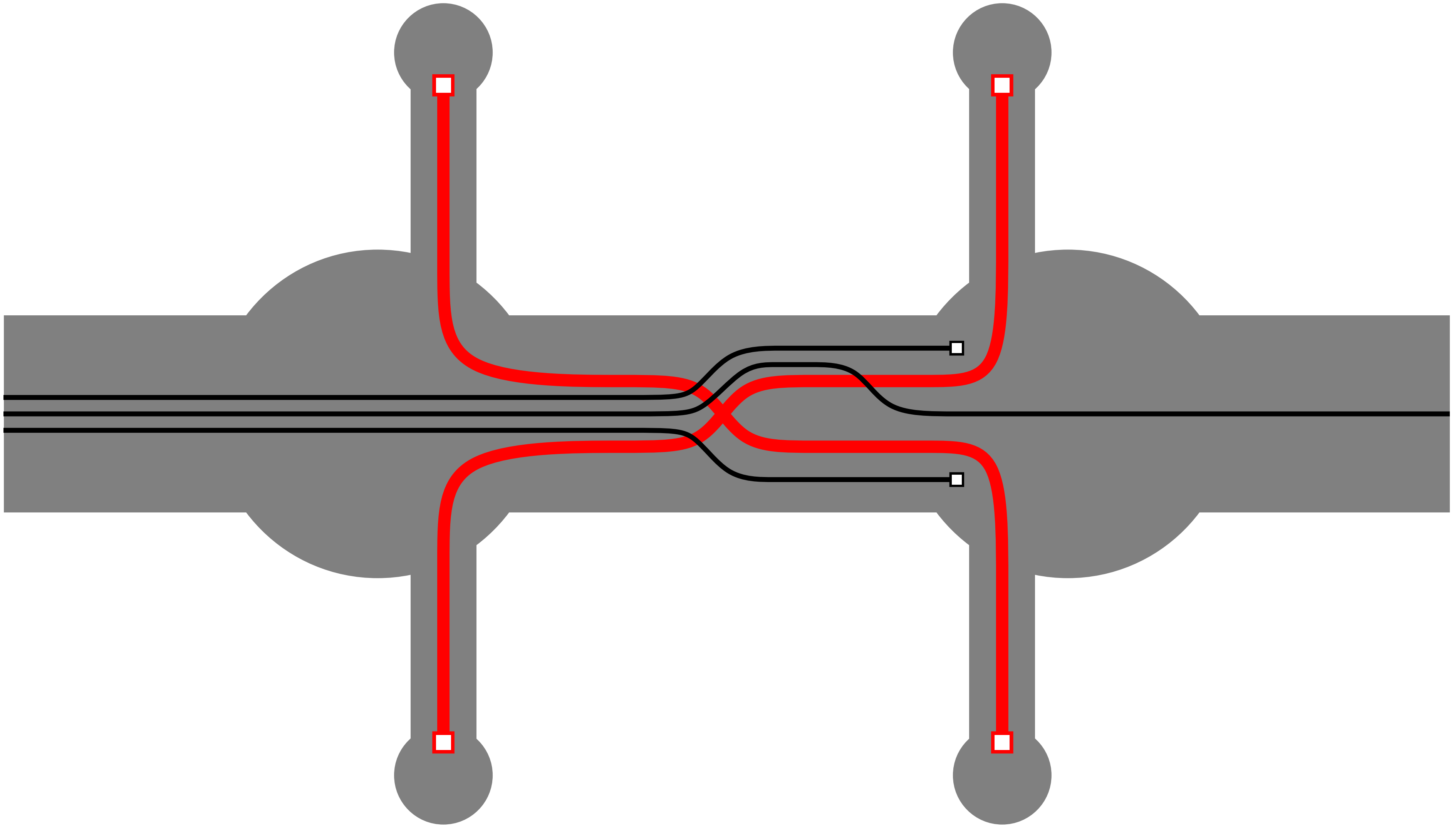

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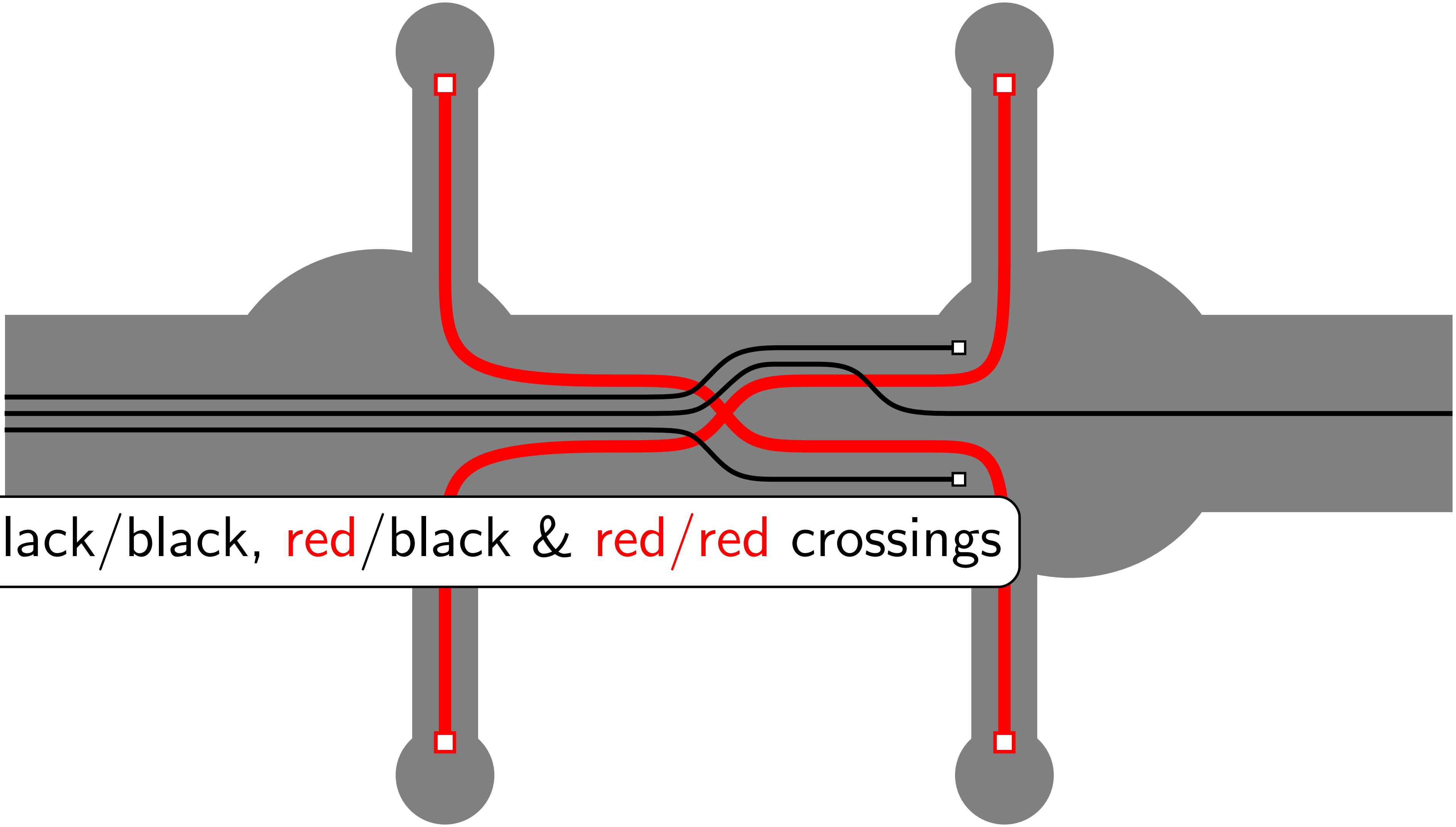

