New BCSV Theorems

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Abstract. In [BCS92] and [Ve93] it has been proved that a language L_1 is polylogtime reducible to a language L_2 if and only if the leaf language class defined by L_1 is included in the leaf language class defined by L_2 under every relativization. This is valid for the leaf language class model using machines with balanced computation trees. In this paper we prove such Bovet-Crescenzi-Silvestri-Vereshchagin theorems (BCSV theorems) also for the leaf language class models using machines with unbalanced computation trees and computation trees with ε-leaves, resp.

1 Machines with Unbalanced Computation Trees

For a finite alphabet Σ, the *initial word relation* $\sqsubseteq$ on Σ^* is defined by $u \sqsubseteq v \Leftrightarrow_{\rm def} \exists w(w \in \Sigma^* \wedge uw = v)$, and we write $u \sqsubset v$ if and only if $u \sqsubseteq v$ and $u \neq v$. In what follows we identify the set $\{0,1\}^*$ with the set $\mathbb{N}$ of natural numbers according to the quasi-lexicographical order on $\{0,1\}^*$ (this is the order $\leq$ defined by $x \leq y \Leftrightarrow_{\rm def} |x| < |y| \vee (|x| = |y| \wedge \exists u(u0 \sqsubseteq x \wedge u1 \sqsubseteq y)) \vee x = y$). Furthermore, we identify a set $A \subseteq \mathbb{N}$ with the infinite sequence $c_A(0)c_A(1)c_A(2)\cdots \in \{0,1\}^\omega$ where c_A is the characteristic function of A. For a set $A \subseteq \{0,1\}^\omega$ and $u \in \{0,1\}^*$ we define the sets $u|A =_{\rm def} uc_A(|u|)c_A(|u|+1)c_A(|u|+2)\ldots$ and $uA =_{\rm def} uc_A(0)c_A(1)c_A(2)\ldots$.

Let Σ be a finite alphabet. We call a triple $t = (T, h, m)$ a Σ-*tree* if $T \subseteq \{0,1\}^*$ is finite, $h : T \to \Sigma$, and $m \in \mathbb{N}$ such that $\forall z \forall u((u \sqsubseteq z \wedge z \in T) \to u \in T)$ and $\forall z(z \in T \to |z| \leq m)$. Let T_Σ be the set of all Σ-trees. A *leaf* of t is a $z \in T$ such that there is no $u \in T$ with $z \sqsubset u$. For a Σ-tree $t = (T, h, m)$, we define the *leaf word* of t as $\beta(t) =_{\rm def} h(z_1)h(z_2)\ldots h(z_s)$ where $\{z_1, z_2, \ldots, z_s\}$ is the set of all leaves of t and $z_1 \prec z_2 \prec \cdots \prec z_s$ with respect to the lexicographical order on $\{0,1\}^*$ (this is the order $\preceq$ defined by $x \preceq y \Leftrightarrow_{\rm def} x \sqsubseteq y \vee \exists u(u0 \sqsubseteq x \wedge u1 \sqsubseteq y)$).

Choose $r \geq 1$ such that $|\Sigma| \leq 2^r$, and let $e : \Sigma \to \{0,1\}^r$ be an injective mapping. A Σ-tree $t = (T, h, m)$ is encoded by the set $B_t =_{\rm def} \{ze(h(z)) : z \in T\}$ and the number $m_t =_{\rm def} m$. On the other hand, an arbitrary set $B \subseteq \{0,1\}^*$ and a number $m \in \mathbb{N}$ define a Σ-tree $t_{B,m} =_{\rm def} (T_{B,m}, h_{B,m}, m)$ where

$$T_{B,m} =_{\rm def} \{z : |z| \leq m \wedge \forall u(u \sqsubseteq z \to \exists v(v \in e(\Sigma) \wedge uv \in B))\} \text{ and}$$
$$h_{B,m}(z) =_{\rm def} e^{-1}(\text{lexicographically first } v \in e(\Sigma) \text{ such that } zv \in B).$$

It is easy to see that $t_{B_t,m_t} = t$ for every Σ-tree t.

