A Reducibility for the Dot-Depth Hierarchy

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Abstract

Hierarchies considered in computability theory and in complexity theory are related to some reducibilities in the sense that levels of the hierarchies are downward closed and have complete sets. In this paper we propose a reducibility having similar relationship to the Brzozowski's dot-depth hierarchy and some its refinements. We prove some basic facts on the corresponding degree structure and discuss relationships of the reducibility to complexity theory (via the leaf-language approach).

1 Introduction

Hierarchies considered in descriptive set theory, computability theory and complexity theory are related to some reducibilities in the sense that levels of the hierarchies are downward closed and have complete sets under the respective reducibility. The complete sets are then used for estimating 'complexity' of other sets and problems. In descriptive set theory this is the Wadge reducibility, in computability theory — the m-reducibility, in complexity theory — the polynomial-time m-reducibility.

In this paper we propose a reducibility having similar relationship to Brzozowski's dot-depth hierarchy (DDH) [BK78]. We establish the relationship of this reducibility to the DDH and some of its refinements, prove some basic facts about the corresponding degree structure and discuss its relationship to complexity theory (via the well-known leaf-language approach). The proofs of this paper demonstrate deep interconnections of this reducibility with automata-theoretic and complexity-theoretic techniques which support our belief that it provides a useful classification of regular star-free languages refining some popular hierarchies within this class.

We use (mostly without definitions here) some standard terminology and notation from automata theory and complexity theory, say the well-known notation of languages by regular expressions or the concept of a polynomial-time non-deterministic Turing machine. The reader may find

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all relevant information e.g. in [BDG88, BDG90, MP71]. Letters A, B, C (sometimes with subscripts) will denote alphabets which are always assumed to contain at least two symbols. By A^+ we denote the set of all non-empty words over A, and by A^* the set of all words (including the empty word ε). Since we use the logical approach to (star-free) regular languages developed in [Bü60, El61, MP71, Th82] we work mostly with languages of non-empty words $L \subseteq A^+$. Correspondingly, the complement $\overline{L}$ of such a language L is defined by $\overline{L} = A^+ \setminus L$. For a class $\mathcal{K}$ of languages, let $\mathrm{BC}(\mathcal{K})$ be the boolean closure of $\mathcal{K}$, i.e., the closure of $\mathcal{K}$ under union and complement.

Relate to any alphabet $A = \{a, b, \ldots\}$ the signature $\sigma_A = \{\leq, Q_a, Q_b, \ldots, \bot, \top, p, s\}$, where $\leq$ is a binary relation symbol, Q_a (for any $a \in A$) is a unary relation symbol, $\bot$ and $\top$ are constant symbols, and p, s are unary function symbols. A word $u = u_0 \ldots u_n \in A^+$ may be considered as a structure $\mathbf{u} = (\{0, \ldots, n\}; <, Q_a, \ldots)$ of the signature σ_A, where $<$ has its usual meaning, $Q_a (a \in A)$ are unary predicates on $\{0, \ldots, n\}$ defined by $Q_a(i) \leftrightarrow u_i = a$, the symbols $\bot$ and $\top$ denote the least and the greatest elements, while p and s are respectively the predecessor and successor functions on $\{0, \ldots, n\}$ (with $p(0) = 0$ and $s(n) = n$). For a sentence ϕ of σ_A, let $L_\phi = \{u \in A^+ | \mathbf{u} \models \phi\}$. In [MP71] it was shown that the class of all languages of the form L_ϕ, where ϕ ranges through first-order sentences of σ_A, coincides with the class of star-free languages. For $n > 0$, let Σ_n be the class of all languages L_ϕ, where ϕ ranges through the Σ_n-sentences of σ_A. In [Th82] it was shown that the sequence $\{\Sigma_n\}_{n>0}$ essentially coincides with a well-known hierarchy of automata theory, namely the Brzozowski's *dot depth hierarchy*. For this reason we call the sequence $\mathrm{DDH} = \{\Sigma_n\}$ here the dot depth hierarchy. (Note that in the literature also the classes $\mathrm{BC}(\Sigma_n)$ belong to the dot depth hierarchy.) Let $\Pi_n = \mathrm{co}(\Sigma_n)$ denote the class of complements of Σ_n-languages, and $\Delta_n = \Sigma_n \cap \Pi_n$. When we want to stress that Σ_n is a level of the DDH over an alphabet A we may use the more exact notation A-Σ_n. We will do in the same way with other classes of languages.

Words correspond bijectively to (isomorphism types of) finite 'colored linear orderings', i.e. finite models of the theory CLO_A of signature σ_A with the following axioms:

- $<$ is a linear ordering,
- any element satisfies exactly one of the predicates $Q_a (a \in A)$,
- $\forall x(\bot \leq x \leq \top)$,
- $\forall x(p(x) \leq x \wedge \neg\exists y(p(x) < y < x))$,
- $\forall x(x \leq s(x) \wedge \neg\exists y(x < y < s(x)))$,
- $\forall x(x > \bot \rightarrow p(x) < x)$ and $\forall x(x < \top \rightarrow x < s(x))$.

Sometimes it is technically more convinient to consider 'relational' versions of the signature σ_A (and of the theory CLO_A). E.g., one could take sigature $\sigma'_A = \{\leq, Q_a, Q_b, \ldots, \bot, \top, S\}$, where $S(x, y)$ is a binary relation symbol interpreted as 'x is an immediate predecessor of y'. The both languages are clearly equivalent, and we will use any of them when appropriate. More information on the logical approach to star-free languages maybe found in [Th82, Se01].

In Section 2 we present the main notions and establish some basic facts about them. In Section 3 complete sets for the levels of the DDH and its refinements are constructed. In Section 4 an interesting property of sets complete for levels of the difference hierarchy over Σ_1 is established. In Section 5 we establish relationships of our reducibility to the well-known polylogtime reducibility playing a crucial role in the leaf-language approach to define complexity classes. These results are applied in Section 6 to investigation of degrees within the second level of the difference hierarchy over Σ_1. In Section 7 we describe a principal ideal of the degree structure under consideration. We conclude in Section 8 with some conjectures and open questions.

