Simple Bases for Boolean Co-Clones

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1 Introduction

This paper arose from the study of the computational complexity of generalized satisfiability problems as begun by Thomas Schaefer in 1978 in his seminal paper [Sch78]. Schaefer considered formulas that are conjunctions of constraints, where the constraints are given by Boolean relations. More formally, let Γ be a non-empty finite set of Boolean relations. A Γ-*formula* F (over some set of variables V) is a finite conjunction of clauses $F = C_1 \wedge \ldots \wedge C_p$, where each clause C_i is an application of some constraint relation R to a vector of variables, i.e., of the form $R(x_1, \ldots, x_n)$ for some $R \in \Gamma$, $x_1, \ldots, x_n \in V$. Let $\mathrm{CNF}(\Gamma)$ be the set of such formulas. As usually, $F \in \mathrm{CNF}(\Gamma)$ is called satisfiable if there is an assignment I of truth values to the variables such that for every clause $R(x_1, \ldots, x_n)$, the vector $(I(x_1), \ldots, I(x_n))$ is in R. Let $\mathrm{SAT}(\Gamma)$ denote the set of satisfiable formulas from $\mathrm{CNF}(\Gamma)$.

Now suppose that we have another set Γ' of relations, where every relation in Γ' can be "implemented" using relations from Γ (the term "implemented" is used in the same meaning in [CKS01]), i.e., for every $R \in \Gamma'$ of arity n, there is a propositional formula $F \in \mathrm{CNF}(\Gamma)$ (with free variables $x_1, \ldots, x_m$, where $m \geq n$) such that $R(x_1, \ldots, x_n) \iff \exists x_{n+1} \ldots \exists x_m F$. Then obviously, every formula from $\mathrm{CNF}(\Gamma')$ can be transformed into a $\mathrm{CNF}(\Gamma)$-formula which is satisfiable if and only if the original formula was satisfiable, and hence $\mathrm{SAT}(\Gamma')$ reduces to $\mathrm{SAT}(\Gamma)$.

For a set of relations Γ, let $\langle\Gamma\rangle$, the *co-clone generated by* Γ, be the set of all relations that can be implemented as above by Σ_1-formulas over relations from Γ. The above considerations then prove that $\mathrm{SAT}(\Gamma)$ and $\mathrm{SAT}(\langle\Gamma\rangle)$ are equivalent under logspace many-one reductions.[1] Hence, the complexity of $\mathrm{SAT}(\Gamma)$ depends only on the co-clone generated by Γ, and to determine the complexity of $\mathrm{SAT}(\Gamma)$ as a function of Γ, it is useful (1) to have a complete list of co-clones, and (2) to have a (simple) base (i.e., generating set of relations) for each co-clone.[2] The list of co-clones is given, as we will explain in the next section, by Post's list of closed classes of Boolean functions. In this short note, we present for each co-clone a simple base. Fig. 2 summarizes our results.

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[1]Strictly speaking, relations from Γ plus a relation EQ for equality are allowed to implement $\langle\Gamma\rangle$, where $\mathrm{EQ}(x, y) \iff x = y$. This subtle point is discussed in detail in [ABI+04], but since it is of no importance for the development of this paper, we do not elaborate it here.

[2]This approach was actually followed in [ABI+04]—that paper makes use of some of the bases given here to obtain complexity results for certain satisfiability problems.

2 Preliminaries

Let R be a Boolean relation of arity n and let f be a Boolean function of arity m. We say that f is a closure property of R if applying f coordinate-wise to m vectors in R, the obtained vector is again in R. More formally, we say that R is *closed* under f (or f *preserves* R, or f is a *polymorphism* of R, or R is an *invariant* of f), if for all $\alpha_1, \ldots, \alpha_m \in R$, where $\alpha_i = (\alpha_i[1], \alpha_i[2], \ldots, \alpha_i[n])$, we have $f(\alpha_1, \ldots, \alpha_m) =_{\mathrm{df}} \big(f(\alpha_1[1], \cdots, \alpha_m[1]), f(\alpha_1[2], \cdots, \alpha_m[2]), \ldots, f(\alpha_1[n], \cdots, \alpha_m[n])\big) \in R$. The set of all polymorphisms of R is denoted by $\mathrm{Pol}(R)$, and for a set Γ of Boolean relations, we define $\mathrm{Pol}(\Gamma)$ to be the set of Boolean functions that are polymorphisms of every relation in Γ. Conversely, if B is a set of Boolean functions, then $\mathrm{Inv}(B)$ is defined to be the set of all Boolean relations which are invariants of the functions in B.