For $L \subseteq \Sigma^*$ and $B \subseteq \{0,1\}^*$ we define $L^B =_{\rm def} \{x : x \in \{0,1\}^* \text{ and } \beta(t_{B-x,|x|}) \in L\}$ and $L^\circ =_{\rm def} \{(B,x) : x \in \{0,1\}^* \text{ and } \beta(t_{B-x,|x|}) \in L\}$ where $B - x =_{\rm def} \{y : x + y \in B\}$ for every integer x.

Proposition 1 *Let $L \in \Sigma^*$, $B, C \subseteq \{0,1\}^*$, and $x, u, v \in \{0,1\}^*$.*

1. *If $|u| = |v| \leq x$ then $x \in L^{uB} \leftrightarrow x \in L^{vB}$.*
2. *If $|u| > 2^{r+2} \cdot x$ then $x \in L^{uB} \leftrightarrow x \in L^{uC}$.*

Proof. To check $x \in L^B$ one has to know $i \in B$ only for $i \in \{x, x+1, \ldots, 2^{r+2} \cdot x\}$. $\square$

A function $f : T_{\Sigma_1} \to T_{\Sigma_2}$ is called a *polynomial time tree function* (for short: ptt function) if and only if there exist functions $g_1 : T_{\Sigma_1} \times \{0,1\}^* \times \mathbb{N} \to \{0,1\}$, $g_2 : T_{\Sigma_1} \times \{0,1\}^* \times \mathbb{N} \to \Sigma_2$, and a $k > 0$ such that

- g_1, g_2 are polynomial time computable in the length of the $\{0,1\}^*$ and $\mathbb{N}$ arguments where a tree $t \in T_\Sigma$ is accessed as the oracle B_t, and
- $f(t) = (T', h', m^k)$ where $T' =_{\text{def}} \{z : g_1^{B_t}(z, m_t) = 1\}$ and $h'(z) =_{\text{def}} g_2^{B_t}(z, m_t)$.

For $L_1 \subseteq \Sigma_1^*$ and $L_2 \subseteq \Sigma_2^*$, we define $L_1 \leq_{\text{m}}^{\text{ptt}} L_2$ (L_1 is ptt-reducible to L_2) if there exists a ptt function $f : T_{\Sigma_1} \to T_{\Sigma_2}$ such that $\beta(t) \in L_1 \leftrightarrow \beta(f(t)) \in L_2$ for all $t \in T_{\Sigma_1}$.

Proposition 2 *$\leq_{\text{m}}^{\text{ptt}}$ is reflexive and transitive.*

Let M be a nondeterministic polynomial time Turing machine (NPTM, for short) which produces on every computation path z on input x a symbol $M(x,z)$ from a finite alphabet Σ, and let k be the smallest natural number such that $n^k + k$ bounds the running time of M. Defining $T_M(x) =_{\text{def}} \{u : \exists z(z \text{ computation path of } M \text{ on } x \text{ and } u \sqsubseteq z)\}$ and $h_M(x)(z) =_{\text{def}} M(x,z)$ for every computation path z of M on input x (for the other $z \in T_M(x)$ the value of $h_M(x)(z)$ is chosen arbitrarily from Σ), we obtain the computation tree $t_M(x) =_{\text{def}} (T_M(x), h_M(x), |x|^k + k)$ of M on input x which is, of course, a Σ-tree. For a nondeterministic polynomial time oracle Turing machine (NPTOM, for short) M, let analogously $t_M^B(x) =_{\text{def}} (T_M^B(x), h_M^B(x), |x|^k + k)$ be the computation tree of M on input x with oracle B.

For a language $L \subseteq \Sigma^*$, define $\text{Leaf}_{\text{u}}(L)$ as the class of all languages A for which there exists a NPTM M such that $x \in A \Leftrightarrow \beta(t_M(x)) \in L$ for all x. For a fixed oracle $B \subseteq \{0,1\}^*$, let $\text{Leaf}_{\text{u}}^B(L)$ be the class of all languages A for which there exists an NPTOM M such that $x \in A \Leftrightarrow \beta(t_M^B(x)) \in L$ for all x. Finally, let $\text{Leaf}_{\text{u}}^{\circ}(L)$ be the class of all sets A for which there exists an NPTOM M such that $(B, x) \in A \Leftrightarrow \beta(t_M^B(x)) \in L$ for all x and all oracles B.

