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A complete characterisation of minimal triangulations of co-square-free graphs

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Abstract. A chordal graph H is a triangulation of a graph G, if H is obtained by adding edges to G. If no proper subgraph of H is a triangulation of G we call H a minimal triangulation of G. The co-square is the complement of the chordless cycle of length 4. Co-square-free graphs do not contain the co-square as induced subgraph. We will show that co-square-free graphs are exactly the graphs whose minimal triangulations are split graphs. We will present an algorithm that efficiently generates all minimal triangulations of a co-square-free graph. We will give a linear-time algorithm that computes a minimal triangulation of a co-square-free graph and $\mathcal{O}(n+m^2)$-time algorithms for computing treewidth and minimum fill-in of co-square-free graphs.

1. Introduction. Lots of well-known graph classes admit a characterisation by giving a set of forbidden induced subgraphs. Some of these classes can be found in Golumbic's book [13], many more in the book by Brandstädt, Le, Spinrad [8]. Gallai, for instance, presented an infinite set of forbidden induced subgraphs to characterise comparability graphs [12]. On the other hand, such sets can be directly used to define graph classes. Maybe the most famous such class is the class of chordal graphs, that is defined to be the class of exactly those graphs that do not contain a chordless cycle of length at least 4. Another well-known class is the class of cographs, which is the set of exactly those graphs that do not contain the P_4 as induced subgraph; we can therefore say that cographs are exactly the P_4-*free* graphs. Here, one single forbidden subgraph guarantees interesting structural properties (see [8] as a good reference). A generalization of this graph class is the class of P_5-free graphs. However, these graphs are very difficult to cope with. We will regard a subclass of P_5-free graphs, in fact the co-square-free graphs. The *square* or C_4 is the chordless cycle on four vertices. The *co-square* is its complement. Since the co-square is P_4-free, not all cographs are co-square-free. But the co-chordal graphs, the complements of chordal graphs, as we will see later, are co-square-free. So, we could regard the class of co-square-free graphs as a generalization of co-chordal graphs.

Given a graph $G = (V, E)$, we call a graph $H = (V, E \cup F)$ a *triangulation of* G if H is chordal; H is a *minimal* triangulation of G if there is no triangulation of G that is a proper subgraph of H. During the last decade, minimal triangulations for some famous graph classes have been studied. One of the most interesting result is that every minimal triangulation of an AT-free graph is AT-free [17] which was extended by Parra and Scheffler to the characterisation of AT-free graphs by means of their minimal triangulations [18].

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Another such class is given by the class of AT-free claw-free graphs which are exactly those graphs whose minimal triangulations are all AT-free claw-free [18]. Finally, the same authors showed that a graph is P_k-free for $k \leq 5$ if and only if all its minimal triangulations are P_k-free [18]. This extends a result for cographs by Bodlaender and Möhring [4]. To continue this series, we will show that co-square-free graphs can also be characterised by their minimal triangulations, in fact, co-square-free graphs are exactly the graphs whose minimal triangulations are all co-square-free. By the way, a co-square-free chrodal graph is a split graph [13]. Furthermore, we will give an easy characterisation of all minimal triangulations of a co-square-free graph only by looking at the graph itself. It turns out that there is a strong correspondence between the minimal triangulations of a co-square-free graph and certain of its maximal independent sets. As an application, we can compute a minimal triangulation of a co-square-free graph in linear time. We will be able to give $\mathcal{O}(n + m^2)$-time algorithms to compute the treewidth and the minimum fill-in of a co-square-free graph. Bouchitté and Todinca showed that treewidth and minimum fill-in for graphs with at most n minimal separators can be computed in time $\mathcal{O}(n^6 + mn^4)$ [5], which is applicable to co-square-free graphs (Lemma 8). The minimum fill-in of a graph is the least number of edges that have to be added to the graph to make it chordal. There is an interesting application of this notion in Gaussian elimination of special sparse matrices [20]. The treewidth of a graph is a parameter that can be defined as one less than the least clique number of all its minimal triangulations. For graphs of small treewidth many NP-complete problems become tractable [3]. The decision problems Minimum Fill-in and Treewidth are known to be NP-complete even on co-bipartite graphs [22], [1].

The rest of the paper is composed as follows. After stating necessary definitions and theorems in the next section, we will give a short report on the method of Parra and Scheffler to describe and compute minimal triangulations using minimal separators, as it was presented in [18]. In Section 4, we will deal with minimal triangulations of co-square-free graphs, and in Section 5, we will present algorithms to compute a minimal triangualtion, the treewidth and the minimum fill-in of co-square-free graphs.

