Fine Hierarchies via Priestley Duality

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Abstract

In applications of the fine hierarchies their characterizations in terms of the so called alternating trees are of principal importance. Also, in many cases a suitable version of many-one reducibility exists that fits a given fine hierarchy. With a use of Priestley duality we obtain a surprising result that suitable versions of alternating trees and of m-reducibilities may be found for any given fine hierarchy, i.e. the methods of alternating trees and m-reducibilities are quite general, which is of some methodological interest.

Along with hierarchies of sets, we consider also more general hierarchies of k-partitions and in this context propose some new notions and establish new results, in particular extend the above-mentioned results for hierarchies of sets.

Key words. Hierarchy, m-reducibility, Boolean algebra, bounded distributive lattice, Stone space, Priestley space, alternating tree, k-partition.

1 Introduction

In applications of the fine hierarchies (see [Se08] for a recent survey), their characterisations in terms of the so called alternating trees are of principal importance. Also, in many cases a suitable version of many-one reducibility exists that fits a given fine hierarchy (FH). Here we establish a surprising result that suitable versions of alternating trees and of m-reducibilities may be found for any given fine hierarchy, i.e. the methods of alternating trees and m-reducibilities are quite general, which is of some methodological interest for the hierarchy theory [Ad62, Se08]. The result is naturally described in terms of Priestley duality [Pr70, DP94].

Note that, similar to [Se08], the term "fine hierarchy" is used in two senses: it denotes either an element of a class of certain hierarchies or a distinguished element of this class — the fine hierarchy over a given ω-base specified below. The term "m-reducibilities" denotes reducibilities having some common features with the well-known m-reducibility from computability theory [Ro67].

For simplicity, we discuss our results in this introduction only for the difference hierarchy (DH) which is the well-known and most important version of a FH; for the DH the alternating trees are simplified to alternating chains. Let $\mathbb{B} = (B; \cup, \cap, \bar{}, 0, 1)$ be a Boolean algebra and L be a sublattice of $(B; \cup, \cap, 0, 1)$. Let $L(k)(k < \omega)$ be the set of all elements representable as $\bigcup_i (a_{2i} \setminus a_{2i+1})$ where $a_i \in L$ satisfy $a_0 \supseteq a_1 \supseteq \cdots$ and $a_k = 0$. The sequence $\{L(k)\}_{k<\omega}$ is called *the difference hierarchy over* L. It is well known that $L(k) \cup \check{L}(k) \subseteq L(k+1)$ and $\bigcup_{k<\omega} L(k)$ is the Boolean algebra generated by L, where $\check{L}(k) = \{\overline{x} \mid x \in L(k)\}$. Most useful results on the DH's (for example, the non-collapse property or decidability properties) are often obtained through their characterization in terms of the so called alternating chains. From many examples in the literature we mention here only a couple:

1. Let $\mathcal{L}$ be the class of open sets in an ω-algebraic domain X. Then $\mathcal{L}(n)$ is the class of approximable

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sets $A \subseteq X$ such that there is no sequence $a_0 \leq \cdots \leq a_n$ of compact elements with $a_{2i} \in A$, $a_{2i+1} \notin A$. Here $\leq$ is the specialization order.

2. Let $\mathcal{L}$ be the level 1/2 of the Straubing-Thérien hierarchy of regular languages over a given alphabet. Then $\mathcal{L}(n)$ is the class of languages A such that there is no sequence $a_0 \leq \cdots \leq a_n$ of words with $a_{2i} \in A$, $a_{2i+1} \notin A$. Here $\leq$ is the subword order.
3. Let L be the set of existential sentences of signature σ. Then $L(n)$ is the set of σ-sentences φ such that there is no sequence $A_0 \subseteq \cdots \subseteq A_n$ of σ-structures with $A_{2i} \models \varphi$, $A_{2i+1} \not\models \varphi$.

Are there similar characterizations for an arbitrary DH? It is not obvious because there are also many examples of DH's in the literature for which the chain-characterisation was not known (even for the thoroughly studied DH's over the computably enumerable sets and over the NP-sets). Nevertheless, the answer is positive, as we show with a heavy use of Priestley duality which is a basic tool in the study of distributive lattices. Thus, the method of alternating chains (the name was coined in [Ad65]) is a general tool for investigating the DH's. Let us stress that the chain-characterisation of a given DH is not unique, and different such characterisations may give sometimes new information about the hierarchy.

In many cases a suitable version of many-one reducibility exists that fits a given DH in the sense that any level of the DH is closed and has a complete set under this reducibility [Se08]. This is of interest because in many cases such reducibilities provide a finer classification than the DH itself. Again, we recall some examples. For the example 1 above, a suitable reducibility is the Wadge reducibility (i.e., the m-reducibility by continuous functions). For the example 2, a suitable reducibility was not considered but for the closely related Brzozowski dot-depth hierarchy a suitable reducibility is the so called quantifier-free reducibility found recently in [SW05]. For the example 3, a suitable reducibility was not known, to our knowledge. For the DH's over the computably enumerable sets (resp. over the NP-sets) a suitable reducibility is the classical m-reducibility (resp. the polynomial-time m-reducibility).

Is there a suitable reducibility that fits arbitrary given DH? Again, the answer is positive, at least for a rather broad natural class of DH's, and it is also proved using the Priestley duality. We present the mentioned results on the DH's in Section 4, after some preliminary information in Section 3 on the Stone and Priestley dualities and some of their extensions. Along with the DH, we establish similar results for a rich class of fine hierarchies. Since the case of arbitrary FH's is technically much more involved, we consider first in Section 5 another particular case of a FH — the so called symmetric difference hierarchy (SDH), and only then in Section 6 — the general case.

Along with the classical hierarchies of sets discussed so far we establish similar results for the DH's of partitions of a given set to $k \geq 2$ parts (called k-partitions here) which were recently considered in different fields of hierarchy theory [Ko00, Ko05, KW00, KW08, Se04]. In Section 7 we extend the results about the DH of sets to the DH of k-partitions. Already this extension is non-trivial and in fact requires a modification of the notion of Boolean hierarchy of k-partitions over posets in [Ko00, Ko05]. The desire to find a natural extension of the chain charactrisation of DH's to k-partitions was a main motivation for finding this modification of the hierarchy from [Ko00, Ko05]. In fact, our modification is simpler and has in general better properties than the hierarchy from [Ko00, Ko05].

The theory may be extended also to the FH of k-partitions which generalizes both the FH of sets and the DH of k-partitions but this requires essential technical complications not always relevant to the mainstream of this paper, so we decided to consider this extension in a separate publication (some impression on this may be obtained from the conference abstract [Se10] which also contains the announcement of the main results of this paper).

Our proofs are short for the DH and SDH but they become more and more technical (though remaining rather elementary) when we move to the FH and the DH of k-partitions. We tried to organize the paper in such a way that technical complications arise step by step. In this paper we consider only finite levels of the FH's. In fact, a good deal of the theory can, under suitable assumptions, be developed also for the transfinite versions of the FH's (including FH's of k-partitions) but this leads to additional technical complications. We hope to consider the transfinite case in a subsequent publication.

2 Notation and Notions

Here we briefly recall some notation and notions used throughout the paper. We use standard logical and set-theoretic notation and assume that the reader is familiar with the notions of structure and substructure of a given signature, distributive lattice, upper and lower semilattice and Boolean algebra. For a subset A of a Boolean algebra $\mathbb{B}$, $\check{A} = \{\bar{a} \mid a \in A\}$ is the dual set for A. For a bounded distributive lattice L (i.e., distributive lattice with a smallest element 0 and a greatest elements 1), let L^c denote the Boolean algebra formed by the elements of L that have a complement in L.

We use some standard notation and terminology on partially ordered sets (posets) which may be found e.g. in [DP94]. Recall that a *preorder* is a structure $(P; \leq)$ satisfying the axioms of reflexivity $\forall x(x \leq x)$ and transitivity $\forall x\forall y\forall z(x \leq y \wedge y \leq z \to x \leq z)$. A *partial order* is a preorder satisfying the antisymmetricity axiom $\forall x\forall y(x \leq y \wedge y \leq x \to x = y)$. A *linear order* (or a chain) is a partial order satisfying the connectivity axiom $\forall x\forall y(x \leq y \vee y \leq x)$. Any partial order $\leq$ on P induces the relation of strict order $<$ on P defined by $a < b \leftrightarrow a < b \wedge a \neq b$ and called the strict order related to $\leq$. By $x|y$ we denote that elements $x, y \in P$ are incomparable, i.e. $x \not\leq y$ and $y \not\leq x$. A subset A of P is *antichain* if any two distinct elements of A are incomparable. A poset $(P; \leq)$ will be often shorter denoted just by P. Any subset of a poset P may be considered as a poset with the induced partial order. In particular, this applies to the "upper cones" $\uparrow x = \{y \in P \mid x \leq y\}$ defined by any $x \in P$.

It is well known that any preorder $(P; \leq)$ induces a partial order $(P^*; \leq^*)$ called *the factorization* or *the quotient* of P. The set P^* is the quotient set $P/\equiv$ of P under the equivalence relation defined by $a \equiv b \leftrightarrow a \leq b \wedge b \leq a$; the set P consists of all equivalence classes $[a] = \{x \mid x = a\}$, $a \in P$. The partial order $\leq^*$ is defined by $[a] \leq^* [b] \leftrightarrow a \leq b$ (in fact, with an abuse of notation we usually use $\leq$ instead of $\leq^*$). For simplicity, we will often apply notions about posets also to preorders; in such cases we mean the corresponding quotient-poset of the preorder. We call two preorders equivalent if the corresponding partial orders are isomorphic.

We assume the reader to be acquainted with elementary notions of topology, like compactness or cartesian product of (topological) spaces. For a subset A of a space X, $\overline{A}$ denotes the complement of A rather than the topological closure of A.

Let ω^* be the set of finite sequences (strings) of natural numbers. The empty string is denoted by $\emptyset$, the concatenation of strings σ, τ by $\sigma^\frown\tau$, by $\sigma \cdot \tau$ or just by $\sigma\tau$, the length of σ by $|\sigma|$. By ω^+ we denote the set of finite non-empty strings in ω. By $\sigma \sqsubseteq \tau$ we denote that the string σ is an initial segment of the string τ (please be careful in distinguishing $\sqsubseteq$ and $\subseteq$). For any n, $1 < n < \omega$, let n^* be the set of finite strings of elements of $\{0, \ldots, n-1\}$, $n^* \subseteq \omega^*$. E.g., 2^* is the set of finite strings of 0's and 1's. In computer science people often consider the sets A^* and A^+ of finite (respectively, finite non-empty) words over a finite alphabet A. Mathematically, these sets are of course the same as n^* and n^+ respectively, where n is the cardinality of A.

We assume the reader to be acquainted with the notions of ordinal (see e.g. [KM67]) and to definitions and proofs by induction on ordinals (and on elements of more complicated well-founded sets). Ordinals are important for the hierarchy theory because levels of many hierarchies are often (almost) well ordered by inclusion. We use some well-known notions and facts about the ordinal arithmetic. As usual, $\alpha + \beta$, $\alpha \cdot \beta$ and α^β denote the addition, multiplication and exponentiation of ordinals α and β, respectively. We will often mention the ordinals ω, ω^2, $\omega^3, \ldots$ and ω^ω. The last ordinal is the order type of finite sequences $(k_1, \ldots, k_n)$ of natural numbers $k_1 \geq \cdots \geq k_n$, ordered lexicographically. Any non-zero ordinal $\alpha < \omega^\omega$ is uniquely representable in the form $\alpha = \omega^{k_1} + \cdots + \omega^{k_n}$ with $\omega > k_1 \geq \ldots \geq k_n$. We will also use the larger ordinal $\varepsilon_0 = sup\{\omega, \omega^\omega, \omega^{(\omega^\omega)}, \ldots\}$. It is well-known that any non-zero ordinal $\alpha < \varepsilon_0$ is uniquely representable in the form $\alpha = \omega^{\gamma_0} + \cdots + \omega^{\gamma_k}$ for a finite sequence $\gamma_0 \geq \cdots \geq \gamma_k$ of ordinals $< \alpha$. The ordinal ε_0 is the smallest solution of the ordinal equation $\omega^\varkappa = \varkappa$.

Since we discuss several hierarchies and relations between them in this paper, let us recall some corresponding notions from [Ad65, Se08].

Definition 2.1 *Let $\mathbb{B}$ be a Boolean algebra and η an ordinal.*

1. *By an η-hierarchy in $\mathbb{B}$ we mean a sequence $\{H_\alpha\}_{\alpha<\eta}$ of subsets of B such that $H_\alpha \cup \check{H} \subseteq H_\beta$ for*

all $\alpha < \beta < \eta$.

2. The sets $H_\alpha \setminus \check{H}_\alpha$ and $\check{H}_\alpha \setminus H_\alpha$ are called non-self-dual constituents of $\{H_\alpha\}$, while the sets $(H_\alpha \cap \check{H}_\alpha) \setminus (\bigcup_{\beta<\alpha} H_\beta \cap \check{H}_\beta)$ are called self-dual constituents of $\{H_\alpha\}$.
3. We say that hierarchy $\{H_\alpha\}$ does not collapse if $H_\alpha \not\subseteq \check{H}_\alpha$ for all $\alpha < \eta$.
4. We call a hierarchy $\{H_\alpha\}$ is non-trivial if $H_\alpha \not\subseteq \check{H}_\alpha$ for some $\alpha < \eta$.
5. A hierarchy $\{H_\alpha\}$ is called perfect in a level β if $\bigcup_{\gamma<\beta}(H_\gamma \cup \check{H}_\gamma) = H_\beta \cap \check{H}_\beta$. A hierarchy is (globally) perfect if it is perfect in all levels.

Note that in most cases $\mathbb{B}$ is the Boolean algebra $P(X)$ of all subsets of X; in this case we also speak of a hierarchy in X. From the Stone representation theorem it follows that this seemingly more restricted notion is in fact equivalent to the definition above. The sets $H_\alpha, \check{H}$ (resp. $H_\alpha \cap \check{H}$) are often called non-self-dual (resp. self-dual) levels of the hierarchy. Note that the constituents of $\{H_\alpha\}$ form a partition of $\bigcup_\alpha H_\alpha$, so the hierarchy $\{H_\alpha\}$ really provides a classification of sets in $\bigcup_\alpha H_\alpha$.

Definition 2.2 *Let $\{H_\alpha\}$ and $\{G_\beta\}$ be hierarchies in X.*

1. $\{H_\alpha\}$ is a refinement of $\{G_\beta\}$ in a given level β if $\bigcup_{\gamma<\beta}(G_\gamma \cup \check{G}_\gamma) \subseteq \bigcup_\alpha H_\alpha \subseteq (G_\beta \cap \check{G}_\beta)$. Such a refinement is called exhaustive if $\bigcup_\alpha (H_\alpha \cup \check{H}_\alpha) = G_\beta \cap \check{G}_\beta$.
2. $\{H_\alpha\}$ is a (global) refinement of $\{G_\beta\}$ if for any β there is an α with $H_\alpha = G_\beta$, and $\bigcup_\alpha H_\alpha = \bigcup_\beta G_\beta$.
3. A hierarchy is called discrete in a given level if it has no non-trivial refinements in this level. A hierarchy is (globally) discrete if it is discrete in each level.
4. $\{H_\alpha\}$ is an extension of $\{G_\beta\}$ if the sequence $\{G_\beta\}$ is an initial segment of the sequence $\{H_\alpha\}$.
5. A hierarchy $\{H_\alpha\}$ is called perfect in a level β if $\bigcup_{\gamma<\beta}(H_\gamma \cup \check{H}_\gamma) = H_\beta \cap \check{H}_\beta$. A hierarchy is (globally) perfect if it is perfect in all levels.

Obviously, if a hierarchy is perfect in some level (resp. globally perfect) then it is discrete in that level (resp. globally discrete).

In descriptive set theory and theoretical computer science, people are interested in different reducibilities, i.e. naturally defined preorders $A \leq_r B$ on subsets of a given set X. The intuitive idea behind this notion is that $A \leq_r B$ is intended to mean that the "complexity" of a set A is less than or equal to that of B. Following a well-established jargon, we call elements of the corresponding quotient-poset *r-degrees*. We say that a pointclass $\mathcal{C} \subseteq P(X)$ is *closed under r-reducibility* if $B \in \mathcal{C}$ and $A \leq_r B$ imply $A \in \mathcal{C}$. A set C is *r-hard for $\mathcal{C}$* (in symbols, $\mathcal{C} \leq^r C$) if $A \leq_r C$ for all $A \in \mathcal{C}$. A set C is *r-complete for $\mathcal{C}$* (in symbols, $C \equiv_r \mathcal{C}$) if $C \in \mathcal{C}$ and C is r-hard for $\mathcal{C}$. Note that $\mathcal{C} = \{A \mid A \leq_r C\}$ (i.e., $\mathcal{C}$ is a principal ideal of $(P(X); \leq_r)$) iff $\mathcal{C}$ is closed under r-reducibility and $C \equiv_r \mathcal{C}$.

The idea may be made precise in many different ways giving rise to a plenty of reducibilities. E.g., we can relate to any hierarchy $\mathcal{H} = \{H_\alpha\}_{\alpha<\eta}$ the reducibility $\leq_\mathcal{H}$ as follows: $A \leq_\mathcal{H} B$ iff $B \in H_\alpha$ (resp. $B \in \check{H}_\alpha$) implies $A \in H_\alpha$ (resp. $A \in \check{H}_\alpha$) for each $\alpha < \eta$. Obviously, the structure of $\mathcal{H}$-degrees captures exactly the information about the inclusions of levels of the hierarchy $\mathcal{H}$.

Among all reducibilities, very simple and useful turned out to be the so called many-one reducibilities (called m-reducibilities for short). Let F be a set of functions on a set X closed under composition and containing the identity function (intuitively, functions in F are considered as "feasible" in some sense). We say that A is F-m-reducible to B (in symbols, $A \leq_m^F B$) if $A = f^{-1}(B)$ for some $f \in F$. Obviously, $\leq_m^F$ is a preorder on $P(X)$.

A reducibility is especially useful if it is related to a hierarchy in the sense of the following definition.

Definition 2.3 *Let $\{H_\alpha\}_{\alpha<\eta}$ be a hierarchy of sets in X and $\leq_r$ be a reducibility on $P(X)$.*

1. We say that the reducibility fits the hierarchy (or, symmetrically, the hierarchy fits the reducibility) if any level H_α is a principal ideal of $(P(X); \leq_m)$. Note that the dual levels will then also have this property.

2. *We say that the reducibility as above perfectly fits the hierarchy if it fits the hierarchy, any non-empty non-self-dual constituent of the hierarchy is an r-degree, and any non-empty self-dual constituent of the hierarchy is an r-degree.*

It is easy to see that the reducibility $\leq_{\mathcal{H}}$ defined above perfectly fits the hierarchy $\mathcal{H}$ (moreover, $\leq_{\mathcal{H}}$ is the weakest among the reducibilities on $\bigcup_\alpha H_\alpha$ that fit $\mathcal{H}$). It is often useful to find a natural notion of m-reducibility that fits a given hierarchy (see e.g. [Se08] for many examples).

3 Duality

In this section we briefly recall some well-known facts about Stone and Priestley dualities and make some relevant observations on a slight extension of Priestley duality which is suitable for our purposes here. We also explain why Priestley duality is useful in the study of fine hierarchies. We assume the reader to be acquainted with basic notions and facts about Stone and Priestley duality (see e.g. [DP94, Si64, RS63]). Moreover, we use in this section elementary notions from category theory, all of them are broadly known and may be found in any text on the subject, see e.g. [BuD70].

3.1 Stone Duality

Let $\mathbf{B}$ be the category formed by the Boolean algebras as objects and the $\{\vee,\wedge,^-,0,1\}$-homomorphisms as morphisms. Recall that a *Stone space* is a compact topological space X such that for any distinct $x,y\in X$ there is a clopen set U with $x\in U\not\ni y$. Let $\mathbf{S}$ be the category formed by the Stone spaces as objects and the continuous mappings as morphisms.

The *Stone duality* [St36, St37] states the dual equivalence between the categories $\mathbf{B}$ and $\mathbf{S}$. Recall that the Stone space $s(\mathbb{B})$ corresponding to a given Boolean algebra $\mathbb{B}$ is formed by the set of prime filters of $\mathbb{B}$ with the base of open (in fact, clopen) sets consisting of the sets $\{F\in s(\mathbb{B})\mid a\in F\}$, $a\in B$. (Note that one could equivalently take ideals in place of filters, as in [DP94].) Conversely, the Boolean algebra $b(X)$ corresponding to a given Stone space X is formed by the set of clopen sets (with the usual set-theoretic operations). By Stone duality, any Boolean algebra $\mathbb{B}$ is canonically isomorphic to the Boolean algebra $b(s(\mathbb{B}))$ (the isomorphism $f:\mathbb{B}\to b(s(\mathbb{B}))$ is defined by $f(a)=\{F\in s(\mathbb{B})\mid a\in F\}$), and any Stone space X is canonically homeomorphic to the space $s(b(X))$.

3.2 Priestley Duality

Recall that *bounded distributive lattice* is a distributive lattice $\mathbb{L}=(L;\vee,\wedge,0,1)$ with a smallest element 0 and a largest element 1. Let $\mathbf{D}$ be the category formed by the bounded distributive lattices as objects and the $\{\vee,\wedge,0,1\}$-homomorphisms as morphisms. A *Priestley space* $(X;\leq)$ is a compact topological space X equipped with a partial order $\leq$ such that for any $x,y\in X$ with $x\not\leq y$ there is a clopen up-set U with $x\in U\not\ni y$ (a subset U of X is up if $x\in U$ and $x\leq y$ imply $y\in U$). Let $\mathbf{P}$ be the category formed by the Priestley spaces as objects and the continuous monotone mappings as morphisms.