black/black, red/black & red/red crossings

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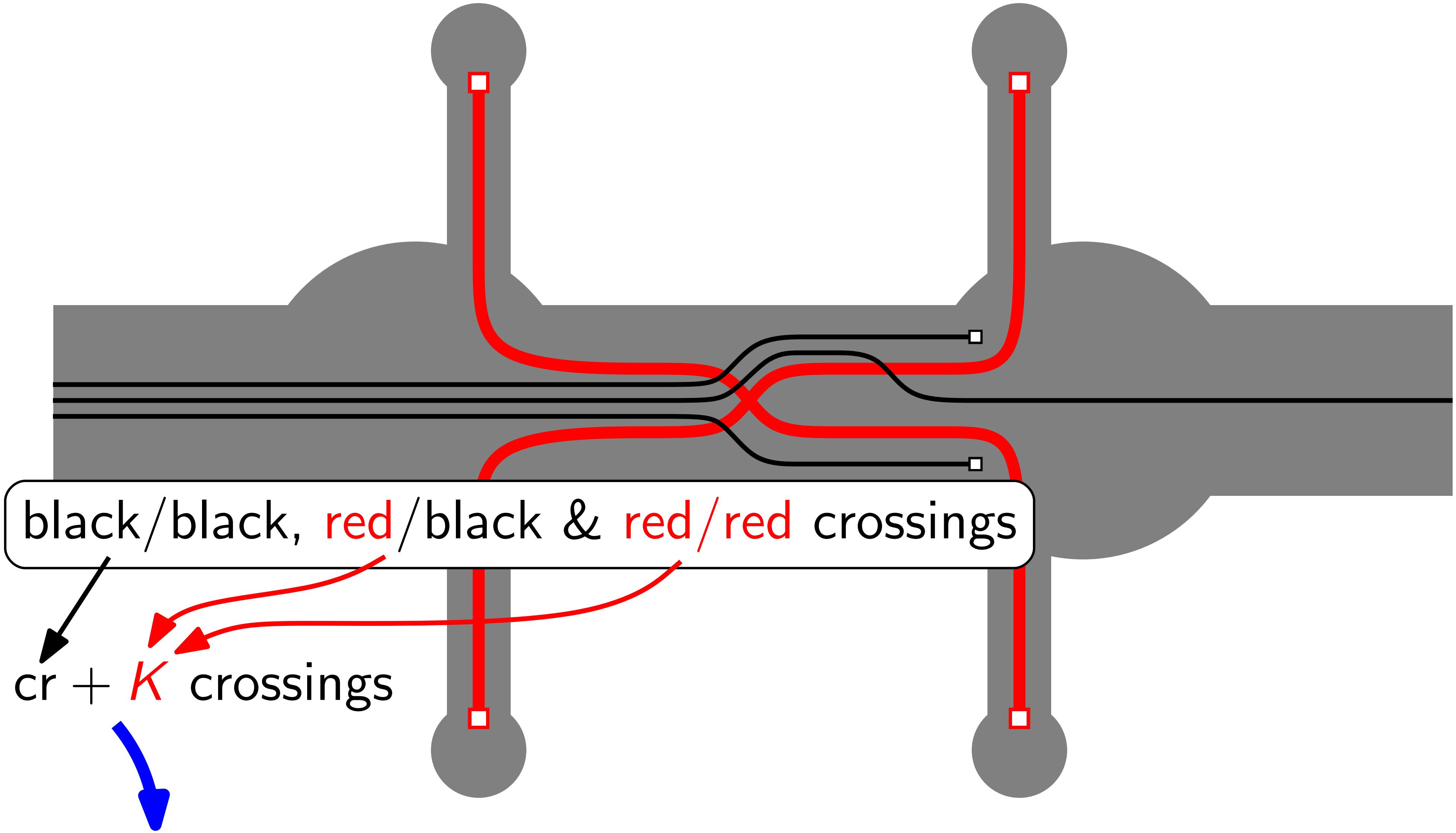


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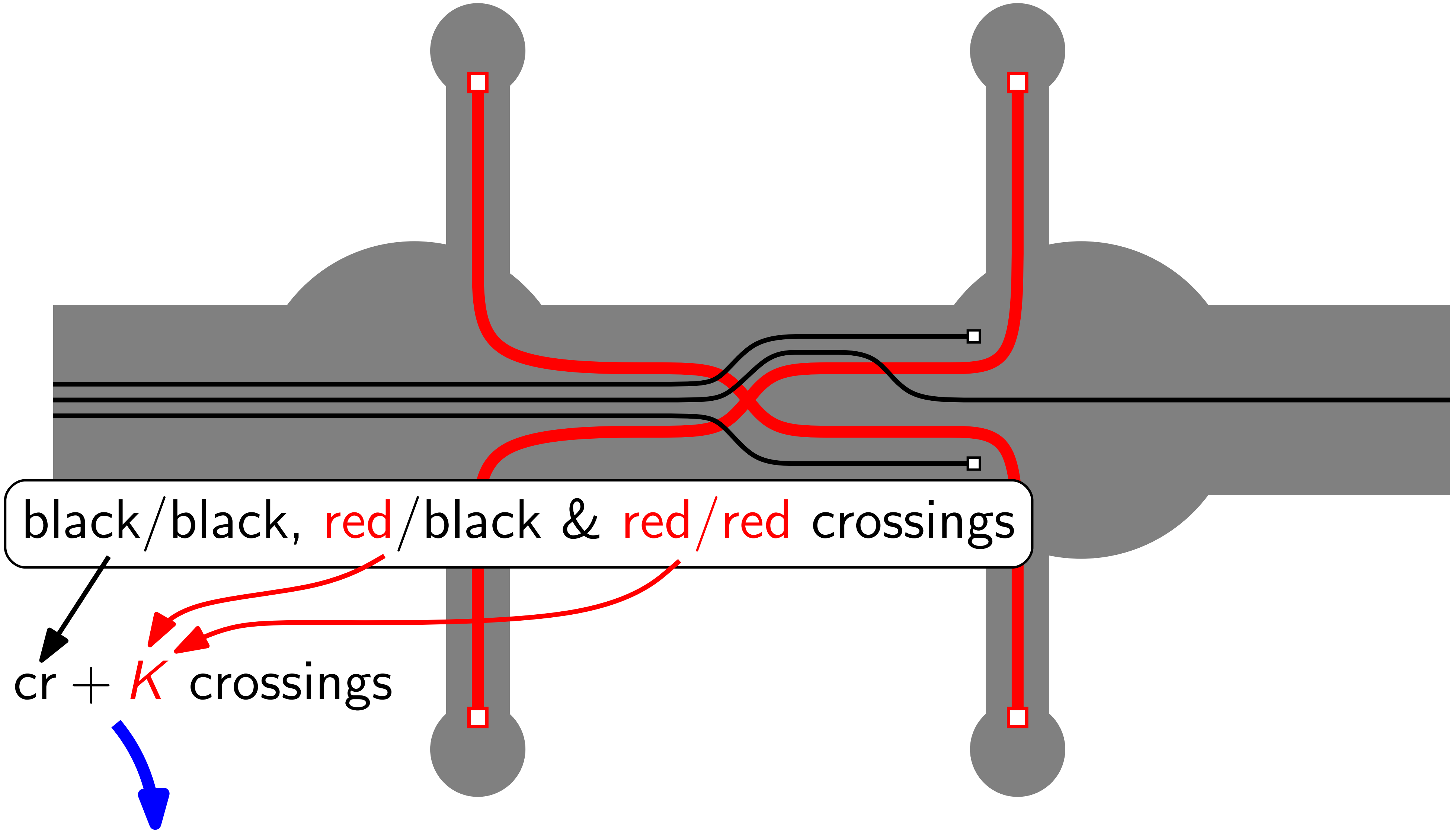


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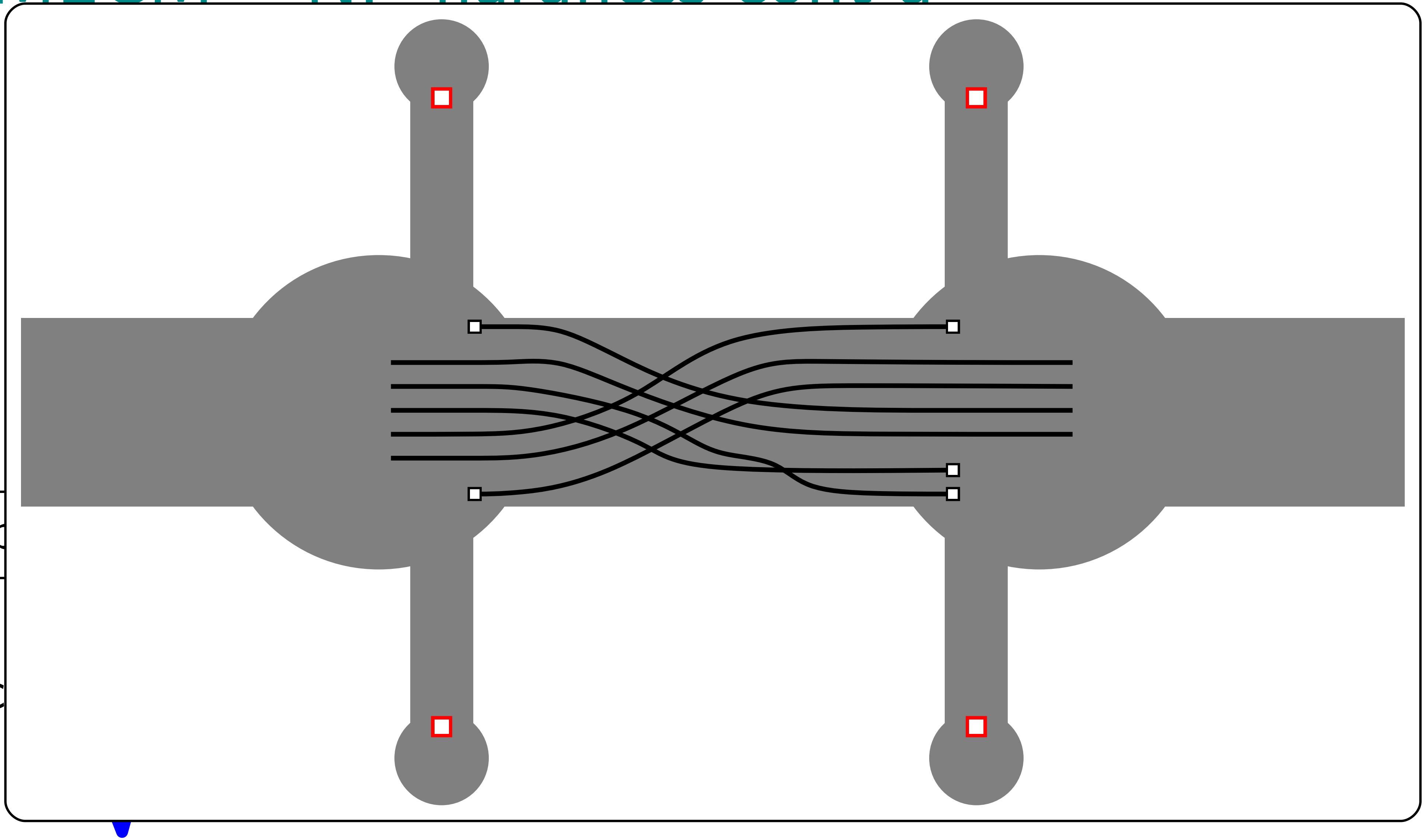