The following is easy to see.

Proposition 3 *Let L be a language.*

1. *$L^B \in \text{Leaf}_{\text{u}}^B(L)$ for every oracle B.*
2. *$L^{\circ} \in \text{Leaf}_{\text{u}}^{\circ}(L)$.*

A language $L \subseteq \Sigma^*$ is called *nontrivial* if $L \neq \emptyset$ and $L \neq \Sigma^*$.

Theorem 4 *For nontrivial $L_1 \subseteq \Sigma_1^*$ and $L_2 \subseteq \Sigma_2^*$ the following are equivalent:*

(1) $L_1 \leq_{\text{m}}^{\text{ptt}} L_2$
(2) $\text{Leaf}_{\text{u}}^{\circ}(L_1) \subseteq \text{Leaf}_{\text{u}}^{\circ}(L_2)$
(3) $L_1^{\circ} \in \text{Leaf}_{\text{u}}^{\circ}(L_2)$

Proof. (1) $\Rightarrow$ (2). Let $L_1 \leq_{\rm m}^{\rm ptt} L_2$ via the ptt function f. For $A \in \mathrm{Leaf}_{\rm u}^{\circ}(L_1)$ there exists an NPTOM M such that $(B,x) \in A \Leftrightarrow \beta(t_M^B(x)) \in L_1$ for all $x \in \Sigma^*$ and oracles B. It is easy to construct an NPTOM M' such that $t_{M'}^B(x) = f(t_M^B(x))$ for all x and B. Consequently,

$$(B,x) \in A \Leftrightarrow \beta(t_M^B(x)) \in L_1 \Leftrightarrow \beta(f(t_M^B(x))) \in L_2 \Leftrightarrow \beta(t_{M'}^B(x)) \in L_2,$$

and hence $A \in \mathrm{Leaf}_{\rm u}^{\circ}(L_2)$.

(2) $\Rightarrow$ (3) is obvious because of Proposition 3.2

(3) $\Rightarrow$ (1). Let $L_1^{\circ} \in \mathrm{Leaf}_{\rm u}^{\circ}(L_2)$. There exists an NPTOM M such that $(B,x) \in L_1^{\circ} \Leftrightarrow \beta(t_M^B(x)) \in L_2$. Let k be the smallest natural number such that $n^k + k$ bounds the running time of M. For a Σ-tree t we obtain

$$\beta(t) \in L_1 \Leftrightarrow \beta(t_{B_t,m_t}) \in L_1 \Leftrightarrow (B_t + 0^{m_t}, 0^{m_t}) \in L_1^{\circ} \Leftrightarrow \beta(t_M^{B_t+0^{m_t}}(0^{m_t})) \in L_2.$$

We define $f(t) =_{\rm def} t_M^{B_t+0^{m_t}}(0^{m_t}) = (T_M^{B_t+0^{m_t}}(0^{m_t}), h_M^{B_t+0^{m_t}}(0^{m_t}), m_t^k + k)$. It is easy to see that there are polynomial time computable functions g_1 and g_2 such that $T_M^{B_t+x}(0^{m_t}) = \{z : g_1^t(z, m_t) = 1\}$ and $h_M^{B_t+x}(0^{m_t}) = g_2^t(z, m_t)$. Hence $L_1 \leq_{\rm m}^{\rm ptt} L_2$. $\square$

Theorem 5 *For nontrivial $L_1 \subseteq \Sigma_1^*$ and $L_2 \subseteq \Sigma_2^*$ the following are equivalent:*
(1) $L_1 \leq_{\rm m}^{\rm ptt} L_2$
(2) $\forall B(\mathrm{Leaf}_{\rm u}^B(L_1) \subseteq \mathrm{Leaf}_{\rm u}^B(L_2))$
(3) $\forall B(L_1^B \in \mathrm{Leaf}_{\rm u}^B(L_2))$

Proof. (1) $\Rightarrow$ (2) follows from (1) $\Rightarrow$ (2) in Theorem 4, and (2) $\Rightarrow$ (3) is obvious because of Proposition 3.1.