It is easy to observe that every set $\mathrm{Pol}(\Gamma)$ is closed under superposition (i.e., generally functional substitution), hence every $\mathrm{Pol}(\Gamma)$ is what is called a *clone* in universal algebra. All Boolean clones were identified by Emil Post in [Pos41], see also [JGK70, Pip97]. Every set $\mathrm{Inv}(B)$ is a co-clone as explained in the introduction, in particular, $\mathrm{Inv}(\mathrm{Pol}(\Gamma)) = \langle\Gamma\rangle$ (cf. [PK79, Pip97]). Thus, a complete list of co-clones and their inclusion structure is obtained by considering all $\mathrm{Inv}(B)$ for clones B and inverting the inclusions among them; see Fig. 1.

To rephrase the preceding paragraph, Pol and Inv form a Galois correspondence (see [Pip97]) between closed sets of relations and closed sets of Boolean functions. This is also surveyed in [BCRV03, BCRV04]; in this paper we will make use of the notation for clones and co-clones developed there.

For a set Γ of Boolean relations and a clone B, we say Γ is a *base* for $\mathrm{Inv}(B)$ if $\mathrm{Pol}(\Gamma) = B$. This definition is sensible, because we can express all relations having B as polymorphisms using only relations from Γ, that is: $\langle\Gamma\rangle = \mathrm{Inv}(B)$.

Similarly, for a set of Boolean functions B, we define $[B] = \mathrm{Pol}(\mathrm{Inv}(B))$ and say B is a *base* for $[B]$. This is the set of functions that can be described by Boolean circuits where each gate computes a function from B. For every clone A a finite B is known such that $[B] = A$.

For a finite set of Boolean relations Γ, let the *order of* Γ be defined as $o(\Gamma) =_{\mathrm{df}} \max\{n : n \text{ is the arity of an } R \in \Gamma\}$. For a co-clone A, the order $o(A)$ of A is defined as the minimal order of a basis for A, if A has a finite base and $o(A) = \infty$ otherwise.

Next we define some Boolean relations which will be important for our list of bases.

$$
\begin{aligned}
\mathrm{OR}^m &=_{\mathrm{df}} \{(a_1, \ldots, a_m) \in \{0,1\}^m : a_1 + \cdots + a_m > 0\} \\
\mathrm{AND}^m &=_{\mathrm{df}} \{(a_1, \ldots, a_m) \in \{0,1\}^m : a_1 + \cdots + a_m = m\} \\
\mathrm{NOR}^m &=_{\mathrm{df}} \{(a_1, \ldots, a_m) \in \{0,1\}^m : a_1 + \cdots + a_m = 0\} \\
\mathrm{NAND}^m &=_{\mathrm{df}} \{(a_1, \ldots, a_m) \in \{0,1\}^m : a_1 + \cdots + a_m \neq m\} \\
\mathrm{EVEN}^m &=_{\mathrm{df}} \{(a_1, \ldots, a_m) \in \{0,1\}^m : \Sigma_{i=1}^m i \text{ is even}\} \\
\mathrm{ODD}^m &=_{\mathrm{df}} \{(a_1, \ldots, a_m) \in \{0,1\}^m : \Sigma_{i=1}^m i \text{ is odd}\} \\
\mathrm{DUP}^3 &=_{\mathrm{df}} \{0,1\}^3 \setminus \{(0,1,0), (1,0,1)\} \\
n\text{-IN-}m &=_{\mathrm{df}} \{(a_1, \ldots, a_3) \in \{0,1\}^n : \Sigma_{i=1}^m i = n\} \\
\mathrm{NAE}^3 &=_{\mathrm{df}} \{0,1\}^3 \setminus \{(0,0,0), (1,1,1)\}
\end{aligned}
$$

Also, we will make use of the familiar correspondence between propositional formulas and Boolean relations and, in order to simplify the notation, not distinguish between relations and their defining formulas. Thus, e.g., x stands for the unary relation $\{1\}$, $x \rightarrow y$ stands for the binary relation $\{(0,0),(0,1),(1,1)\}$, and $x \oplus y$ stands for the binary relation $\{(0,1),(1,0)\}$.