2 Definitions and Basic Properties

Let us recall a well-known logical notion of interpretation (see e.g. [Sho67]); actually we use a particular case of this notion (namely, interpretation by quantifier-free formulas) which seems sufficient for our intensions here. The *interpretation* I *over alphabets* $A = \{a, \ldots\}$ *and* $B = \{b, \ldots\}$ is given by a tuple

$$(\phi_U(\bar{x}), \phi_<(\bar{x}, \bar{y}), \phi_b(\bar{x}), \ldots, \phi_\bot(\bar{x}), \phi_\top(\bar{x}), \phi_S(\bar{x}, \bar{y})),$$

where $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_n)$ are sequences of different variables of the same length $n > 0$ (n is fixed in advance) and $\phi_U(\bar{x}), \ldots, \phi_S(\bar{x}, \bar{y})$ are quantifier-free formulas of σ_A with the following properties. Let $u = u_0 \cdots u_l$ be any word over A of length $|u| = l+1$. Then the set $T = \{\bar{x} \in \{0, \ldots, l\}^n | \mathbf{u} \models \phi_U(\bar{x})\}$ should be non-empty and $\phi_<(\bar{x}, \bar{y}), \phi_b(\bar{x}), \ldots, \phi_\bot(\bar{x}), \phi_\top(\bar{x}), \phi_S(\bar{x}, \bar{y}))$ interpreted in $\mathbf{u}$ should define a model of CLO_B with the universe T (the formulas $\phi_\bot(\bar{x}), \phi_\top(\bar{x})$ should be true exactly on the first and the last element, respectively). Since finite models of CLO_B are in a bijective correspondence with elements of B^+, any interpretation I induces a function $u \mapsto u^I$ from A^+ into B^+.

Examples. 1. Let $\phi_U(\bar{x})$ be a valid formula, let $\phi_<(\bar{x}, \bar{y})$ define the lexicographic ordering between $\bar{x}$ and $\bar{y}$, let $\phi_\bot(\bar{x}), \phi_\top(\bar{x}), \phi_S(\bar{x}, \bar{y})$ be defined in the obvious way according to their intended interpretations, and let $\phi_b(\bar{x}, \ldots)$ be choosen in a way to get a model of CLO_B on this ordering. Then u^I is of length $|u|^n$, and letters of the word u^I are easily computed from the interpretation.

2. Let $\phi_U(x)$ be a valid formula of one variable x, let $\phi_<(x, y)$ be $x > y$, let $\phi_a(x)$ be $Q_a(x)$ for any $a \in A$, let $\phi_\bot(\bar{x}), \phi_\top(\bar{x})$ be $x = \top$, $x = \bot$ respectively, and let $\phi_S(x, y)$ be $S(y, x)$. Then we get an interpretation I over A and A such that u^I is the reverse of the word $u \in A^+$.

3. Let $u \mapsto pu$ be the function on A^+ which adds a fixed prefix $p \in A^*$ to a word u. Is there an interpretation I over A and A such that $u^I = pu$ for any u? For $p = \varepsilon$ the answer is of course positive, otherwise it is negative (since any interpretation sends words of length 1 to words of length 1). But it is easy to see that there is an interpretation I such that $u^I = pu$ for any u of length > 1. The same of course applies to the operation of adding a suffix to a word.

4. Let $h : A^+ \to B^+$ be a semigroup morphism. Such functions are defined by their values $h : A \to B^+$ on the letters of A (i.e., words of length 1) because we have $h(a_0 \cdots a_l) = h(a_0) \cdots h(a_l)$, where $a_i \in A$. It is easy to see that for any such h there is an interpretation I over A and B such that $u^I = h(u)$ for almost all $u \in A^+$ (i.e. for all but finitely many of words).

We are ready to give the main definition of this paper.

Definition 2.1
- *A function $f : A^+ \to B^+$ is called an qf-function if there is an interpretation I over A and B such that $u^I = f(u)$ for almost all $u \in A^+$.*
- *Let $L \subseteq A^+$ and $K \subseteq B^+$. We say that L is qf-reducible to K (in symbols $L \leq_{\mathrm{m}}^{\mathrm{qf}} K$) if $L = f^{-1}(K)$ for some qf-function $f : A^+ \to B^+$.*

When we check some property of an qf-function $f(u)$ we usually for simplicity identify it with the corresponding function $u \mapsto u^I$. This is possible in the cases (which are mostly considered here) when the finite set of 'exceptional' words does not destroy the property under consideration.

We start the investigation of the introduced notion with a lemma.

Lemma 2.2 *1. For any $U \subseteq A^+$, $U \in \Delta_1$ iff $U = L_\phi$ for some quantitier-free sentence ϕ of σ_A iff U is a finite union of languages of the form u or uA^*v where $u, v \in A^+$.*

2. If $U \in \Delta_1$ and $L \subseteq A^+$ is non-trivial then $U \leq_{\mathrm{m}}^{\mathrm{qf}} L$.

3. If $L \leq_{\mathrm{m}}^{\mathrm{qf}} M$ and $M \in \Delta_1$ then $L \in \Delta_1$.

Proof. The assertion 1 is known (see e.g. [BKS99] or the proof of Proposition 4.2 below), and the assertion 2 is an immediate corollary of 1.