(3) $\Rightarrow$ (1). Because of Theorem 4 it suffices to prove $L_1^{\circ} \in \mathrm{Leaf}_{\rm u}^{\circ}(L_2)$. Let $M_0, M_1, M_2, \ldots$ be an enumeration of all NPTOMs. Assume $L_1^{\circ} \notin \mathrm{Leaf}_{\rm u}^{\circ}(L_2)$, i.e., for every k there exist an oracle B_k and a word x_k such that

$$x_k \in L_1^{B_k} \leftrightarrow \beta(t_{M_k}^{B_k}(x_k)) \notin L_2. \tag{1}$$

Claim. There exist $u_0, u_1, u_2, \ldots \in \{0,1\}^*$ and an $z_0, z_1, z_2, \ldots \in \mathbb{N}$ such that
- u_{k-1} is a proper initial word of u_k, and
- $z_k \in L_1^B \leftrightarrow \beta(t_{M_k}^B(z_k)) \notin L_2$ for all $B \in u_k \cdot \{0,1\}^\omega$.

Proof of the Claim. By induction on k. For $k = 0$, let $z_0 =_{\rm def} x_0$, and let u_0 be the shortest initial word of $B_0 \in \{0,1\}^\omega$ such that $|u_0| \geq 2^{r+2} \cdot x_0$ and $\{0, 1, \ldots, |u_0|\}$ contains every query of M_0 to the oracle B_0 during its work on x_0. For every $B \in u_0 \cdot \{0,1\}^\omega$ we obtain using Proposition 1.2 and (1)

$$\begin{aligned} z_0 \in L_1^B &\Leftrightarrow x_0 \in L_1^B \Leftrightarrow x_0 \in L_1^{B_0} \Leftrightarrow \beta(t_{M_0}^{B_0}(x_0)) \notin L_2 \\ &\Leftrightarrow \beta(t_{M_0}^B(x_0)) \notin L_2 \Leftrightarrow \beta(t_{M_0}^B(z_0)) \notin L_2. \end{aligned}$$

Now assume that we have already constructed $u_0, u_1, \ldots, u_k \in \{0,1\}^*$ and an $z_0, x_1, \ldots, x_k \in \mathbb{N}$ which fulfill the claim. To construct u_{k+1} and z_{k+1} which fulfill the claimwe consider an NPTOM M_r which works on an arbitrary input x and oracle B as follows: If $x \geq |u_k|$ then M_r with oracle B on input x works as M_{k+1} with oracle $u_k|B$ on input x. If $x < |u_k|$ then M_r works with oracle B on input x in such a way that $x \in L_1^B \leftrightarrow \beta(t_{M_r}^B(x)) \in L_2$.

Because of (1) we know that $x_r \in L_1^B \leftrightarrow \beta(t_{M_r}^B(x_r)) \notin L_2$. Hence $x_r \geq |u_k|$, and consequently, using Proposition 1.1 and (1)

$$x_r \in L_1^{u_k|B_r} \Leftrightarrow x_r \in L_1^{B_r} \Leftrightarrow \beta(t_{M_r}^{B_r}(x_r)) \notin L_2 \Leftrightarrow \beta(t_{M_{k+1}}^{u_k|B_r}(x_r)) \notin L_2. \tag{2}$$

Now define $z_{k+1} =_{\text{def}} x_r$ and u_{k+1} as the shortest initial word of $u_k|B_r$ such that $|u_{k+1}| > |u_k|$, $|u_{k+1}| > 2^{r+2} \cdot x_r$, and $\{0, 1, \ldots, |u_{k+1}|\}$ contains every query of M_{k+1} to the oracle $u_k|B_r$ during its work on input x_r. Hence u_k is a proper initial word of u_{k+1}, and by Proposition 1.2 and (2) we obtain for all $B \in u_{k+1} \cdot \{0,1\}^\omega$

$$\begin{aligned} z_{k+1} \in L_1^B &\Leftrightarrow x_r \in L_1^B \Leftrightarrow x_r \in L_1^{u_k|B_r} \Leftrightarrow \beta(t_{M_{k+1}}^{u_k|B_r}(x_r)) \notin L_2 \\ &\Leftrightarrow \beta(t_{M_{k+1}}^B(x_r)) \notin L_2 \Leftrightarrow \beta(t_{M_{k+1}}^B(z_{k+1})) \notin L_2. \end{aligned}$$

This completes our induction.