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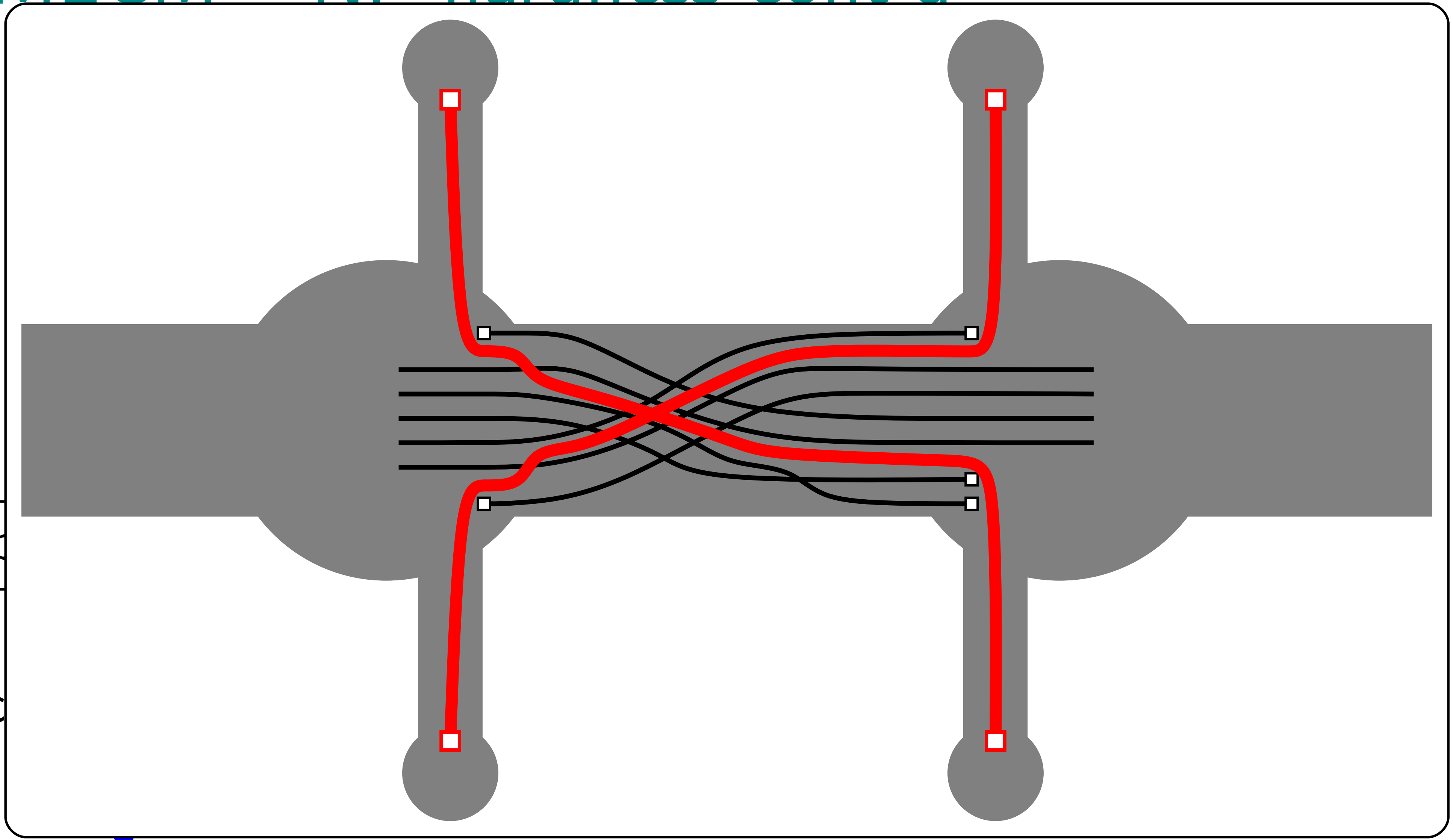

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c

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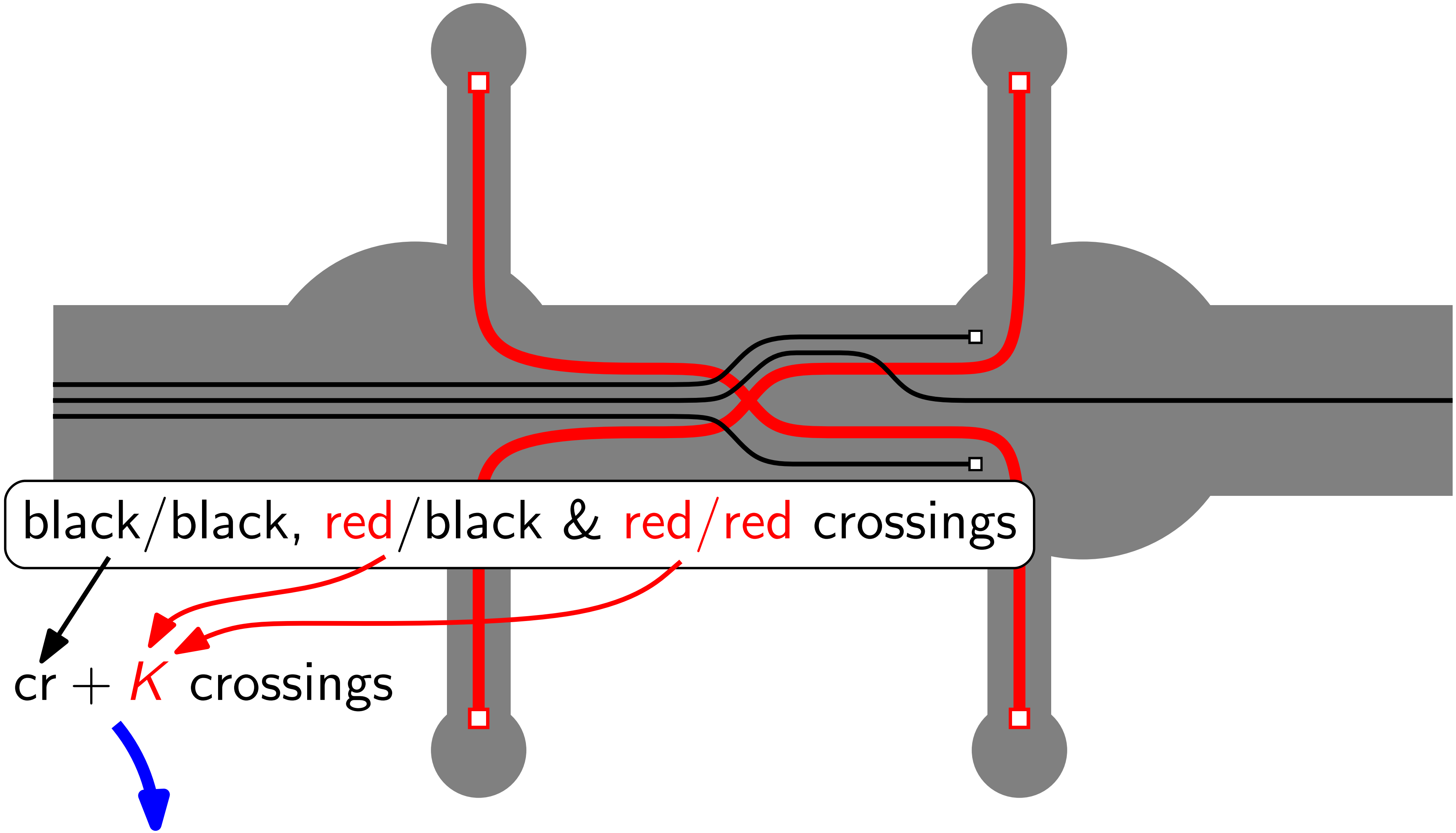