3. We prove a more general assertion. Let $L \in B$-Σ_n, we will show that for any interpretation I over A and B the language $I^{-1}(L) = \{u | u^I \in L\}$ is in A-Σ_n. Let ϕ be a Σ_n-sentence of σ_B satysfying $L = L_\phi$. As is well-known [Sho67], the interpretation I induces a natural syntactic map $\psi \mapsto \psi^I$ sending σ_B-formulas to σ_A-formulas in such a way that $\mathbf{u}^I \models \psi$ iff $\mathbf{u} \models \psi^I$. Since I consists of quantifier-free formulas only, this map respects classes of formulas like $\Sigma_n, \Pi_n, \Delta_n$. Therefore, ϕ^I is a Σ_n-sentence and $u \in I^{-1}(L)$ iff $\mathbf{u} \models \phi^I$, hence $I^{-1}(L) = L_{\phi^I} \in \Sigma_n$.

We have shown the downward closure of Σ_n under the qf-reducibility. A similar argument proves the downward closure of the classes Π_n, Δ_n. □

Next we establish an important technical property of the relation $\leq_{\mathrm{m}}^{\mathrm{qf}}$.

Theorem 2.3 *The relation $\leq_{\mathrm{m}}^{\mathrm{qf}}$ is reflexive and transitive.*

Proof. Reflexivity is obvious since the identity function is a qf-function. Now assume that languages $L \subseteq A^+$, $M \subseteq B^+$, $K \subseteq C^+$ satisfy $L \leq_{\mathrm{m}}^{\mathrm{qf}} M$ and $M \leq_{\mathrm{m}}^{\mathrm{qf}} K$. Let $f : A^+ \to B^+$ and $g : B^+ \to C^+$ be qf-functions witnessing these reductions and let I, J be interpretations satisfying $f(u) = u^I$ and $g(v) = v^J$ for almost all $u \in A^+, v \in V^+$. From the well-known properties of interpretations [Sho67] it follows that the map $u \mapsto (u^I)^J$ maybe defined by an interpretation (by quantifier-free formulas). Unfortunately, this map may differ from the function $u \mapsto g(f(u))$ on an infinite set of words, hence we cannot conclude that the last function (which of course reduces L to K) is a qf-function.

Let $\{u_1, \ldots, u_k\}$ be the set of exceptional words for f and $\{v_1, \ldots, v_l\}$ be the set of exceptional words for g. Define

$$V_0 = \{v_j | g(v_j) \notin K\},\ V_1 = \{v_j | g(v_j) \in K\},$$

and

$$U_0 = f^{-1}(V_0) \cup \{u_i | gf(u_i) \notin K\},\ U_1 = f^{-1}(V_1) \cup \{u_i | gf(u_i) \in K\}.$$

The sets U_0, U_1 are disjoint and, by Lemma 2.2, are in Δ_1. Note that $gf(u) = (u^I)^J$ for $u \in \overline{U_0 \cup U_1}$. Define a function $h : A^+ \to C^+$ by

$$h(u) =_{\mathrm{def}} \begin{cases} w_0 & \text{if } u \in U_0 \\ w_1 & \text{if } u \in U_1 \\ gf(u) & \text{otherwise}, \end{cases}$$

where $w_0 \in \overline{K}$ and $w_1 \in K$ (we may assume K to be non-trivial because otherwise the assertion is obvious). It is not hard to see that h is a qf-function reducing L to K. □

Let $\mathcal{P}(A^+)$ be the power-set of A^+. The preorder $\leq_{\mathrm{m}}^{\mathrm{qf}}$ on $\mathcal{P}(A^+)$ induces in the usual way an equivalence relation on $\mathcal{P}(A^+)$ denoted by $\equiv_{\mathrm{m}}^{\mathrm{qf}}$. The corresponding quotient partial order is denoted $(\mathbf{D}_{\mathrm{qf}}; \leq)$ and is called here *the structure of qf-degrees*. Our next goal is to get some information about this ordering. Let $\mathbf{D}'_{\mathrm{qf}}$ consist of degrees of proper (or non-trivial) subsets of A^+.

We need one more lemma. Let $L, K, M \subseteq A^+$. If $L \subseteq M \subseteq \overline{K}$ then we say that M *separates* L from K (note that this is equivalent to saying that $\overline{M}$ separates K from L). We define *the join* of L and K by $L \oplus K = aL \cup (A \setminus a)K$, where a is a fixed letter from A.

Lemma 2.4 *Let L, K be non-trivial languages over A.*

1. *For any $U \in \Delta_1$, $U \cap L \leq_{\mathrm{m}}^{\mathrm{qf}} L$.*
2. *For all $u, v \in A^*$, $uLv \equiv_{\mathrm{m}}^{\mathrm{qf}} L$.*
3. *If L is separated from K by a Δ_1-language U and $L \cup K \neq A^+$ then $L \cup K \equiv_{\mathrm{m}}^{\mathrm{qf}} L \oplus K$.*
4. *For any language M, $M \leq_{\mathrm{m}}^{\mathrm{qf}} L \oplus K$ iff $M = (U \cap L') \cup (\overline{U} \cap K')$ for some $U \in \Delta_1$, $L' \leq_{\mathrm{m}}^{\mathrm{qf}} L$ and $K' \leq_{\mathrm{m}}^{\mathrm{qf}} K$.*

Proof. 1. Let ϕ be a quantifier-free sentence satisfying $U = L_\phi$ and let $y \in \overline{L}$. Define a function $f : A^+ \to A^+$ by

$$f(u) =_{\mathrm{def}} \begin{cases} u \text{ if } \mathbf{u} \models \phi \\ y \text{ otherwise} \end{cases}.$$

Then f is an qf-function reducing $U \cap L$ to L.

2. The reducibility $L \leq_{\mathrm{m}}^{\mathrm{qf}} uLv$ is witnessed by the qf-function $w \to uwv$. The converse reduction is witnessed by the qf-function

$$f(w) =_{\mathrm{def}} \begin{cases} x \text{ if } \mathbf{w} \models \phi \\ y \text{ otherwise} \end{cases}$$

where ϕ is a quantifier-free sentence satisfying $L_\phi = uA^+v$, $y \in \overline{L}$, and x satisfies $w = uxv$.