The *Priestley duality* [Pr70] states the dual equivalence between the categories $\mathbf{D}$ and $\mathbf{P}$. Recall that the Priestley space $(p(\mathbb{L});\subseteq)$ corresponding to a given bounded distributive lattice $\mathbb{L}$ is formed by the set of prime filters of $\mathbb{L}$ with the prebase consisting of the sets $\{F\in p(L)\mid a\in F\}$, $a\in L$, and their complements. (Note that one could equivalently take ideals in place of filters, as in [DP94].) Conversely, the bounded distributive lattice $d(X;\leq)$ corresponding to a given Priestley space $(X;\leq)$ is formed by the set $\mathcal{L}$ of clopen up-sets under set inclusion.

If $\mathbb{B}$ is a Boolean algebra and $L\subset B$, let (L) denote the subalgebra of $\mathbb{B}$ generated by L. From Priestley duality it follows that for any bounded distributive lattice $\mathbb{L}$ there is a Boolean algebra $(\mathbb{L})$ such that $\mathbb{L}$ is a substructure of $((\mathbb{L});\vee,\wedge,0,1)$ and $(\mathbb{L})$ is generated by L. Moreover, $(\mathbb{L})$ is a unique (in a natural exact sense [Gr78]) Boolean algebra with these properties, and $p(\mathbb{L})$ is homeomorphic to $s((\mathbb{L}))$. We call $(\mathbb{L})$ the Boolean algebra generated by the lattice $\mathbb{L}$.

For purposes of this paper, it is convenient to slightly weaken the notion of Priestley space. Namely, by a *pre-Priestley space* we mean a pair $(X;\leq)$ formed by a compact topological space X and a preorder $\leq$ on X such that for any $x,y\in X$ with $x\not\leq y$ there is a clopen up-set U with $x\in U\not\ni y$; as above, $\mathcal{L}$ denotes the set of clopen up-sets in $(X;\leq)$. The relation between two notions is explained by the following

Lemma 3.1 *1. If $(X;\leq)$ Priestley then it is pre-Priestley.*

2. If $(X;\leq)$ is a pre-Priestley space then $(X/\equiv;\leq)$ is a Priestley space in the quotient-topology and the set of clopen $\equiv$-saturated subsets of X coincides with $(\mathcal{L})$.

Proof. Assertion 1 is obvious.

2. Clearly, $(X/\equiv;\leq)$ is Priestley. Since the class of clopen $\equiv$-saturated sets forms a Boolean algebra and contains $\mathcal{L}$, it contains also $(\mathcal{L})$. Conversely, let $A\subseteq X$ be clopen and $\equiv$-saturated. Then for all $a\in A$ and $b\in\overline{A}$ we have $a\not\equiv b$, hence $a\not\leq b$ or $b\not\leq a$, and therefore $a\in S_{a,b}\not\ni b$ for some $S_{a,b}\in(\mathcal{L})$ (in fact, for some $S_{a,b}\in\mathcal{L}\cup\check{\mathcal{L}}$). For any fixed $a\in A$, $\{\overline{S_{a,b}}\mid b\in\overline{A}\}$ is then an open cover of $\overline{A}$. Since $\overline{A}$ is clopen, there is a finite subcover $\{\overline{S_{a,b_i}})\}_{i\leq k}$, for some $k<\omega$ and $b_0,\dots,b_k\in\overline{A}$. For the set $S_a=\bigcap_{i\leq k}S_{a,b_i}\in(\mathcal{L})$ we then have $a\not\in\overline{S_a}\supseteq\overline{A}$, hence $a\in S_a\subseteq A$. Then $\{S_a\mid a\in A\}$ is an open cover of A. Since A is clopen, there is a finite subcover $\{S_{a_i})\}_{i\leq k}$, for some $k<\omega$ and $a_0,\dots,a_k\in A$. For the set $S=\bigcup_{i\leq k}S_{a_i}\in(\mathcal{L})$ we then have $A\subseteq S\subseteq A$, hence $A=S\in(\mathcal{L})$. □

Pre-Priestley spaces naturally arise in the following situation. Let L be a sublattice (in signature $\{\vee,\wedge,0,1\}$) of a Boolean algebra $\mathbb{B}$. Define the preorder $\leq$ on $s(\mathbb{B})$ by: $F\leq G$ iff $F\cap L\subseteq G\cap L$. Then we have:

Lemma 3.2 *1. The structure $(s(\mathbb{B});\leq)$ is a pre-Priestley space.*

2. The restriction $f|_L$ of the canonical Stone isomorphism $f:\mathbb{B}\to b(s(\mathbb{B}))$ to L is a lattice isomorphism between L and $\mathcal{L}$.

3. The Priestley space $(s(B)/\equiv;\leq)$ is homeomorphic with $(p(L);\subseteq)$.

Proof. 1. Let $F,G\in s(\mathbb{B})$ and $F\not\leq G$. Then $a\in F\setminus G$ for some $a\in L$, hence $F\in f(a)\not\ni G$. Since $f(a)$ is an up-set w.r.t. $\leq$, $f(a)\in\mathcal{L}$.

2. By Stone duality and the proof of 1, $f|_L$ is an isomorphic embedding of L into $\mathcal{L}$, so it remains to check that for any $A\in\mathcal{L}$ there is $a\in L$ with $f(a)=A$. Since $\forall F\in A\forall G\in\overline{A}(F\not\leq G)$, there is a family $\{a_{F,G}\mid F\in A,G\in\overline{A}\}$ of elements of L such that $a_{F,G}\in F\setminus G$, i.e. $F\in f(a_{F,G})\not\ni G$ and hence $F\not\in f(a_{F,G})\ni G$, for all $F\in A,G\in\overline{A}$. For any fixed $F\in A$, $\{f(\overline{a_{F,G}})\mid G\in\overline{A}\}$ is then an open cover of $\overline{A}$. Since $\overline{A}$ is clopen, there is a finite subcover $\{f(\overline{a_{F,G_i}})\}_{i\leq k}$, for some $k<\omega$ and $G_0,\dots,G_k\in\overline{A}$. For the element $a_F=\bigcap_{i\leq k}a_{F,G_i}\in L$ we then have $F\not\in f(\overline{a_F})\supseteq\overline{A}$, hence $F\in f(a_F)\subseteq A$. Then $\{f(a_F)\mid F\in A\}$ is an open cover of A. Since A is clopen, there is a finite subcover $\{f(a_{F_i})\}_{i\leq k}$, for some $k<\omega$ and $F_0,\dots,F_k\in A$. For the element $a=\bigcup_{i\leq k}a_{F_i}\in L$ we then have $A\subseteq f(a)\subseteq A$, hence $A=f(a)$ as desired.

3. Follows from Priestley duality since L is isomorphic to any of $\mathcal{L}(p(L);\subseteq)$, $\mathcal{L}(s(B)/\equiv;\leq)$. □

We conclude this subsection with establishing some additional properties of pre-Priestley spaces which will be used in the sequel. For a pre-Priestley space $(X;\leq)$ and a set $A\subseteq X$, let $\downarrow A=\{x\mid\exists a\in A(x\leq a)\}$ and $\uparrow A=\{x\mid\exists a\in A(a\leq x)\}$.

Lemma 3.3 *Let $(X;\leq)$ be a pre-Priestley space and $A,B,B_0,\dots,B_n$ be closed subsets of X.*

1. If $\forall a\in A\forall b\in B(a\not\leq b)$ then $A\subseteq U\subseteq\overline{B}$ for some $U\in\mathcal{L}$.

2. If $\forall a\in A\bigvee_{i\leq n}\forall b\in B_i(a\not\leq b)$ then there exist $U_0,\dots U_n\in\mathcal{L}$ such that $A\subseteq U_0\cup\cdots\cup U_n$ and $A\cap U_i\subseteq\overline{B}_i$ for all $i\leq n$.

3. The sets $\downarrow A$ and $\uparrow A$ are closed.

Proof. 1. For any $a \in A$ and $b \in B$, choose $U_{a,b} \in \mathcal{L}$ such that $a \in U_{a,b} \not\ni b$, so $a \notin \overline{U}_{a,b} \ni b$. Then, for any fixed $a \in A$, $\{\overline{U}_{a,b}\}_{b \in B}$ is an open cover of B. Since B is closed and X is compact, there is a finite subcover $\{\overline{U}_{a,b_i}\}_{i \leq k}$, for some $k < \omega$ and $b_0, \ldots, b_k \in B$. For the set $U_a = \bigcap_{i \leq k} U_{a,b_i} \in \mathcal{L}$ we then have $a \notin \overline{U}_a \supseteq B$, so $a \in U_a \subseteq \overline{B}$. Then $\{U_a\}_{a \in A}$ is an open cover of A. Since A is closed, there is a finite subcover $\{U_{a_i}\}_{i \leq k}$, for some $k < \omega$ and $a_0, \ldots, a_k \in A$. For the set $U = \bigcup_{i \leq k} U_{a_i} \in \mathcal{L}$ we then have $A \subseteq U \subseteq \overline{B}$, as desired.

2. For any $i \leq n$, define the family $\{U_i^a\}_{a \in A}$ as follows: for a given $a \in A$, if $\forall b \in B_i (a \not\leq b)$, then let U_i^a be any element of $\mathcal{L}$ (existing by the proof of item 1) with $a \in U_i^a \subseteq \overline{B}_i$, otherwise let $U_i^a = \emptyset$ (note that $U_i^a \subseteq \overline{B}_i$ for all $i \leq n$ and $a \in A$). Then $\{U_0^a \cup \cdots \cup U_n^a\}_{a \in A}$ is an open cover of A. Since A is closed, there is a finite subcover $(\bigcup_{i \leq n} U_i^{a_0}, \ldots, \bigcup_{i \leq n} U_i^{a_k})$, for some $k < \omega$ and $a_0, \ldots, a_k \in A$. For any $i \leq n$, let $U_i = \bigcup_{j \leq k} U_i^{a_j} \in \mathcal{L}$. Then $U_0, \ldots, U_n$ have the desired properties because $A \cap U_i = \bigcup_{j \leq k} (A \cap U_i^{a_j}) \subseteq \overline{B}_i$.

3. We consider only the set $\downarrow A$ because for the other set the proof is similar. Let $x \in X \setminus \downarrow A$, then $\forall a \in A (x \not\leq a)$. Since $\{x\}$ and A are closed, by item 1 there is $U \in \mathcal{L}$ with $x \in U \subseteq \overline{A}$. We even have $U \subseteq \overline{\downarrow A}$ (otherwise, $y \in U \cap \downarrow A$ for some y; since $y \leq a$ for some $a \in A$, $a \in U \cap A$ which is a contradiction). Thus, U is a neighborhood of x disjoint with A. Therefore, A is closed. $\square$

3.3 α-Bases

Here we recall a technical notion of a base. Bases serve as a starting point to build hierarchies we are interested in (see [Se08] for examples), and have nothing in common with bases in topology or linear algebra.

Definition 3.4 *For any ordinal $\alpha \geq 1$, by an α-base we mean a sequence $L = \{L_\beta\}_{\beta < \alpha}$ of bounded distributive lattices such that, for all $\beta < \gamma < \alpha$, L_β is a sublattice (in signature $\{\vee, \wedge, 0, 1\}$) of the Boolean algebra f L_γ^c of all elements of L_γ which have complements in L_γ.*

Note that a 1-base is essentially a bounded distributive lattice. More generally, for any $n < \omega$ the $(n+1)$-bases are sequences of bounded distributive lattices of the form $(L_0, \ldots, L_n)$, with the corresponding inclusions. Note also that any $(n+1)$-base $(L_0, \ldots, L_n)$ may be extended to the ω-base $\{L_k\}_{k < \omega}$ by setting $L_k = (L_n)$ for all $k > n$. Similarly, any α-base may be extended to a β-base for $\beta > \alpha$. In the sequel we deal mostly with 1-bases, 2-bases and ω-bases.

For an α-base L, let L^* be a smallest Boolean algebra that contains the lattice $\bigcup_{\beta < \alpha} L_\beta$ as a sublattice (clearly, $L^* = \bigcup_{\beta < \alpha} L_\beta$ if α is a limit ordinal and $L^* = (L_\beta)$ if $\alpha = \beta + 1$ is a successor ordinal). Let L and $M = \{M_\beta\}_{\beta < \alpha}$ be α-bases. By a homomorphism $f : L \to M$ we mean a morphism $f : L^* \to M^*$ in the category $\mathbf{B}$ of Boolean algebras such that $f(L_\beta) \subseteq M_\beta$ for each $\beta < \alpha$. Let $\mathbf{B}_\alpha$ be the category formed by α-bases as objects and by homomorphisms of α-bases as morphisms.

Next we define some special types of ω-bases which are interesting for the further discussion. But first we recall definitions of some "structural properties" which are important for the hierarchy theory (see e.g. [Ke94]). We will often mention the following simplest versions of two such properties.

Definition 3.5 *Let B be a Boolean algebra and $C \subseteq B$.*

1. *The set C has the separation property iff for any $a, b \in C$ with $a \cap b = 0$ there is $c \in C \cap \check{C}$ with $a \subseteq c \subseteq \overline{b}$. We say that c separates a from b (note that it is equivalent to say that $\overline{c}$ separates b from a).*
2. *The set C has the reduction property iff for all $c_0, c_1 \in C$ there are disjoint $c_0', c_1' \in C$ such that $c_i' \subseteq c_i$ for both $i < 2$ and $c_0 \cup c_1 = c_0' \cup c_1'$. The pair (c_0', c_1') is called a reduct for the pair (c_0, c_1) in C.*

It is well-known and easy to see that if C has the reduction property then the dual set $\check{C}$ has the separation property, but not vice versa. Also, if C has the reduction property then for any finite sequence $(c_0, \ldots, c_n)$ of elements of C there is a reduct $(c_0', \ldots, c_n')$, $c_i' \in C$ for $(c_0, \ldots, c_n)$ (i.e., c_i' are pairwise disjoint, $c_i' \leq c_i$

and $c_0' \cup \cdots \cup c_n' = c_0 \cup \cdots \cup c_n$). The following types of ω-bases will be frequently mentioned in the sequel.

Definition 3.6 *Let L be an ω-base.*

1. *L is reducible iff any L_n has the reduction property.*
2. *L is interpolable if for each $n < \omega$ any two disjoint elements in $\check{L}_{n+1}$ are separable by an element of (L_n) (equivalently, for any $n < \omega$ $\check{L}_{n+1}$ has the separation property and $(L_n) = L_{n+1} \cap \check{L}_{n+1}$).*

3.4 α-Spaces

Here we introduce and study α-spaces which are slight generalisation of Priestley spaces.

Definition 3.7 *For any ordinal $\alpha \geq 1$, by an α-space we mean a compact topological space X equipped with a sequence $\{\leq_\beta\}_{\beta<\alpha}$ of preorders such that:*

1. *$(X; \leq_\beta)$ is a pre-Priestley space for each $\beta < \alpha$;*
2. *for all $\gamma < \beta < \alpha$, $x \leq_\beta y$ implies $x \equiv_\gamma y$;*
3. *if $x \equiv_\beta y$ for all $\beta < \alpha$ then $x = y$.*

We denote such a space as $(X; \leq_\beta)_{\beta<\alpha}$ or, abusing notation, just by X. Relate to any α-space $(X; \leq_\beta)_{\beta<\alpha}$ the sequence $\{\mathcal{L}_\beta\}_{\beta<\alpha}$ where $\mathcal{L}_\beta$ is the set of clopen $\leq_\beta$-up subsets of X.

Lemma 3.8 *Let $(X; \leq_\gamma)_{\gamma<\alpha}$ be an α-space.*

1. *The sequence $\{\mathcal{L}_\gamma\}_{\gamma<\alpha}$ is an α-base.*
2. *If $\alpha = \beta + 1$ is a successor ordinal then $(X; \leq_\beta)$ is a Priestley space and X is homeomorphic to the Stone space $s(\mathcal{L}_\beta)$.*
3. *If α is a limit ordinal then X is homeomorphic to the Stone space $s(\bigcup_{\gamma<\alpha} \mathcal{L}_\gamma)$.*

Proof. 1. Since $(X; \leq_\gamma)$ is a pre-Piestley space for each $\gamma < \alpha$ by item 1 of Definition 3.7, $\mathcal{L}_\gamma$ is a bounded distributive lattice. Item 2 of Definition 3.7 implies that, for all $\gamma < \beta < \alpha$, $\mathcal{L}_\gamma$ is a sublattice of the Boolean algebra $\mathcal{L}_\beta^c$.

2. Follows from item 3 of Definition 3.7.

3. By Stone duality, it suffices to check that the set of clopen subsets of X coincides with $(\mathcal{L}) = \bigcup_{\gamma<\alpha} \mathcal{L}_\gamma$. The inclusion from right to left follows from item 1. Conversely, let A be a clopen subset of X. Since α is limit, by item 3 of Definition 3.7 we have: for all $a \in A$ and $b \in \overline{A}$ there is $\beta = \beta(a,b) < \alpha$ with $a \not\leq_\beta b$, so $a \in U \not\ni b$ for some $U = U_{a,b,\beta} \in (\mathcal{L})$. For any fixed $a \in A$, $\{\overline{U_{a,b,\beta(a,b)}} \mid b \in \overline{A}\}$ is then an open cover of $\overline{A}$. Since $\overline{A}$ is clopen, there is a finite subcover $\{\overline{U_{a,b_i,\beta(a,b_i)}}\}_{i\leq k}$, for some $k < \omega$ and $b_0, \ldots, b_k \in \overline{A}$. For the set $U_a = \bigcap_{i\leq k} U_{a,b_i,\beta(a,b_i)} \in (\mathcal{L})$ we then have $a \notin \overline{U_a} \supseteq \overline{A}$, hence $a \in U_a \subseteq A$. Then $\{U_a \mid a \in A\}$ is an open cover of A. Since A is clopen, there is a finite subcover $\{U_{a_i}\}_{i\leq k}$, for some $k < \omega$ and $a_0, \ldots, a_k \in A$. For the set $U = \bigcup_{i\leq k} U_{a_i} \in (\mathcal{L})$ we then have $A \subseteq U \subseteq A$, hence $A = U \in (\mathcal{L})$. $\square$

We will need the following easy fact on subspaces of α-spaces.

Lemma 3.9 *Let $(X; \leq_\beta)_{\beta<\alpha}$ be an α-space, $\gamma < \alpha$ and let Y be a closed $\leq_\gamma$-up subset of X. Then $(Y; \leq_{\gamma+\delta})_{\delta<\alpha-\gamma}$ (with the induced topology and preorders) is an $(\alpha-\gamma)$-space.*

Proof. Since Y is closed, the space Y is compact. The only item of Definition 3.7 which is not obvious is 2. Let $y, z \in Y$ and $y \not\leq_\beta z$, $\beta = \gamma + \delta$; we have to find $U \in \mathcal{L}_\beta(Y)$ with $y \in U \not\ni z$. Since $(X; \leq_\gamma)_{\gamma<\alpha}$ is an α-space, there is $V \in \mathcal{L}_\beta(X)$ with $y \in V \not\ni z$. Clearly, $U = V \cap Y$ is clopen in Y and $y \in U \not\ni z$, so it remains to check that U is $\leq_\beta$-up. Let $u \in U$ and $u \leq_\beta v$ in Y, in particular $v \in Y$. Since Y is $\leq_\gamma$-up in X and $\gamma \leq \beta$, Y is $\leq_\beta$-up by item 2 of Definition 3.7. Therefore, U is $\leq_\beta$-up. $\square$

For any ordinal $\alpha \geq 1$, let $\mathbf{S}_\alpha$ denote the category with the α-spaces as objects and the continuous functions between α-spaces which are $\leq_\beta$-monotone for each $\beta < \alpha$, as morphisms.

Lemma 3.10 *For any ordinal $\alpha \geq 1$, the category $\mathbf{S}_\alpha$ has products and coproducts.*

Proof. Let $\{(X_i; \leq^i_\beta)_{\beta<\alpha} \mid i \in I\}$ be an indexed family of α-spaces. Let $X = \prod_i X_i$ be the cartesian product of sets X_i equipped with the standard Tychonoff topology and with preorders $\leq_\beta$ which are the products of the preorders $\leq^i_\beta$, $i \in I$. It is straightforward to check that $(X; \leq_\beta)_{\beta<\alpha}$ (together with the projection morphisms) is a product of $\{(X_i; \leq^i_\beta)_{\beta<\alpha} \mid i \in I\}$ in the category $\mathbf{S}_\alpha$.

Let $X = \coprod_i X_i$ be the disjoint union of sets X_i equipped with the standard coproduct topology and with preorders $\leq_\beta$ which are the coproducts of the preorders $\leq^i_\beta$, $i \in I$. It is straightforward to check that $(X; \leq_\beta)_{\beta<\alpha}$ (together with the injection morphisms) is a coproduct of $\{(X_i; \leq^i_\beta)_{\beta<\alpha} \mid i \in I\}$ in the category $\mathbf{S}_\alpha$. $\square$

Remark 3.11 One can slightly relax Definition 3.7 by omitting item 3. Let us call (in this remark) such a space *pre-α-space*. It is easy to see that pre-α-spaces are related to α-spaces in the same way as pre-Priestley spaces are related to Priestley spaces (see Lemma 3.1).