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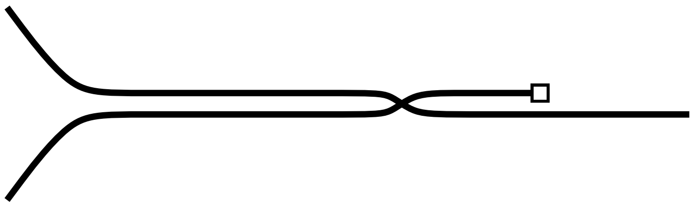

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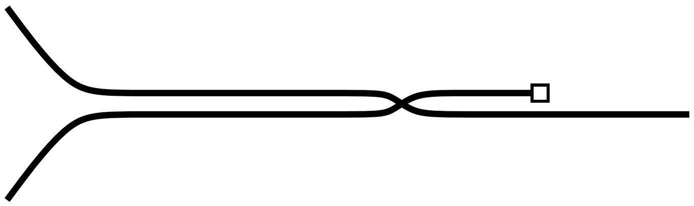

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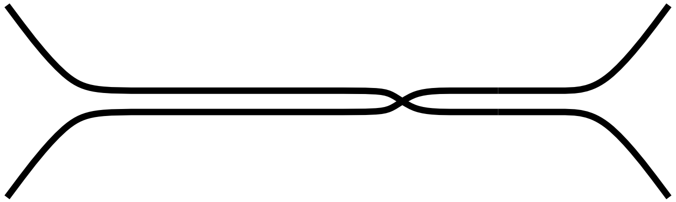

# MLCM – Crossing-free solutions

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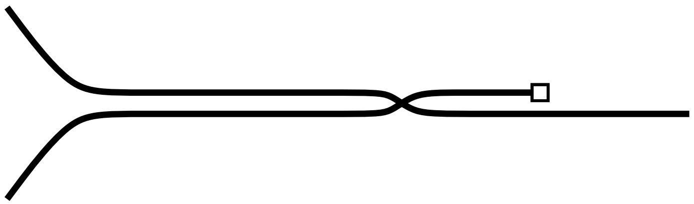

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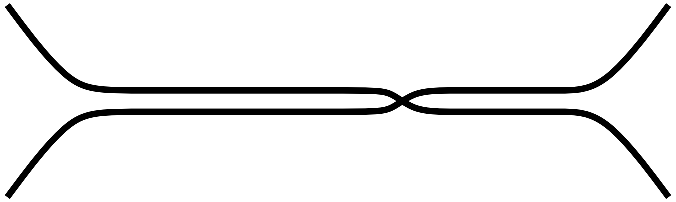
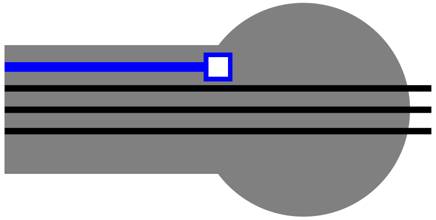


- We show:
  crossing-free solution exists $\Leftrightarrow$ no unavoidable crossing

- Main results:
  - **MLCM** is NP-hard.
  - $O(\sqrt{\log |L|})$-approximation for **MLCM-P**
- Additional results:
  - **MLCM** + **MLCM-P**: check for crossing-free solution
  - **MLCM-P**: Checking for solution with $\leq k$ crossings is fixed-parameter tractable.
  - a restricted version of MLCM-P can be solved efficiently

# MLCM-P – 2SAT model

- variable $l_v$ models side of line end at node $v$



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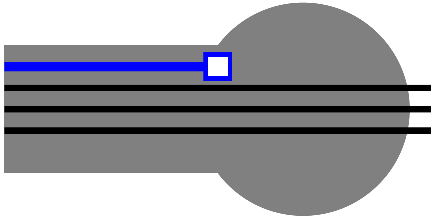


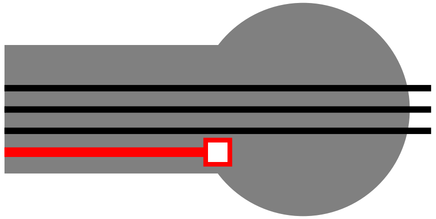

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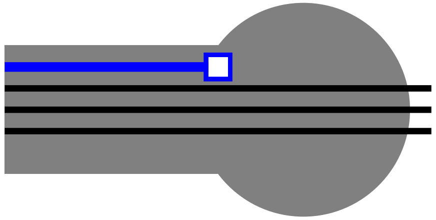

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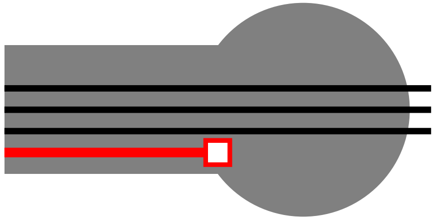

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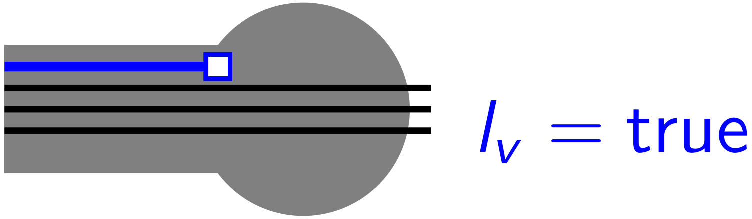


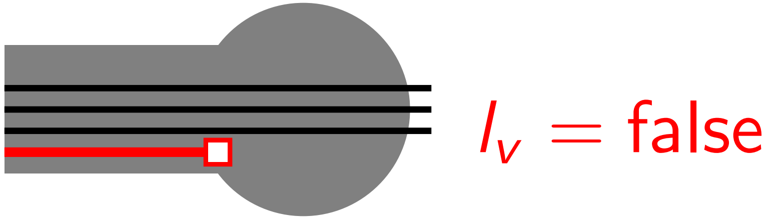


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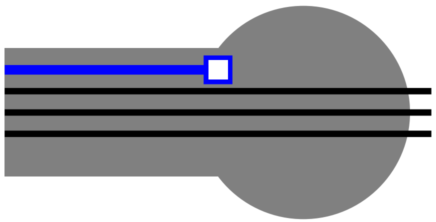

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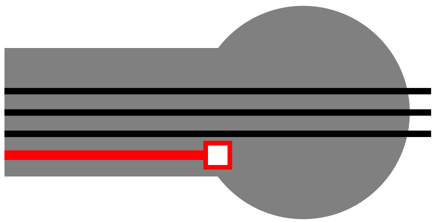

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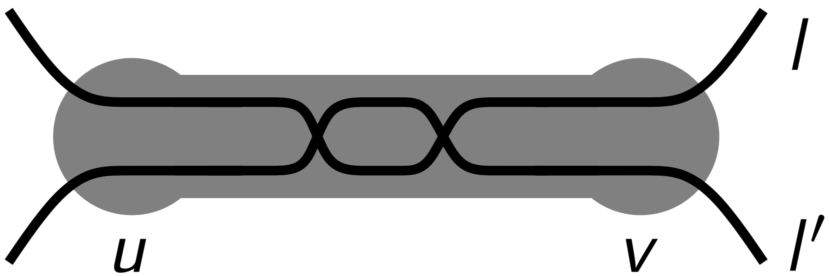


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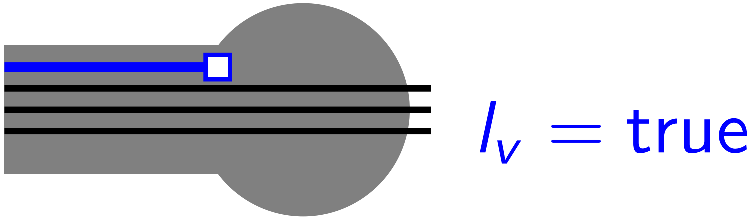