3. The relation $L \cup K \leq_{\mathrm{m}}^{\mathrm{qf}} L \oplus K$ is witnessed by the qf-function

$$f(u) =_{\mathrm{def}} \begin{cases} au \text{ if } \mathbf{u} \models \phi \\ bu \text{ otherwise} \end{cases}$$

where ϕ is a quantifier-free sentence satisfying $L_\phi = U$ and $b \in A \setminus \{a\}$.

For the converse reduction, consider the function

$$g(u) =_{\mathrm{def}} \begin{cases} u' \text{ if } u = cu' \wedge (c = a \leftrightarrow u' \in U) \\ z \text{ otherwise} \end{cases}$$

where $c \in A, u' \in A^+$ and $z \in \overline{L \cup K}$. One easily checks that g is an qf-function and

$$u \in L \oplus K \leftrightarrow g(u) \in L \cup K$$

for all words u of length > 1. Hence, a finite modification of g reduces $L \oplus K$ to $L \cup K$.

4. Let $M = f^{-1}(L \oplus K)$ where f is an qf-function. Take

$$U = f^{-1}(aA^+),\ L' = f^{-1}(aL) \text{ and } K' = f^{-1}((A \setminus a)K).$$

Since

$$L \oplus K = (aA^+ \cap aL) \cup (\overline{aA^+} \cap (A \setminus a)K),$$

we have $M = (U \cap L') \cup (\overline{U} \cap K')$. Since $aA^+ \in \Delta_1$, $aL \leq_{\mathrm{m}}^{\mathrm{qf}} L$ and $(A \setminus a)K \leq_{\mathrm{m}}^{\mathrm{qf}} K$, we have $U \in \Delta_1$, $L' \leq_{\mathrm{m}}^{\mathrm{qf}} L$ and $K' \leq_{\mathrm{m}}^{\mathrm{qf}} K$.

Conversely, let M be represented as $(U \cap L') \cup (\overline{U} \cap K')$ and let f, g be qf-functions reducing respectively L' to L and K' to K. Define a function $h : A^+ \to A^+$ by

$$h(u) =_{\text{def}} \begin{cases} af(u) \text{ if } u \in U \\ bg(u) \text{ otherwise} \end{cases},$$

where $b \in A \setminus \{a\}$. One easily checks that h is an qf-function reducing M to $L \oplus K$. $\square$

The next theorem is parallel to corresponding properties of the structures of m-degrees and of polynomial-time m-degrees.

Theorem 2.5 *1. $\{\emptyset\}$ and $\{A^+\}$ are two distinct minimal elements of $\mathbf{D}_{\text{qf}}$ which are below any other element.*

2. The structure $(\mathbf{D}_{\text{qf}}; \leq)$ is an upper semilattice where the supremum operation $\cup$ is induced by the join operation.

3. The structure $(\mathbf{D}'_{\text{qf}}; \leq, \cup)$ is a distributive upper semilattice with a least element which consists exactly of non-trivial Δ_1-languages.

Proof. Statement 1 is clear.

2. Let $L, K \leq_{\text{m}}^{\text{qf}} L \oplus K$ (the qf-functions $u \mapsto au$ and $u \mapsto bu$, where $b \in A \setminus \{a\}$, witness the reductions). Now let $L, K \leq_{\text{m}}^{\text{qf}} M$; we have to show that $L \oplus K \leq_{\text{m}}^{\text{qf}} M$. If at least one of L, K is trivial, the assertion is evident. Otherwise, by Lemma 2.4.2, both aL and $(A \setminus a)K$ are qf-reducible to M. Take interpretations $I_1 = (\phi_U(\bar{x}), \ldots)$ and $I_2 = (\psi_U(\bar{x}), \ldots)$ witnessing the reductions of aL and $(A \setminus a)K$ to M, respectively. It is easy to see that w.l.o.g. we may assume that the corresponding formulas of these interpretations have the same free variables. Define a tuple of formulas $I = (\theta_U(\bar{x}), \ldots)$ by

$$\theta_U(\bar{x}) \leftrightarrow (Q_a(\perp) \wedge \phi_U(\bar{x})) \vee (\neg Q_a(\perp) \wedge \psi_U(\bar{x})),$$

and similarly for the remaining pairs of formulas. One easily gets that I is an interpretation witnessing the reducibility $L \oplus K \leq_{\text{m}}^{\text{qf}} M$.

3. The least element in $\mathbf{D}'_{\text{qf}}$ is clearly the sup of the two minimal elements in $\mathbf{D}_{\text{qf}}$. By Lemma 2.2.2, the least degree consists exactly of non-trivial Δ_1-languages. It remains to show the distributivity, i.e. if $M \leq_{\text{m}}^{\text{qf}} L \oplus K$ and L, K, M are non-trivial, then $M \equiv_{\text{m}}^{\text{qf}} L_1 \oplus K_1$ for some $L_1 \leq_{\text{m}}^{\text{qf}} L$ and $K_1 \leq_{\text{m}}^{\text{qf}} K$. By Lemma 2.4.4, $M = (U \cap L') \cup (\overline{U} \cap K')$ for some $U \in \Delta_1$, $L' \leq_{\text{m}}^{\text{qf}} L$ and $K' \leq_{\text{m}}^{\text{qf}} K$. Let $L_1 = (U \cap L')$ and $K_1 = (\overline{U} \cap K')$. By Lemma 2.4.3, $M \equiv_{\text{m}}^{\text{qf}} L_1 \oplus K_1$ (if some of L_1, K_1 is trivial, correct it by joining, say, with the non-trivial set aA^* the degree of which is the least element of $\mathbf{D}'_{\text{qf}}$). $\square$

The last theorem gives some information about $\mathbf{D}_{\text{qf}}$ which is shown in Figure 1.