3.5 α-Duality

We are now ready to formulate the following extension of Priestley duality:

Theorem 3.12 *For any ordinal $\alpha \geq 1$, the categories $\mathbf{B}_\alpha$ and $\mathbf{S}_\alpha$ are dually equivalent.*

Proof sketch. By definition of dual equivalence [BuD70], we have to find contravariant functors $s : \mathbf{B}_\alpha \to \mathbf{S}_\alpha$ and $b : \mathbf{S}_\alpha \to \mathbf{B}_\alpha$ such that the functor $b \circ s$ (resp. $s \circ b$) is naturally isomorphic to the identity functor on $\mathbf{B}_\alpha$ (resp. $\mathbf{S}_\alpha$).

For an α-base $L = \{L_\beta\}_{\beta<\alpha}$, let $s(L) = (X; \leq_\beta)_{\beta<\alpha}$ where $X = s((L))$ is the Stone space of the Boolean algebra (L) and, for all $\beta < \alpha$ and $F, G \in X$, $F \leq_\beta G$ means $F \cap L_\beta \subseteq G \cap L_\beta$. It is easy to check (using, in particular, Lemmas 3.2 and 3.8) that $s(L)$ is an α-space. For a morphism $f : L \to M$ of α-bases $L = \{L_\beta\}_{\beta<\alpha}$ and $M = \{M_\beta\}_{\beta<\alpha}$, define the function $s(f) : s((M)) \to s((L))$ by $s(f)(F) = f^{-1}(F)$. It is easy to check that $s(f)$ is a morphism between the α-spaces $s(M)$ and $s(L)$, and s is a contravariant functor.

For an α-space $(X; \leq_\beta)_{\beta<\alpha}$, let $b(X) = \{\mathcal{L}_\beta\}_{\beta<\alpha} = \mathcal{L}$ be the α-base from Lemma 3.8. For a morphism $f : X \to$ of α-spaces $(X; \leq_\beta)_{\beta<\alpha}$ and $(Y; \leq_\beta)_{\beta<\alpha}$, define the function $b(f) : \bigcup_{\beta<\alpha} \mathcal{M}_\beta \to \bigcup_{\beta<\alpha} \mathcal{L}_\beta$, where $\mathcal{M} = b(Y)$, by $b(f)(A) = f^{-1}(A)$. It is easy to check that $b(f)$ is a morphism between the α-bases $b(Y)$ and $b(X)$, and b is a contravariant functor. Similar to the arguments of Priestley duality [Pr70, DP94] one can check that functors s and b have the desired properties. $\square$

From Theorem 3.12 and Lemma 3.10 we immediately obtain

Corollary 3.13 *For any ordinal $\alpha \geq 1$, the category $\mathbf{B}_\alpha$ has products and coproducts.*

Remark 3.14 Priestley duality is a particular case of Theorem 3.12 (for $\alpha = 1$) while Stone duality is a particular case of Priestley duality [DP94].

3.6 ω-Bases and Fine Hierarchies

We conclude this section by defining the notion of fine hierarchy (FH) and by explaining why Priestley duality is useful for the study of FH's.

In the study of finite levels of FH's, the notion of ω-base is central because such bases arise naturally in different fields (the quantifier-alternation hierarchy in logic, the arithmetical hierarchy in computability theory, the polynomial-time hierarchy in complexity theory and so on), and FH's are refinements of such bases (which are themselves hierarchies). Let us recall the notion of FH from [Se08].

Definition 3.15 *Let L be an ω-base and let H be a hierarchy in a Boolean algebra $\mathbb{B}$. We call H a fine hierarchy (w.r.t. L) if it is a refinement of L in some level or a global refinement of L.*

By Theorem 3.12, any ω-base L is isomorphic (in the category $\mathbf{B}_\omega$) to the ω-base $\mathcal{L} = \{\mathcal{L}_n\}_{n<\omega}$ in the dual ω-space $s(L) = (X; \leq_0, \leq_1, \ldots)$ of L. Since levels of FH's are defined in the signature of Boolean algebras, the corresponding levels of FH's over L and $\mathcal{L}$ (with the inclusion relation) are isomorphic and we may reduce the study of FH's over abstract bases L to the study of FH's over the bases $\mathcal{L}$. Moreover, the structure of ω-spaces provides important tools for a deeper study of FH's. This is why Priestley duality is useful in the study of FH's (as well as in the study of many other fields related to distributive lattices).

We show below that the levels of FH's over $\mathcal{L}$ have nice characterisations in terms of chains and trees formed from preorders $\leq_0, \leq_1, \ldots$, and the reducibility by morphisms of the category of ω-spaces (called M-reducibility) behaves in a sense similarly to the classical Wadge reducibility [Wad84]. Let $(X; \leq_0, \leq_1, \ldots)$ be an ω-space and $A, B \subseteq X$. We say that A is M-reducible to B (in symbols, $A \leq_M B$) if $A = f^{-1}(B)$ for some morphism $f : X \to X$.

As already mentioned in the Introduction, for pedagogical reasons we will consider first technically easier cases of some important concrete FH's, then proceed to the general FH of sets and finally to the more involved hierarchies of k-partitions.

4 Difference Hierarchy

In this section we discuss difference hierarchies (DH) which form the simplest and most important class of the fine hierarchies. DH's were first introduced and studied by F. Hausdorff [Ha14, Ha27] in an abstract setting and in the topological context. In the 1960-s, DH's were studied by J. Addison [Ad65] in the context of logic and by Yu.L. Ershov [Er68] in the context of computability theory. Later, DH's were considered by many authors working in different fields of mathematics and computer science.

4.1 Preliminaries on Difference Hierarchy

We start with recalling well-known facts about DH. For a sublattice $(L; \cup, \cap, 0, 1)$ of a Boolean algebra $\mathbb{B}$ and for each $k < \omega$, let $L(k)$ be the set of elements $\bigcup_i (a_{2i} \setminus a_{2i+1})$ where $a_i \in L$ satisfy $a_0 \supseteq a_1 \supseteq \cdots$ and $a_k = 0$. The sequence $\{L(k)\}_{k<\omega}$ is called *the difference hierarchy over L*. The following basic fact is well known:

Proposition 4.1 *For any $k < \omega$, $L(k) \cup \check{L}(k) \subseteq L(k+1)$ and $\bigcup_{k<\omega} L(k) = (L)$ is the Boolean algebra generated by L.*

The following characterization of the DH is due to several people (for additional information on this see [Se95]). Let T be the set of finite Boolean terms (i.e. terms of signature $\{\cup, \cap, \ , 0, 1\}$) with variables $v_k (k < \omega)$. Relate to any $t \in T$ the set $t(L)$ of all values of t when its variables range over L.

Proposition 4.2 *For any L specified above, $\{t(L) \mid t \in T\} = \{L(n), \check{L}(n) \mid n < \omega\}$.*

The last result explains why many characterizations of the DH are possible. We mention the following well-known characterizations:

Proposition 4.3
1. *$L(n) = t_n(L)$, where $t_1 = v_0$, $t_2 = v_0 \setminus v_1$, $t_3 = (v_0 \setminus v_1) \cup v_2$, $t_4 = (v_0 \setminus v_1) \cup (v_2 \setminus v_3)$, and so on [Er68];*
2. *$L(n) = L \ominus L(n-1) = L \ominus (\cdots \ominus L)$ (n summands, parenthesis to the right) where $A \ominus C = \{a \setminus c \mid a \in A, c \in C\}$ for $A, C \subseteq B$ [KSW87];*

3. $L(n) = L \oplus \cdots \oplus L$ *(n summands) where* $A \oplus C = \{a \triangle c \mid a \in A, c \in C\}$ *for* $A, C \subseteq B$ *and* $a \triangle c = (a \setminus c) \cup (c \setminus a)$ *is the symmetric difference of a and c [KSW87].*

The next fact from [Se95, Se09] characterizes the self-dual levels of the DH over bases with the separation property.

Proposition 4.4 *Let L have the separation property and* $k < \omega$. *Then* $L(k+1) \cap \check{L}(k+1)$ *coincides with the set of elements of the form* $(u \cap a) \cup (\overline{u} \cap b)$, *where* $u \in L \cap \check{L}$, $a \in L(k)$ *and* $b \in \check{L}(k)$.

It is easy to check the following sufficient condition for the perfectness of the DH:

Proposition 4.5 *Let 1 be join-irreducible in L. Then the DH over L is perfect and hence has no non-trivial refinements.*

Let $(X; \leq)$ be the pre-Priestley space corresponding to the pair (B, L) fixed above, and let $\mathcal{L}$ be the set of clopen up-sets in X. By Priestley duality (see Lemma 3.2), there is a canonical isomorphism f between the Boolean algebras (L) and $(\mathcal{L})$ that sends L onto $\mathcal{L}$. The next corollary of Priestley duality shows that all questions about the DH's over L and $\mathcal{L}$ are equivalent.

Proposition 4.6 *The isomorphism f of Boolean algebras* (L) *and* $(\mathcal{L})$ *induces isomorphisms of posets* $(L(n); \subseteq)$ *and* $(\mathcal{L}(n); \subseteq)$ *for each* $n < \omega$, *as well as between posets* $\{L(n), \check{L}(n) \mid n < \omega\}; \subseteq)$ *and* $(\{\mathcal{L}(n), \check{\mathcal{L}}(n) \mid n < \omega\}; \subseteq)$.

Proof. By Lemma 3.2, the first isomorphism is given by the restriction map $f|_{L(n)}$. Since f is a bijection between (L) and $(\mathcal{L})$, the image map $A \mapsto f(A)$ is an isomorphism of the powerset Boolean algebras $P((L))$ and $P((\mathcal{L}))$, hence it remains to check that this map sends $L(n)$ (resp. $\check{L}(n)$) to $\mathcal{L}(n)$ (resp. $\check{\mathcal{L}}(n)$) for each $n < \omega$. But this again follows from Lemma 3.2. □

4.2 Chain Characterisation of Difference Hierarchy

Let again $(X; \leq)$ be a pre-Priestley space. Of course, the DH over $\mathcal{L}$ may collapse (e.g., if $\mathcal{L}$ is a Boolean algebra then $\mathcal{L} = \check{\mathcal{L}}$ and the DH over $\mathcal{L}$ collapses to the first level). But many concrete DH's do not collapse, and now we discuss a notion variants of which are often used in proving the non-collapse property as well as other non-trivial facts on the DH. Namely, the DH's are closely related to the so called alternating chains.

Definition 4.7 *By an* alternating chain *of length k for* $A \subseteq X$ *we mean a sequence* $(x_0, \ldots, x_k)$ *of elements of X such that* $x_0 \leq \cdots \leq x_k$ *and* $x_i \in A$ *iff* $x_{i+1} \notin A$ *for each* $i < k$. *Such a chain is called a* 1-chain *if* $x_0 \in A$, *otherwise it is called a 0-chain.*

For $A \subseteq X$, $i \leq 1$ and $k \geq 0$, let $A^1 = A$, $A^0 = \overline{A}$, and let A^i_k be the set of all $x \in X$ such that $x \leq x_0$ for some i-alternating chain $(x_0, \ldots, x_k)$ for A.

Lemma 4.8 *Let* $(X; \leq)$ *be a pre-Priestley space and A a clopen subset of X. Then* A^i_k *is closed for all* $i \leq 1$ *and* $k \geq 0$.

Proof. By induction on k. For $k = 0$ the assertion holds by item 3 of Lemma 3.3. For $k \geq 1$ we have $A^i_k = \downarrow (A^i \cap A^{1-i}_{k-1})$, hence A^i_k is closed by induction and Lemma 3.3. □

The main result of this section is the following

Theorem 4.9 *Let* $(X; \leq)$ *be a pre-Priestley space,* $A \subseteq X$ *and* $k \geq 0$. *Then* $A \in \mathcal{L}(k)$ *iff* $A \in (\mathcal{L})$ *and A has no 1-alternating chain of length k.*

Proof. Let $A \in \mathcal{L}(k)$, then $A \in (\mathcal{L})$ and $A = \bigcup_i (A_{2i} \setminus A_{2i+1})$, where $A_0 \supseteq A_1 \supseteq \cdots$ is a descending sequence of $\mathcal{L}$-sets with $A_k = \emptyset$. Toward a contradiction, suppose that $(x_0, \ldots, x_k)$ is an 1-alternating chain for A. Since $x_0 \in A$, $x_0 \in A_0$. Since A_0 is an up-set and $x_0 \leq x_1$, $x_1 \in A_0$. Since $x_1 \notin A$, $x_1 \in A_1$. Continuing in this manner, we obtain $x_k \in A_k = \emptyset$, a contradiction.

The converse implication is checked by induction on k, the cases $k = 0, 1$ being trivial. For $k \geq 2$, we have $\forall a \in A \forall b \in A^0_{k-1} (a \not\leq b)$. By Lemma 4.8 and item 1 of Lemma 3.3, $A \subseteq U \subseteq \overline{A^0_{k-1}}$ for some $U \in \mathcal{L}$. The set $U \setminus A$ is clopen and has no 1-chain of length $k-1$ (indeed, if $(x_0, \ldots, x_{k-1})$ were such a chain it would be also a 0-chain for A, hence $x_0 \in U \cap A^0_{k-1}$ which is a contradiction). By induction $U \setminus A \in \mathcal{L}(k-1)$, hence $A = U \setminus (U \setminus A) \in \mathcal{L} \ominus \mathcal{L}(k-1)$. By item 2 of Proposition 4.3 $A \in \mathcal{L}(k) = \mathcal{L} \ominus \mathcal{L}(k-1)$. □

Note that if 1 is join-irreducible in a bounded distributive lattice L then $\{1\}$ is the smallest element in the Priestley space $(p(L) \leq)$, so the following corollary extends Proposition 4.5.

Corollary 4.10 *Let $(X; \leq)$ be a pre-Priestley space with a smallest element $\perp$ or a greatest element $\top$. Then the DH over $\mathcal{L}$ is perfect.*

Proof. We have to show that $\mathcal{L}(k) \cup \check{\mathcal{L}}(k) = \mathcal{L}(k+1) \cap \check{\mathcal{L}}(k+1)$ for each $k \geq 0$. The inclusion from left to right is trivial, so it remain to check if a set $A \in (\mathcal{L})$ is not in $\mathcal{L}(k) \cup \check{\mathcal{L}}(k)$ then it is not in $\mathcal{L}(k+1) \cap \check{\mathcal{L}}(k+1)$. By Theorem 4.9, for any $i \leq 1$ there is an i-chain $(x^i_0, \ldots, x^i_k)$ for A. Then one of $(\perp, x^i_0, \ldots, x^i_k)$, $i \leq 1$, or one of $(x^i_0, \ldots, x^i_k, \top)$, $i \leq 1$, is an alternating chain for A. By Theorem 4.9, A is not in $\mathcal{L}(k+1) \cap \check{\mathcal{L}}(k+1)$. □

Remark 4.11 For a given DH over L, the chain characterisation is not unique, i.e. there might exist a poset $(Y; \preceq)$ and a sublattice $\mathcal{M}$ of $P(Y)$ such that the DH over L and $\mathcal{M}$ are isomorphic, chain characterisation for DH over $\mathcal{M}$ holds, and $(Y; \preceq)$ is not isomorphic to the Priestley space $(X; \leq)$ of L. E.g., it is easy to check that $(A^*; \subseteq)$ is not isomorphic to the Priestly poset for the level 1/2 of the Straubing-Thérien hierarchy (see Introduction, Example 1). Different chain characterisations of a given DH may provide useful information on the hierarchy. E.g., in [GS01, GSS08] two different chain characterisations of the DH over the level 1/2 of the Brzozowski's dot-depth hierarchy were found which yield, respectively, polynomial-space and nondeterministic log-space algorithms deciding the levels of the hierarchy.

4.3 Difference Hierarchy and M-Reducibility

Here we show that M-reducibility introduced in Subsection 3.6 is closely related to the DH. For the context of pre-Pristley spaces definition of M-reducibility looks as follows. Let $(X; \leq)$ be a pre-Pristley space and $A, B \subseteq X$. We say that A is M-reducible to B (in symbols, $A \leq_M B$) if $A = f^{-1}(B)$ for some monotone continuous function $f : X \to X$.

Theorem 4.12 *Let $(X; \leq)$ be a pre-Priestley space.*

1. *For any $n \geq 0$, $\mathcal{L}(n)$ is closed under M-reducibility.*
2. *If $\mathcal{C} = \mathcal{L}(n) \setminus \check{\mathcal{L}}(n)$ is non-empty then $\mathcal{L}(n)$ has an M-complete set and $\mathcal{C}$ forms an M-degree.*
3. *If $\check{\mathcal{L}}$ has the separation property and $\mathcal{C} = (\mathcal{L}(n+1) \cap \check{\mathcal{L}}(n+1)) \setminus (\mathcal{L}(n) \cup \check{\mathcal{L}}(n))$ is non-empty then $\mathcal{L}(n+1) \cap \check{\mathcal{L}}(n+1)$ has an M-complete set and $\mathcal{C}$ forms an M-degree.*

Proof. 1. Since $\mathcal{L}$ is clearly closed under M-reducibility, so is also $\mathcal{L}(n)$.

2. It suffices to show that for all $A \in \mathcal{L}(n)$ and $B \in (\mathcal{L}) \setminus \mathcal{L}(n)$ we have $A \leq_M B$. By Theorem 4.9, there is a 0-chain $(b_0, \ldots, b_n)$ for B. Since $A \in \mathcal{L}(n)$, $A = \bigcup_i (A_{2i} \setminus A_{2i+1})$ for a descending sequence $A_0 \supseteq A_1 \supseteq \cdots$ of $\mathcal{L}$-sets with $A_n = \emptyset$. Define $f : X \to X$ by $f(x) = b_{m(x)}$ where $m(x) = \mu k (x \notin A_k)$ (so in fact $f : X \to \{b_0, \ldots, b_n\}$). Since $x \leq y$ and $x \in A_k$ imply $y \in A_k$, f is monotone. Since $f^{-1}(C)$ is a Boolean combination of $A_0, \ldots, A_{n-1}$ for each $C \subseteq X$, f is continuous. Since $x \in A$ iff $m(x)$ is odd iff $b_{m(x)} \in B$, $A = f^{-1}(B)$ and therefore $A \leq_M B$.

3. It suffices to show that for all $A \in \mathcal{L}(n+1) \cap \check{\mathcal{L}}(n+1)$ and $C \in (\mathcal{L}) \setminus (\mathcal{L}(n) \cup \check{\mathcal{L}}(n))$ we have $A \leq_M C$. By Theorem 4.9, there are a 0-chain $(c_0, \ldots, c_n)$ and a 1-chain $(d_0, \ldots, d_n)$ for C. Since $\check{\mathcal{L}}$ has the separation property and $A \in \mathcal{L}(n+1) \cap \check{\mathcal{L}}(n+1)$, by Proposition 4.4 there is $U \in \mathcal{L} \cap \check{\mathcal{L}}$ such that $A \cap U \in \mathcal{L}(n)$ and $A \cap \overline{U} \in \check{\mathcal{L}}(n)$, hence $\overline{A} \cap \overline{U} \in \mathcal{L}(n)$. Let $A_0 \supseteq A_1 \supseteq \cdots$ and $B_0 \supseteq B_1 \supseteq \cdots$ be descending sequences of $\mathcal{L}$-sets such that $A_n = B_n = \emptyset$, $A \cap U = \bigcup_i (A_{2i} \setminus A_{2i+1})$ and $\overline{A} \cap \overline{U} = \bigcup_i (B_{2i} \setminus B_{2i+1})$. Define $f : X \to X$ as follows: if $x \in U$ then $f(x) = c_{a(x)}$, otherwise $f(x) = d_{b(x)}$ where $a(x) = \mu k(x \notin A_k)$ and $b(x) = \mu k(x \notin B_k)$ (so in fact $f : X \to \{c_0, \ldots, c_n, d_0, \ldots, d_n\}$). Similar to the proof of 2, $A \leq_M C$ via f. $\square$

Remark 4.13 For a given DH over L, there might exist different m-reducibilities that fit isomorphic copies of the hierarchy. E.g., qf-reducibility from [SW05] fits the DH over the level 1/2 of the Brzozowski's dot-depth hierarchy (denoted as $\mathcal{L}$ in this remark) but it is distinct from the corresponding M-reducibility (because M-reducibility perfectly fits the isomorphic copy of the hierarchy by Proposition 4.12 and the result in [SW05] that $\check{\mathcal{L}}$ has the separation property while qf-reducibility does not). If $\check{\mathcal{L}}$ has the separation property, M-reducibility is the weakest among such reducibilities on $(\mathcal{L})$ because it induces the reducibility that perfectly fits the DH over $\mathcal{L}$.

4.4 Typed Difference Hierarchy

Here we mention a simple natural global refinement of an arbitrary ω-base $\mathcal{L}$ (recall that we are interested in natural refinements of $\mathcal{L}$). For each $n < \omega$, we can of course form the DH $\{\mathcal{L}_n(m)\}_m$ over $\mathcal{L}_n$; this is a refinement of $\mathcal{L}$ in the $(n+1)$-st level. We can also define a global refinement of $\mathcal{L}$, namely the sequence $\{\mathcal{L}_n(m)\}_{n,m<\omega}$ which we call here the typed DH over $\mathcal{L}$ (actually, in order to obtain a hierarchy in the sense of Definition 2.1, we have to renumerate the levels by ordinals $\alpha < \omega^2$ in the obvious way). This hierarchy has the length at most ω^2 because, in case it does not collapse, we have: $\mathcal{L}_n(m) \subseteq \mathcal{L}_{n_1}(m_1)$ iff $n < n_1$ or $n = n_1 \wedge m \leq m_1$, for all $n, m, n_1, m_1 < \omega$, $m, m_1 > 0$.