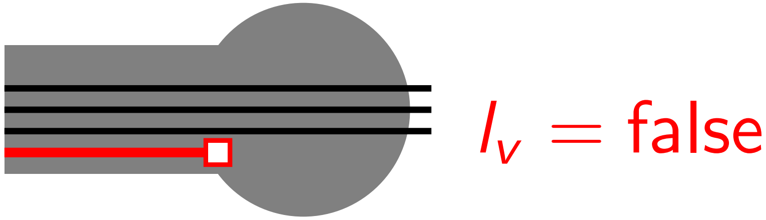


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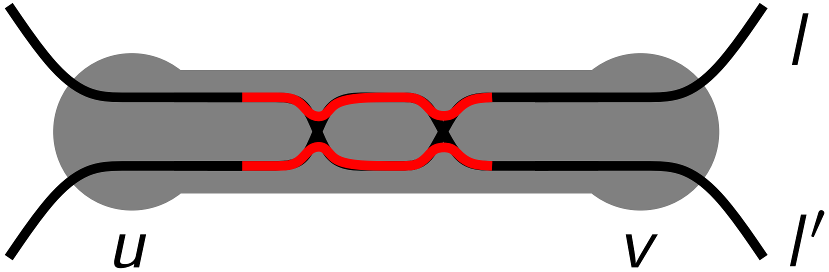


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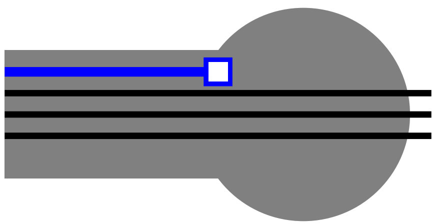

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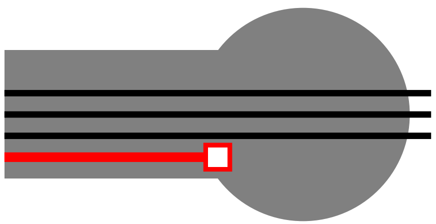

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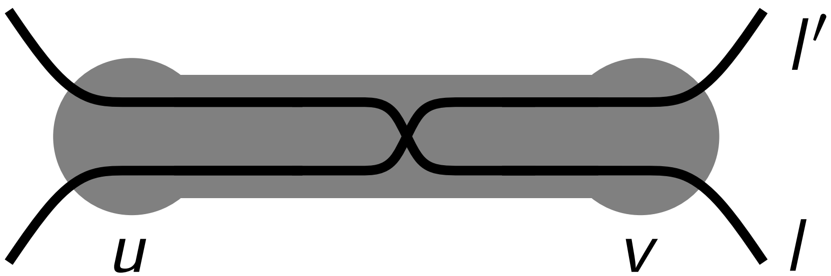


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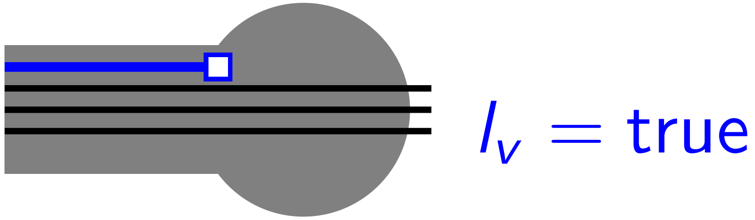


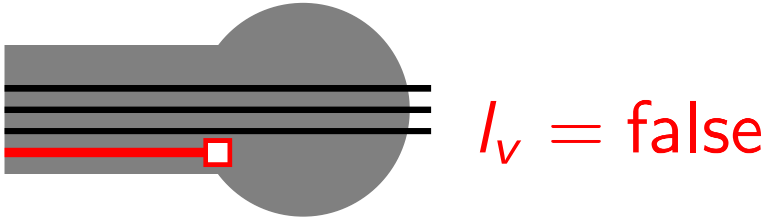


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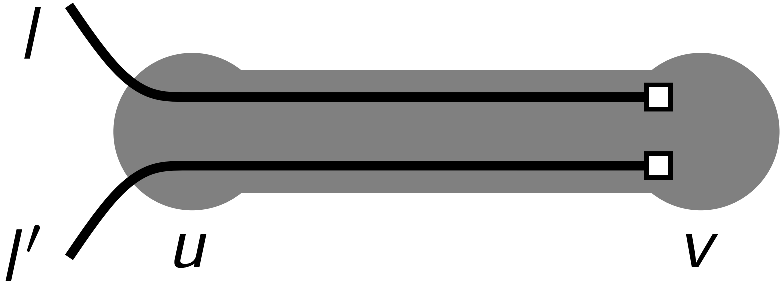

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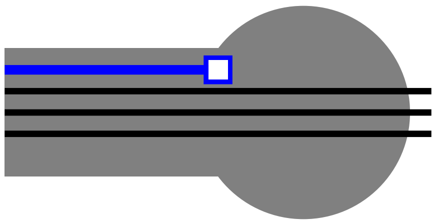

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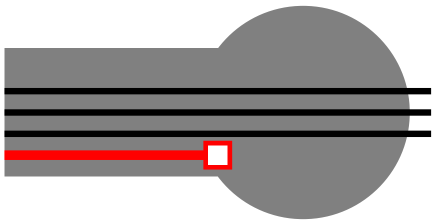

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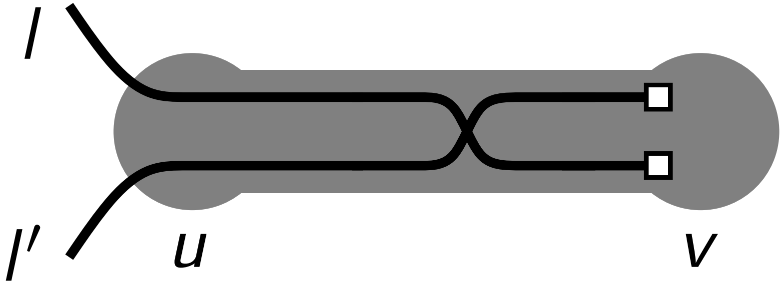

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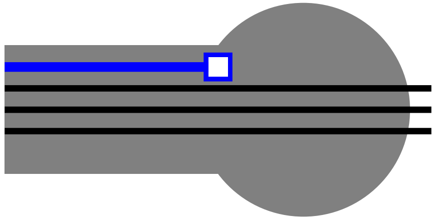


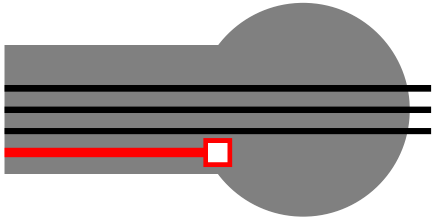


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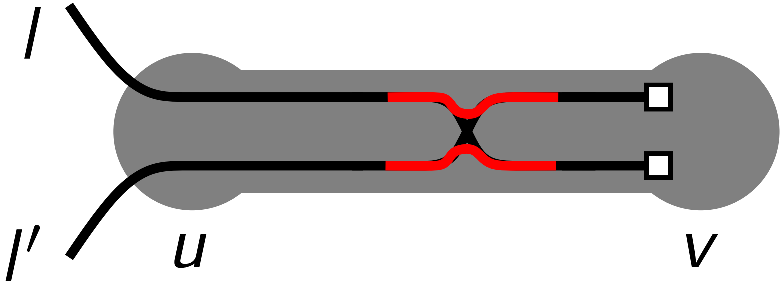


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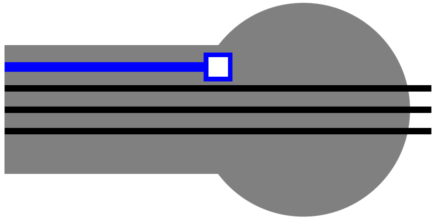

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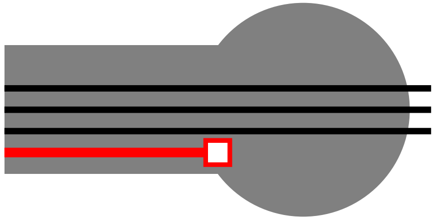

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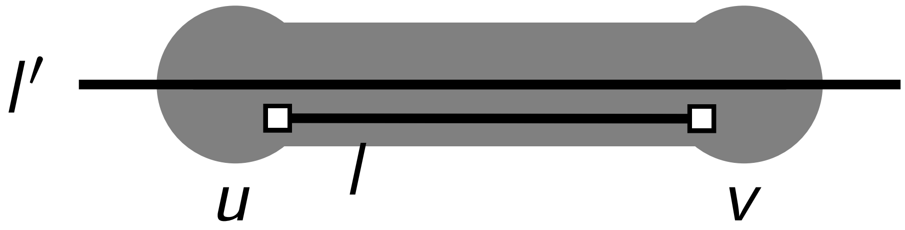

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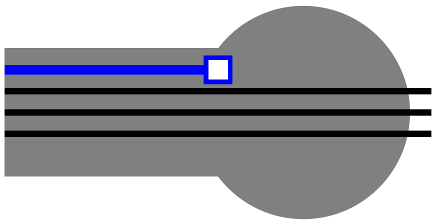