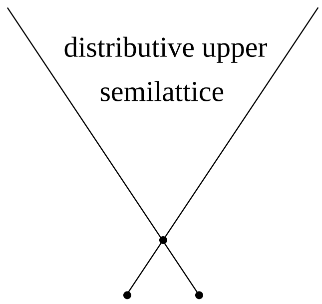

Figure 1. The distributive upper semilattice of qf-degrees.

Below we will get additional information about qf-degrees which will help to understand the structure near the minimal elements. Here we will show that some natural classes of languages correspond to ideals of qf-degrees. Recall that an *ideal* of an upper semilattice is a nonempty downward closed set of its elements which is closed under supremum. An ideal is *principal* if it contains a biggest element. Let $\mathcal{R}$ and $\mathcal{SF}$ denote the class of regular and of regular star-free languages, respectively.

Theorem 2.6 *All the classes $\Sigma_n, \Pi_n, \Delta_n (n > 0)$, as well as the classes $\mathcal{SF}$ and $\mathcal{R}$, correspond to ideals of $(\mathbf{D}_{\mathrm{qf}}; \leq)$.*

Proof. We have shown the downward closure of $\Sigma_n, \Pi_n, \Delta_n$ under the qf-reducibility in the proof of Lemma 2.2. The downward closure of the class $\mathcal{SF}$ follows from $\mathcal{SF} = \cup_n \Sigma_n$. For the class $\mathcal{R}$ the proof is the same (one may use the result from [Bü60, El61] which characterizes the regular languages as those defined by monadic second-order sentences).

That the classes $\Sigma_n, \Pi_n, \ldots$ are closed under join is a well-known fact. □

Next we consider some refinements of the DDH introduced in [Se02]. Let $(B; \cup, \cap, \bar{}, 0, 1)$ be a boolean algebra (b.a.). Without loss of generality, one may think that B is class of subsets of some set. By *a base in B* we mean any sequence $\mathcal{L} = \{\mathcal{L}_n\}_{n<\omega}$ of sublattices of $(B; \cup, \cap, 0, 1)$, satisfying the inclusions $\mathcal{L}_n \cup \mathrm{co}(\mathcal{L}_n) \subseteq \mathcal{L}_{n+1}$ (here $\mathrm{co}(\mathcal{L}_n)$ denotes the dual set $\{\bar{x} | x \in \mathcal{L}_n\}$ for $\mathcal{L}_n$). With any base $\mathcal{L} = \{\mathcal{L}_n\}_{n<\omega}$ one can associate a family of new subsets of B as follows. Let T be the set of terms of signature $\{\cup, \cap, \bar{}, 0, 1\}$ with variables $v_k^n (k, n < \omega)$. We call $v_k^n (k < \omega)$ *variables of type n*, and elements of T—*typed boolean terms*. Relate to any term $t \in T$ the set $t(\mathcal{L})$ of all its values when variables of type n range over $\mathcal{L}_n$. We call the family $\{t(\mathcal{L})\}_t$ *the typed boolean hierarchy over $\mathcal{L}$*. Note that DDH $= \{\Sigma_{n+1}\}$ form a base, hence we may (and will) consider levels $t(\mathrm{DDH})$ of the typed boolean hierarchy over this base.

In [Se02] also the so called *fine hierarchy* over the DDH was considered. The levels of the fine hierarchy are denoted S_α, where α ranges through the ordinals less that the least epsilon number ε_0 [KM67]; we do not recall the exact definition of the fine hierarchy because we only rarely mention it here.

From definitions in [Se02] and the last theorem we immediately get

Corollary 2.7 *All levels of the typed boolean hierarchy and of the fine hierarchy over the DDH correspond to ideals of $(\mathbf{D}_{\mathrm{qf}}; \leq)$.*

The most important particular case of the mentioned refinements is the difference hierarchy $\{\mathcal{D}_k(\Sigma_n)\}_{k<\omega}$ over any level Σ_n of the DDH. The k-th level $\mathcal{D}_k(\Sigma_n)$ of this hierarchy consists of languages of the form $\bigcup_i (L_{2i} \setminus L_{2i+1})$, where $L_0 \supseteq L_1 \supseteq \cdots$ is a descending sequence of Σ_n-languages and $L_i = \emptyset$ for $i \geq k$. For the case of difference hierarchy over Σ_1 we simplify notation of the levels to $\mathcal{D}_k$. As is well-known, $\mathcal{D}_k \cup \mathrm{co}(\mathcal{D}_k) \subseteq \mathcal{D}_{k+1}$, and similarly for the difference hierarchy over any level. The class $\mathrm{BC}(\Sigma_1) = \bigcup_k \mathcal{D}_k$ is well-known in automata theory as the class of languages *of dot-depth one*. The last corollary applies of course to the levels of the difference hierarchies.

In the next section we will show that some of ideals discussed above are indeed principal, i.e. contain complete sets w.r.t. qf-reducibility.

3 Complete Sets

First we show the existence of complete sets in the levels of the dot-depth hierarchy. For $n > 0$ let $A_n =_{\text{def}} \{0, 1, \ldots, n\}$. Define $H_1 =_{\text{def}} 0^*1\{0,1\}^*$, and $H_n =_{\text{def}} A_n^* n(A_{n-1}^* \setminus H_{n-1})nA_n^*$ for $n > 1$.

Lemma 3.1 *Let $n \geq 0$, and let A be a finite alphabet. There holds $H_n \in A_n\text{-}\Sigma_n$, and $L \leq_{\mathrm{m}}^{\mathrm{qf}} H_n$ for every $L \in A\text{-}\Sigma_n$.*

Proof. By a simple induction we obtain $H_n \in A_n\text{-}\Sigma_n$ for $n \geq 0$.