Let us formulate the chain characterisation of the typed DH. This is mainly for methodical reasons, in order to make preparations for similar technically more complicated notions for the finer hierarchies to be discussed later. Proposition 4.8 is obviously extended to the following:

Proposition 4.14 *Let $(X; \leq_0, \ldots)$ be an ω-space. For all $n, m < \omega$ the class $\mathcal{L}_n(m)$ coincides with the class of clopen subsets of X that have no 1-alternating chain in $(X; \leq_n)$ of length m.*

For subsets A, B of an ω-space X, the notion of M-reducibility is modified as follows: A is M-reducible to B (in symbols, $A \leq_M B$) if $A = f^{-1}(B)$ for some morphism $f : X \to X$ of ω-spaces. For this notion we have the following obvious corollary of Theorem 4.12.

Proposition 4.15 *Let $(X; \leq_0, \ldots)$ be an ω-space.*

1. *For any $m, n \geq 0$, $\mathcal{L}_n(m)$ is closed under M-reducibility.*
2. *If $\mathcal{C} = \mathcal{L}_n(m) \setminus \check{\mathcal{L}}_n(m)$ is non-empty then $\mathcal{L}_n(m)$ has an M-complete set and $\mathcal{C}$ forms an M-degree.*

In many concrete examples of the typed DH the DH over $\mathcal{L}_0$ is discrete. In contrast, the DH's over $\mathcal{L}_n$ for $n > 0$ usually have natural refinements (this is the reason why the analog of item 3 of Theorem 4.12 does not hold for those levels). We discuss some of them in the next sections.

5 Symmetric Difference Hierarchy

Here we discuss a fine hierarchy that is a bit less obvious to discover than the long DH. It was introduced in [Se94, Se95, Se99] under the name "plus-hierarchy" and renamed in [Wag98] to the "symmetric-difference hierarchy" (SDH); we use the last name in this paper.

5.1 Preliminaries on Symmetric Difference Hierarchy

Let L be an ω-base. By item 3 of Proposition 4.3, for all $n \geq 0$ and $m \geq 1$ we have $L_n(m) = L_n \oplus \cdots \oplus L_n$ (m summands). It is natural to ask what classes do we get if we also add the levels L_n for different n. Let *Alg* denote the collection of classes obtained in this way. Let *Seq* be the set of finite non-empty strings $\sigma = (n_0, \ldots, n_k)$ of natural numbers satisfying $n_0 \geq \cdots \geq n_k$, and let $<$ be the lexicographic order on *Seq*. For $\sigma = (n_0, \ldots, n_k) \in Seq$, let $P_\sigma = L_{n_0} \oplus \cdots \oplus L_{n_k}$; we call the sets P_σ *levels of the symmetric-difference hierarchy over L*. In other words, the non-zero levels of the SDH are sets of the form $L_{n_0}(k_0) \oplus \cdots \oplus L_{n_l}(k_l)$, for some $l \geq 0$, $n_0 > \cdots > n_l$ and $k_0, \ldots, k_l > 0$. Let us recall some easy properties of the defined objects from [Se99].

Proposition 5.1 *1. The structure $(Seq; <)$ is well-ordered with the corresponding ordinal ω^ω.*

2. *$\{P_\sigma \mid \sigma \in Seq\} = Alg$.*
3. *The SDH is a refinement of the typed DH.*
4. *For all $\sigma, \tau \in Seq$, if $\sigma < \tau$ then $P_\sigma \cup \check{P}_\sigma \subseteq P_\tau$, i.e. the SDH is a hierarchy in the sense of Definition 2.1 (after the obvious enumeration of levels by ordinals $< \omega^\omega$).*

It is easy to check that the analog of Proposition 4.6 holds for the SDH, i.e. the SDH over a given ω-base L is isomorphic to the SDH over the ω-base $\mathcal{L}$ in the dual ω-space $(X; \leq_0, \ldots)$, hence in most cases it suffices to investigate the second hierarchy. We also recall two straightforward characterisations of SDH from [Se95, Se99]. The first one is as follows:

Proposition 5.2 *Let $(X; \leq_0, \ldots)$ be an ω-space, $C \subseteq X$, $l > 0$, $n_0 > \cdots > n_l$, $k_0, \ldots, k_l > 0$, $\mathcal{C} = \mathcal{L}_{n_0}(k_0) \oplus \cdots \oplus \mathcal{L}_{n_l}(k_l)$ and $\mathcal{D} = \mathcal{L}_{n_0}(k_0) \oplus \cdots \oplus \mathcal{L}_{n_{l-1}}(k_{l-1})$. The following conditions are equivalent:*

1. *$C \in \mathcal{C}$.*
2. *There are $\mathcal{L}_{n_l}$-sets $E_0 \supseteq E_1 \supseteq \cdots$ such that $E_{k_l} = \emptyset$, $C \cap \overline{E}_0 \in \mathcal{D}$ and $C \cap (E_{2i+1} \setminus E_{2i+2}) \in \mathcal{D}$, $C \cap (E_{2i} \setminus E_{2i+1}) \in \check{\mathcal{D}}$ for all $i \geq 0$.*
3. *There are $\mathcal{L}_{n_l}$-sets $E_0 \supseteq E_1 \supseteq \cdots$ and $\mathcal{D}$-sets $D_0, D_1, \ldots$ such that $E_{k_l} = \emptyset$ and*

$$C = \bigcup_i ((D_0 \cap E_0) \cup (D_{2i+1} \cap (E_{2i+1} \setminus E_{2i+2}) \cup ((\overline{D}_{2i} \cap (E_{2i} \setminus E_{2i+1}).$$

The second characterisation is in terms of the following operation (introduced by W. Wadge [Wad84, Lo83]) on subsets of a Boolean algebra $\mathbb{B}$: $Sep(A, B, C) = \{(a \cap b) \cup (\overline{a} \cap c) \mid a \in A, b \in B, c \in C\}$.

Proposition 5.3 *Let $(X; \leq_0, \ldots)$ be an ω-space, $C \subseteq X$, $l > 0$, $n_0 > \cdots > n_l$, $k_0, \ldots, k_l > 0$, $\mathcal{C} = \mathcal{L}_{n_0}(k_0) \oplus \cdots \oplus \mathcal{L}_{n_l}(k_l)$ and $\mathcal{D} = \mathcal{L}_{n_0}(k_0) \oplus \cdots \oplus \mathcal{L}_{n_{l-1}}(k_{l-1})$. Then we have:*

1. *$\mathcal{L}_n(0) = \{\emptyset\}$ and $\mathcal{L}_n(k+1) = Sep(\mathcal{L}_n, \check{\mathcal{L}}_n(k), \mathcal{L}_n(0))$ for all $n, k \geq 0$.*
2. *$\mathcal{D} \oplus \mathcal{L}_{n_l} = Sep(\mathcal{L}_n, \check{\mathcal{D}}, \mathcal{L}_n(0))$ and $\mathcal{C} \oplus \mathcal{L}_{n_l} = Sep(\mathcal{L}_n, \check{\mathcal{C}}, \mathcal{L}_n(0))$.*

5.2 Chain Characterisation of Symmetric Difference Hierarchy

Here we adapt the alternating chains to the context of SDH. This is mainly also for methodical reasons, as a particular case of a more general notion in the next section.

Definition 5.4 *Let $(X; \leq_0, \ldots)$ be an ω-space and $\mathcal{L}$ the corresponding ω-base. Define chains of type $((n_0, k_0), \ldots, (n_l, k_l))$, for all $l < \omega$, $n_0 > \cdots > n_l$ and $k_0, \ldots, k_l > 0$, by induction on l as follows:*

1. *Chain of type (n_0, k_0) is a sequence $(x_0, \ldots, x_{k_0})$ in X satisfying $x_0 \leq_{n_0} \cdots \leq_{n_0} x_{k_0}$. Atoms of such a chain are by definition the components $x_0, \ldots, x_{k_0}$.*
2. *For $l > 0$, chain of type $((n_, k_0), \ldots, (n_l, k_l))$ is by definition a sequence $(X_0, \ldots, X_{k_l})$ of chains of type $((n_0, k_0), \ldots, (n_{l-1}, k_{l-1}))$ satisfying $X_0 \leq_{n_l} \cdots \leq_{n_l} X_{k_l}$ where $X_i \leq_{n_l} X_j$ means that for some (equivalently, for all) atoms x of X_i and y of X_j it holds $x \leq_{n_l} y$. Atoms of $(X_0, \ldots, X_{k_l})$ are the atoms of the components.*

Note that chains of type $((1,k_0),(0,k_1))$ essentially coincide with the corresponding "superchains" introduced in [Wag79]. From induction on l and Definition 5.4 we immediately obtain

Lemma 5.5 *If $(X_0,\ldots,X_{k_l})$ is a chain of type $((n_0,k_0),\ldots,(n_l,k_l))$ then there is a $\leq_{n_l}$-smallest atom in the set of all atoms in $X_0,\ldots,X_{k_l}$ (hence $a\equiv_n b$ for all atoms a,b and all $n<n_l$).*

Next we generalize the notion of 1-alternating chain from Subsection 4.2.

Definition 5.6 *Let $A\subseteq X$, $l<\omega$, $n_0>\cdots>n_l$ and $k_0,\ldots,k_l>0$. We define 1-alternating chains of type $((n_0,k_0),\ldots,(n_l,k_l))$ for A by induction on l as follows:*

1. *1-Alternating chain of type (n_0,k_0) for A is a chain $(x_0,\ldots,x_{k_0})$ of type (n_0,k_0) in X such that $x_{2i}\in A$ and $x_{2i+1}\notin A$.*
2. *For $l>0$, 1-alternating chain of type $((n_0,k_0),\ldots,(n_l,k_l))$ for A is a chain $(X_0,\ldots,X_{k_l})$ of type $((n_0,k_0),\ldots,(n_l,k_l))$ such that X_{2i} are 1-alternating chains for A and X_{2i+1} are 1-alternating chains for $\overline{A}$.*

The next result extends Theorem 4.9 to the context of the SDH.

Theorem 5.7 *Let $(X;\leq_0,\ldots)$ be an ω-space, $\mathcal{L}$ the corresponding ω-base, $l<\omega$, $n_0>\cdots>n_l$ and $k_0,\ldots,k_l>0$. Then the level $\mathcal{C}=\mathcal{L}_{n_0}(k_0)\oplus\cdots\oplus\mathcal{L}_{n_l}(k_l)$ of the SDH over $\mathcal{L}$ coincides with the class of clopen subsets of X that have no 1-alternating chains of type $((n_0,k_0),\ldots,(n_l,k_l))$.*

Proof. By induction on l. The case $l=0$ holds by Theorem 4.9, so let $l>0$. Let $C\in\mathcal{C}$, then of course C is clopen. Toward a contradiction, suppose that $(X_0,\ldots,X_{k_l})$ is a 1-chain for C of type $((n_0,k_0),\ldots,(n_l,k_l))$. Represent C as in item 2 of Proposition 5.2. We have $X_0\in E_0$ (i.e. any atom of X_0 is in E_0) because otherwise X_0 would be a 1-chain for $C\setminus E_0\in\mathcal{D}$, contradicting the inductive hypothesis. Since $X_0\leq_{n_l}X_1$, $X_1\in E_0$. We even have $X_1\in E_1$ (otherwise, X_1 would be a 1-chain for $\overline{C\cap(E_0\setminus E_1)}\in\mathcal{D}$, contradicting the inductive hypothesis). Continuing in this manner, we obtain a contradiction $X_{n_l}\in E_{n_l}=\emptyset$.

Conversely, let C be clopen and have no 1-alternating chain of type $((n_0,k_0),\ldots,(n_l,k_l))$. We argue by induction on k_l. For $k_l=1$, consider the set A (resp. B) of all elements $a\in X$ such that $a\leq_{n_{l-1}}Y_0$ (resp. $a\leq_{n_{l-1}}Y_1$) for some 1-chain Y_0 (resp. 0-chain Y_1) for C of type $((n_0,k_0),\ldots,(n_{l-1},k_{l-1}))$. Here $a\leq_{n_{l-1}}Y$ of course means that $a\leq_{n_{l-1}}y$ for each atom y of Y. By the analog of Lemma 4.8, the sets A,B are closed, and they satisfy $\forall a\in A\forall b\in B(a\not\leq_{n_l}b)$ because otherwise (Y_0,Y_1) would be a 1-alternating chain for C of type $((n_0,k_0),\ldots,(n_l,k_l))$. By item 1 of Lemma 3.3, $A\subseteq U\subseteq\overline{B}$ for some $U\in\mathcal{L}_{n_l}$. The set $C\cap\overline{U}$ has no 1-chain of type $((n_0,k_0),\ldots,(n_{l-1},k_{l-1}))$ (if $(X_0,\ldots,X_{k_{l-1}})$ were such a chain then, by Lemma 5.5, $X_0,\ldots,X_{k_{l-1}}\in\overline{U}$, so $(X_0,\ldots,X_{k_{l-1}})$ is also a 1-chain for C, hence $X_0\in A\subseteq U$, a contradiction). Similarly, $C\cap U$ has no 0-chain of type $((n_0,k_0),\ldots,(n_{l-1},k_{l-1}))$. By induction on l, $C\cap\overline{U}\in\mathcal{D}$ and $C\cap U\in\check{\mathcal{D}}$. By item 2 of Proposition 5.3, $C\in\mathcal{D}\oplus\mathcal{L}_{n_l}$, as desired.

Now let $k_l>1$. Let A be the same set as in the previous paragraph and B be the set of all $b\in X$ such that $b\leq_{n_{l-1}}Y_1$ for some 0-chain Y_1 for C of type $((n_0,k_0),\ldots,(n_{l-1},k_{l-1}),(n_l,k_l-1))$. As above, the sets A,B are closed, and they satisfy $\forall a\in A\forall b\in B(a\not\leq_{n_l}b)$ because otherwise (Y_0,Y_1) would be a 1-alternating chain for C of type $((n_0,k_0),\ldots,(n_l,k_l))$. By Lemma 3.3, $A\subseteq U\subseteq\overline{B}$ for some $U\in\mathcal{L}_{n_l}$. The set $C\cap\overline{U}$ has no 1-chain of type $((n_0,k_0),\ldots,(n_{l-1},k_{l-1},(n_l,k_l-1)))$ (if $(X_0,\ldots,X_{k_l})$ were such a chain then, by Lemma 5.5, $X_0,\ldots,X_{k_l}\in\overline{U}$, so $(X_0,\ldots,X_{k_l})$ is also a 1-chain for C, hence $X_0\in A\subseteq U$, a contradiction). Similarly, $C\cap U$ has no 0-chain of type $((n_0,k_0),\ldots,(n_{l-1},k_{l-1}))$. By induction on k_l, $C\cap\overline{U}\in\mathcal{D}\oplus\mathcal{L}_{n_l}(k_l-1)$ and $C\cap U\in\check{\mathcal{D}}$. By item 2 of Proposition 5.3, $C\in\mathcal{D}\oplus\mathcal{L}_{n_l}$, as desired. $\square$

5.3 Symmetric Difference Hierarchy and M-Reducibility

Here we show that M-reducibility (i.e. the many-one reducibility by morphisms of ω-spaces) is closely related to the SDH.

Theorem 5.8 *Let $(X;\leq_0,\ldots)$ be an ω-space and $\mathcal{L}$ the corresponding ω-base.*

1. *Any level of the SDH over $\mathcal{L}$ is closed under M-reducibility.*
2. *If $\mathcal{C}$ is a non-selfdual level of the SDH such that $\mathcal{C}\setminus\check{\mathcal{C}}\neq\emptyset$ then $\mathcal{C}$ has an M-complete set and $\mathcal{C}\setminus\check{\mathcal{C}}$ forms an M-degree.*

Proof. 1. Since any level of the ω-base $\mathcal{L}$ is clearly closed under M-reducibility, so is also any level of the SDH over $\mathcal{L}$.

2. To simplify notation, let us consider only the typical particular case $l=1$ (see Proposition 5.2), i.e. $\mathcal{C}=\mathcal{L}_{n_0}(k_0)\oplus\mathcal{L}_{n_1}(k_1)$ for some $n_0>n_1$ and $k_0,k_1>0$. It suffices to show that for any $A\in\mathcal{C}$ and any clopen set $B\not\in\check{\mathcal{C}}$ we have $A\leq_M B$. By Theorem 5.5, there is a 0-chain $(X_0,\ldots,X_{k_1})$ of type $((n_0,k_0),(n_1,k_1))$ for B. Let $X_i=(x^i_0,\ldots,x^i_{k_0})$ for each $i\leq k_1$, so in particular $x^i_0\leq_{n_0}\cdots\leq_{n_0}x^i_{k_0}$. Since $A\in\mathcal{C}$, by Proposition 5.2 there are $\mathcal{L}_{n_1}$-sets $A_0\supseteq A_1\supseteq\cdots$, $A_{k_1}=\emptyset$, and, for each i, $\mathcal{L}_{n_0}$-sets $F^i_0\supseteq F^i_1\supseteq\cdots$, $F^i_{k_0}=\emptyset$, such that

$$F^0_0\subseteq\overline{A}_0,\ F^{i+1}_0\subseteq A_i\setminus A_{i+1},\ A\cap\overline{A}_0=\bigcup_j(F^0_{2j}\setminus F^0_{2j+1}),$$

$$A\cap(A_{2i+1}\setminus A_{2i+2})=\bigcup_j(F^{2i+2}_{2j}\setminus F^{2i+2}_{2j+1}),\ A\cap(A_{2i}\setminus A_{2i+1})=\overline{F}^{2i+2}_0\cup\bigcup_j(F^{2i+1}_{2j+1}\setminus F^{2i+1}_{2j+2}).$$

Define $f:X\to X$ by $f(x)=x^i_j$ where i is the smallest number with $x\not\in A_i$ (note that $i\leq k_1$ because $A_{k_1}=\emptyset$) and j is the smallest number with $x\not\in F^i_j$ (note that $i\leq k_0$ because $F^i_{k_0}=\emptyset$). One easily checks that f is an endomorphism of $(X;\leq_0,\cdots)$ and $A\leq_M B$ via f. $\square$

As for the typed DH, the SDH is not discrete in most of its levels, which is the main reason why the analog of item 3 of Theorem 4.12 does not hold for those levels. This defect disappears for a further refinement of the SDH.

6 Fine Hierarchy

Are there other natural refinements of a given ω-base L? The answer is positive, and in principle we could continue the sequence typed DH, SDH,... indefinitely. Since any next element of this sequence would have more and more involved definitions, we choose another possibility. Namely, in this section we introduce a refinement which is in many cases the richest one, i.e. it refines all other reasonable (in a sense) refinements. We call this richest refinement *the fine hierarchy* (FH) over L. It was first discovered by the author in the context of computability theory [Se83] in terms of some jump operations. After acquaintance with some set-theoretic operations introduced by W. Wadge [Wad84, Lo83], the author [Se89, Se95] characterized the FH in terms of (some versions) of these operations and developed the abstract version of the FH.

6.1 Preliminaries on Fine Hierarchy

Here we recall some notions and results from [Se89, Se95, Se08]. We need the following operation *Bisep* [Se89, Se95] (a version of the corresponding operation of W. Wadge [Wad84, Lo83]) on subsets of a Boolean algebra $\mathbb{B}$:

$$Bisep(A,B_0,B_1,B_2)=\{(a_0\cap a_0)\cup(a_1\cap a_1)\cup(\overline{a}_0\cap\overline{a}_1\cap B_2)\mid a_i\in A,a_j\in B_j,a_0\cap a_1\cap a_0=a_0\cap a_1\cap a_1\}.$$

Definition of the fine hierarchy below uses the ordinal $\varepsilon_0=sup\{\omega,\omega^\omega,\omega^{\omega^\omega},\ldots\}$ (see Subsection 2).

Definition 6.1 *Let L be an ω-base. By the fine hierarchy over L we mean the sequence $\{S_\alpha\}_{\alpha<\varepsilon_0}$, where $S_\alpha=S^0_\alpha$ and the sets S^n_α, $n<\omega$, are defined by induction on α as follows:*

1. *$S^n_0=\{0\}$; $S^n_{\omega^\gamma}=S^{n+1}_\gamma$ for $\gamma>0$;*

2. $S^n_{\beta+1} = Bisep(L_n, S^n_\beta, \check{S}^n_\beta, S^n_0)$ *for all* $\beta < \varepsilon_0$*, and*
3. $S^n_{\beta+\omega^\gamma} = Bisep(L_n, S^n_\beta, \check{S}^n_\beta, S^n_{\omega^\gamma})$ *for* $\gamma > 0$ *and* β *of the form* $\beta = \omega^\gamma \cdot \beta_1 > 0$.

Recall that any non-zero ordinal $\alpha < \varepsilon_0$ is uniquely representable in the form $\alpha = \omega^{\gamma_0} + \cdots + \omega^{\gamma_k}$ for a finite non-empty sequence $\gamma_0 \geq \cdots \geq \gamma_k$ of ordinals $< \alpha$. Applying Definition 6.1 we subsequently get $S^n_{\omega^{\gamma_0}}$, $S^n_{\omega^{\gamma_0}+\omega^{\gamma_1}}, \ldots$, S^n_α. The sets S^n_γ for $n > 0$ play only a technical role, they are all among the levels S_α of the FH.