3 Results

We can use the following well-known algorithms from universal algebra to test whether a given set of relations is a base of a co-clone and whether a given relation is contained in a co-clone.

Proposition 3.1 *1. A set Γ is a base for a co-clone A if and only if for all $R \in \Gamma$ holds $R \in A$ and for every co-clone $C \subsetneq A$ there is an $R' \in \Gamma$ such that $R' \notin C$.*

2. Let R be a relation, A be a co-clone, and B be a base of $\mathrm{Pol}(A)$. Then $R \in A$ if and only if R is preserved by every f from B.

Proof. 1. Obviously, $\langle\Gamma\rangle$ is a co-clone, and $\langle\Gamma\rangle \subseteq A$. Since for every $C \subset A$ there is an R in Γ that is not in C, we have $\langle\Gamma\rangle \neq C$.

2. If $R \in A$ then, per definition, R is preserved by every $f \in B$. If R is closed under every $f \in B$, it is also an invariant for every $g \in \mathrm{Pol}(A)$ and hence is in A.

□

A central concept in our study is that of *duality*. For a Boolean relation R, we define $\mathrm{dual}(R) =_{\mathrm{df}} \{\overline{\alpha} \mid \alpha \in R\}$. For example, $\mathrm{dual}(\mathrm{OR}^m) = \mathrm{NAND}^m$. The dual of a set of Boolean relations Γ is defined by $\mathrm{dual}(\Gamma) =_{\mathrm{df}} \{\mathrm{dual}(R) : R \in \Gamma\}$. For an n-ary Boolean function f, the dual one is defined by $\mathrm{dual}(f)(a_1,\ldots,a_n) =_{\mathrm{df}} \overline{f(\overline{a_1},\ldots,\overline{a_n})}$. For a set of Boolean functions B, let $\mathrm{dual}(B) =_{\mathrm{df}} \{\mathrm{dual}(f) : f \in B\}$. In the graphical representation of the lattices of clones (and co-clones), it is easy to find the dual of a (co-)clone: It is the mirror image of the vertical line drawn through the middle of the graph. For example, V_2 is the dual of E_2 and M_2 is its own dual.

Now, statement 1 of the following proposition is obvious from the definition, and statement 2 follows easily from 1.

Proposition 3.2 *1. For any Boolean relation R, $\mathrm{Pol}(\mathrm{dual}(R)) = \mathrm{dual}(\mathrm{Pol}(R))$.*

2. Let B_1, B_2 be clones and Γ_1, Γ_2 bases for $\mathrm{Inv}(B_1), \mathrm{Inv}(B_2)$. Then $\Gamma_1 \cup \Gamma_2$ is a base for $\mathrm{Inv}(B_1 \cap B_2)$ and $\mathrm{dual}(\Gamma_1)$ is a base for $\mathrm{Inv}(\mathrm{dual}(B_1))$.

Our complete list of bases for co-clones will follow, using the above proposition, from the following seven basic cases.

Lemma 3.3 *1.* $\mathrm{Pol}(\{\mathrm{OR}^m\}) = \mathrm{S}_0^m$.

2. $\mathrm{Pol}(\{x\}) = \mathrm{R}_1$.

3. $\mathrm{Pol}(\{x \oplus y\}) = \mathrm{D}$.

4. $\mathrm{Pol}(\{\mathrm{EVEN}^4\}) = \mathrm{L}$.

5. $\mathrm{Pol}(\{x \rightarrow y\}) = \mathrm{M}$.

6. $\mathrm{Pol}(\{\mathrm{DUP}^3\}) = \mathrm{N}$.

7. $\mathrm{Pol}(\{x \vee y \vee \overline{z}\}) = \mathrm{V}$.

Proof. Let f be a k-ary Boolean function. Then $f \in \mathrm{S}_0^m$ iff $f^{-1}(\{0\})$ is 0-separating of degree m, i.e., by definition, if $\alpha_1, \ldots, \alpha_m \in f^{-1}(\{0\})$ then $\{\alpha_1, \ldots, \alpha_m\} \subseteq \{0,1\}^{i-1} \times \{0\} \times \{0,1\}^{k-1}$ for some i. This is the same as saying that, if $f(\alpha_1[1], \ldots, \alpha_1[k]) = 0$, $f(\alpha_2[1], \ldots, \alpha_2[k]) = 0$, $\ldots$, $f(\alpha_m, [1], \ldots, \alpha_m[k]) = 0$, then there is some i such that $(\alpha_1[i], \alpha_2[i], \ldots, \alpha_m[i]) = (0, 0, \ldots, 0)$; and this is equivalent to $f \in \mathrm{Pol}(\{\mathrm{OR}^m\})$.