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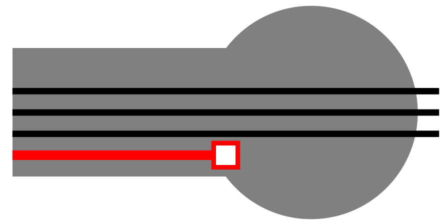

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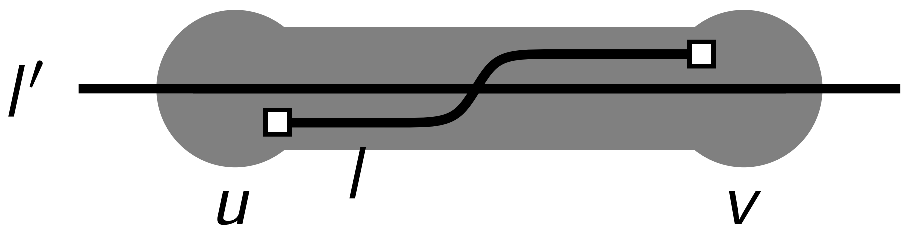


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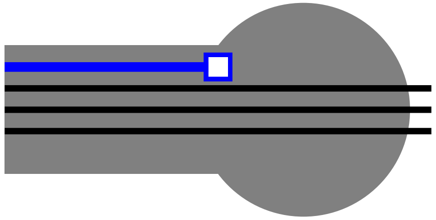


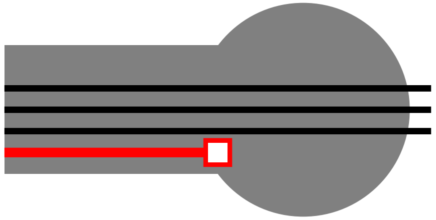


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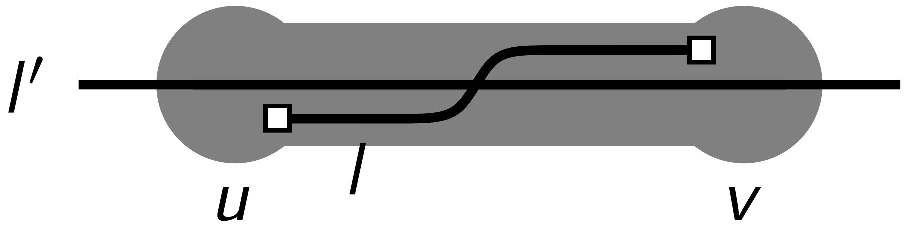


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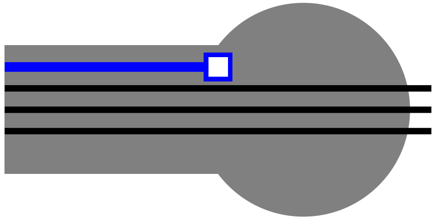

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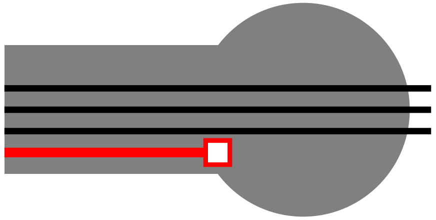

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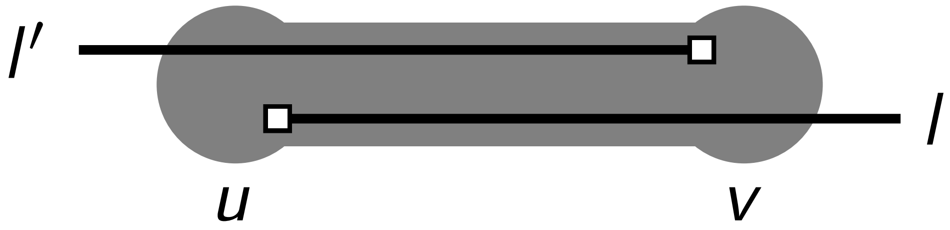

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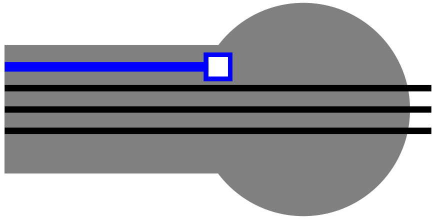

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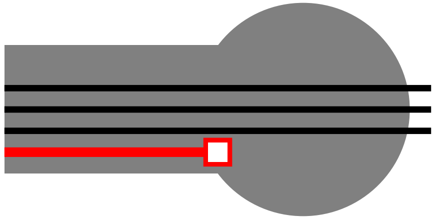

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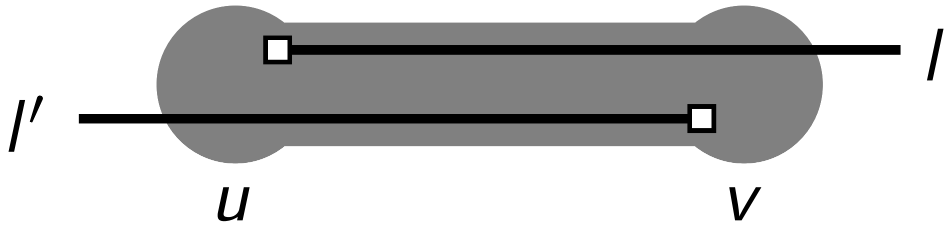

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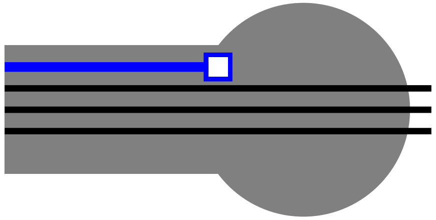

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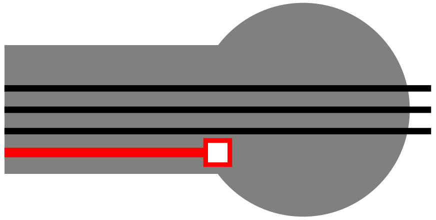

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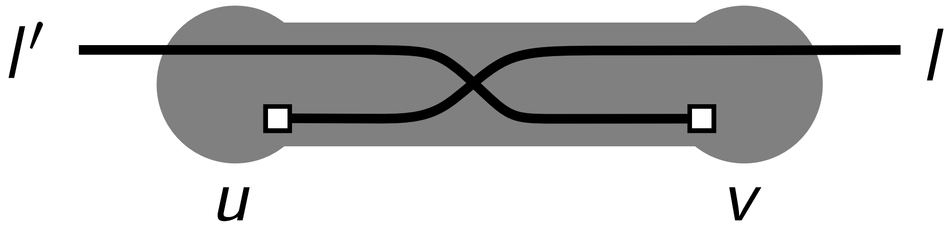


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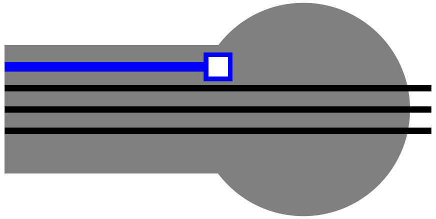

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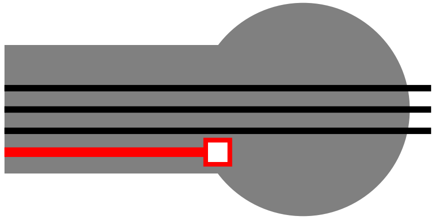

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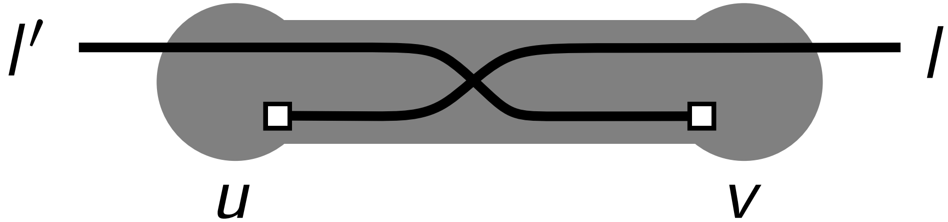


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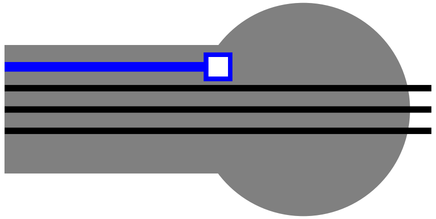

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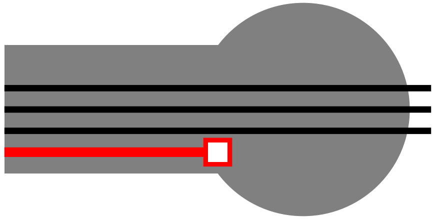