It is easy to check that the analog of Proposition 4.6 holds for the FH, i.e. the FH over a given ω-base L is isomorphic to the FH over the ω-base $\mathcal{L}$ in the dual ω-space $(X; \leq_0, \ldots)$, hence in most cases it suffices to investigate the second hierarchy. Let us recall some properties of the FH.

Proposition 6.2 *Let* $(X; \leq_0, \ldots)$ *be an* ω*-space and* $\mathcal{L}$ *the corresponding* ω*-base.*

1. *The FH over* $\mathcal{L}$ *is a hierarchy, i.e.* $\mathcal{S}_\alpha \cup \check{\mathcal{S}}_\alpha \subseteq \mathcal{S}_\beta$ *for all* $\alpha < \beta < \varepsilon_0$.
2. *For any limit ordinal* $\beta < \varepsilon_0$ *and any* $n < \omega$, $\mathcal{S}^n_{\beta+1}$ *coincides with the class of sets* $A \subseteq X$ *such that for some* $\mathcal{L}_n$*-sets* U_0, U_1 *we have* $A \subseteq U_0 \cap U_1$ *and* $\overline{A \cap U_0}, A \cap U_1$ *are in* $\mathcal{S}^n_\beta$.
3. *Let* $n < \omega, 1 \leq \gamma < \varepsilon_0$ *and* $\alpha = \beta + \omega^\gamma$ *for some non-zero* β *of the form* $\beta = \omega^\gamma \cdot \beta_1$. *Then* $\mathcal{S}^n_\alpha$ *coincides with the class of sets* $A \subseteq X$ *such that for some* $\mathcal{L}_n$*-sets* U_0, U_1 *we have:* $A \cap \overline{U_0} \cap \overline{U_1}$ *is in* $\mathcal{S}^n_{\omega^\gamma}$ *and* $\overline{A \cap U_0}, A \cap U_1$ *are in* $\mathcal{S}^n_\beta$.

Next we recall [Se89] characterisations of some levels of the FH in terms of the operation *Sep* from the previous section (cf. Proposition 5.3).

Proposition 6.3 *Let* $(X; \leq_0, \ldots)$ *be an* ω*-space and* $\mathcal{L}$ *the corresponding* ω*-base.*

1. *For all* $n, m < \omega$, $\mathcal{S}^n_{m+1} = Sep(\mathcal{L}_n, \check{\mathcal{S}}^n_m, \mathcal{S}^n_0)$. *Hence,* $\mathcal{S}^n_m = \mathcal{L}_n(m)$.
2. *For any successor ordinal* $\beta > \omega$ *and any* $n < \omega$, $\mathcal{S}^n_{\beta+1} = Sep(\mathcal{L}_n, \check{\mathcal{S}}^n_\beta, \mathcal{S}^n_0)$. *Furthermore,* $\mathcal{S}^n_{\beta+1}$ *coincides with the class of sets* $A \subseteq X$ *such that for some* $\mathcal{L}_n$*-set* U *we have* $A \subseteq U$ *and* $U \setminus A \in \mathcal{S}^n_\beta$.
3. *For all* $n < \omega$, $1 \leq m < \omega$ *and* $1 \leq \gamma < \varepsilon_0$, $\mathcal{S}^n_{\omega^\gamma(m+1)} = Sep(\mathcal{L}_n, \check{\mathcal{S}}^n_{\omega^\gamma m}, \mathcal{S}^n_{\omega^\gamma})$. *Furthermore,* $\mathcal{S}^n_{\omega^\gamma(m+1)}$ *coincides with the class of sets* $A \subseteq X$ *such that for some* $\mathcal{L}_n$*-set* U *we have* $A \setminus U \in \mathcal{S}^n_{\omega^\gamma}$ *and* $U \setminus A \in \mathcal{S}^n_{\omega^\gamma m}$.
4. *For all* $n < \omega$, $1 \leq m < \omega$, $1 \leq \gamma < \varepsilon_0$ *and* $\beta = \omega^\gamma \cdot \beta_1 > 0$, $\mathcal{S}^n_{\beta+\omega^\gamma(m+1)} = Sep(\mathcal{L}_n, \check{\mathcal{S}}^n_{\beta+\omega^\gamma m}, \mathcal{S}^n_{\omega^\gamma}$. *Furthermore,* $\mathcal{S}^n_{\beta+1}$ *coincides with the class of sets* $A \subseteq X$ *such that for some* $\mathcal{L}_n$*-set* U *we have* $A \setminus U \in \mathcal{S}^n_{\omega^\gamma}$ *and* $U \setminus A \in \mathcal{S}^n_\beta$.

As an immediate corollary of the last proposition and Proposition 5.3 we see that the FH over $\mathcal{L}$ is a refinement of the SDH over $\mathcal{L}$. Properties of the FH strongly depend on the properties of the corresponding ω-base. First we consider the interpolable ω-bases (see Subsection 3.3). It turns out that in this case the FH is often the finest possible.

Proposition 6.4 *Let the* ω*-base* $\mathcal{L}$ *in a given* ω*-space be interpolable. Then the fine hierarchy over* $\mathcal{L}$ *is perfect in all limit levels, i.e.* $\mathcal{S}_\alpha \cap \check{\mathcal{S}}_\alpha = \bigcup_{\beta<\alpha} \mathcal{S}_\beta$ *for all limit ordinals* $\alpha < \varepsilon_0$. *If, in addition,* X *is join-irreducible in* $(\mathcal{L}_0; \cup)$ *then the fine hierarchy over* $\mathcal{L}$ *is perfect and, consequently, has no non-trivial refinements.*

The FH as defined above seems rather artificial. It turns out that the FH's over reducible ω-bases have a nice characterization similar to the characterization of the DH by Boolean terms (cf. Proposition 4.2). Let T^* be the set of terms of signature $\{\cup, \cap, \bar{}, 0, 1\}$ with variables $v^n_k (k, n < \omega)$; we call them *typed Boolean terms.* Relate to any $t \in T^*$ the set $t(\mathcal{L})$ of values of t when the variables v^n_k $(k < \omega)$ of type n range through $\mathcal{L}_n$, for each $n < \omega$.

Proposition 6.5 *Let the base* $\mathcal{L}$ *in a given* ω*-space be reducible. Then* $\{\mathcal{S}_\alpha, \check{\mathcal{S}}_\alpha \mid \alpha < \varepsilon_0\} = \{t(\mathcal{L}) \mid t \in T^*\}$ *and there are algorithms that compute from any ordinal* $\alpha < \varepsilon_0$ *the corresponding Boolean term* $t \in T$ *and vice versa.*

Now let us return to the general case and recall a characterization of the FH in terms of trees. For any string $\tau \in \omega^*$ and any ω-base $\mathcal{L}$ as above, by a *τ-tree in $\mathcal{L}$* we mean a family $\{A_\sigma\}_{\sigma\in 2^*}$ of sets such that $A_\sigma = \emptyset$ for $|\sigma| > |\tau|$, $A_{\sigma k} \in \mathcal{L}_{\tau(|\sigma|)}$ for $|\sigma| < |\tau|$ and $k < 2$, and $A_\sigma \supseteq A_{\sigma k}$. A tree is *reduced*, if $A_{\sigma 0} \cap A_{\sigma 1} = \emptyset$ for all σ. We say that a set A *is defined* by a tree $\{A_\sigma\}$ as above, if $A \subseteq A_0 \cup A_1$, $A \cap A_{\sigma 0} \subseteq A_{\sigma 00} \cup A_{\sigma 01}$ and $\overline{A} \cap A_{\sigma 1} \subseteq A_{\sigma 10} \cup A_{\sigma 11}$. This notion does not depend on $A_\emptyset$; applying it we usually think that $A_\emptyset = X$ (if not, just replace $A_\emptyset$ by X). Let T_τ (resp. R_τ) denote the class of sets defined by the τ-trees (resp. by the reduced τ-trees) in $\mathcal{L}$.

Define strings $\tau^n_\alpha (n < \omega)$ by induction on α as follows: $\tau^n_0 = \emptyset$, $\tau^n_{\alpha+1} = n\tau^n_\alpha$, $\tau^n_{\omega^\gamma} = \tau^{n+1}_\gamma$ for $\gamma > 0$, and $\tau^n_{\delta+\omega^\gamma} = \tau^n_{\omega^\gamma} n \tau^n_\delta$ for $\delta = \omega^\gamma \cdot \delta' > 0$, $\gamma > 0$. Let $\tau_\alpha = \tau^0_\alpha$. Then we have the following characterization of the FH:

Proposition 6.6 *Let X be an ω-space and $\alpha < \varepsilon_0$. Then $\mathcal{S}_\alpha = T_\alpha$. If, in addition, $\mathcal{L}$ is reducible then $\mathcal{S}_\alpha = R_\alpha$ and, furthermore, $\mathcal{S}_{\alpha+1} \cap \check{\mathcal{S}}_{\alpha+1}$ is the class of sets defined by the reduced $\tau_{\alpha+1}$-trees $\{A_\sigma\}$ with $A_0 \cup A_1 = X$.*

6.2 Tree Characterisation of Fine Hierarchy

Here we extend the characterisation of SDH from Theorem 5.7 to a similar chracterisation of the FH. The alternating chains are now extended to alternating trees as follows. Let $(X; \leq_0, \ldots)$ be an ω-space, $A \subseteq X$ and $\tau \in \omega^*$. By a *τ-alternating tree* for A we mean a family $\{p_\sigma \mid \sigma \in 2^*, |\sigma| \leq |\tau|\}$ of elements of X such that $p_\emptyset \notin A$ and $p_{\sigma 0} \notin A$, $p_{\sigma 1} \in A$, $p_\sigma \leq_{\tau(|\sigma|)} p_{\sigma k}$ for $|\sigma| < |\tau|$ and $k < 2$.

Theorem 6.7 *Let $(X; \leq_0, \ldots)$ be an ω-space and $\alpha < \varepsilon_0$. Then the level $\mathcal{S}_\alpha$ of the fine hierarchy over $\mathcal{L}$ coincides with the class of clopen $A \subseteq X$ such that $\overline{A}$ does not have τ_α-alternating trees.*

Proof. Obviously, any $A \in \mathcal{S}_\alpha$ is clopen. To show the absence of alternating trees, by Proposition 6.6 it suffices to show that for all $\mu \in \omega^*$ and $A \in T_\mu$ there is no μ-alternating tree for $\overline{A}$. This is proved by induction on $|\mu|$. Let $|\mu| = 0$, then μ-alternating tree for $\overline{A}$ has the form $\{u_\emptyset\}$, for $u_\emptyset \in A$. Nonexistence of such a tree is equivalent to $A = \emptyset$, i.e. to $A \in T_\mu$. Let $|\mu| > 0$, then μ is uniquely representable in the form $\mu = \nu n \xi$ for some $\nu, \xi \in \omega^*$ and $n \in \omega$ with $\forall i < |\nu| (n < \nu(i))$ and $\forall i < |\xi| (n \leq \xi(i))$. Toward a contradiction, suppose that A is defined by a μ-tree $\{A_\sigma\}$ over $\mathcal{L}$ and there is a μ-alternating tree $\{u_\sigma\}_{\sigma \in 2^{\leq \mu}}$ for $\overline{A}$, so in particular $u_\emptyset, u_{\sigma 0} \in A, u_{\sigma 1} \notin A$. Let $B = \bigcap \{\overline{A}_{\delta k} | \delta \in 2^{|\nu|}, k < 2\}$, $C_\emptyset = X$ and $C_\sigma = A_\sigma B$ for $\sigma \neq \emptyset$. For any $\delta \in 2^{|\nu|}$ let $D^\delta_\emptyset = E^\delta_\emptyset = X$ and $D^\delta_\tau = A_{\delta 0 \tau}, E^\delta_\tau = A_{\delta 1 \bar{\tau}}$ for $\tau \neq \emptyset$. From the choice of ν, n, ξ it follows that $\{C_\sigma\}$ is a ν-tree that defines $A \cap B$, and for any $\delta \in 2^{|\nu|}$ the families $\{D^\delta_\tau\}$ and $\{E^\delta_\tau\}$ are ξ-trees that define respectively $A \cap A_{\delta 0}$ and $\overline{A} \cap A_{\delta 1}$.

Now consider the following three cases: $u_{\delta k} \notin A_{\delta k}$ for all $\delta \in 2^{|\nu|}, k < 2$; $u_{\delta 0} \in A_{\delta 0}$ for some $\delta \in 2^{|\nu|}$; $u_{\delta 1} \in A_{\delta 1}$ for some $\delta \in 2^{|\nu|}$. In the first case, $u_\sigma \in B$ for all $\sigma \in 2^{\leq |\nu|}$ (because $A_{\delta k} \in \mathcal{L}_n$, $u_\sigma \leq_n u_{\delta k}$ for $\sigma \subseteq \delta \in 2^{|\nu|}$ and $u_\sigma \equiv_n u_\rho$ for $\sigma, \rho \in 2^{\leq |\nu|}$), hence $\{u_\sigma\}_{\sigma \in 2^{\leq |\nu|}}$ is a ν-alternating tree for $A \cap B \in T_\nu$. In the second case, $u_{\delta 0 \tau} \in A_{\delta 0}$ for all $\tau \in 2^{\leq |\xi|}$ (because $u_{\delta 0} \in A_{\delta 0} \in L_n$ and $u_{\delta 0} \leq_n u_{\delta 0 \tau}$), hence $\{u_{\delta 0 \tau}\}_{\tau \in 2^{\leq |\xi|}}$ is a ξ-alternating tree for $A \cap A_{\delta 0} \in T_\xi$. In the third case, $\{u_{\delta 1 \bar{\tau}}\}_{\tau \in 2^{\leq |\xi|}}$ is similarly a ξ-alternating tree for $\overline{A} \cap A_{\delta 1} \in T_\xi$. In all cases we get contradictions with the induction hypothesis, because $|\nu|, |\xi| < |\mu|$. This completes the proof of one direction.

In the other direction, it suffices to show that if A is a clopen subset of X and $\overline{A}$ does not have τ^n_α-alternating trees then $A \in \mathcal{S}^n_\alpha$. We argue by induction on α and distinguish several cases. Let first $\alpha < \omega$, then τ^n_α is the sequence of n's of length α. Then A has no 1-alternating chain of length α w.r.t. $\leq_n$. By Theorem 4.9 and Proposition 6.3, $A \in \mathcal{L}_n(\alpha) = \mathcal{S}^n_\alpha$.

Let now $\alpha = \beta + 1$ where β is a limit ordinal, then $\tau^n_\alpha = n^\frown \tau^n_\beta$ and the first element m of τ^n_β is larger than n. Let C_0 (resp. C_1) be the set of all $c \in X$ such that there is a τ^n_β-tree Y for $\overline{A}$ (resp. for A) such that $c \leq_n Y$. As in Lemma 4.8, C_0 and C_1 are closed. We also have $\forall a \in A (\forall c \in C_0 (a \not\leq_n c) \vee \forall c \in C_0 (a \not\leq_n c))$ (otherwise, there are $a \in A, c_0 \in C_0, c_1 \in C_1$, a τ^n_β-tree Y_0 for $\overline{A}$ and a τ^n_β-tree Y_1 for A with $a \leq_n c_0 \leq_n Y_0$, $a \leq_n c_1 \leq_n Y_1$, yielding a τ^n_α-tree (a, Y_0, Y_1) for A which is a contradiction). By item 2 of Lemma 3.3, for some $\mathcal{L}_n$-sets U_0, U_1 we have $A \subseteq U_0 \cup U_1$, $A \cap U_0 \subseteq \overline{C}_0$ and $A \cap U_1 \subseteq \overline{C}_1$. Note that $\overline{A \cap U_0}$ has no

τ^n_β-tree. Indeed, suppose $Y_0 = \{p_\sigma \mid \sigma \in 2^*, |\sigma| \leq |\tau|\}$ is such a tree then (since $n < m$ and all elements of τ^n_β are $\geq n$) all elements of Y are in U_0, hence Y_0 is also a τ^n_β-tree for $\overline{A}$, so $p_0 \equiv_n p_\emptyset \in C_0$. Then $p_0 \in A \cap U_0 \subseteq \overline{C}_0$, a contradiction. Similarly, $A \cap U_1$ has no τ^n_β-tree. By induction, $\overline{A \cap U_0}$ and $A \cap U_1$ are in $\mathcal{S}^n_\beta$. By item 2 of Proposition 6.2, $A \in \mathcal{S}^n_\alpha$.

Let now $\alpha = \beta + 1$ where $\beta > \omega$ is a successor ordinal, then $\tau^n_\alpha = n^\frown\tau^n_\beta$ and the first element of τ^n_β is n. Let C be the set of all $c \in X$ such that there is a τ^n_β-tree Y for $\overline{A}$ with $c \leq_n Y$. As in Lemma 4.8, C is closed. We also have $\forall a \in A \forall c \in C(a \not\leq_n c)$ (otherwise, there are $a \in A, c \in C_0$ and a τ^n_β-tree Y for $\overline{A}$ with $a \leq_n c \leq_n Y$, yielding a τ^n_α-tree (a, Y) for A which is a contradiction). By Lemma 3.3, for some $\mathcal{L}_n$-set U we have $A \subseteq U \subseteq \overline{C}$. The set $U \setminus A$ has no τ^n_β-tree (if Y were such a tree then (since all elements of τ^n_β are $\geq n$) all elements of Y are in U, hence Y is also a τ^n_β-tree for $\overline{A}$, so $p_\emptyset \in C$, contradicting to $p_\emptyset \in U \subseteq \overline{C}$). By induction, $U \setminus A$ is in $\mathcal{S}^n_\beta$. By item 4 of Proposition 6.3, $A \in \mathcal{S}^n_\alpha$.

It remains to consider the case when α is a limit ordinal. If $\alpha = \omega^\gamma$ for some $\gamma \geq 1$, the assertion holds by induction, so let $\alpha = \beta + \omega^\gamma$ for some non-zero β of the form $\beta = \omega^\gamma \cdot \beta_1$. We have $\tau^n_\alpha = \tau^{n\frown}_{\omega^\gamma} n^\frown\tau^n_\beta$ and the first element m of τ^n_β is larger than n. Let B be the set of all $b \in X$ such that there is a $\tau^n_{\omega^\gamma}$-tree Z for A with $b \leq_{n+1} Z$. Let C_0 (resp. C_1) be the set of all $c \in X$ such that there is a τ^n_β-tree Y for $\overline{A}$ (resp. for A) such that $c \leq_n Y$. As in Lemma 4.8, B, C_0 and C_1 are closed. We also have $\forall b \in B(\forall c \in C_0(a \not\leq_n c) \vee \forall c \in C_0(a \not\leq_n c))$. Indeed, suppose the contrary: there are $b \in B, c_0 \in C_0, c_1 \in C_1$, a $\tau^n_{\omega^\gamma}$-tree $Z = \{z_\sigma \mid \sigma \in 2^*, |\sigma| \leq |\tau^n_{\omega^\gamma}|\}$ for A with $b \leq_{n+1} z_\varepsilon$, a τ^n_β-tree $Y_0 = \{y^0_\rho \mid \rho \in 2^*, |\sigma| \leq |\tau^n_\beta|\}$ for $\overline{A}$ with $b \leq_n c_0 \leq_n y^0_\varepsilon$, and a τ^n_β-tree $Y_1 = \{y^1_\sigma \mid \rho \in 2^*, |\rho| \leq |\tau^n_\beta|\}$ for A with $b \leq_n c_1 \leq_n y^1_\varepsilon$. Define the family $T = \{t_\sigma \mid \sigma \in 2^*, |\sigma| \leq |\tau^n_\alpha|\}$ as follows: $t_\sigma = z_\sigma$ for each $\sigma \in 2^*, |\sigma| \leq |\tau^n_{\omega^\gamma}|$; $t_\sigma i\rho = y^i_\rho$ for all $\sigma \in 2^*, |\sigma| = |\tau^n_{\omega^\gamma}|$, $i \leq 1$, and $\rho \in 2^*, |\rho| \leq |\tau^n_\beta|$. Then T is a τ^n_α-tree for A which is a contradiction. By Lemma 3.3, for some $\mathcal{L}_n$-sets U_0, U_1 we have $B \subseteq U_0 \cup U_1$, $B \cap U_0 \subseteq \overline{C}_0$ and $B \cap U_1 \subseteq \overline{C}_1$. As above, $\overline{A \cap U_0}$ has no τ^n_β-tree and $A \cap U_1$ has no τ^n_β-tree. Similarly, $A \cap \overline{U_0} \cap \overline{U_1}$ has no $\tau^n_{\omega^\gamma}$-tree. By induction, $A \cap \overline{U_0} \cap \overline{U_1}$ is in $\mathcal{S}^n_{\omega^\gamma}$, and $\overline{A \cap U_0}$ and $A \cap U_1$ are in $\mathcal{S}^n_\beta$. By item 3 of Proposition 6.2, $A \in \mathcal{S}^n_\alpha$. $\square$

Remark 6.8 One may wonder which levels of the FH can be characterised by alternating chains and which can not. With some additional efforts (and using Proposition 6.3 and alternating trees from [Se89]) it can be shown that a level of the FH over $\mathcal{L}$ can be characterised by alternating chains iff it is a level if the SDH over $\mathcal{L}$. All other levels do need alternating trees, though the width of the trees may be made less than of the trees used above.

6.3 Fine Hierarchy and M-Reducibility

Here we extend Theorems 4.12 and 5.8 to the FH. Of course, by M-reducibility we mean here the reducibility by morphisms of ω-spaces.