Now define $B' =_{\text{def}} \lim_{k\to\infty} u_k \cdot 0^\omega$, from which we get $B' \in u_k \cdot \{0,1\}^\omega$ for all $k \geq 0$. By the claim we obtain $z_k \in L_1^{B'} \leftrightarrow \beta(t_{M_k}^{B'}(z_k)) \notin L_2$ for every $k \geq 0$. This means $L_1^{B'} \notin \text{Leaf}_\text{u}^{B'}(L_2)$. □

2 Machines with Computation Trees Having ε-Leaves

It turns out that the quantifier-free reducibility $\leq_\text{m}^\text{qf}$ studied in [SW04] is not a special case of the ptt-reducibility (for example, $0^*1(0\cup 1)^* \leq_\text{m}^\text{qf} 1$ but $0^*1(0\cup 1)^* \not\leq_\text{m}^\text{ptt} 1$). In this section we relax the notion of ptt-reducibility to cover also the quantifier-free reducibility. We start with relaxing the notion of leaf language definability.

Let $L \subseteq \Sigma^*$. The class $\text{Leaf}_\text{e}(L)$ consists of those languages A for which there exists a nondeterministic polynomial time machine M producing on every computation path a symbol from $\Sigma \cup \{\varepsilon\}$ (where ε is the empty word) such that $x \in A \leftrightarrow \beta(t_M(x)) \in L$. The classes $\text{Leaf}_\text{e}(L)$ can be defined equivalently as follows:

Proposition 6 *For every language $L \subseteq \Sigma^*$, a language $A \subseteq \Delta^*$ is in* $\text{Leaf}_\text{e}(L)$ *if and only if there exist polynomial time computable functions $f : \Delta^* \times \mathbb{N} \to \Sigma \cup \{\varepsilon\}$ and $g : \Delta^* \to \mathbb{N}$ such that $x \in A \leftrightarrow f(x,0)f(x,1)\ldots f(x,g(x)) \in L$ for every $x \in \Delta^*$.*

By the definition of $\text{Leaf}_\text{e}(L)$ we obtain immediately

Proposition 7 *For every language L,* $\text{Leaf}_\text{b}(L) \subseteq \text{Leaf}_\text{u}(L) \subseteq \text{Leaf}_\text{e}(L)$.

Since $\text{Leaf}_\text{u}(1(0\cup 1)^*) = \text{P}$ and $\text{Leaf}_\text{e}(1(0\cup 1)^*) = \text{NP}$, it is unlikeley that $\text{Leaf}_\text{u}(L)$ and $\text{Leaf}_\text{e}(L)$ coincide for every L. However:

Proposition 8 *If L is paddable then* $\text{Leaf}_\text{b}(L) = \text{Leaf}_\text{u}(L) = \text{Leaf}_\text{e}(L)$.

For $L \subseteq \Sigma^*$ and $a \notin \Sigma$ define $L_a =_{\text{def}} \{a^{m_0} b_1 a^{m_1} b_2 a^{m_2} \dots a^{m_{n-1}} b_n a^{m_n} : b_1 b_2 \dots b_n \in L\}$.

Proposition 9 *For every language L, $\mathrm{Leaf}_\mathrm{e}(L) = \mathrm{Leaf}_\mathrm{b}(L_a) = \mathrm{Leaf}_\mathrm{u}(L_a) = \mathrm{Leaf}_\mathrm{e}(L_a)$.*

Theorem 10 *If L is regular then $\mathrm{Leaf}_\mathrm{e}(L) \subseteq \exists \cdot (\mathrm{Leaf}_\mathrm{b}(L) \wedge \mathrm{coNP})$.*

Now we define our reducibility. For a finite alphabet Σ and $a \notin \Sigma$, the homomorphism $h_{\Sigma,a} : (\Sigma \cup \{\varepsilon\})^* \to \Sigma^*$ is defined by $h_{\Sigma,a}(b) =_{\text{def}} b$ for $b \in \Sigma$ and $h_{\Sigma,a}(a) =_{\text{def}} \varepsilon$. For $L \subseteq \Sigma_1^*$ and $K \subseteq \Sigma_2^*$ define: $L \leq_\mathrm{m}^\mathrm{pte} K$ if and only if there exist a $a \notin \Sigma_1 \cup \Sigma_2$ and a plt-function $f : (\Sigma_1 \cup \{a\})^* \to (\Sigma_2 \cup \{a\})^*$ such that $h_{\Sigma_1,a}(x) \in L \leftrightarrow h_{\Sigma_2,a}(f(x)) \in K$ for all $x \in (\Sigma_1 \cup \{a\})^*$.