The remaining points can easily be verified using the method given in Proposition 3.1.

□

As already mentioned, the basic cases just proven together with Proposition 3.2 allow us to compile a complete list of bases for all co-clones, see Fig. 2. For every class not appearing among the basic cases in Lemma 3.3, we indicate in the third column if it is obtained as dual of another co-clone or as invariants of an intersection of other clones.

This systematic approach making use of Proposition 3.2 yields quite complicated bases for some co-clones as e.g. the important ones $\mathrm{Inv}(\mathrm{N}_2)$ and $\mathrm{Inv}(\mathrm{I}_2)$. So we additionally give at least one base of minimal order for each co-clone. That these are in fact bases can be verified using Proposition 3.1.

To determine the order of a co-clone, we proceed as follows. First, note that $\langle\{\emptyset\}\rangle = \mathrm{Inv}(\mathrm{BF})$, $\langle\{x\}\rangle = \mathrm{Inv}(\mathrm{R}_1)$, $\langle\{\overline{x}\}\rangle = \mathrm{Inv}(\mathrm{R}_0)$, and $\langle\{x, \overline{x}\}\rangle = \mathrm{Inv}(\mathrm{R}_0)$. Since we have thus indentified all co-clones that have order of at most one, every other co-clone has to have an order of at least two. The next proposition tells us something about the nature of co-clones of order two.

Proposition 3.4 *For all co-clones A that are not a subset of* $\mathrm{Inv}(\mathrm{D}_2)$ *there is no base for A that contains just relations with arity less than 3.*

Proof. $\mathrm{Inv}(\mathrm{D}_2)$ is the set of relations that are expressible via bijunctive (i.e., 2-CNF) formulas (see [BCRV04]), and every 2-ary relation can be expressed by such a formula. □

Hence every non-bijunctive co-clone has an order greater than two. It turns out, that we find a base of order three for nearly all of them. Propositions 3.5 and 3.6 handle the remaining cases.

Proposition 3.5 *The order of* $\mathrm{Inv}(\mathrm{L})$ *and* $\mathrm{Inv}(\mathrm{L}_3)$ *is 4.*

Proof. Observe, that $\{\mathrm{EVEN}^4\}$ is a base for $\mathrm{Inv}(\mathrm{L})$ and $\{\mathrm{ODD}^4\}$ is a base for $\mathrm{Inv}(\mathrm{L}_3)$ and therefore 4 is an upper bound for $\mathrm{Inv}(\mathrm{L})$ and $\mathrm{Inv}(\mathrm{L}_3)$. Since all 2-ary relations are bijunctive, it suffices to show in the case of $\mathrm{Inv}(\mathrm{L})$ that all 3-ary relations in $\mathrm{Inv}(\mathrm{L})$ are also in $\mathrm{Inv}(\mathrm{BF})$ and in the case of $\mathrm{Inv}(\mathrm{L}_3)$ that all 3-ary relations in $\mathrm{Inv}(\mathrm{L}_3)$ are in $\mathrm{Inv}(\mathrm{D})$. This is an easy exercise, since we have to check only a small number of relations: If $R \in \mathrm{Inv}(\mathrm{L}_3)$, then R is complementive and affine. It is a well-known fact, that affine relations also have a size equal to a power of two. Hence we have to check at most $\Sigma_{i=0}^{2}\binom{4}{2^i} = 11$ relations. □

Proposition 3.6 *If $n \geq 2$ and $a \in \{0,1\}$, then the order of* $\mathrm{Inv}(\mathrm{S}_a^n)$, $\mathrm{Inv}(\mathrm{S}_{a2}^n)$, $\mathrm{Inv}(\mathrm{S}_{a1}^n)$, *and* $\mathrm{Inv}(\mathrm{S}_{a0}^n)$ *is n.*

Proof. It suffices to show the claim for $a = 1$, because the case $a = 0$ follows by duality. We define $h_n(x_1, \ldots, x_{n+1}) =_{\mathrm{df}} \bigvee_{i=1}^{n+1} x_1 \wedge x_2 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_{n+1}$ to be the $n+1$-ary function that maps inputs to 1 if and only if at least n of the inputs are set to 1. We show that, for $2 \leq m < n$, every m-ary relation that is closed under h_n is already closed under h_{n-1}. From this follows that if a relation of arity less than n is in S_1^n then it is in fact already included in S_1^{n-1} since for all $n \geq 2$ the set $\{x \wedge \overline{y}, h_n\}$ is a base for S_1^n.