2. Preliminaries. We will consider only simple, finite, undirected graphs $G = (V, E)$ and denote by n and m the numbers of vertices and edges in G, respectively. Edges are denoted as uv, which means that vertices u and v are *adjacent*. Two edges uv and $u'v'$ are adjacent if $u \in \{u', v'\}$ or $v \in \{u', v'\}$. A vertex is *isolated* if it is not adjacent to any other vertex. For $S \subseteq V$, the subgraph of G *induced* by S, denoted by $G[S]$, is the graph containing only the vertices of S, and two vertices are adjacent in $G[S]$ if and only if they are adjacent in G; we write $G \setminus S$ instead of $G[V \setminus S]$. Adding a set F of edges to G will be denoted by $G \cup F$, which is short for $(V, E \cup F)$; deleting an edge $e \in E$ will be denoted by $G - e$ instead of $(V, E \setminus \{e\})$. The *neighbourhood* of a vertex $u \in V$, denoted by $N_G(u)$, is the set of all vertices adjacent to u in G; $N_G[u] =_{\text{def}} N_G(u) \cup \{u\}$ denotes the *closed* neighbourhood of u. For $A \subseteq V$, the neighbourhood $N_G(A)$ of A is the union of the neighbourhoods of the vertices in A outside A, i.e. $N_G(A) =_{\text{def}} \bigcup_{v \in A} N_G(v) \setminus A$. For further definitions, we refer to [13]. A graph is *connected* if there is a path between every two vertices; a *connected component* of G is a maximal connected subgraph of G. A *clique* in G is a set of pairwise adjacent vertices of G, whereas an *independent set* in G is a set of pairwise non-adjacent vertices. By $\omega(G)$ and $\alpha(G)$ we denote the numbers of vertices of a maximum clique and independent set of G, respectively.

Definition 1. *The* ***co-square*** *or* $2K_2$ *is the smallest graph on four vertices containing* *two non-adjacent edges.* ***Co-square-free*** *or* $2K_2$***-free*** *graphs are the graphs that do not contain the co-square as induced subgraph.*

Fact 1. *The graph* $G = (V, E)$ *is co-square-free if and only if for every set* $S \subseteq V$ *the graph* $G \setminus S$ *has at most one connected component containing two vertices.*

Proof: If there are a set $S \subseteq V$ and two connected components in $G \setminus S$ containing two vertices, we have a co-square in G. If u, v, x, z induce a co-square in G, than $G \setminus S$ for $S = V \setminus \{u, v, x, z\}$ contains two connected components on two vertices. ∎

A *path* of length k is a sequence $(x_1, \ldots, x_{k+1})$ of $k+1$ different vertices of $G = (V, E)$ where x_i and x_{i+1} are adjacent for every $i \in \{1, \ldots, k\}$. The *diameter* of G, $\mathrm{diam}(G)$, is the length of the longest shortest path between two vertices of G; if G is disconnected then $\mathrm{diam}(G) = \infty$. The *linegraph* $L(G)$ of G contains a vertex for every edge of G, and two vertices of $L(G)$ are adjacent if they correspond to adjacent edges in G.

Fact 2. *A graph* G *is co-square-free if and only if* $\mathrm{diam}(L(G)) \leq 2$.

Proof: Let $G = (V, E)$ be co-square-free. Let $e_1, e_2 \in E$ be non-adjacent edges. Then, there is an edge $e \in E$ that is adjacent to e_1 and e_2, which gives a path of length 2 between e_1 and e_2 in $L(G)$. Let u, v, x, z induce a co-square in G where $uv, xz \in E$. Then, uv and xz have no common neighbour in $L(G)$, hence $\mathrm{diam}(L(G)) > 2$. ∎

A path $(x_1, \ldots, x_k)$ is a *cycle of length* k of G if $x_1x_k \in E$. A *chord* in a cycle C is an edge of G between two non-consecutive vertices in C; a chord is *unique* in C if it is the only edge in G that is a chord in C. C is *chordless* if there is no edge in G that is a chord in C. G is *chordal* if there is no chordless cycle of length at least 4 in G. The complement of a chordal graph is *co-chordal*. A *split graph* $G = (V, E)$ is a graph whose vertex set can be partitioned into a clique and an independent set.