The quantifier-free reducibility $\leq_\mathrm{m}^\mathrm{qf}$ studied in [SW04] now turns out to be a special case of the pte-reducibility.

Theorem 11 *If $L \leq_\mathrm{m}^\mathrm{qf} K$ then $L \leq_\mathrm{m}^\mathrm{pte} K$.*

Proof. Let $L \leq_\mathrm{m}^\mathrm{qf} K$ where $L \subseteq \Sigma_1^*$ and $K \subseteq \Sigma_2^*$, and let $a \notin \Sigma_1 \cup \Sigma_2$. Hence there exists a qf-function f such that $x \in L \leftrightarrow f(x) \in K$ for all $x \in \Sigma_1^*$. Since $h_{\Sigma_1,a}$ is a qf-function $f \circ h_{\Sigma_1,a}$ is a qf-function too. Consequently, there exists a plt-function g such that $f \circ h_{\Sigma_1,a} = h_{\Sigma_2,a} \circ g$. Now we conclude $h_{\Sigma_1,a}(x) \in L \leftrightarrow f(h_{\Sigma_1,a}(x)) \in K \leftrightarrow h_{\Sigma_2,a}(g(x)) \in K$ for all $x \in (\Sigma_1 \cup \{a\})^*$. Hence $L \leq_\mathrm{m}^\mathrm{pte} K$. $\square$

Proposition 12 *For $L \subseteq \Sigma_1^*$, $K \subseteq \Sigma_2^*$ and $a \notin \Sigma_1 \cup \Sigma_2$, there holds $L \leq_\mathrm{m}^\mathrm{pte} K$ if and only if $L_a \leq_\mathrm{m}^\mathrm{plt} K_a$.*

Now we establish an analogon of the BCSV theorem for the leaf language classes of type $\mathrm{Leaf}_\mathrm{e}(L)$.

Theorem 13 *The following are equivalent for the languages L and K:*

(1) $L \leq_\mathrm{m}^\mathrm{pte} K$

(2) $\mathrm{Leaf}_\mathrm{e}^B(L) \subseteq \mathrm{Leaf}_\mathrm{e}^B(K)$ *for all oracles B.*

Proof. Let $L \subseteq \Sigma_1^*$, $K \subseteq \Sigma_2^*$ and $a \notin \Sigma_1 \cup \Sigma_2$. We conclude:

$$\begin{array}{rcll} L \leq_\mathrm{m}^\mathrm{pte} K & \Leftrightarrow & L_a \leq_\mathrm{m}^\mathrm{plt} K_a & \text{Proposition 12} \\ & \Leftrightarrow & \forall B(\mathrm{Leaf}_\mathrm{b}^B(L_a) \subseteq \mathrm{Leaf}_\mathrm{b}^B(K_a)) & \text{classical BCSV theorem} \\ & \Leftrightarrow & \forall B(\mathrm{Leaf}_\mathrm{e}^B(L) \subseteq \mathrm{Leaf}_\mathrm{e}^B(K)) & \text{Proposition 9} \end{array}$$ $\square$

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References

[BCS92] D.P. Bovet, P. Crescenzi and R. Silvestri. A uniform approach to define complexity classes. *Theoret. Comp. Sci*, 104 (1992), 263–283.

[SW04] V.L. Selivanov, K.W. Wagner. A reducibility for the dot-depth hierarchy. Proc. of the 29th Intern. Symp. on Mathematical Foundations of Computer Science, Lecture Notes in Computer Science Vol. 3153, 2004, 783–793.

[Ve93] N.K. Vereshchagin. Relativizable and non-relativizable theorems in the polynomial theory of algorithms. *Izvestiya Rossiiskoi Akademii Nauk*, 57 (1993), 51–90 (in Russian).