Let R be such a relation that is closed under h_n and suppose it were not closed under h_{n-1}. Hence, there are $\alpha_1, \dots, \alpha_n$ such that $\delta =_{\mathrm{df}} h_{n-1}(\alpha_1, \dots, \alpha_n) \notin R$. For $i \in \{1, \dots, n\}$, let $\beta_i =_{\mathrm{df}} h_n(\alpha_1, \dots, \alpha_{i-1}, \alpha_i, \alpha_i, \alpha_{i+1}, \dots, \alpha_n)$. Since h_n preserves R, all of the β_i are in R and therefore for all of them holds $\beta_i \neq \delta$. We have a look at the coordinates of β_i. For that let $k \in \{1, \dots, m\}$. If $\alpha_i[k] = 1$, then $\beta_i[k] = \delta[k]$. If $\alpha_i[k] = 0$ and there is a $j \neq i$ such that $\alpha_j[k] = 0$ then again $\beta_i[k] = \delta[k]$. But if $\alpha_i[k] = 0$ and $\alpha_j[k] = 1$ for all $j \neq i$, then $\beta_i[k] = 0$ and $\delta[k] = 1$.

So, if $\beta_i[k]$ differs from $\delta[k]$ then $\alpha_i[k] = 0$ and $\alpha_j[k] = 1$ for all $j \neq i$. Therefore for all $j \neq i$ we have $\beta_j[k] = \delta[k]$. Following the pigeon hole principle, there has to be a j such that $\beta_j = \delta$, which is a contradiction.

The proof for $\mathrm{Inv}(\mathrm{S}_{12}^n)$, $\mathrm{Inv}(\mathrm{S}_{11}^n)$, and $\mathrm{Inv}(\mathrm{S}_{10}^n)$ is analogous. □

For some co-clones, Fig. 2 gives no finite bases. For completeness, we prove in the next lemma, that for these cases there actually is no finite basis.

Lemma 3.7 *There is no finite constraint language* Γ *such that* $\mathrm{Pol}(\Gamma) \in \{\mathrm{S}_0, \mathrm{S}_{02}, \mathrm{S}_{01}, \mathrm{S}_{00}, \mathrm{S}_{10}, \mathrm{S}_{11}, \mathrm{S}_{12}, \mathrm{S}_1\}$.

Proof. The proof establishes the standard argumentation from universal algebra. Assume $\Gamma =_{\mathrm{df}} \{R_1, \dots, R_n\}$ is a finite set of Boolean relations such that $\mathrm{Pol}(\Gamma)$ is one of the above. Since every clone S_{ab} is the union of all S_{ab}^k, for every $i \in \{1, \dots, n\}$ there exists a k_i such that $R_i \in \mathrm{S}_{ab}^{k_i}$. Since the S_{ab}^k-classes are a chain, there exists some k such that $\Gamma \subseteq \mathrm{Inv}(\mathrm{S}_{ab}^k) \subsetneqq \mathrm{Inv}(\mathrm{S}_{ab})$. □

For the affine co-clone it is quite easy to find a large number of bases.

Proposition 3.8 *For every odd* $n \geq 3$*, the sets* $\{\mathrm{EVEN}^n, \{(1)\}\}$ *and* $\{\mathrm{ODD}^n, \{(0)\}\}$ *are bases for* $\mathrm{Inv}(\mathrm{L}_2)$*. For all* $m \geq 3$*, the sets* $\{\mathrm{EVEN}^m, \{(0)\}, \{(1)\}\}$ *and* $\{\mathrm{ODD}^m, \{(0)\}, \{(1)\}\}$ *are bases for* $\mathrm{Inv}(\mathrm{L}_2)$*.*