Theorem 3. **[13]** *Let* $G = (V, E)$ *be a graph. The following statements are equivalent:*

i. *G is a split graph,*

ii. *G is chordal and co-chordal,*

iii. *G is* $\{2K_2, C_4, C_5\}$*-free.*

We can, therefore, say that split graphs are exactly the chordal co-square-free graphs, the co-square-free graphs that do not contain C_4 and C_5 as induced subgraphs. A graph is *weakly chordal* if it and its complement do not contain a chordless cycle of length at least 5. Obviously, weakly chordal graphs are closed under complementation. Chordal and co-chordal graphs are weakly chordal.

Fact 4. *Let* $G = (V, E)$ *be a graph. Then,* G *is co-square-free and weakly chordal if and only if* G *is co-chordal.*

Proof: If G is co-square-free and weakly chordal, the complement of G cannot contain a chordless cycle of length at least 4. ∎

Figure 1 shows relations between the class of co-square-free graphs and other graph classes such as split, chordal and weakly chordal graphs. Observe that co-square-free graphs are P_5-free graphs. The class of P_5-free graphs is an interesting object of research but difficult to deal with. Recently, a lot of research has been devoted to examine subclasses of P_5-free graphs that can be defined by an additional forbidden induced subgraph [2], [6], [7].

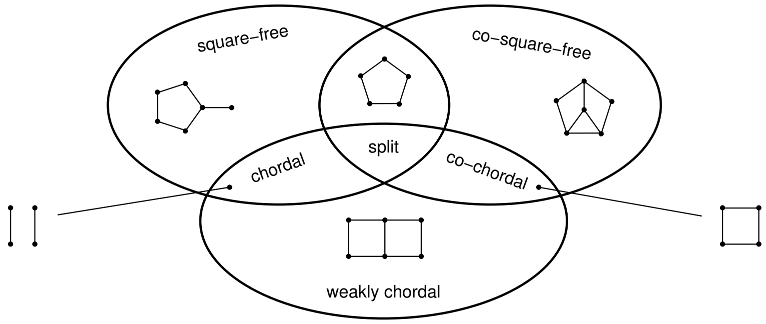

Figure 1. The class of co-square-free graphs relative to well-known graph classes.

3. Separators and triangulations. To compute and charaterise minimal triangulations, Parra and Scheffler revealed an interesting correspondence between the minimal triangulations of a graph and the maximal cliques of its separator graphs [18]. This correspondence will be the core part of our results in Section 4.

Let $G = (V, E)$ be a graph. A *triangulation* of G is a chordal graph H on the same vertex set as G such that G is a subgraph of H. H is a *minimal* triangulation of G if there is no triangulation of G that is a proper subgraph of H. Rose, Tarjan and Lueker could characterise minimal triangulations of a graph.

Theorem 5. **[21]** *Let $G = (V, E)$ and $H = G \cup F$, $E \cap F = \emptyset$, be two graphs. The following statements are equivalent:*

i. *H is a minimal triangulation of G,*

ii. *For every $e \in F$, e is the unique chord in a cycle of length 4 in H,*

iii. *For every $e \in F$, the graph $H - e$ is not chordal.*

Let $G = (V, E)$ be a graph and $S \subseteq V$. We call S an *a, b-separator* of G for two non-adjacent vertices $a, b \in V$ if a and b are in different connected components of $G \setminus S$. S is a *minimal* a, b-separator of G if there is no a, b-separator S' such that $S' \subset S$. S is a *minimal separator* of G if S is a minimal a, b-separator for some non-adjacent vertices $a, b \in V$.

Lemma 6. **[13]** *Let $G = (V, E)$ be a graph and $S \subseteq V$. S is a minimal separator in G if and only if $G \setminus S$ contains two connected components such that every vertex in S has a neighbour in both components.*

Let $S, T \subseteq V$ be two minimal separators of G. We say that S *crosses* T, if there are two connected components in $G \setminus T$ that have a non-empty intersection with S. If S crosses T then T crosses S [18]. S and T are *parallel*, $S \parallel T$, if they do not cross. The *separator graph* $\mathcal{S}_G$ of G contains a vertex for every minimal separator of G and two vertices are adjacent if they correspond to parallel minimal separators. A maximal clique in $\mathcal{S}_G$ corresponds to a maximal set of pairwise parallel minimal separators of G. A motivation for the following theorem can be found in Dirac's famous characterisation of chordal graphs: a graph is chordal if and only if all its minimal separators are cliques [11].