Theorem 6.9 *Let $(X; \leq_0, \ldots)$ be an ω-space and $\mathcal{L}$ the corresponding ω-base.*

1. *Any level $\mathcal{S}_\alpha$ of the FH over $\mathcal{L}$ is closed under M-reducibility.*
2. *If $\mathcal{L}$ is reducible and $\mathcal{C} = \mathcal{S}_\alpha \setminus \check{\mathcal{S}}_\alpha$ is non-empty then $\mathcal{S}_\alpha$ has an M-complete set and $\mathcal{C}$ forms an M-degree.*
3. *If $\mathcal{L}$ is reducible and $\mathcal{C} = (\mathcal{S}_{\alpha+1} \cap \check{\mathcal{S}}_{\alpha+1}) \setminus (\mathcal{S}_\alpha \cup \check{\mathcal{S}}_\alpha)$ is non-empty then $\mathcal{S}_{\alpha+1} \cap \check{\mathcal{S}}_{\alpha+1}$ has an M-complete set and $\mathcal{C}$ forms an M-degree.*

Proof. 1. Since $\mathcal{L}$ is closed under the M-reducibility, so is $\mathcal{S}_\alpha$ (use induction and Definition 6.1).

2. It suffices to show that for any $A \in \mathcal{S}_\alpha$ and any clopen set $B \notin \mathcal{S}_\alpha$ we have $A \leq_M B$. By Theorem 6.7, there is a τ_α-tree $\{t_\sigma \mid \sigma \in 2^*, |\sigma| \leq |\tau_\alpha|\}$ for B. Since $A \in \mathcal{S}_\alpha$, by Proposition 6.6 A is defined by a reduced τ_α-tree $\{A_\sigma\}$. Define $f : X \to X$ by $f(x) = t_\sigma$ where σ is the unique sting with $x \in \tilde{A}_\sigma$. It is easy to check that f is a morphism of ω-spaces and $A = f^{-1}(B)$, hence $A \leq_M B$.

3. It suffices to show that for any $A \in \mathcal{S}_{\alpha+1} \cap \check{\mathcal{S}}_{\alpha+1}$ and any clopen set $B \notin \mathcal{S}_\alpha \cup \check{\mathcal{S}}_\alpha$ we have $A \leq_M B$. By Theorem 6.7, there are τ_α-trees $\{t_\sigma \mid \sigma \in 2^*, |\sigma| \leq |\tau_\alpha|\}$ and $\{s_\sigma \mid \sigma \in 2^*, |\sigma| \leq |\tau_\alpha|\}$ for $\overline{C}$ and C,

respectively. By Proposition 6.6 A is defined by a reduced τ_α-tree $\{A_\sigma\}$ such that $A_0 \cup A_1 = X$. Define $f : X \to X$ by $f(x) = t_\sigma$ where σ is the unique string with $x \in \tilde{A}_\sigma$. It is easy to check that f is a morphism of ω-spaces and $A = f^{-1}(B)$, hence $A \leq_M B$. $\square$

Remark 6.10 As for the DH, for a given FH over an ω-base L there might exist different m-reducibilites that fit isomorphic copies of the hierarchy. If $\mathcal{L}$ is reducible and interpolable then M-reducibility is the weakest among such reducibilities on $(\mathcal{L})$ because, by the last theorem and Proposition 6.4, it induces the reducibility that perfectly fits the FH over $\mathcal{L}$. There are examples when our M-reducibility induces the preorders on $\bigcup_n \mathcal{L}_n$ which were already known in the literature. E.g., if we apply our construction to the 2-base $(\mathbf{\Sigma}^0_1 \cap \mathcal{R}, \mathbf{\Sigma}^0_2 \cap \mathcal{R})$ (more precisely, to the corresponding ω-base, see Definition 3.4 and the next paragraph) we obtain the Wagner's DA-reducibility on the class $\mathcal{R}$ of regular ω-languages [Wag79], and if we apply our construction to the 2-base $(\mathbf{\Sigma}^0_1 \cap \mathcal{A}, \mathbf{\Sigma}^0_2 \cap \mathcal{A})$ we obtain the author's AA-reducibility on the class $\mathcal{A}$ of regular aperiodic ω-languages [Se08a].

7 Difference Hierarchies of k-partitions

In this section we extend the DH of sets to the DH of k-partitions. Note that in general our definition of the DH of k-partitions is distinct from the definition of Boolean hierarchy of k-partitions over posets introduced and studied in [Ko00, Ko05].

7.1 Hierarchies and m-Reducibilities of k-Partitions

By Definition 2.1, levels of hierarchies of sets are semi-well-ordered by inclusion, in particular there are no three levels which are pairwise incomparable by inclusion. It turns out that the structure of hierarchies of k-partitions for $k \geq 3$ is usually more complicated than the structure of the hierarchies of sets, in particular, for $k \geq 3$ the poset of levels of hierarchies of k-partitions under inclusion usually has antichains with any finite number of elements.

In this subsection we propose a definition of a hierarchy of k-partitions that covers all hierarchies we discuss below and extends Definition 2.1. We start with the following very general notions:

Definition 7.1 1. *For any poset P and any set A, by a P-hierarchy in A we mean a family $\{H_p\}_{p\in P}$ of subsets of A such that $p \leq q$ implies $H_p \subseteq H_q$.*

2. *Levels (resp. constituents) of a P-hierarchy $\{H_p\}$ are the sets $H_{p_0} \cap \cdots \cap H_{p_n}$ (resp. the sets $C_{p_0,\ldots,p_n} = (H_{p_0} \cap \cdots \cap H_{p_n}) \setminus \bigcup\{H_q \mid q \in P \setminus \uparrow \{p_0, \ldots, p_n\}\}$) where $n \geq 0$ and $\{p_0, \ldots, p_n\}$ is an antichain in P.*
3. *A P-hierarchy $\{H_p\}$ is precise if $p \leq q$ is equivalent to $H_p \subseteq H_q$.*
4. *A P-hierarchy $\{H_p\}$ is perfect if $\{C_p\}_{p\in P}$ is a partition of $\bigcup_{p\in P} H_p$.*

Note that hierarchies in Definition 2.1 are obtained from the above definition if $B = A$ and $P = \bar{2} \cdot \eta$ is the poset obtained by replacing any element of the ordinal η by an antichain with two elements, and that the notion of preciseness extends the non-collapse property of hierarchies in Subsection 2. As was already mentioned, for hierarchies of k-partitions we cannot hope to deal only with semi-well-ordered posets $P = \bar{2} \cdot \eta$ in the definition above. Fortunately, a slight weakening of this property is sufficient for our purposes: we can confine ourselves with the so called well posets (wpo) or, more generally well preorders (wqo). Recall that a wqo is a preorder P that has neither infinite descending chains nor infinite antichains. The theory of wqo (widely known as the wqo-theory) is a well developed field with several deep results and applications, see e.g. [Kru72]. It is also of great interest to hierarchy theory.

Note that if P is a wpo then the structure $(\{H_p \mid p \in P\}; \subseteq)$ of levels of a P-hierarchy under inclusion is also a wpo, hence some important features of the hierarchies of sets hold also for the hierarchies of partitions. Moreover, for such hierarchies we have the following important properties of consituents (especially important is property 1 without which the hierarchy $\{H_p\}$ does not provide a classification of sets in $\bigcup_p H_p$):

Lemma 7.2 *Let P be a wpo and $\{H_p\}$ a P-hierarchy.*

1. *The constituents of $\{H_p\}$ form a partition of $\bigcup_p H_p$.*
2. *$\{H_p\}$ is perfect iff $H_p \cap H_q = \bigcup\{H_r \mid r \leq p, r \leq q\}$ for all $p, q \in P$.*

Proof. 1. First we check that constituents $C_{p_0,\ldots,p_n}$ and $C_{q_0,\ldots,q_m}$ are disjoint for all distinct antichains $\{p_0,\ldots,p_n\}$ and $\{q_0,\ldots,q_m\}$. Suppose x in the both constituents. Then $\uparrow\{p_0,\ldots,p_n\} = \uparrow\{q_0,\ldots,q_m\}$, hence also $\{p_0,\ldots,p_n\} = \{q_0,\ldots,q_m\}$. Indeed, by symmetry it suffices to show that $p_i \in \{q_0,\ldots,q_m\}$ for each $i \leq n$. Since $p_i \in \uparrow\{p_0,\ldots,p_n\} = \uparrow\{q_0,\ldots,q_m\}$, $q_j \leq p_i$ for some $j \leq m$. By symmetry, $p_l \leq q_j$ for some $l \leq n$. Then $p_l \leq p_i$, hence $l = i$ and $p_i = q_j \in \{q_0,\ldots,q_m\}$. It remains to check that any $x \in \bigcup_p H_p$ is in some constituent. Let $B = \{p \mid x \in H_p\}$. Since P is a wpo, there is an antichain $\{p_0,\ldots,p_n\}$ such that $B = \uparrow\{p_0,\ldots,p_n\}$. Then $x \in C_{p_0,\ldots,p_n}$.

2. Follows from 1. $\square$

If $\{H_p\}$ is a P-hierarchy and $Q \subset P$ is such that $\{H_p \mid p \in P\} = \{H_q \mid q \in Q\}$ then $\{H_q\}_{q\in Q}$ is a Q-hierarchy which is in a sense "more precise" than $\{H_p\}_{p\in P}$. For this reason we are interested in finding as small such Q as possible. But reducing of P to Q may change constituents which is clearly not desirable. Next two lemmas gives sufficient conditions for the reducing procedure being safe in this sense.

Lemma 7.3 *Let P be a poset, $\{H_p\}$ a P-hierarchy, $Q \subseteq P$, $f : P \to Q$ a monotone function such that $f(p) \leq p$ and $H_p = H_{f(p)}$ for each $p \in P$, and $f(q) = q$ for each $q \in Q$. Then any constituent of $\{H_q\}_{q\in Q}$ is a constituent of $\{H_p\}_{p\in P}$.*

Proof. We claim that any constituent $C_{q_0,\ldots,q_n}(Q)$ of $\{H_q\}$ coincides with the constituent $C_{q_0,\ldots,q_n}(P)$ of $\{H_p\}$. The inclusion $C_{q_0,\ldots,q_n}(P) \subseteq C_{q_0,\ldots,q_n}(Q)$ is obvious. Conversely, take $x \in C_{q_0,\ldots,q_n}(G)$ and suppose, toward a contradiction, that $x \notin C_{q_0,\ldots,q_n}(P)$. Then $x \in H_r$ for some $r \in P$ with $q_0 \not\leq r, \ldots, q_n \not\leq r$. Then $x \in H_{f(r)}$, $f(r) \in Q$ and $q_0 \not\leq f(r), \ldots, q_n \not\leq f(r)$, a contradiction. $\square$

Lemma 7.4 *Let P be a wpo which is a lower semilattice w.r.t. $\sqcap$, let Q be a subset of P which generates P under $\sqcap$, and let $\{H_p\}$ be a P-hierarchy such that $H_p \cap H_r = H_{p\sqcap r}$ for all $p, r \in P$. Then $\{H_q\}$ has the same levels as $\{H_p\}$ and any constituent of $\{H_q\}_{q\in Q}$ is a constituent of $\{H_p\}_{p\in P}$.*

Proof. The first assertion is obvious. We claim that any constituent $C_{q_0,\ldots,q_n}(Q)$ of $\{H_q\}$ coincides with the constituent $C_p(P)$, $p = q_0 \sqcap \ldots \sqcap q_n$ of $\{H_p\}$. The inclusion $C_p(P) \subseteq C_{q_0,\ldots,q_n}(Q)$ is obvious. Conversely, let $x \in C_{q_0,\ldots,q_n}(Q)$, then $x \in H_p$ and $x \notin H_r$ for all $r \in Q$ with $q_0 \not\leq r, \ldots, q_n \not\leq r$. Suppose, toward a contradiction, that $x \notin C_p(P)$. Then $x \in H_s$ for some $s \in P$ with $p \not\leq s$. For some $m \geq 0$ and $r_0, \ldots, r_m \in Q$ we have $s = r_0 \sqcap \cdots \sqcap r_m$, hence $p \not\leq r_i$ for some $i \leq m$. Then $q_0 \not\leq r_i, \ldots, q_n \not\leq r_i$ which is a contradiction. $\square$

We conclude this subsection by a discussion of some specific features of hierarchies and m-reducibilities of k-partitions. Notions related to m-reducibilities are extended to k-partitions in a straightforward way. Let F be a set of functions on a set X closed under composition and containing the identity function. For k-partitions $A, B \in k^X$, $A \leq_m^F B$ means that $A = B \circ f$ for some $f \in F$. As for the F-m-reducibility on subsets of X, $\leq_m^F$ is a preorder on k^X. Standard notions related to reducibilities on k-partitions (like completeness) are defined similarly to the case of sets. Observe that if C is F-m-complete in $\mathcal{C} \subseteq k^X$ and $h \in \mathbf{S}_k$ is a permutation of $k = \{0, \ldots, k-1\}$ then $h \circ C$ is F-m-complete in $\mathcal{C}^h = \{h \circ A \mid A \in \mathcal{C}\}$.

To capture some essential properties of hierarchies of k-partitions, we introduce the notion of *k-symmetric poset*: this is a triple $(P; \leq, \varphi)$ consisting of a poset $(P; \leq)$ and an isomorphic embedding $\varphi : \mathbf{S}_k \to Aut(P)$ of the symmetric group $\mathbf{S}_k$ into the automorphism group of $(P; \leq)$. Simplifying notation, we sometimes denote $(P; \leq, \varphi)$ just by P. The idea comes from the fact that the group $\mathbf{S}_k$ acts on the set k^X of k-partitions of X according to the rule $h \mapsto \lambda A.h \circ A$, and P-hierarchies of k-partitions should somehow respect this fact. An example of a k-symmetric poset is $(\overline{k}\cdot\alpha;\leq,\varphi)$ where α is an ordinal, $(\overline{k}\cdot\alpha;<)$ is obtained by replacing any point of α by the antichain with k elements and $\varphi(h)=\varphi_h$ permutes the elements of each copy of the antichain according to h. We use the bar in $\overline{k}$ in order to distinguish the antichain with k elements from the ordinal k which is a chain with k elements.

Definition 7.5 *1. Let $(P;\leq,\varphi)$ be a k-symmetric poset. A P-hierarchy of k-partitions of X ia a P-hierarchy $\{H_p\}$ in k^X such that $H_{\phi_h(p)}=\{h\circ A\mid A\in H_p\}$ for all $h\in\mathbf{S}_k$ and $p\in P$.*

2. *A Q-hierarchy $\{G_q\}$ of k-partitions of X is called a refinement of a P-hierarchy of k-partitions of X in a level $p\in P$ if $\bigcup\{H_r\mid\forall p_1\in Orb(p)(r<p_1)\}\subseteq\bigcup_{q\in Q}G_q\subseteq\bigcap_{p_1\in Orb(p)}H_{p_1}$ where $Orb(p)$ is the orbit of p under the action φ.*
3. *The hierarchy $\{H_p\}$ is discrete in a level p if it has no non-trivial refinement in this level.*

Because of Lemma 7.2 we are mostly interested in P-hierarchies of k-partitions for the case when P is a wpo. Note that for $k=2$ and $P=\overline{2}\cdot\eta$ the notion of P-hierarchy of 2-partitions essentially coincides with the notion of η-hierarchy of sets. As for hierarchies of sets, any perfect hierarchy of k-partitions is discrete in any level.

For a k-symmetric poset $(P;\leq,\varphi)$, we say that F-m-reducibility *fits* a P-hierarchy $\{H_p\}$ of k-partitions if any level H_p is a principal ideal of $(k^X;\leq^F_m)$. Other notions from Subsection 2 relating hierarchies and reducibilities are extended to k-partition in the straightforward way.

Remark 7.6 Similar to Subsection 4.1, instead of k-partitions of a set we could consider the more abstract case of k-partitions of the greatest element of a Boolean algebra $\mathbb{B}$, i.e. of sequences $(a_0,\ldots,a_{k-1})$ of pairwise disjoint elements of B with $a_0\cup\cdots\cup a_{k-1}=1$. The notions related to hierarchies of k-partitions are generalized to this case in a straightforward way. We do not do it explicitly because such "abstract" hierarchies are again isomorphic to the "concrete" hierarchies, via the duality theorems. Note that the notions related to m-reducibilities do not generalize to the abstract case directly.

7.2 Preliminaries on h-Preorders

In this subsection we recall some facts on the so called h-preorders studied in [Her93, Ko00, Ko05, KW00, KW08, Se04, Se07] in relation to the DH's and the Wadge reducibility of k-partitions, and make some additional remarks.

By *a finite forest* we mean a finite nonempty poset P in which every upper cone $\uparrow x$, $x\in P$, is a chain. *A finite tree* is a finite forest having the largest element (called *the root* of the tree). Since $(\omega^*;\sqsupseteq)$ is the infinite branching tree, any finite forest $(P;\leq)$ is isomorphic to a forest $(Q;\sqsupseteq)$ where Q is an initial segment of $(\omega^+;\sqsubseteq))$ such that $\forall\sigma\in\omega^*\forall n<\omega\forall m<n(\sigma^\frown n\in Q\to\sigma^\frown m\in Q)$. We will use this obvious fact a couple of times which sometimes simplifies notation because we can use standard notation related to strings.

Let $k\geq 2$. A *k-poset* is a triple $(P,\leq,c)$ consisting of a finite nonempty poset $(P;\leq)$, $P\subseteq\omega$, and a labeling $c:P\to k$. A *morphism* $f:(P,\leq,c)\to(P',\leq',c')$ between k-posets is a monotone function $f:(P;\leq)\to(P';\leq')$ satisfying $c=c'\circ f$. Let $\mathcal{P}_k$, $\mathcal{F}_k$, $\mathcal{T}_k$ and $\mathcal{T}^i_k$ (where $i<k$) denote the sets of all finite k-posets, k-forests, k-trees, and k-trees carrying label i on the root, respectively. The *h-preorder* $\leq_h$ on $\mathcal{P}_k$ is defined as follows: $(P,\leq,c)\leq(P',\leq',c')$, if there is a morphism from $(P,\leq,c)$ to $(P',\leq',c')$. The *h-equivalence* $\equiv_h$ on $\mathcal{P}_k$ is the equivalence relation induced by the h-preorder. The quotient-posets of $\mathcal{P}_k$, $\mathcal{F}_k$, $\mathcal{T}_k$ and $\mathcal{T}^i_k$ with the induced partial order $\leq$ are denoted $\mathbb{P}_k$, $\mathbb{F}_k$, $\mathbb{T}_k$ and $\mathbb{T}^i_k$, respectively.

We need a couple of results from [Se04] which are formulated below without proofs or with short proof hints (when it helps to explain some related things). For a finite poset P, $h(P)$ will denote *the height of* P, i.e. the number of elements of the longest chain in P. For any $i, 1\leq i\leq h(P)$, let $P(i)=\{x\in P\mid h(\uparrow x)=i\}$. Then $P(1),\ldots,P(h(P))$ is a partition of P on "levels"; note that $P(1)$ is the set of all maximal elements of P. For any $x\in P$, let $x\downarrow$ denote the set of all immediate predecessors of x in P, i.e. $x\downarrow=\{y<x\mid\neg\exists z(y<z<x)\}$. Note that $x\downarrow=\emptyset$ iff x is minimal in P.

Proposition 7.7 *1. For any finite poset P there exist a finite forest F and a monotone function f from F onto P so that $h(F) = h(P)$, f establishes a bijection between $F(1)$ and $P(1)$, and for any $x \in F$ function f establishes a bijection between $x \downarrow$ and $f(x) \downarrow$.*

2. For any finite k-poset $(P, \leq, c)$ there exist a finite k-forest $(F, \leq, d)$ and a morphism f from F onto P with the properties specified in item 1. Moreover, F is a largest element in $(\{G \in \mathcal{F}_k \mid G \leq_h P\}; \leq_h)$.

Proof hint. 1. The forest $F = F(P)$ is constructed by a natural top-down unfolding of P (for additional details see [Se04]).

2. For the first assertion, it suffices to expand the forest $F = F(P)$ by the labeling $d = c \circ f$. The second assertion (the analog of which was not included in [Se04]) is checked by a straightforward top-down induction: from a morphism $g : G \to P$, $G \in \mathcal{F}_k$, one constructs a morphism $h : G \to F$ with $g = f \circ h$. $\square$

Recall that a *minimal k-forest* is a finite k-forest not h-equivalent to a k-forest of lesser cardinality. As observed in [Se04], any finite k-forest is equivalent to a unique (up to isomorphism) minimal k-forest. The next characterisation of the minimal k-forests from [Se04] is often of use. This characterisation is a kind of inductive definition (by induction on the cardinality) of the minimal k-forests. The proof is also by induction.

Proposition 7.8 *1. Any singleton k-forest is minimal.*

2. A non-singleton k-tree $(T, \leq, c)$ is minimal iff the h-forest $(T \setminus T(1), c)$ is minimal and $c(x) \neq c(y)$ for all $x \in T(1)$ and $y \in T(2)$.

3. A proper (i.e., not h-equivalent to a k-tree) k-forest is minimal iff all its k-trees are minimal and pairwise incomparable under $\leq_h$.

As observed in [Ko00, Se04], the structure $(\mathcal{P}_k; \leq_h)$ (and even its substructure $(\mathcal{L}_k; \leq)$ formed by finite k-lattices) has for $k \geq 3$ infinite antichains and infinite descending chains (moreover, any countable poset is embeddable in $(\mathbb{P}_k; \leq)$ [Le08]). In contrast, the structure of $(\mathcal{F}_k; \leq_h)$ is much simpler as the following result (depending on a fact of wqo-theory [Kru72]) from [Se04] shows. Let $\mathbb{F}'_k$ be the structure obtained from $\mathbb{F}_k$ by adjoining the bottom element $\perp$ (which corresponds to the empty forest). By a *canonical representation* of $x \in \mathbb{F}'_k$ we mean representation $x = \sqcup \mathcal{Y}$ for some finite antichain $\mathcal{Y} \subseteq \mathbb{T}_k$. For h-posets P and R, let $P \sqcup R$ be their disjoint union, $P \sqcup R \in \mathcal{P}_Q$.