Proof. Let $n \geq 3$ and $B =_{\mathrm{df}} \{\mathrm{EVEN}^n, \{(1)\}\}$. We first fix $f(x, y, z) =_{\mathrm{df}} x \oplus y \oplus z$ and note that $\{f\}$ is a base for L_2. Observe, that $\{(1)\}$ is closed under f and is hence in $\mathrm{Inv}(\mathrm{L}_2)$. Now let $f(x_1, x_2, x_3) = (\alpha_1, \dots, \alpha_n) =_{\mathrm{df}} a$ for $x_1, x_2, x_3 \in \mathrm{EVEN}^n$. Since $\alpha_i = 1$ can happen only if there is an odd number of ones in $x_1[i], x_2[i], x_3[i]$, and since there occur an even number of ones in x_1, x_2, and x_3 together, there is an even number of ones in a.

It remains to show that B is not contained in any of the precomplete co-clones. Since n is odd, EVEN^n is not 1-valid and $\{(1)\}$ is not 0-valid, hence $B \notin \mathrm{Inv}(\mathrm{I}_1)$ and $B \notin \mathrm{Inv}(\mathrm{I}_0)$. Furthermore, $(0, \dots, 0) \in \mathrm{EVEN}^n$, so $\mathrm{EVEN}^n \notin \mathrm{Inv}(\mathrm{N}_2)$. Also, the tuples $a =_{\mathrm{df}} (1, 1, 0, \dots, 0)$ and $b =_{\mathrm{df}} (1, 0, 1, 0, \dots, 0)$ are in EVEN^n but the bitwise or of a and b contains 3 ones, and the bitwise and of a and b contains 1 one, hence EVEN^n is not in $\mathrm{Inv}(\mathrm{V}_2)$ and not in $\mathrm{Inv}(\mathrm{E}_2)$. Finally, examine the bitwise combination of $a =_{\mathrm{df}} (1, 1, 0, \dots, 0)$, $b =_{\mathrm{df}} (1, 0, 1, 0, \dots, 0)$ and $c =_{\mathrm{df}} (0, \dots, 0)$ with the function $h(x, y, z) =_{\mathrm{df}} xy \vee xz \vee yz$ which is a base of D_2. Here we get $h(a, b, c) = (1, \dots, 0) \notin \mathrm{EVEN}^n$.

The argumentation for $\{\mathrm{ODD}^n, \{(0)\}\}$ is analogous.

For the sets $\{\mathrm{EVEN}^m, \{(0)\}, \{(1)\}\}$ and $\{\mathrm{ODD}^m, \{(0)\}, \{(1)\}\}$ observe that in the proof above, we needed n to be odd just so that $(1, \dots, 1) \notin \mathrm{EVEN}^n$. Observe, that $\{(0)\}$ has this property. □

Finally, it is often convenient to have a base for a co-clone that contains only one relation. From the following proposition it is clear that such a base exists for every co-clone of finite order.

Proposition 3.9 *Let* $B = \{R_1, R_2, \dots, R_k\}$ *be a base for a co-clone* A*. Then* $R_1 \times R_2 \times \cdots \times R_k$ *is a base of* A*.*