Theorem 7. **[18]** *Let $G = (V, E)$ be a graph and $\mathcal{S}_G$ its separator graph.*

i. *Let C be a maximal clique in $\mathcal{S}_G$. All minimal separators of G that correspond to vertices in C are completed into cliques; we obtain the graph H. Then, H is a minimal triangulation of G.*

ii. *Let H be a minimal triangulation of G. Then, there is a maximal clique C in $\mathcal{S}_G$*

such that H originates from G by completing into cliques all minimal separators of G corresponding to the vertices in C.

Kloks and Kratsch presented an algorithm that generates all minimal separators of a given graph in time $\mathcal{O}(n^5 \cdot R)$ where R is the number of minimal separators [14]. If R is of polynomial size in n for a graph class the algorithm runs in polynomial time. Furthermore, if we can find all maximal cliques in the separator graph in polynomial time we can solve problems defined by minimal triangulations efficiently. As it was pointed out by Parra and Scheffler, there has to be some knowledge about the separator graph to be able to compute all its maximal cliques fast [18]. Fortunately, for co-square-free graphs, we can characterise the separator graphs. Moreover, we can list all its maximal cliques in polynomial time.

4. Minimal triangulations of co-square-free graphs. A chordal graph on n vertices has at most n minimal separators. We will see that co-square-free graphs on n vertices also have at most n minimal separators. This bounds the size of the separator graph of a co-square-free graph.

Lemma 8. *Let $G = (V, E)$ be a co-square-free graph. For every minimal separator $S \subseteq V$ of G, there is a vertex $u \in V$ such that $S = N_G(u)$.*

Proof: Let $S \subseteq V$ be a minimal separator of G. By Lemma 6, $G \setminus S$ contains two connected components such that every vertex in S has a neighbour in both components. One of these components must contain exactly one vertex u due to Fact 1. Hence, $S \subseteq N_G(u)$, and since u has no neighbour in $G \setminus S$ we conclude $N_G(u) \subseteq S$. ■

Hence, minimal separators of co-square-free graphs are easy to find since only neighbourhoods of vertices have to be examined. It is also easy to determine whether two minimal separators are parallel.

Lemma 9. *Let $G = (V, E)$ be a co-square-free graph. Let $u, v \in V$ be two vertices such that $S =_{\mathrm{def}} N_G(u)$ and $T =_{\mathrm{def}} N_G(v)$ are minimal separators of G. Then, $S \parallel T \leftrightarrow uv \notin E$.*

Proof: Let $uv \in E$. Since $u \in T$ and T is a minimal separator in G, there are two connected components in $G \setminus T$ that contain a neighbour of u. Hence, S has a non-empty intersection with two connected components of $G \setminus T$. If $uv \notin E$, there is at most one connected component in $G \setminus T$ that contains vertices from S. ■

Corollary 10. *Let $G = (V, E)$ be a co-square-free graph. Then, its separator graph $\mathcal{S}_G$ is isomorphic to an induced subgraph of $\overline{G}$.*

It follows that the separator graph of a co-square-free graph G can be built in polynomial time, even in time $\mathcal{O}(nm)$ since for every vertex u of G it must be tested whether $G \setminus N_G(u)$ contains two connected components whose neighbourhoods are exactly $N_G(u)$. Furthermore, the separator graph of G is small and square-free. The class of co-square-free graphs is one of the rare graph classes for which the structure of the separator graphs can be determined.

Several authors could characterise a small variety of graph classes in terms of minimal triangulations. By $\mathrm{an}(G)$ for graph G we denote the *asteroidal number* of G which is the largest cardinality of a set A of vertices of G such that $G \setminus N_G(a)$ contains a connected component containing $A \setminus \{a\}$ for every $a \in A$. *AT-free* graphs do not have asteroidal number greater than 2. The *claw* is the tree on four vertices with three pairwise non-adjacent vertices. AT-free claw-free graphs have asteroidal number not exceeding 2 and do not contain the claw as induced subgraph.

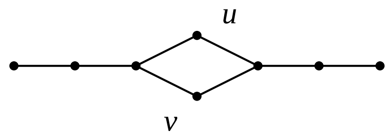

Figure 2. A $3K_2$-free graph having a minimal triangulation that is not $3K_2$-free.