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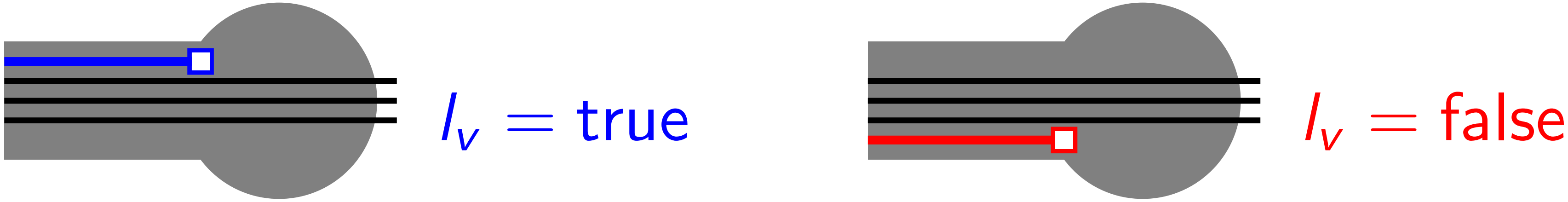


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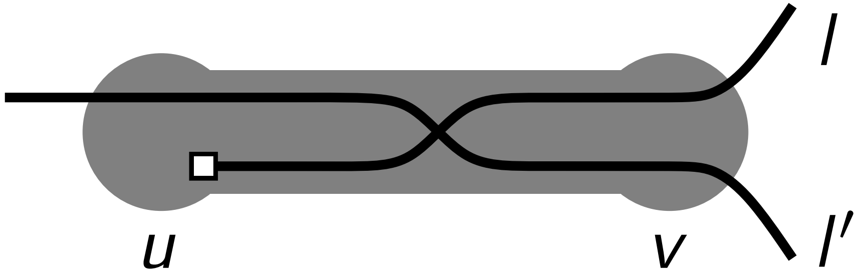


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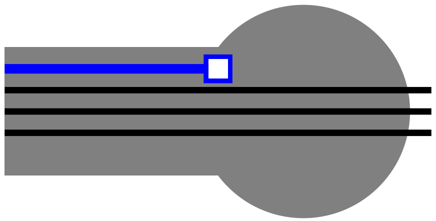

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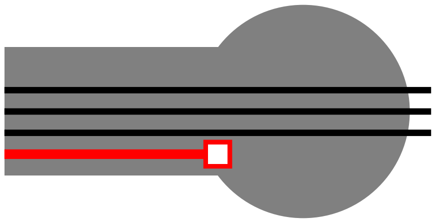

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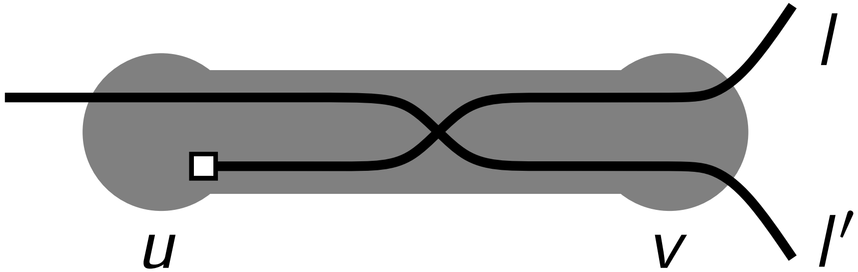


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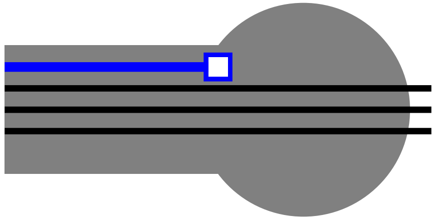

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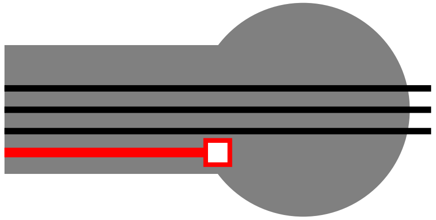

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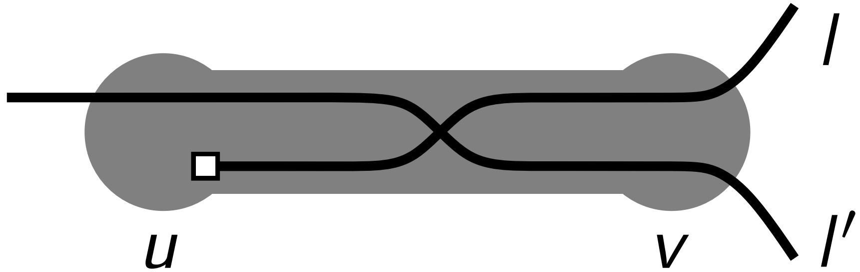


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- checking for crossing-free solution in $O(|L|^2)$ time

note: replace $l_u$ by $\neg l_u$ if necessary for consistent meaning of "top"

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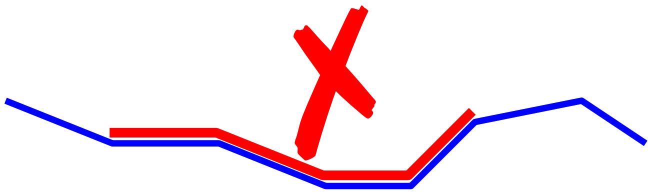
- $O(\sqrt{\log n})$-approximation for **Min 2CNF Deletion** $\Rightarrow$ $O(\sqrt{\log |L|})$-approximation for **MLCM-P**
- similarly: MLCM-P is fixed-parameter tractable.

- Main results:
  - **MLCM** is NP-hard.
  - $O(\sqrt{\log |L|})$-approximation for **MLCM-P**
- Additional results:
  - **MLCM** + **MLCM-P**: check for crossing-free solution
  - **MLCM-P**: Checking for solution with $\leq k$ crossings is fixed-parameter tractable.
  - a restricted version of MLCM-P can be solved efficiently

# Proper MLCM-P

- no line is subpath of another line



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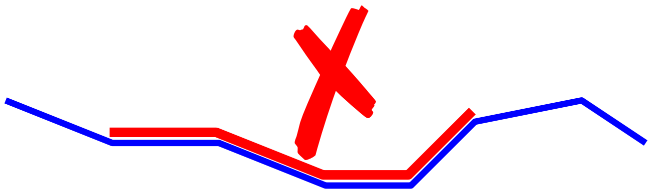
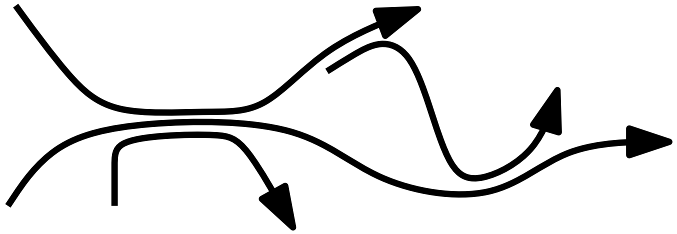


- consistent line directions



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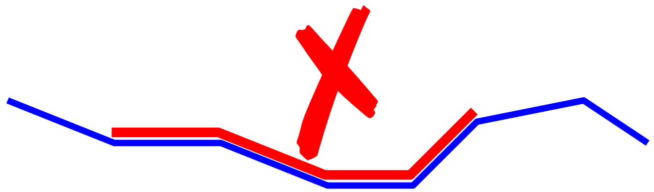

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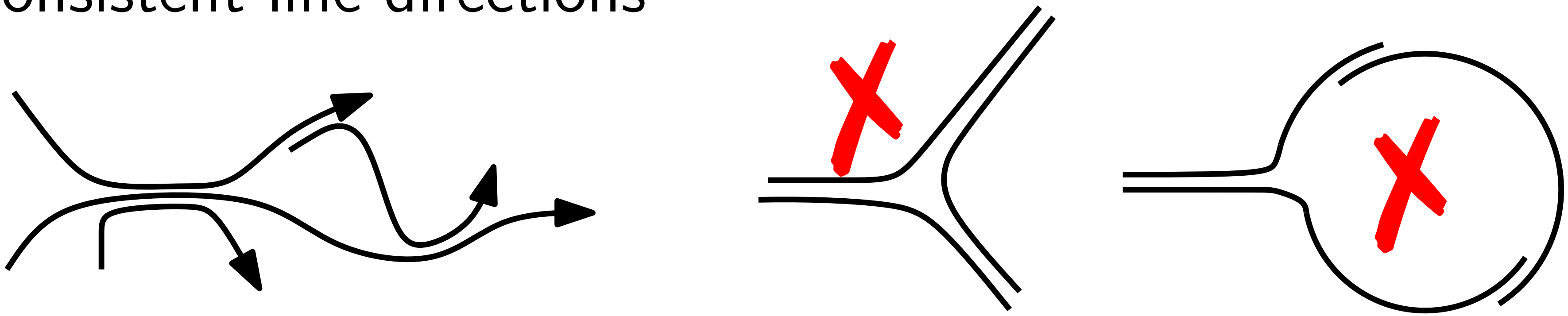