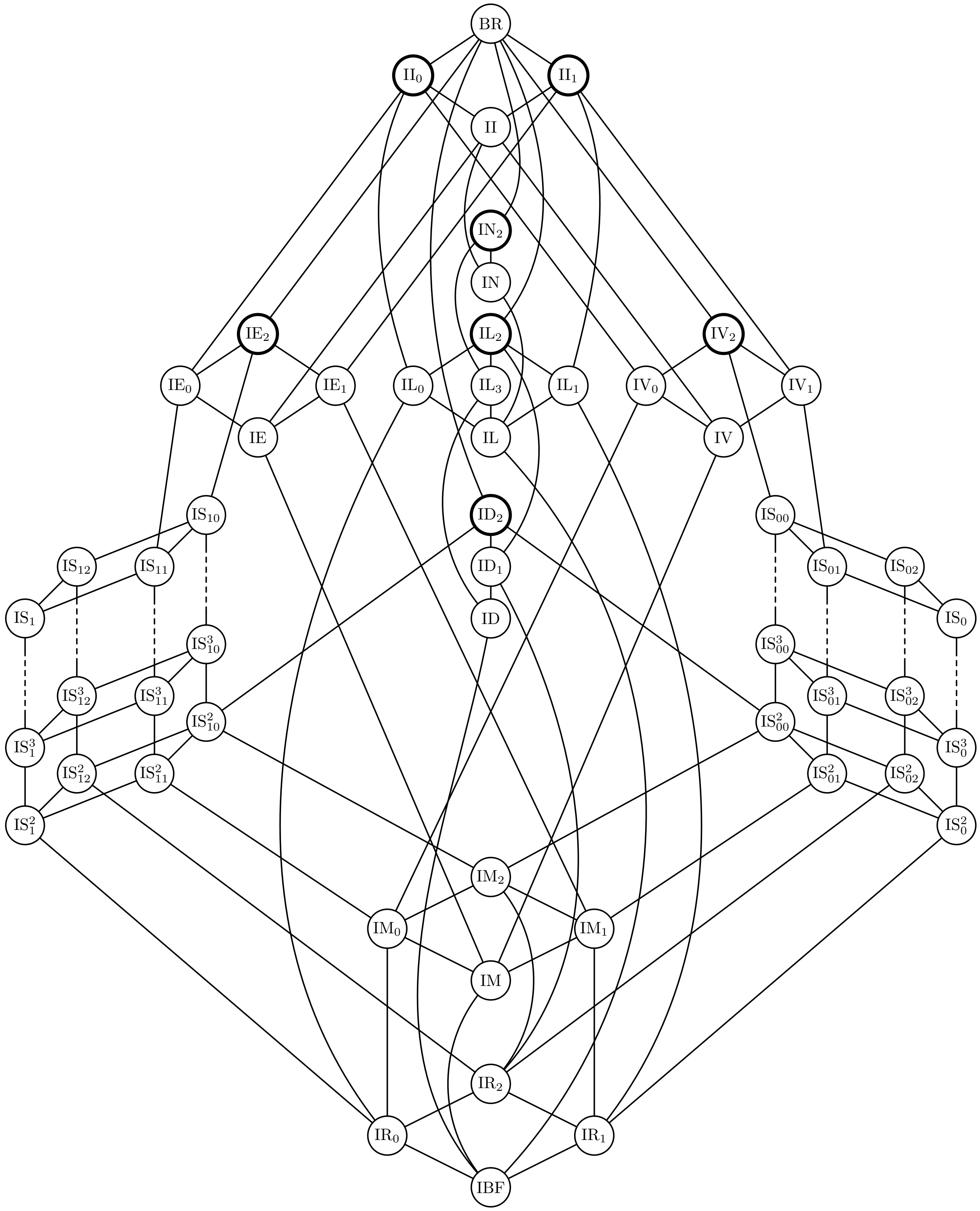

Figure 1: The co-clone lattice. Here, IB for a clone B denotes $\mathrm{Inv}(B)$.