Theorem 11. **[4], [15], [17], [18]** *Let G be a graph, and let $k \geq 1$.*

i. $\mathrm{an}(G) \leq k$ *if and only if* $\mathrm{an}(H) \leq k$ *for every minimal triangulation H of G,*

ii. *G is AT-free claw-free if and only if every minimal triangulation of G is AT-free claw-free,*

iii. *If $k \leq 5$, then G is P_k-free if and only if every minimal triangulation if G is P_k-free.*

The following theorem shows that minimal triangulations of co-square-free graphs are co-square-free, and that this is the largest class of graphs with this property. A *simplicial* vertex is a vertex whose neighbourhood is a clique. The graph obtained from G by making a vertex u simplicial is the smallest graph G' such that G is a subgraph of G' and u is simplicial.

Lemma 12. *Let $G = (V, E)$ be a co-square-free graph, and let $u \in V$. If G' emerges from G by making u simplicial then G' is co-square-free.*

Proof: If G' contains a co-square induced by $U \subseteq V$ then $|U \cap N_G(u)| = 2$. But then G contains a co-square with vertex u. ■

Theorem 13. *A graph is co-square-free if and only if all its minimal triangulations are co-square-free.*

Proof: Let $G = (V, E)$ contain a co-square, induced by $U \subseteq V$. Then, $H =_{\mathrm{def}} (V, E \cup F)$ where $F =_{\mathrm{def}} \{uv \notin E : \{u, v\} \not\subseteq U\}$ is a triangulation of G. Hence, there is a minimal triangulation H' of G containing a co-square. Let G be co-square-free and let H be a minimal triangulation of G. Due to Theorem 7, there is a maximal clique C in $\mathcal{S}_G$ such that H originates from G by making the minimal separators of G corresponding to the vertices of C into cliques. Let $X =_{\mathrm{def}} \{u_1, \ldots, u_k\}$, $k =_{\mathrm{def}} |C|$, be a set of vertices such that for every minimal separator S of G corresponding to a vertex of C there is $i \in \{1, \ldots, k\}$ such that $S = N_G(u_i)$, and vice versa. Since X is an independent set in G by Lemma 9, for every u_i, $i \in \{1, \ldots, k\}$, holds $N_G(u_i) = N_H(u_i)$. Obviously, H originates from G by making each vertex in X simplicial. We define H_i, $i \in \{1, \ldots, k\}$, to originate from H_{i-1} by making u_i simplicial, and $H_0 =_{\mathrm{def}} G$. By Lemma 12, every graph H_i is co-square-free, hence $H = H_k$ is co-square-free. ■

Applying Theorem 3 we can reformulate the last result as follows: a graph is co-square-free if and only if all its minimal triangulations are split graphs. The first question that may come into ones mind concerning a possible generalization of our result must be negated. Consider graph G depicted in Figure 2: G is $3K_2$-free but not co-square-free. The $3K_2$ is the graph on six vertices and three edges all pairs of which are non-adjacent. Adding edge uv to G yields a minimal triangulation of G that contains a $3K_2$. Thus, our result can be considered optimal.

We want to continue our study of minimal triangulations of co-square-free graphs by characterising the minimal triangulations themselves. We shall see that there is a strong correspondence between maximal independent sets of a co-square-free graph and its minimal triangulations. Let $G = (V, E)$ be a co-square-free graph that contains either only isolated vertices or a connected component of at least three vertices. In this case, we say that G is not isomorphic to a $K_2 + rK_1$ for any $r \geq 0$, i.e. to a graph on $r+2$ vertices and with exactly one edge. We define a collection $\mathcal{U}_G$ of independent sets of G in the following way. For every $U \subseteq V$, $U \in \mathcal{U}_G$ if and only if U satisfies all three conditions:

i. U is a maximal independent set in G,

ii. $\{u \in V : d_G(u) \leq 1\} \subseteq U$,

iii. if there is a vertex $v \in V \setminus U$ satisfying $|N_G(v) \cap U| = 1$ and $V \setminus U \not\subseteq N_G[v]$ then there is no vertex $u \in U$ such that $N_G(u) = V \setminus U$.

We observe that no set of vertices of a graph containing a connected component of two vertices can satisfy conditions i and ii simultaneously. However, graphs of the isomorphism classes $K_2 + rK_1$ are easily indentifiable, and since they are chordal they can be omitted from the following considerations. Hence, the graphs to be regarded subsequently do not contain adjacent vertices of degree 1.