To show $L \leq_{\mathrm{m}}^{\mathrm{qf}} H_n$ for every $L \in A\text{-}\Sigma_n$, let $\phi(i_1, \ldots, i_n)$ a quantifier-free formula of σ_A where $i_j = (i_{j1}, \ldots, i_{jm})$ for $j = 1, \ldots, n$. We define

$$f_\phi(x, i_1, \ldots, i_n) =_{\text{def}} \begin{cases} 1 \text{ if } \mathbf{x} \models \phi(i_1, \ldots, i_n) \\ 0 \text{ otherwise} \end{cases},$$

$$f_\phi(x, i_1, \ldots, i_{n-1}) =_{\text{def}} f_\phi(x, i_1, \ldots, i_{n-1}, 1) f_\phi(x, i_1, \ldots, i_{n-1}, 2) \ldots f_\phi(x, i_1, \ldots, i_{n-1}, |x|^m),$$

and, for $r = 0, 1, \ldots, n-2$,

$$f_\phi(x, i_1, \ldots, i_r) =_{\text{def}} (n-r) f_\phi(x, i_1, \ldots, i_r, 1)(n-r) \ldots (n-r) f_\phi(x, i_1, \ldots, i_r, |x|^m)(n-r).$$

We prove by induction on $r = 1, \ldots, n$

$$\mathbf{x} \models \exists i_{n-r+1} \forall i_{n-r+2} \ldots \exists i_n \phi(i_1, \ldots, i_n) \Leftrightarrow f_\phi(x, i_1, \ldots, i_{n-r}) \in H_r \text{ if } r \text{ is odd and} \tag{1}$$

$$\mathbf{x} \models \forall i_{n-r+1} \exists i_{n-r+2} \ldots \exists i_n \phi(i_1, \ldots, i_n) \Leftrightarrow f_\phi(x, i_1, \ldots, i_{n-r}) \in \overline{H_r} \text{ if } r \text{ is even.} \tag{2}$$

For $r = 1$ we have obviously $\mathbf{x} \models \exists i_n \phi(i_1, \ldots, i_n) \Leftrightarrow f_\phi(x, i_1, \ldots, i_{n-1}) \in H_1$.

For even $r > 1$ we conclude

$$\begin{aligned} \mathbf{x} \models \exists i_{n-r} \forall i_{n-r+1} \ldots \exists i_n \phi(i_1, \ldots, i_n) &\Leftrightarrow \exists i_{n-r}(\mathbf{x} \models \forall i_{n-r+1} \ldots \exists i_n \phi(i_1, \ldots, i_n)) \Leftrightarrow \\ &\Leftrightarrow \exists i_{n-r}(f_\phi(x, i_1, \ldots, i_{n-r}) \in \overline{H_r}) \Leftrightarrow f_\phi(x, i_1, \ldots, i_{n-r-1}) \in H_{r+1}. \end{aligned}$$

For odd $r > 1$ we get analogously

$$\mathbf{x} \models \forall i_{n-r} \exists i_{n-r+1} \ldots \exists i_n \phi(i_1, \ldots, i_n) \Leftrightarrow f_\phi(x, i_1, \ldots, i_{n-r-1}) \in \overline{H_{r+1}}.$$

This completes the induction. Furthermore, we conclude from (2) for even $r > 1$

$$\mathbf{x} \models \exists i_{n-r+1} \forall i_{n-r+2} \ldots \forall i_n \phi(i_1, \ldots, i_n) \Leftrightarrow f_{\neg\phi}(x, i_1, \ldots, i_{n-r}) \in H_r. \tag{3}$$

Now, for $L \in A\text{-}\Sigma_n$ there exists a quantifier-free formula $\phi(i_1, \ldots, i_n)$ of σ_A such that

$$x \in L \Leftrightarrow \mathbf{x} \models \exists i_1 \forall i_2 \ldots Q i_n \phi(i_1, \ldots, i_n).$$

For odd n we set $r = n$ in (1), and we obtain $x \in L \Leftrightarrow f_\phi(x) \in H_r$. For even n we set $r = n$ in (3), and we obtain $x \in L \Leftrightarrow f_{\neg\phi}(x) \in H_r$. Finally, it is not hard to see that that f_ϕ and $f_{\neg\phi}$ are qf-functions. □

Theorem 3.2 *For any $n > 0$ and any alphabet B, the class $B\text{-}\Sigma_n$ has a qf-complete set.*

Proof. It clearly suffices to consider only the binary alphabet $B = \{0, 1\}$ to find a Σ_n-set $C \subseteq B^+$ such that $H_n \leq_{\mathrm{m}}^{\mathrm{qf}} C$.

The idea of proof is evident: to code symbols of the bigger alphabet $A = A_n$ by sequences of symbols of the smaller alphabet B using the presence of the successor function in the signature

σ_B. Define a function $f:\omega\to\{0,1\}^+$ by $f(k)=01\ldots10$, where the sequence of 1's is of length $k+1$. With the alphabet A we associate a semigroup morphism $f=f_A:A^+\to\{0,1\}^+$ induced by the restriction of f to A. E.g., for $A=\{0,1,2\}$ and $w=0212$ we get $f(w)=01001110011001110$. In general, if $w=a_0\cdots a_k$ for $a_j\in A$ then $f(w)$ is the superposition $f(a_0)\cdots f(a_k)$.

It is easy (see [Se02]) to relate to any sentence ϕ of σ_A a sentence ϕ' of σ_B such that the following conditions hold:

(i) for any $w\in A^+$, $\mathbf{w}\models\phi$ iff $\mathbf{f}(\mathbf{w})\models\phi'$;

(ii) if ϕ is Σ_n then so is also ϕ'.

Now let ϕ be a Σ_n-sentence of σ_A with $A_n=L_\phi$, and let $C=L_{\phi'}$. Then $C\in B\text{-}\Sigma_n$ and $H_n=f^{-1}(C)$. But f is an qf-function, hence $H_n\leq_{\mathrm{m}}^{\mathrm{qf}} C$. □

Next we show that many levels of the typed boolean hierarchy also have complete sets.