Clone	O.	Remark	Base(s) of corresponding co-clone
BF	0		$\{=\}$, $\{\emptyset\}$
R_0	1	dual of R_1	$\{\overline{x}\}$
R_1	1		$\{x\}$
R_2	1	$R_0 \cap R_1$	$\{x, \overline{x}\}$,$\{x\overline{y}\}$
M	2		$\{x \to y\}$
M_1	2	$M \cap R_1$	$\{x \to y, x\}$,$\{x \wedge (y \to z)\}$
M_0	2	$M \cap R_0$	$\{x \to y, \overline{x}\}$,$\{\overline{x} \wedge (y \to z)\}$
M_2	2	$M \cap R_2$	$\{x \to y, x, \overline{x}\}$,$\{x \to y, \overline{x \to y}\}$, $\{x\overline{y} \wedge (u \to v)\}$
S_0^m	m		$\{\mathrm{OR}^m\}$
S_1^m	m	dual of S_0^m	$\{\mathrm{NAND}^m\}$
S_0	∞	$\cap_{m \geq 2} S_0^m$	$\{\mathrm{OR}^m \mid m \geq 2\}$
S_1	∞	dual of S_0	$\{\mathrm{NAND}^m \mid m \geq 2\}$
S_{02}^m	m	$S_0^n \cap R_2$	$\{\mathrm{OR}^m, x, \overline{x}\}$
S_{02}	∞	$S_0 \cap R_2$	$\{\mathrm{OR}^m \mid m \geq 2\} \cup \{x, \overline{x}\}$
S_{01}^m	m	$S_0^n \cap M$	$\{\mathrm{OR}^m, x \to y\}$
S_{01}	∞	$S_0 \cap M$	$\{\mathrm{OR}^m \mid m \geq 2\} \cup \{x \to y\}$
S_{00}^m	m	$S_0^n \cap R_2 \cap M$	$\{\mathrm{OR}^m, x, \overline{x}, x \to y\}$
S_{00}	∞	$S_0 \cap R_2 \cap M$	$\{\mathrm{OR}^m \mid m \geq 2\} \cup \{x, \overline{x}, x \to y\}$
S_{12}^m	m	dual of S_{02}^m	$\{\mathrm{NAND}^m, x, \overline{x}\}$
S_{12}	∞	dual of 02	$\{\mathrm{NAND}^m \mid m \geq 2\} \cup \{x, \overline{x}\}$
S_{11}^m	m	dual of S_{01}^m	$\{\mathrm{NAND}^m, x \to y\}$
S_{11}	∞	dual of S_{01}	$\{\mathrm{NAND}^m \mid m \geq 2\} \cup \{x \to y\}$
S_{10}^m	m	dual of S_{00}^m	$\{\mathrm{NAND}^m, x, \overline{x}, x \to y\}$
S_{10}	∞	dual of S_{00}	$\{\mathrm{NAND}^m \mid m \geq 2\} \cup \{x, \overline{x}, x \to y\}$
D	2	$\{x \oplus y\}$	
D_1	2	$D \cap R_1$	$\{x \oplus y, x\}$, every $R \in \{\{\alpha, \beta\} : \exists\gamma(\{\alpha, \beta, \gamma\} \in \{\text{1-IN-3}, \text{2-IN-3}\})\}$
D_2	2	$D \cap M$	$\{x \oplus y, x \to y\}$,$\{x\overline{y} \vee \overline{x}yz\}$
L	4		$\{\mathrm{EVEN}^4\}$
L_0	3	$L \cap R_0$	$\{\mathrm{EVEN}^4, \overline{x}\}$,$\{\mathrm{EVEN}^3\}$
L_1	3	$L \cap R_1$	$\{\mathrm{EVEN}^4, x\}$,$\{\mathrm{ODD}^3\}$
L_2	3	$L \cap R_2$	$\{\mathrm{EVEN}^4, x, \overline{x}\}$, every $\{\mathrm{EVEN}^n, \{(1)\}\}$, $n \geq 3$ is odd
L_3	4	$L \cap D$	$\{\mathrm{EVEN}^4, x \oplus y\}$,$\{\mathrm{ODD}^4\}$
V	3		$\{x \vee y \vee \overline{z}\}$
V_0	3	$V \cap R_0$	$\{x \vee y \vee \overline{z}, \overline{x}\}$
V_1	3	$V \cap R_1$	$\{x \vee y \vee \overline{z}, x\}$
V_2	3	$V \cap R_2$	$\{x \vee y \vee \overline{z}, x, \overline{x}\}$
E	3	dual of V	$\{\overline{x} \vee \overline{y} \vee z\}$
E_1	3	dual of V_0	$\{\overline{x} \vee \overline{y} \vee z, x\}$
E_0	3	dual of V_1	$\{\overline{x} \vee \overline{y} \vee z, \overline{x}\}$
E_2	3	dual of V_2	$\{\overline{x} \vee \overline{y} \vee z, x, \overline{x}\}$
N	3		$\{\mathrm{DUP}^3\}$
N_2	3	$N \cap L_3$	$\{\mathrm{DUP}^3, \mathrm{EVEN}^4, x \oplus y\}$, $\{\mathrm{NAE}^3\}$
I	3	$L \cap M$	$\{\mathrm{EVEN}^4, x \to y\}$
I_0	3	$L \cap M \cap R_0$	$\{\mathrm{EVEN}^4, x \to y, \overline{x}\}$,$\{\mathrm{DUP}^3, x \to y\}$
I_1	3	$L \cap M \cap R_1$	$\{\mathrm{EVEN}^4, x \to y, x\}$,$\{x \vee (x \oplus z)\}$
I_2	3	$L \cap M \cap R_2$	$\{\mathrm{EVEN}^4, x \to y, x, \overline{x}\}$, $\{\text{1-IN-3}\}$,$\{x \to (y \oplus z)\}$

Figure 2: Bases for all Boolean co-clones

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