Recall some natural objects related to any wpo P. There are a unique ordinal $rk(P)$ and a unique *rank function* rk_P from P onto $rk(P)$ satisfying $a < b \to rk_P(a) < rk_P(b)$. It is defined by induction $rk_P(x) = sup\{rk_P(y) + 1 \mid y < x\}$. The ordinal $rk(P)$ is called the *rank* of P, and the ordinal $rk_P(x)$ is called the rank of the element $x \in P$ in P. By the *spectrum function* on P we mean the function $sp : rk(P) \to \omega$ where $sp(\alpha)$ is the number of elements of rank $\alpha < rk(P)$ in P. The *width* $w(P)$ of P defined as follows: if P has antichains with any finite number of elements, then $w(P) = \omega$, otherwise $w(P)$ is the greatest natural number n for which P has an antichain with n elements.

Proposition 7.9 *1. For any $k \geq 2$, $\mathbb{F}'_k$ is a distributive lattice, the set of non-zero join-irreducible elements of which coincides with $\mathbb{T}_k$.*

2. $(\mathbb{T}^0_k, \ldots, \mathbb{T}^{k-1}_k)$ is a partition of $\mathbb{T}_k$.

3. For any $k \geq 2$, $\mathbb{F}'_k$ is a wpo of rank ω.

4. $w(\mathbb{F}'_k) = 2$ and $w(\mathbb{F}'_k) = \omega$ for $k \geq 3$

5. Any element of $\mathbb{F}'_k$ has a unique canonical representation.

6. The symmetric group $\mathbf{S}_k$ acts on the automorphisms of $\mathbb{F}'_k$ by permuting the labels.

Proof hint. 1. The operation $\sqcup$ induces the operation of least upper bound on $\mathbb{P}_k$ and $\mathbb{F}_k$.

The greatest lower bound of h-posets $(P, \leq, c)$ and $(R, \leq, d)$ in $(\mathcal{P}'_k; \leq_h)$ (denoted by $P \sqcap R$) is defined in [Le06]. It is induced by the k-poset $(S; e)$ where S is the set of pairs $(p, r) \in P \times R$ such that $c(p) = d(r)$, the partial order in S is induced by the product partial order on $P \times R$, and the label $e(p, r)$ is $c(p)$.

The greatest lower bound of Q-forests $(P, \leq, c)$ and $(R, \leq, d)$ in $\mathbb{F}'_k$ is induced by $F(S)$ (see the proof of Proposition 7.7) where S is the k-poset defined above. $\Box$

For any $F \in \mathcal{F}'_k$ and and $i < k$, let $p_i(F)$ denote a k-tree obtained from F by adding a new smallest element with the label i. Operations p_i on $\mathcal{F}'_k$ induce operations on $\mathbb{F}'_k$ which we denote again by p_i.

Lemma 7.10 *For all $x, y \in \mathbb{F}'_k$ and for all distinct $i, j < k$ we have: $x \leq p_i(x)$, $x \leq y \to p_i(x) \leq p_i(y)$, $p_i(p_i(x)) = p_i(x)$, $p_i(x) \leq p_j(y) \to p_i(x) \leq y$, and $p_i(\mathbb{F}'_k) = \mathbb{T}^i_k$. Moreover, for any $x \in \mathbb{F}'_k$ there are $y_0, y_1 \in \mathbb{T}_k$ such that $x = y_0 \sqcap y_1$.*

Proof. For the first assertion, see [Se04]. For the second one, take $y_0 = p_0(x)$ and $y_1 = p_1(x)$. We have to check that $z \leq y_0, y_1$ implies $z \leq x$. For $z = \bot$ this is obvious, otherwise consider the canonical representation $z = z_0 \sqcup \cdots \sqcup z_n$ where $z_i \in \mathbb{T}_k = \mathbb{T}^0_k \cup \cdots \cup \mathbb{T}^{k-1}_k$. From the first assertion it follows that $z_i \leq x$ for all $i \leq n$, hence $z \leq x$. $\Box$

For any $x \in \mathbb{F}'_k$, let $M(x)$ be the set of minimal elements in $A_x = \{y \in \mathbb{F}'_k \mid y \not\leq x\}$, so $M(x)$ is a finite antichain and $\uparrow M(x) = A_x$. Let also $S(x) = \{y \in \mathbb{T}_k \mid y \not\leq x\}$.

Lemma 7.11 *1. For any $x \in \mathbb{F}'_k$, $M(x) \subseteq \mathbb{T}_k$.*

2. For all $x, y \in \mathbb{F}'_k$, $S(x \sqcap y) = S(x) \cup S(y)$ and $S(x \sqcup y) = S(x) \cap S(y)$.

Proof. 1. Let $y \in M(x)$, then clearly $y \neq \bot$. Assume $y = y_1 \sqcup y_2$. Since $y \not\leq x$, $y_1 \not\leq x$ or $y_2 \not\leq x$. By minimality of y, $y = y_1$ or $y = y_2$, hence y is join-irreducible and so $y \in \mathbb{T}_k$ by Proposition 7.9.

2. Follows from 1 because for any $t \in \mathbb{T}_k$ we have $t \not\leq x \sqcap y$ (resp. $t \not\leq x \sqcup y$) iff $t \not\leq x \vee t \not\leq y$ (resp. $t \not\leq x \wedge t \not\leq y$). $\Box$

Any word $w \in \{0, \ldots, k-1\}^*$ is naturally identified with a chain in $\mathcal{F}_k$ of length $|w|$. From Proposition 7.8 it follows that such a chain is minimal iff w is repetition-free. It is easy to see that the minimal (in the sense of Proposition 7.8) forests in $\mathcal{F}_2$ are in the bijective correspondence with the elements of $\{t_n, \bar{t}_n, t_n \sqcup \bar{t}_n \mid n \geq 1\}$ where $t_1 = 0, t_{n+1} = 0 \cdot \bar{t}_n$ and $\bar{w} = (1 - i_0) \cdots (1 - i_k)$ for $w = i_0 \cdots i_k$, $i_l \leq 1$. Thus, $\mathbb{F}_2$ is very simple, in particular its rank is ω, its width is 2 and its monadic second order theory is decidable. Also, the function M on $\mathbb{F}'_2$ is easy to compute: $M(\bot) = \{t_1, \bar{t}_1\}$, and, for each $n \geq 1$, $M(t_n) = \{\bar{t}_n\}$, $M(\bar{t}_n) = \{t_n\}$, $M(t_n \sqcup \bar{t}_n) = \{t_{n+1}, \bar{t}_{n+1}\}$.

In contrast, the poset $\mathbb{F}_k$ for $k \geq 3$ is much more complicated. E.g., its first order theory is computably isomorphic to the first-order arithmetic [KS07]. Nevertheless, the structure $\mathbb{F}_k$ is clearly computably presentable, and some natural functions and predicates on the structure are computable. E.g., it is easy to see that the rank and spectrum functions are computable. Below we will need the following information on the function M on $\mathbb{F}'_k$. For simplicity of notation we formulate it in terms of k-forests rather that for the corresponding classes of h-equivalence.

Lemma 7.12 *1. $M(\bot) = \{0, \ldots, k-1\}$ where $i < k$ is identified with the singleton tree labeled by i.*

2. For any $i < k$, $M(i) = k \setminus \{i\}$.

3. If $F = F_0 \sqcup \cdots \sqcup F_n$ is a minimal (in the sense of Proposition 7.8) k-forest which is not a tree then $M(F) \subseteq \{p_j(G_0 \sqcup \cdots \sqcup G_n) \mid j < k, G_0 \in M(F_0), \ldots, G_n \in M(F_n)\}$.

4. If $F = p_i(G)$ is a minimal (in the sense of Proposition 7.8) k-tree and G is nonempty then $M(F) \subseteq \{p_j(K) \mid j \in k \setminus \{i\}, K \in M(G)\}$.

5. The function $F : \mathbb{F}'_k \to P(\mathbb{T}_k)$ is computable.

Proof. Assertions 1 and 2 are obvious.

3. Let $Z \in M(F)$, then $Z \equiv_h p_j(Z)$ for some $j < k$ by Lemmas 7.11 and 7.10, and $Z \not\leq_h F$. Then $Z \not\leq F_l$ for all $l \leq n$, hence for any $l \leq n$ there is $G_l \in M(F_l)$ with $G_l \leq_h Z$. By Lemma 7.10, $p_j(G_0 \sqcup \cdots \sqcup G_n) \leq_h Z$. But, again by Lemma 7.10, $p_j(G_0 \sqcup \cdots \sqcup G_n) \not\leq_h F$, so in fact $p_j(G_0 \sqcup \cdots \sqcup G_n) \equiv_h Z$.

4. Let $Z \in M(F)$, then $Z \equiv_h p_j(Y)$ for some minimal (in the sense of Proposition 7.8) k-tree $p_j(Y)$, and $Z \not\leq_h F$. Then $j = i$ (otherwise, $Y <_h p_i(Y)$, so $Y \leq_h F$ and hence $Z \equiv_h p_i(Y) \leq_h F$ by Lemma 7.10,

a contradiction). Since $Z \not\leq_h G$, $K \leq_h Z$ for some $K \in M(G)$, hence $p_j(K) \leq_h Z$ by Lemma 7.10. But $p_j(K) \not\leq_h F$ (otherwise, $K \leq_h p_j(K) \leq_h G$ by Lemma 7.10, a contradiction), so in fact $p_j(K) \equiv_h Z$.

5. Note that item 1 takes care of the zero element, items 2 and 4 — of the nonzero join-irreducible elements, and item 3 — of all other elements of $\mathbb{F}'_k$. Altogether, items 1 — 4 provide an algorithm which produces (by induction on the rank) for any given $x \in \mathbb{F}'_k$, a finite set $Y \subseteq \mathbb{T}_k$ with $M(x) \subseteq Y$. Since for any $y \in Y$ we can effectively check whether $y \in M(x)$, $M(x)$ is computable. $\square$

7.3 Definition and Basic Facts

Let again $(X; \leq)$ be a pre-Priestley space and $\mathcal{L}$ the class of clopen up-sets in X. By a k-partition of X we mean a function $A : X \to k$ often written as a tuple $(A_0, \ldots, A_{k-1})$ where $A_i = \{x \in X \mid A(x) = i\}$. By a *partial k-partition of X* we mean a function $A : Y \to k$ for some $Y \in \mathcal{L}$. Let $P \in \mathcal{P}_k$. We say that a partial k-partition A is *defined by a P-family* $\{B_p\}_{p\in P}$ of $\mathcal{L}$-sets if $A_i = \bigcup_{p\in P_i} \tilde{B}_p$ for each $i < k$ where $\tilde{B}_p = B_p \setminus \bigcup_{q<p} B_q$ and $P_i = c^{-1}(i)$; note that in this case $A \in k^Y$ where $Y = \bigcup_{p\in P} B_p$.

Lemma 7.13 *Let $P, Q \in \mathcal{P}_k$, $P \leq_h Q$ via $\varphi : P \to Q$, let A be defined by a P-family $\{B_p\}$ of $\mathcal{L}$-sets, and let $\{C_q\}$ be a Q-family of $\mathcal{L}$-sets such that $\bigcup_{q\in Q} C_q = \bigcup_{p\in P} B_p$ and $\tilde{C}_q \subseteq \bigcup\{\tilde{B}_p \mid p \in \varphi^{-1}(q)\}$. Then A is defined by $\{C_q\}$.*

Proof. We have to check that $A_i = \bigcup_{q\in Q_i} \tilde{C}_q$ for each $i < k$. Let $x \in \tilde{C}_q$ for some $q \in Q_i$, $i < k$. Then $x \in \tilde{B}_p$ for some $p \in \varphi^{-1}(q) \subseteq P_i$, hence $x \in A_i$ and we have checked the inclusion $\bigcup_{q\in Q_i} \tilde{C}_q \subseteq A_i$. In the other direction, let $x \in A_i$. Then $x \in \tilde{B}_p$ for some $p \in P_i$. Since $\bigcup_{q\in Q} C_q = \bigcup_{p\in P} B_p$, $x \in \tilde{C}_q$ for some $q \in Q$, hence $x \in \tilde{B}_{p_1}$ for some $p_1 \in \varphi^{-1}(q)$. Since $A_i = \bigcup\{\tilde{B}_p \mid p \in P_i\}$, the components of A are pairwise disjoint, $\tilde{B}_p \subseteq A_i$ and $\tilde{B}_p \cap \tilde{B}_{p_1} \neq \emptyset$, $\tilde{B}_{p_1} \subseteq A_i$. Therefore, $q \in Q_i$ and $x \in \bigcup_{q\in Q_i} \tilde{C}_q$. $\square$

We denote by $\mathcal{L}^Y(P)$ the set of partitions $A : Y \to k$ defined by some P-family $\{B_p\}_{p\in P}$ of $\mathcal{L}$-sets. In case $Y = X$ we omit the superscript X and call (temporarily) the family $\{\mathcal{L}(P)\}_{P\in\mathcal{P}_k}$ *the DH of k-partitions over $\mathcal{L}$*. The idea behind definition of $\mathcal{L}(P)$ is to take the most liberal generalization of levels $\mathcal{L}(n)$ of the DH of sets, with the chain of n elements replaced by a poset P.

Lemma 7.14 *1. If $A \in \mathcal{L}^Y(P)$ then $A|_Z \in \mathcal{L}^Z(P)$ for each $Z \subseteq Y$, $Z \in \mathcal{L}$.*

2. *Any $A \in \mathcal{L}^Y(P)$ is defined by a monotone P-family $\{C_p\}$ (monotonicity means that $C_q \subseteq C_p$ for $q \leq p$).*
3. *Let f be a function on X such that $f^{-1}(A) \in \mathcal{L}$ for each $A \in \mathcal{L}$. Then $A \in \mathcal{L}^Y(P)$ implies $f^{-1}(A) = (f^{-1}(A_0), \ldots, f^{-1}(A_{k-1})) \in \mathcal{L}^{f^{-1}(Y)}(P)$.*
4. *If $P \leq_h Q$ then $\mathcal{L}^Y(P) \subseteq \mathcal{L}^Y(Q)$.*
5. *$\mathcal{L}^Y(P) = \mathcal{L}^Y(F(P))$ where $F(P)$ is the unfolding of P to a k-forest from Proposition 7.7.*
6. *The collection $\{\mathcal{L}(P) \mid P \in \mathcal{P}_k\}$ is well partially ordered by inclusion.*

Proof. 1. If A is defined by a P-family $\{B_p\}$ of $\mathcal{L}$-sets then the restriction $A|_Z = (A_0 \cap Z, \ldots, A_{k-1} \cap Z)$ is defined by the P-family $\{B_p \cap Z\}$ of $\mathcal{L}$-sets.

2. Let A be defined by a P-family $\{B_p\}$ of $\mathcal{L}$-sets. then A is defined also by the monotone P-family $\{C_p\}$ of $\mathcal{L}$-sets where $C_p = \bigcup_{q\leq p} B_q$.

3. Let $A \in \mathcal{L}^Y(P)$, so A is defined by a P-family $\{B_p\}_{p\in P}$ of $\mathcal{L}$-sets. Then $f^{-1}(A)$ is defined by the P-family $\{f^{-1}(B_p)\}_{p\in P}$ of $\mathcal{L}$-sets, hence $f^{-1}(A) \in \mathcal{L}^{f^{-1}(Y)}(P)$.

4. Let $\varphi : P \to Q$ be a monotone function such that $P_i = \varphi^{-1}(Q_i)$ for each $i < k$. Let $A \in \mathcal{L}^Y(P)$, so A is defined by a P-family $\{B_p\}_{p\in P}$ of $\mathcal{L}$-sets. Define the Q-family $\{C_q\}_{q\in Q}$ of $\mathcal{L}$-sets by $C_q = \bigcup\{B_p \mid p \in \varphi^{-1}(q)\}$, so in particular $C_q = \emptyset$ for $q \notin rng(\varphi)$. It remains to check that A is defined by $\{C_q\}$ (then $A \in \mathcal{L}^Y(Q)$). Since $\bigcup_{q\in Q} C_q = \bigcup_{p\in P} B_p$, by Lemma 7.13 it suffices to check that $\tilde{C}_q \subseteq \bigcup\{\tilde{B}_p \mid p \in \varphi^{-1}(q)\}$. Let $x \in \tilde{C}_q$, i.e. $x \in C_q$ and $x \notin C_r$ for all $r < q$. Then $x \in B_p$ for

some $p \in \varphi^{-1}(q)$; choose a minimal such p. Then $x \notin B_r$ for all $r < p$ (if $x \in B_r$, then $x \in C_{\varphi(r)}$ and $\varphi(r) \leq \varphi(p) = q$, hence $\varphi(r) = q$, contradicting to minimality of p). Therefore, $x \in \tilde{B}_p$, as desired.

5. Let $F = F(P)$. Since $F \leq_h P$ via the function $\varphi = \varphi_P : F \to P$ in Proposition 7.7, $\mathcal{L}^Y(F) = \mathcal{L}^Y(P)$ by 4. Conversely, let $A \in \mathcal{L}^Y(P)$, then A is defined by a P-family $\{B_p\}_{p \in P}$ of $\mathcal{L}$-sets. Define the F-family $\{C_q\}_{q \in F}$ of $\mathcal{L}$-sets by $C_q = B_{\varphi(q)}$. By construction of F and φ, $rng(\varphi) = P$ and $\tilde{C}_q = \tilde{B}_{\varphi(q)}$ for each $q \in F$. By Lemma 7.13, A is defined by $\{C_q\}$, hence $A \in \mathcal{L}^Y(F)$.

6. By 4 and 5, $F \mapsto \mathcal{L}(F)$ is a monotone function from $(\mathcal{F}_k; \leq_h)$ onto $\mathcal{D} = (\{\mathcal{L}(P) \mid P \in \mathcal{P}_k\}; \subseteq)$. By a well-known fact of wqo-theory, $\mathcal{D}$ is a wpo. $\Box$

By the last lemma and Subsection 7.1, it is better to denote the DH of k-partitions over $\mathcal{L}$ by $\{\mathcal{L}(F)\}_{P \in \mathcal{F}_k}$, so from now on by DH of k-partitions over $\mathcal{L}$ we usually mean the last family. By Lemmas 7.14 and 7.3, this reduction of $\mathcal{P}_k$ to $\mathcal{F}_k$ is safe.

For $F \in \mathcal{F}_k$, by a *reduced F-family of $\mathcal{L}$-sets* we mean a monotone F-family $\{B_p\}_{p \in F}$ of $\mathcal{L}$-sets such that $B_p \cap B_q = \emptyset$ for all incomparable $p, q \in F$. Let $A \in \mathcal{L}_r^Y(F)$ be the set of partial k-partitions defined by reduced F-families $\{B_p\}_{p \in F}$ of $\mathcal{L}$-sets such that $\bigcup_p B_p = Y$. The next result is similar to Theorem 3.1 in [Se04] and is proved by essentially the same argument.

Proposition 7.15 *Let $\mathcal{L}$ have the reduction property, $Y \in \mathcal{L}$ and $F \in \mathcal{F}_k$. Then $\mathcal{L}^Y(F) = \mathcal{L}_r^Y(F)$.*

Proof. For the non-trivial inclusion, let $A \in \mathcal{L}^Y(F)$, so by item 2 of Lemma 7.14 A is defined by a monotone F-family $\{B_p\}_{p \in F}$ of $\mathcal{L}$-sets such that $\bigcup_p B_p = Y$. Assuming w.l.o.g. the standard embedding of F in ω^+ (see the beginning of Subsection 7.2), we define C_p for $p \in F$ by induction on $|p|$ as follows. Let $\{0, \ldots, n\}$ be the set of minimal elements in $(F; \sqsubseteq)$. Let $(C_0, \ldots, C_n)$ be any reduct of $(B_0, \ldots, B_n)$ in $\mathcal{L}$ (see Definition 3.5). Suppose by induction that C_p is already defined and let p be not a maximal element in $(F; \sqsubseteq)$. Let $\{p0, \ldots, pm\}$ be the set of all immediate successors of p in $(F; \sqsubseteq)$. Let $(C_{p0}, \ldots, C_{pm})$ be any reduct of $(C_p \cap B_{p0}, \ldots, C_p \cap B_{p0})$ in $\mathcal{L}$.