Lemma 14. *Let $G = (V, E)$ be a co-square-free graph not isomorphic to a $K_2 + rK_1$ for any $r \geq 0$, and let $U \in \mathcal{U}_G$. Then, $H =_{\text{def}} G \cup F$ where $F =_{\text{def}} \{uv \notin E : u, v \in V \setminus U\}$ is a minimal triangulation of G.*

Proof: Of course, U is an independent set and $V \setminus U$ is a clique in H, thus H is a triangulation of G. Due to Theorem 5 we will show that every edge in F is the unique chord in a cycle of length 4 in H. For every vertex in $V \setminus U$, there is a neighbour in U due to the maximality of U. If two vertices in $V \setminus U$ each have a neighbour in U that is non-adjacent to the other vertex then they must be adjacent in G. Let $uv \in F$. Without loss of generality, we assume $N_G(u) \cap U \subseteq N_G(v) \cap U$. If u has two neighbours $x, z \in U$ then they are also neighbours of v, and uv is the unique chord in the cycle (x, u, z, v). If u has exactly one neighbour x in U then, by condition iii for U, there is a vertex $w \in V \setminus U$ that is not adjacent to x. Then, w is adjacent to u and v and non-adjacent to x in H, and uv is the unique chord in the cycle (u, x, v, w) in H. ∎

Lemma 15. *Let $G = (V, E)$ be a co-square-free graph not isomorphic to a $K_2 + rK_1$ for any $r \geq 0$. Let $H = G \cup F$, $E \cap F = \emptyset$, be a minimal triangulation of G. Let $\langle U, V \setminus U \rangle$ be a partition of V into a maximal independent set U in H and a clique $V \setminus U$ in H. Then, $U \in \mathcal{U}_G$.*

Proof: The partition $\langle U, V \setminus U \rangle$ is well-defined due to Theorem 13. Of course, $N_H(u) = N_G(u)$ for every vertex $u \in U$, so that U is a maximal independent set in G, too. An isolated vertex of G must be an isolated vertex of H. A vertex of degree 1 in G has degree 1 in H. Hence, all vertices of degree less than 2 must be contained in U. Finally, if $uv \in F$, uv is the unique chord in a cycle of length 4 due to Theorem 5. Then u and v are non-adjacent in H, thus $U \in \mathcal{U}_G$. ∎

Theorem 16. *Let $G = (V, E)$ be a co-square-free graph which is not isomorphic to a $K_2 + rK_1$ for any $r \geq 0$, and let H be a triangulation of G. Then, H is a minimal triangulation of G if and only if there is $U \in \mathcal{U}_G$ such that $H = G \cup \{uv \notin E : u, v \in V \setminus U\}$.*

We will see that this characterisation of minimal triangulations of co-square-free graphs leads to a polynomial-time algorithm that generates all minimal triangulations of a co-square-free graph.

5. Algorithmic implications. In this last section, we address three interesting problems associated with minimal triangulations. We will show that a minimal triangulation of a co-square-free graph can be computed in linear time. We will also show that the treewidth tw and the minimum fill-in mfi of a co-square-free graph can be computed in time $\mathcal{O}(n+m^2)$. We will finally show that computing the treewidth of a co-square-free graph is not harder than computing its independent set number.

Minimal triangulations of arbitrary graphs can be computed in time $\mathcal{O}(nm)$ [21]. For some restricted graph classes there are even linear-time algorithms to compute minimal triangulations: planar graphs [9], AT-free claw-free graphs and co-comparability graphs [16]. For graphs of bounded degree a minimal triangulation can be computed in almost-linear time [10]. The two graph parameters treewidth and minimum fill-in can be defined in terms of minimal triangulations as follows. Let $G = (V, E)$ be a graph.

$$\mathrm{tw}(G) =_{\mathrm{def}} \min\{\omega(H) - 1 : H \text{ a minimal triangulation of } G\}$$
$$\mathrm{mfi}(G) =_{\mathrm{def}} \min\{|F| : G \cup F \text{ a minimal triangulation of } G\}$$

Hence, the treewidth problem asks for a triangulation of smallest clique number and minimum fill-in asks for a triangulation of smallest number of edges. The corresponding decision problems Treewidth and Minimum Fill-in are NP-complete even on co-bipartite graphs, the complements of bipartite graphs [1], [22].