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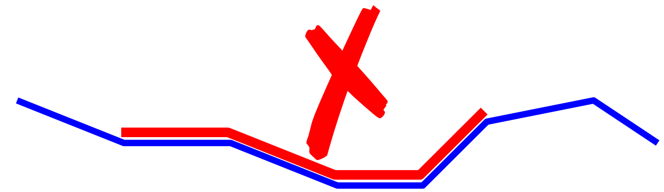

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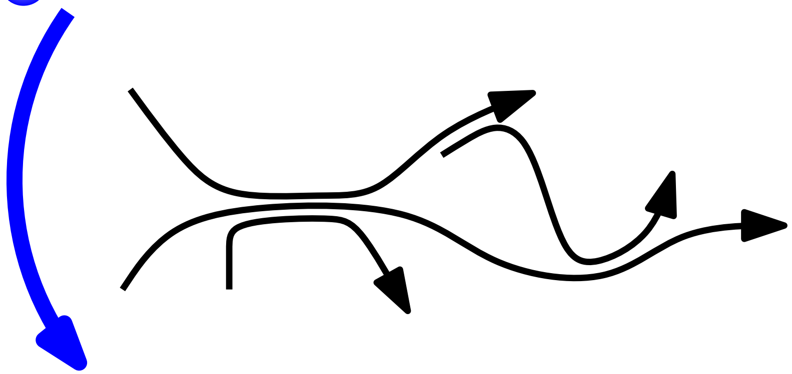

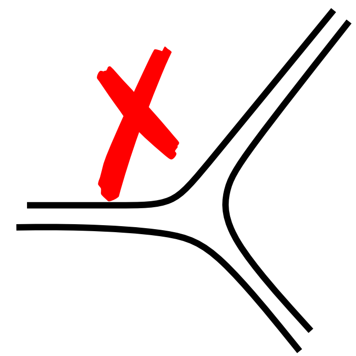

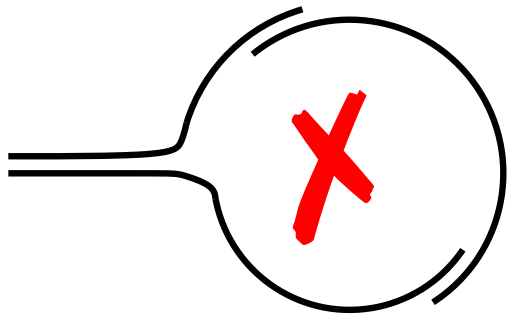

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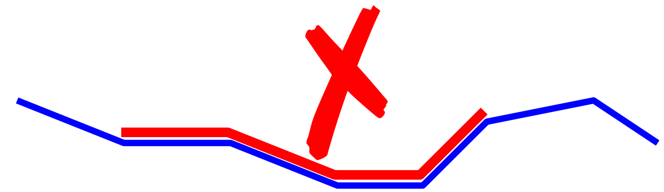

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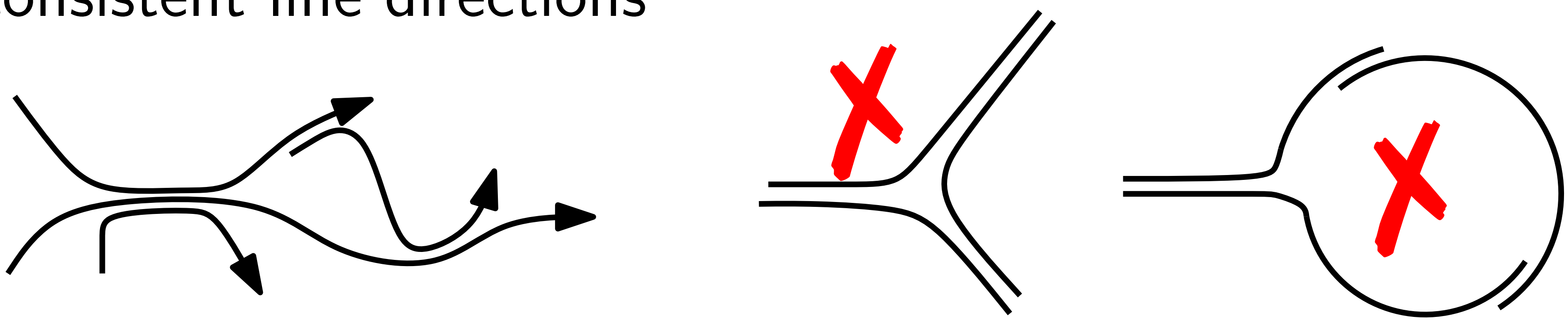

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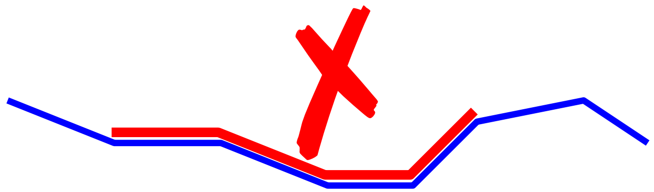

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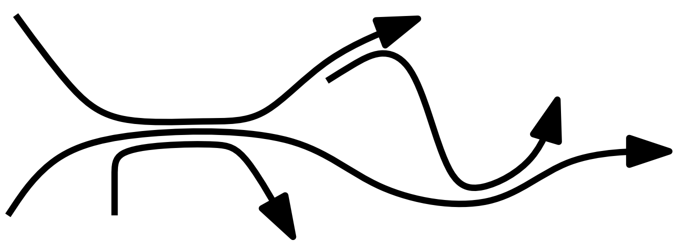

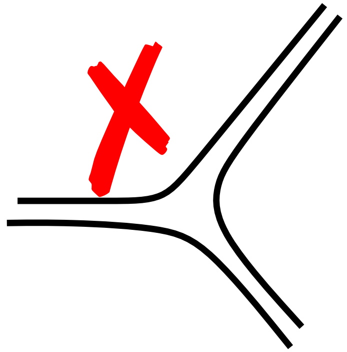

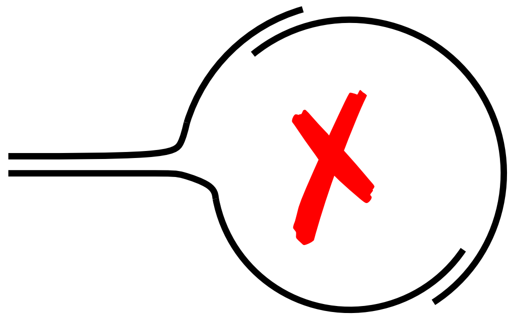

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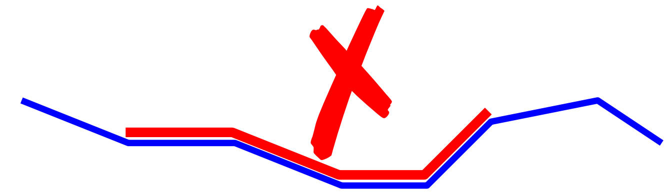

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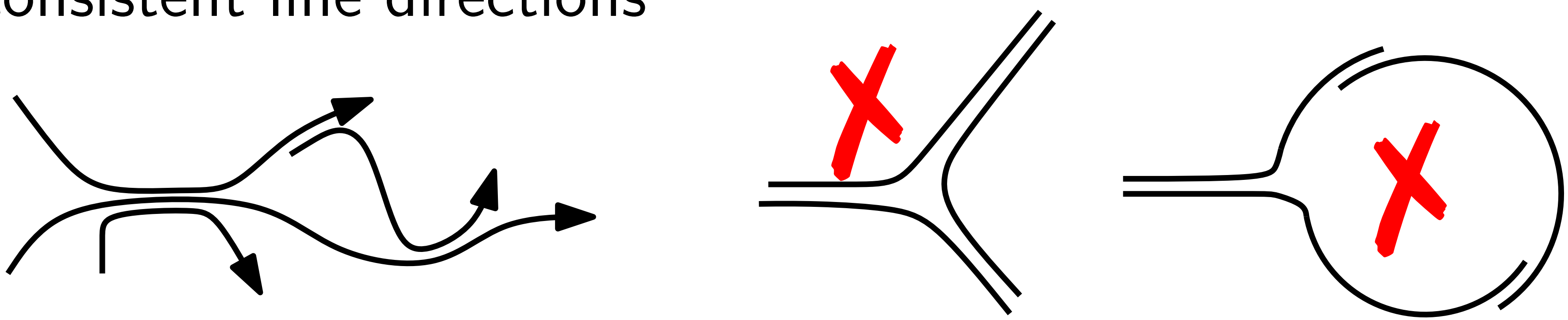

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- $\Rightarrow$ polynomial-time algorithm for Proper MLCM-P

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- **MLCM** is NP-hard.
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Thank you!