Theorem 3.3 *For any alphabet A, all levels of the typed boolean hierarchy over $\{A\text{-}\Sigma_n\}_{n\geq2}$ have qf-complete sets.*

Proof. The map $\phi\mapsto\phi'$ from the preceding proof respects actually not only levels of the DDH but also levels of the typed boolean hierarchy (see again [Se02]), hence the coding technique of the proof above applies also in this case and we do not have to worry about the size of the alphabets. Hence, it remains to find for any typed boolean term $t(x_0,\ldots,x_k)$ an alphabet A and a set C complete in $A\text{-}t(\{\Sigma_{n+2}\})$. For any $i\leq k$, choose a set C_i which is complete in $B\text{-}\Sigma_{n+2}$ whenever x_i is of type n (B is again the binary alphabet). Let A be the alphabet $\{0,1,2\}$. For any $i\leq k$, let

$$C_i'=\{u_02u_12\cdots2u_k|u_0,\ldots,u_k\in B^+,u_i\in C_i\}$$

and $C=t(C_0',\ldots,C_k')$. One easily checks that $C_i'\in\Sigma_{n+2}$ whenever x_i is of type n, hence $C\in A\text{-}t(\{\Sigma_{n+2}\})$.

It remains to show that any $L\in A\text{-}t(\{\Sigma_{n+2}\}$ is reducible to C. We have $L=t(L_0,\ldots,L_k)$ for some sets L_i such that $L_i\in A\text{-}\Sigma_{n+2}$ whenever x_i is of type n. Hence, there are qf-functions $f_i:A^+\mapsto B^+$ with $L_i=f_i^{-1}(C_i),i\leq k$. Define a function $f:A^+\mapsto A^+$ by $f(u)=f_0(u)2\cdots2f_k(u)$. It is easy to see that f is an qf-function reducing L to C. □

The last theorem leaves open the question of existence of complete sets for some natural classes, say for some levels of the plus-hierarchy [Se94, Se02] over the DDH (in [Wa98] the hierarchy is called the symmetric-difference hierarchy). We guess that using a cleverer coding one may prove the existence of complete sets in any level of the typed boolean hierarchy over the DDH. As an important example, we show the existence of complete sets in levels of the difference hierarchy over Σ_1 (the last theorem implies of course that all levels of the difference hierarchy over any $\Sigma_n,n>1$, have complete sets).

Theorem 3.4 *For any alphabet A and any $k<\omega$, the class $A\text{-}\mathcal{D}_k$ has a qf-complete set.*

Proof. Again, by using the trick with the map $\phi\mapsto\phi'$, we may simplify the assertion to the following one: for any k there is an alphabeth A_k such that the class $A_k\text{-}\mathcal{D}_k$ has a complete set. The case $k=0$ is trivial since $\mathcal{D}_0=\{\emptyset\}$. For $k>0$, it is convinient to take the alphabet $A_k=\{0,\ldots,k\}$. Let

$$D_k=\{u\in A_k^+|\max(u)\text{ is odd}\},$$

where $\max(u)$ denotes the biggest number from A_k occuring as a letter in u. We claim that D_k is qf-complete in A_k-$\mathcal{D}_k$. It is clear that $D_k \in A_k$-$\mathcal{D}_k$, hence it remains to show that $L \leq_{\mathrm{m}}^{\mathrm{qf}} D_k$ for any $L \in \mathcal{D}_k$.

We have $\bigcup_i (L_{2i} \setminus L_{2i+1})$, for some B-Σ_n-languages (B is some alphabet) $L_0 \supseteq L_1 \supseteq \cdots$ with $L_k = \emptyset$. Note that $u \in L$ iff $u \in L_0$ and the maximum number j satisfying $w \in L_j$ is even. Let ϕ_i be Σ_1-sentences satisfying $L_i = L_{\phi_i}$. Let ψ_i be quantifier-free formulas satisfying $\phi_i = \exists \bar{x} \psi_i(\bar{x})$; w.l.o.g. we may assume that the list $\bar{x} = (x_1, \ldots, x_n)$ of variables is the same for all $i \leq k$.

For all $u = u_0 \cdots u_l \in B^+$, $i \leq k$ and $\bar{x} \in \{0, \ldots, l\}^n$, let

$$\psi_i^u(\bar{x}) =_{\mathrm{def}} \begin{cases} i+1 \text{ if } \mathbf{u} \models \psi_i(\bar{x}) \\ 0 \text{ otherwise} \end{cases} .$$

Define a function $f : B^+ \to A_k^+$ by

$$f(u) = \prod_{x_1=0}^{l} \cdots \prod_{x_n=0}^{l} \psi_0^u(\bar{x}) \cdots \psi_k^u(\bar{x})$$

where the product means concatenation. Then $u \in L_i$ iff the letter $i+1$ occurs in $f(u)$, hence $L = f^{-1}(D_k)$. But f is an qf-function, hence $L \leq_{\mathrm{m}}^{\mathrm{qf}} D_k$. □

A natural question is whether complete sets constructed above are reducible to their compliments. The analogy with computability theory [Se95] prompts the negative answer, the analogy with complexity theory envokes some doubts... We can prove the corresponding result at least for the levels of the difference hierarchies. And actually this answer follows immediately from results of Section 2, results from [BK78, Th82, Shu98, SS00] on the non-collapse of the difference hierarchies and from the existence of complete sets established above.

Corollary 3.5 *If C is qf-complete in some level of the difference hierarchy over $\Sigma_n, n > 0$, then $C \not\leq_{\mathrm{m}}^{\mathrm{qf}} \overline{C}$.*

We guess that the last result generalizes to all levels of the typed boolean hierarchy over the DDH.

4 A Property of $\mathcal{D}_k$-Complete Sets

Let D_k be the qf-complete language in the level $\mathcal{D}_k$ of the difference hierarchy over Σ_1 defined in the proof of Theorem 3.4, and let $\mathbf{d}_k, \bar{\mathbf{d}}_k$ be the qf-degrees of D_k and $\overline{D}_k$, respectively. From results of the last section it follows that $\mathbf{d}_k$ and $\bar{\mathbf{d}}_k$ are incomparable and both of them are below $\mathbf{d}_{k+1}$ and $\bar{\mathbf{d}}_{k+1}$.