By construction, $\{C_p\}$ is a reduced F-family of $\mathcal{L}$-sets and $C_0 \cup \cdots \cup C_n = B_0 \cup \cdots \cup B_n$. By Lemma 7.13 (taken for the identity function φ on F), it suffices to check (by induction on $|p|$) that $\tilde{C}_p \subseteq \tilde{B}_p$ for each $p \in F$. Let $|p| = 1$, so $p \leq n$. Let $x \in \tilde{C}_p$, so $x \in C_p$ and $x \notin C_q$ for $q \sqsupset p$. Since $(C_0, \ldots, C_n)$ is a reduct of $(B_0, \ldots, B_n)$, $x \in B_p$. Toward a contradiction, suppose that $x \in B_q$ for some $q \sqsupset p$. Since $\{B_p\}$ is monotone, $x \in B_{pi}$ for some $i \leq m$ where $\{p0, \ldots, pm\}$ is the set of all successors of p in $(F; \sqsubseteq)$. Then $x \in C_p \cap B_{pi}$. Since $(C_{p0}, \ldots, C_{pm})$ is a reduct of $(C_p \cap B_{p0}, \ldots, C_p \cap B_{p0})$, $x \in C_{pj}$ for some $j \leq m$ which is a contradiction. The same proof works for $|p| > 1$. $\Box$

Remark 7.16 Our DH of k-partitions is a modification of the "generalized Boolean hierarchy of k-partitions over posets" from [Ko00, Ko05] denoted in this remark by $\{\mathcal{L}'(P)\}_{P \in \mathcal{P}_k}$. Recall from [Ko00, Ko05] that $\mathcal{L}'(P)$ is the set of k-partitions defined by the monotone P-families $\{B_p\}_{p \in P}$ of $\mathcal{L}$-sets such that $B_p \cap B_q = \bigcup\{B_r \mid r \leq p \wedge r \leq q\}$ for all $p, q \in P$. Note that the last condition implies that $\{\tilde{B}_p\}_{p \in P}$ is a partition of X which is not in general the case for our P-families. Since $\{\mathcal{L}'(P)\}_{P \in \mathcal{P}_k}$ is a $\mathcal{P}_k$-hierarchy in the sense of Definition 7.5 [Ko00, Ko05], items 4 and 5 of Lemma 7.14 imply that $\bigcup\{\mathcal{L}'(Q) \mid F(Q) \leq_h P\} \subseteq \mathcal{L}(P)$ for each $P \in \mathcal{P}_k$, i.e. our hierarchy is in general much coarser than the hierarchy in [Ko00, Ko05]. The collection of levels of our DH of k-partitions is well partial ordered by inclusion, in contrast to the hierarchy in [Ko00, Ko05]. Another advantage of our definition of the DH compared with the definition from [Ko00, Ko05] is that the results of Section 4 may be naturally extended to our DH of k-partitions, as we show in the next subsection. On the other hand, the hierarchy from [Ko00, Ko05] over NP enables to make fine classification of some problems in complexity theory. For the important particular case when $\mathcal{L}$ has the reduction property both definitions are equivalent (this follows from Proposition 7.15 and a similar fact about the hierarchy $\{\mathcal{L}'(P)\}_{P \in \mathcal{P}_k}$ established in Theorem 3.1 of [Se04]).

7.4 Tree Characterisation of DH of k-Partitions

We again work with an arbitrary pre-Priestley space $(X;\leq)$ and the class $\mathcal{L}$ of clopen up-sets in X. For a partial k-partition A and a class $\mathcal{C}$ of subsets of X, let $A\in\mathcal{C}$ mean that all components $A_0,\ldots,A_{k-1}$ of A are in $\mathcal{C}$. The next result extends Theorem 4.9 to the DH hierarchy of k-partitions (to see this note that the second conjunct below is equivalent to "there is no $T\in M(F)$ with $T\leq_h(X;\leq,A)$" and look at the description of the operation M for $k=2$ in the example before Lemma 7.12).

Theorem 7.17 *Let $(X;\leq)$ be a pre-Priestley space, $k\geq 2$, $A\in k^X$ and $F\in\mathcal{F}_k$. Then $A\in\mathcal{L}(F)$ iff $A\in(\mathcal{L})$ and $T\leq_h F$ for all $T\in\mathcal{T}_k$ with $T\leq_h(X;\leq,A)$.*

Proof. Let $A\in\mathcal{L}(F)$, then A is defined by an F-family $\{B_p\}_{p\in F}$ of $\mathcal{L}$-sets, i.e. $A_i=\bigcup_{p\in F_i}\tilde{B}_p$ for each $i<k$. Then $A_i\in(\mathcal{L})$ for each $i<k$, hence $A\in(\mathcal{L})$.

Let now $(T,\leq,c)\in\mathcal{T}_k$ and $T\leq_h(X;\leq,A)$, i.e. there is a monotone function $\varphi:T\to(X;\leq)$ such that $\varphi^{-1}(A_i)=T_i$ for each $i<k$ where $T_i=c^{-1}(i)$. We have to show that $T\leq_h F$, i.e. to find a monotone function $\psi:T\to F$ such that $\psi^{-1}(F_i)=T_i$ for each $i<k$ (or, equivalently, $\psi(T_i)\subseteq F_i$ for each $i<k$). Assuming w.l.o.g. that T is embedded in ω^+ in the standard way explained in Subsection 7.2, we define $\psi(t)$, $t\in T$, by induction on $|t|$ as follows. Let first $|t|=1$, i.e. $t=0$, and let i be the unique number with $t\in T_i$. Then $\varphi(t)\in A_i$, so $\varphi(t)\in\tilde{B}_p$ for some $p\in F_i$. Choose any such p and set $\psi(t)=p$, so $\psi(t)\in F_i$ and $\varphi(t)\in\tilde{B}_{\psi(t)}$. Let now $s\in T$ and $|s|>1$, then $s=tl$ and $s\in T_j$ for unique $t\in T$, $l<\omega$, $j<k$, and assume by induction that $\psi(t)$ is defined and $\varphi(t)\in\tilde{B}_{\psi(t)}$. Since $t\sqsubset s$, $\varphi(s)\geq\varphi(t)\in B_{\psi(t)}$. Since $B_{\psi(t)}\in\mathcal{L}$, $\varphi(s)\in B_{\psi(t)}$. Then there is $p\leq\psi(t)$ with $\varphi(s)\in A_j$ (if $j=i$, $p=\psi(t)$, otherwise $p<\psi(t)$). Choose any such p and set $\psi(s)=p$. Then ψ has the desired properties.

In the other direction, let $A\in(\mathcal{L})$ and $T\leq_h F$ for all $T\in\mathcal{T}_k$ with $T\leq_h(X;\leq,A)$. The second conjunct implies that $T\not\leq_h(X;\leq,A)$ for all $T\in M(F)$. We prove $A\in\mathcal{L}(F)$ by induction on the cardinality of the forest F (assuming w.l.o.g. that F is minimal in the sense of Proposition 7.8). For singleton forests F the assertion follows from item 2 of Lemma 7.12. Let now the cardinality of F be ≥ 2 and F be not a tree, i.e. $F=F_0\sqcup\cdots\sqcup F_n$ for some $n\geq 1$ and minimal k-trees $F_0,\ldots,F_n$. By item 3 of Lemma 7.12, for all $G_0\in M(F_0),\ldots,G_n\in M(F_n)$ and $j<k$ we have $H=p_j(G_0\sqcup\cdots\sqcup G_n)\not\leq_h(X,\leq,A)$, i.e. there is no monotone function $\varphi:H\to X$ with $\varphi(H_i)\subseteq A_i$ for $i<k$. For any $l\leq n$. let C_l be the set of all $y\in X$ such that for some $K\in M(F_l)$ there is a morphism $\psi:K\to(X,\leq,A)$ with $y\leq\psi(g_l)$ where g_l is the root of G_l. Then we have $\forall x\in X(\forall y\in C_0(x\not\leq y)\vee\cdots\vee\forall y\in C_n(x\not\leq y))$ (otherwise it is easy to obtain a morphism from H to $(X,\leq,A)$ which is a contradiction). Similar to Lemma 4.8, $C_0,\ldots,C_n$ are closed. By item 2 of Lemma 3.3, there are $\mathcal{L}$-sets $U_0,\ldots,U_n$ such that $X\subseteq U_0\cup\cdots\cup U_n$ and $U_l\subseteq\overline{C}_l$ for each $l\leq n$. Then for the partial k-partitions $A|_{U_l}$, $l\leq n$ we have $\forall K\in M(F_l)(K\not\leq(U_l;\leq,A|_{U_l})$ because U_l are up-sets. By induction (applied to the pre-Priestley spaces $(U_l,\leq)$, $l\leq n$, see Lemma 3.9), $A|_{U_l}\in\mathcal{L}(U_l,F_l)=\mathcal{L}^{U_l}(F_l)$ for each $l\leq n$, i.e. $A|_{U_l}$ is defined by an F_l-family $\{B^l_p\}$ of $\mathcal{L}$-sets. Then A is defined by the F-family $\{C_q\}$ of $\mathcal{L}$-sets where $C_q=B^l_q$ for all $q\in F_l$, $l\leq n$.

Finally, let $F=p_i(G)$ for some i and nonempty k-forest G. By item 4 of Lemma 7.12, $S\not\leq_h(X;\leq,A)$ for all S in the set $\{p_j(K)\mid j\in k\setminus\{i\},K\in M(G)\}$. Let C be the set of all $y\in X$ such that there are $K\in M(G)$ and a morphism $\psi:K\to(X,\leq,A)$ with $y\leq\psi(r)$ where r is the root of K. Then we have $\forall x\in\overline{A}_i\forall y\in C(x\not\leq y)$ (if there are $x\in\overline{A}_i,y\in C$ with $x\not\leq y$, it is easy to obtain an h-morphism from $S=p_j(K)$, $x\in A_j$, to $(X,\leq,A)$ which is a contradiction). By Lemmas 4.8 and 3.3, there is an $\mathcal{L}$-set U such that $\overline{A}_i\subseteq U\subseteq\overline{C}$. Then for the partial k-partition $A|_U$ we have $\forall K\in M(G)(K\not\leq(U;\leq,A|_U)$ because U is an up-set. By induction (applied to the pre-Priestley space $(U;\leq)$, see Lemma 3.9), $A|_U\in\mathcal{L}^U(G)$, i.e. $A|_{U_l}$ is defined by a G-family $\{B_p\}$ of $\mathcal{L}$-sets. Then A is defined by the F-family $\{C_q\}$ of $\mathcal{L}$-sets where $C_p=B_p$ for all $p\in G$, and $C_r=X$ for the root r of F. $\Box$

The last theorem implies the following nice properties of the DH of k-partitions over $\mathcal{L}$.

Corollary 7.18 *Let $(X;\leq)$ be a pre-Priestley space and $k\geq 2$.*

1. *For all $F,G\in\mathcal{F}_k$, $\mathcal{L}(F)\cap\mathcal{L}(G)=\mathcal{L}(F\sqcap G)$. Here we assume that $\mathcal{L}(\perp)=\emptyset$.*
2. *If X has a bottom element $\perp$ then $\mathcal{L}(F)\cup\mathcal{L}(G)=\mathcal{L}(F\sqcup G)$ for all $F,G\in\mathcal{F}_k$, and the hierarchy $\{\mathcal{L}(T)\}_{T\in\mathcal{T}_k}$ is perfect.*

Proof. 1. The inclusion from right to left follows from item 4 of Lemma 7.14. Conversely, let $A \in \mathcal{L}(F) \cap \mathcal{L}(G)$ and suppose, toward a contradiction, that $A \notin \mathcal{L}(F \sqcap G)$. Then, in particular, F and G are h-incomparable. By Theorem 7.17, $T \leq_h (X, \leq, A)$ for some $T \in S(F \sqcap G)$. By item 2 of Lemma 7.11, $T \in S(F)$ or $T \in S(G)$. By Theorem 7.17, $A \notin \mathcal{L}(F)$ of $A \notin \mathcal{L}(G)$, a contradiction.

2. The inclusion $\mathcal{L}(F) \cup \mathcal{L}(G) \subseteq \mathcal{L}(F \sqcup G)$ holds by item 4 of Lemma 7.14. Conversely, let $A \in \mathcal{L}(F \sqcup G)$ and suppose, toward a contradiction, that $A \notin \mathcal{L}(F) \cup \mathcal{L}(G)$. By Theorem 7.17, there are $T_1 \in S(F)$ and $T_2 \in S(G)$ with $T_1, T_2 \leq_h (X, \leq, A)$. Let $T = p_i(T_1 \sqcup T_2)$ where $i = A(\perp)$, then $T \in S(F) \cap S(G)$ and $T \leq_h (X, \leq, A)$. By item 2 of Lemma 7.11, $T \in S(F \sqcup G)$. By Theorem 7.17, $A \notin \mathcal{L}(F \sqcup G)$, a contradiction.

For the second assertion, we have to show that $\mathcal{L}(F) \cap \mathcal{L}(G) \subseteq \bigcup\{\mathcal{L}(T) \mid T \in \mathcal{T}_k, T \leq_h F \sqcap G\}$ for all $F, G \in \mathcal{T}_k$. For some $n < \omega$ and some $T_i \in \mathcal{T}_k$ we have $F \sqcap G = \bigsqcup_{i<n} T_i$. By the previous two paragraphs, $\mathcal{L}(F) \cap \mathcal{L}(G) = \mathcal{L}(F \sqcap G) = \bigcup_{i<n} \mathcal{L}(T_i)$. $\square$

Corollary 7.19 *Let $(X; \leq)$ be a pre-Priestley space, $k \geq 2$ and $A \in (\mathcal{L})$ a k-partition of X. Then there is an $\leq_h$-greatest element F_A in $I_A = \{G \in \mathcal{F}_k \mid G \leq_h (X, \leq, A)\}$, and A is in the F_A-constituent of the DH over $\mathcal{L}$ (see Lemma 7.2), i.e. $A \in \mathcal{L}(F_A) \setminus \bigcup\{\mathcal{L}(H) \mid H \in \mathcal{F}_k, F_A \not\leq_h H\}$.*

Proof. By [Ko00], $A \in \mathcal{L}'(C)$ for some k-chain $C \in \mathcal{C}_k$, hence $A \in \mathcal{L}(F)$ for some $F \in \mathcal{F}_k$. By Theorem 7.17, $G \leq_h F$ for each $G \in I_A$, hence I_A is finite modulo $\equiv_h$. Then $F_A = \bigsqcup I_A$ is an $\leq_h$-greatest element in I_A. Now, $G \leq_h F_A$ for all $G \leq_h (X, \leq, A)$. By Theorem 7.17, $A \in \mathcal{L}(F_A)$. It remains to show that $A \notin \mathcal{L}(H)$ for each $H \in \mathcal{F}_k, F_A \not\leq_h H$, i.e. $A \in \mathcal{L}(H)$ implies $F_A \leq_h H$. This again follows from Theorem 7.17 because $F_A \leq_h (X, \leq, A)$. $\square$

Remark 7.20 Corollary 7.18 shows that levels $\mathcal{L}(T)$, for $T \in \mathcal{T}_k$, are analogs of the non-self-dual levels $\mathcal{L}(n), \check{\mathcal{L}}(n)$, $n \leq \omega$, of the DH of sets: any other ("self-dual") level is an intersection of finitely many of the levels $\mathcal{L}(T)$ (interestingly, for hierarchies of k-partitions the distinction of self-dual and non-self-dual levels is not related to automorphisms, as for hierarchies of sets). Therefore, the precise analog of the DH of sets is in fact the hierarchy $\{\mathcal{L}(T)\}_{T \in \mathcal{T}_k}$ rather than the hierarchy $\{\mathcal{L}(F)\}_{F \in \mathcal{F}_k}$. Note that, by Lemmas 7.10 and 7.4, this reduction from $\mathcal{F}_k$ to $\mathcal{T}_k$ is safe. The perfectness property of the DH of k-partitions extends Proposition 4.5 (interestingly, the additional sufficient condition of perfectness of the DH of sets in terms of $\top$ from Corollary 4.10 is not extended to the DH of k-partitions for $k \geq 3$). For some natural examples of $\mathcal{L}$ the hierarchy $\{\mathcal{L}(F)\}_{F \in \mathcal{F}_k}$ is exact and discrete for each $k \geq 2$ [Se04, Se07a].

7.5 DH of Partitions and M-Reducibility

Here we find an extension of Theorem 4.12 to the DH of k-partitions.

Theorem 7.21 *Let $(X; \leq)$ be a pre-Priestley space and $k \geq 2$.*

1. *For any $F \in \mathcal{F}_k$, $\mathcal{L}(F)$ is closed under the M-reducibility.*
2. *If $\mathcal{L}$ has the reduction property then $A \leq_M C$ for all $A \in \mathcal{L}(F)$ and $C \in (\mathcal{L})$ with $F \leq_h (X, \leq, C)$.*
3. *If $\mathcal{L}$ has the reduction property then $A \equiv_M \mathcal{L}(F_A)$ for all $A \in (\mathcal{L})$.*
4. *If $\mathcal{L}$ has the reduction property and $C \in \mathcal{L}(F) \setminus \bigcup\{\mathcal{L}(H) \mid H \in \mathcal{F}_k, F \not\leq_h H\}$ then $C \equiv_M \mathcal{L}(F)$.*
5. *If $\mathcal{L}$ has the reduction property then the quotient-structure of $(\{A \in k^X \mid A \in (\mathcal{L})\}; \leq_M)$ is embeddable into $(\{\mathcal{L}(F) \mid F \in \mathcal{F}_k\}; \subseteq)$.*
6. *If $\mathcal{L}$ has the reduction property and all constituents of $\{\mathcal{L}(F)\}_{F \in \mathcal{F}_k}$ are nonempty then the quotient-structure of $(\{A \in k^X \mid A \in (\mathcal{L})\}; \leq_M)$ is isomorphic to $(\{\mathcal{L}(F) \mid F \in \mathcal{F}_k\}; \subseteq)$ and to $(\mathbb{F}_k; \leq)$.*

Proof. 1. Since $\mathcal{L}$ is closed under M-reducibility, the assertion follows from item 3 of Lemma 7.14.

2. By Proposition 7.15, A is defined by a reduced F-family $\{B_p\}$ of $\mathcal{L}$-sets. Let $\varphi : F \to X$ be a witness for $F \leq_h (X, \leq, C$. Define $f : X \to X$ by $f(x) = \varphi(p)$ where p is the unique element of F with $x \in \tilde{B}_p$. Since $f^{-1}(Y)$ is clopen for each $Y \subseteq X$, f is continuous. If $x \leq y$ and $x \in \tilde{B}_p$, $y \in \tilde{B}_q$ then

$p \le q$, hence $f(x) = \varphi(p) \le \varphi(q) = f(y)$ and f is monotone. If $x \in A_i$ and $x \in \tilde{B}_p$ then $p \in F_i$, hence $f(x) = \varphi(p) \in C_i$. Therefore, $A \le_M C$ via f.

3. By Corollary 7.19, $A \in \mathcal{L}(F_A)$ and $F_A \le_h (X, \le, A)$. By 2, $A \equiv_M \mathcal{L}(F_A)$.

4. By Corollaries 7.18 and 7.19, $F_C \equiv_h F$. By item 4 of Proposition 7.14, $\mathcal{L}(F_C) = \mathcal{L}(F)$. By 3, $C \equiv_M \mathcal{L}(F)$.

5. We show that $A \mapsto F_A$ induces a desired embedding, i.e. $A \le_M B$ iff $\mathcal{L}(F_A) \subseteq \mathcal{L}(F_B)$. Let $A \le_M B$ and $C \in \mathcal{L}(F_A)$. By 3, $C \le_M A$, hence $C \le_M B$. Since $B \in \mathcal{L}(F_B)$, $C \in \mathcal{L}(F_B)$ by 1. Therefore, $\mathcal{L}(F_A) \subseteq \mathcal{L}(F_B)$. Conversely, let us assume the last inclusion. Since $A \in \mathcal{L}(F_A)$, $A \in \mathcal{L}(F_B)$. By 3, $A \le_M B$.

6. By 4 and 5, the quotient-structure of $(\{A \in k^X \mid A \in (\mathcal{L})\}; \le_M)$ is isomorphic to $(\{\mathcal{L}(F) \mid F \in \mathcal{F}_k\}; \subseteq)$. Since all constituents are nonempty, the DH $\{\mathcal{L}(F)\}_{F] \in \mathcal{F}_k}$ is exact, hence $(\{\mathcal{L}(F) \mid F \in \mathcal{F}_k\}; \subseteq)$ is isomorphic to $(\mathbb{F}_k; \le)$. □

Remark 7.22 For some F the assertion 2 above holds also for $\mathcal{L}$ without the reduction property. E.g., this is the case when F is a k-chain or if any label in F occurs only once. In both cases any $x \in X$ is in exactly one of the sets. $\tilde{B}_p$, $p \in F$.

8 Conclusion

The theory of hierarchies originated in the work of A. Mostowski and S. Kleene who discovered deep analogies between hierarchies in descriptive set theory and in computability theory. In the work of J. Addison [Ad62, Ad65] more analogies between different hierarchies were discovered, the name "hierarchy theory" was coined, and some basic definitions of the theory were proposed. In [Se83] and subsequent publications of the author some further notions of hierarchy theory were developed in the context of FH's. Many interesting and useful concrete hierarchies were studied in different parts of theoretical computer science (partially systematized in [Se08]).

But "theory" should probably be something more than a collection of analogies between many objects with similar properties. In hierarchy theory two methods were used frequently: the method of alternating chains (and trees) and the method of m-reducibilities. This paper shows that these two methods apply essentially to arbitrary hierarchy and they may be even treated uniformly using an extension of Priestley duality. For this reason we hope that this paper is of some methodological interest to the theory of hierarchies of sets.

Along with hierarchies of sets, more general hierarchies of k-partitions were considered recently in the literature (in fact, implicitly some related objects were implicitly considered in computability theory earlier). The first examples of Boolean hierarchy of k-partitions were studied in [Ko00, KW00, Se04]. As the discussion in Subsection 7.3 shows, even the basic concept of a DH of k-partitions is far from obvious. We hope that the properties of DH of k-partitions in Section 7 show that the corresponding notion is "right". In fact, notions of Subsection 7.1 are sufficient for development of the FH of k-partition, as we hope to demonstrate in a subsequent publication. Thus, this paper is also a step in development of the theory of hierarchies of k-partitions.

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