We first show that an easy greedy algorithm can compute a minimal triangulation of a co-square-free graph. Our algorithm will not compute the graph itself since outputting the additional edges could be too time-consuming. Therefore, the algorithm will compute a partition of the vertices of the input graph into an independent set and a clique that represents a minimal triangulation in the sense of Theorem 16.

Theorem 17. *There is a linear-time algorithm that computes a minimal triangulation of a co-square-free graph represented by a split partition.*

Proof: Let $G = (V, E)$ be a co-square-free graph. In linear time the degree of each vertex can be computed and the vertices can be ordered accordingly. If the maximal degree is less than 2, G contains at most one edge, thus is chordal. Otherwise, compute a maximal independent set U of G in the greedy way starting with a vertex of smallest degree. Since U can be computed in linear time, our algorithm runs in linear time. Due to Theorem 16, it suffices to show that $U \in \mathcal{U}_G$. All vertices of degree 0 are contained in U. Suppose there is a vertex of degree 1 that is not in U. A neighbour must have been selected by the algorithm earlier, but this could only be a vertex of degree 1, which would imply that G contains a connected component of exactly two vertices and which is not possible. Suppose there is a vertex $u \in U$ that is adjacent to every vertex in $V \setminus U$. If there is a vertex v in $V \setminus U$ such that u is its only neighbour in U and that is not adjacent to all vertices in $V \setminus U$, v must have smaller degree than u. But the algorithm would have chosen v before u. Hence, $U \in \mathcal{U}_G$. ■

Consider two adjacent vertices u and v of a co-square-free graph $G = (V, E)$. Their common non-neighbourhood, i.e. $V \setminus (N_G(u) \cup N_G(v))$, is an independent set. Prisner showed that co-square-free graphs with m edges have at most $m + 1$ maximal independent sets [19].

Lemma 18. *There is an $\mathcal{O}(n+m^2)$-time algorithm that generates all maximal independent sets of a co-square-free graph.*

Proof: Let $G = (V, E)$ be a co-square-free graph, and let $x_1, \ldots, x_n$ be the vertices of

G. An algorithm that generates the maximal independent sets of $G[\{x_1, \ldots, x_{i+1}\}]$, $i < n$, from the set of the maximal independent sets of $G[\{x_1, \ldots, x_i\}]$ can be implemented to run in time $\mathcal{O}(n + m^2)$. ∎

Corollary 19. *For G co-square-free, $\mathcal{U}_G$ and $\alpha(G)$ can be computed in time $\mathcal{O}(n+m^2)$.*

Combining the results of Theorem 16 and Corollary 19 we obtain the following theorem.

Theorem 20. *Let G be a co-square-free graph. Then, $\mathrm{mfi}(G)$ and $\mathrm{tw}(G)$ can be computed in time $\mathcal{O}(n + m^2)$.*

Using this approach we can solve in polynomial time every problem on co-square-free graphs that can be formulated as a property of minimal triangulations, that can be decided for every minimal triangulation in polynomial time and that can be computed from the results for each minimal triangulation in polynomial time.

In case of co-square-free graphs there seems to be a strong relationship between computing the treewidth and computing the independent set number. Our last result will show that an improvement of the compution time of the independent set number directly translates into an improvement of the computation time of the treewidth. We say that a problem X is *at most as hard as* a problem Y, if for every algorithm A_Y that solves Y in time $t_{A_Y}(n)$ there is an algorithm A_X that solves X in time $\mathcal{O}(n + t_{A_Y}(n))$, where n is the length of the input.

Proposition 21. *Computing the treewidth of a co-square-free graph is at most as hard as computing its independent set number.*

Proof: We reduce the problem treewidth to the problem independent set number. Let $G = (V, E)$ be a co-square-free graph. Let $W =_{\mathrm{def}} \{u \in V : d_G(u) \geq n - \alpha(G)\}$. If $\alpha(G) = \alpha(G \setminus W)$ then $\mathrm{tw}(G) = n - \alpha(G) - 1$; otherwise $\mathrm{tw}(G) = n - \alpha(G)$. If $t_\alpha(n, m)$ and $t_{\mathrm{tw}}(n, m)$ are the best running times of algorithms that compute the independent set number and the treewidth of a co-square-free graph on n vertices and with m edges, respectively, then $t_{\mathrm{tw}}(n, m) = \mathcal{O}(n + m + t_\alpha(n, m))$. The correctness follows from Theorem 16 and the fact that an independent set and a clique of a split graph share at most one vertex. ∎

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