In this section we establish the following relationship between these degrees which is similar to the corresponding property [Se83] of the difference hierarchy over recursively enumerable sets.

Theorem 4.1 *For any $k < \omega$, $\mathbf{d}_k \cup \bar{\mathbf{d}}_k = \mathbf{d}_{k+1} \cap \bar{\mathbf{d}}_{k+1}$.*

For proving this we need the following result from [SS00]. Since the result was not published and its proof is short, we reproduce it here.

Lemma 4.2 *For any $n > 0$, any disjoint Σ_1-languages L, K are separable by a Δ_1-language.*

Proof. As is well known, any Σ_1-language is a finite union of languages of the form

$$u_1A^*u_2A^*\cdots A^*u_n \text{ where } n>0 \text{ and } u_i\in A^+. \tag{1}$$

Note that if L is a finite union of Σ_1-languages L_i ($i\in I$), K is a finite union of Σ_1-languages K_j ($j\in J$) and for all i,j there is a Δ_1-language $M_{i,j}$ separating L_i from K_j then the Δ_1-language $\bigcup_i\bigcap_j M_{i,j}$ separates L from K (because $L_i\subseteq M_{i,j}\subseteq\overline{K}_j$ imply $L_i\subseteq\bigcap_j M_{i,j}\subseteq\bigcap_j\overline{K}_j$ for all i, hence $L=\bigcup L_i\subseteq\bigcup_i\bigcap_j M_{i,j}\subseteq\bigcap_j\overline{K}_j=\overline{K}$).

This shows that both L,K may be assumed to be of the form (1), so let $L=u_1A^*\cdots A^*u_n$ and $K=v_1A^*\cdots A^*v_m$. If $n=1$ then $L=\{u_1\}\in\Delta_1$ separates L from K. The same applies of course to m, hence we may assume that $m,n>1$.

We claim that the condition

$$(u_1\in v_1A^*\vee v_1\in u_1A^*)\wedge(u_n\in A^*v_m\vee v_m\in A^*u_n) \tag{2}$$

is false. Suppose the contrary and consider the four cases $(u_1\in v_1A^*\wedge u_n\in A^*v_m)$, $(u_1\in v_1A^*\wedge v_m\in A^*u_n)$ and so on. In the first case we have $u_1=v_1x$ and $u_n=yv_m$ for some $x,y\in A^*$. Then $u_1u_n=v_1xyv_m\in L\cap K$, which is a contradiction. The same contradiction is also obtained in the remaining three cases.

Therefore, the negation of (2) holds, i.e.

$$(u_1\notin v_1A^*\wedge v_1\notin u_1A^*)\vee(u_n\notin A^*v_m\wedge v_m\notin A^*u_n).$$

Then the Δ_1-language $u_1A^*u_n$ separates L from K. □

Proof of Theorem 4.1. We need only to show that if M is reducible to both D_{k+1} and $\overline{D}_{k+1}$ (i.e., $M,\overline{M}\in\mathcal{D}_{k+1}$) then M is reducible to $D_k\oplus\overline{D}_k$ (i.e., by Lemma 2.4.4, that M is representable in the form $M=(U\cap L)\cup(\overline{U}\cap K)$, for some $U\in\Delta_1$, $L\in\mathcal{D}_k$ and $K\in\mathrm{co}(\mathcal{D}_k)$).

Simplifying notation, consider only the typical particular case $k=2$. We have

$$M=\overline{L}_0\cup(L_1\setminus L_2) \text{ and } \overline{M}=\overline{K}_0\cup(K_1\setminus K_2), \tag{3}$$

for some Σ_1-languages $L_0\supseteq L_1\supseteq L_2$, $K_0\supseteq K_1\supseteq K_2$. Taking complements from (3), we get

$$\overline{M}=(L_0\setminus L_1)\cup L_2 \text{ and } M=(K_0\setminus K_1)\cup K_2, \tag{4}$$

hence L_2 and K_2 are disjoint. By Lemma 4.2, $L_2\subseteq U\subseteq\overline{K}_2$ for some $U\in\Delta_1$. Intersecting the first equality in (3) with $\overline{U}$ we get

$$M\cap\overline{U}=(\overline{L}_0\cap\overline{U})\cup(L_1\cap\overline{U})=(\overline{L}_0\cup L_1)\cap\overline{U}.$$

Intersecting the second equality in (4) with U we get $M\cap U=(K_0\setminus K_1)\cap U$, hence $M=(U\cap L)\cup(\overline{U}\cap K)$, where $L=(K_0\setminus K_1)\in\mathcal{D}_2$ and $K=(\overline{L}_0\cup L_1)\in\mathrm{co}(\mathcal{D}_2)$. □

Theorem 4.1 implies that the difference hierarchy over Σ_1 is discrete, i.e., it has no refinement in any level (under a natural exact definition of the refinement [Se83]). But (as is also the case in computability theory) this does not exclude the existence of rich degree structures inside intervals like $(\bar{\mathbf{d}}_k,\mathbf{d}_{k+1})$. In Section 6 below we will show that such intervals are really rather complicated.

5 Leaf Languages and Polylogtime Reducibility

In this section we will see how the leaf language approach to complexity classes (well-known from literature) will help us to get new insights into the structure of qf-reducibility.

Let us recall some relevant definitions. Consider a polynomial-time nondeterministic Turing machine M working on an input word x over some alphabet B and printing a letter from another alphabet A after finishing any computation path. These values are the leaves of the binary tree defined by the nondeterministic choices of M on input x. An ordering of the tuples in the program of M determines a left-to-right ordering of all the leaves. In this way, M may be considered as a deterministic transducer computing a total function $M : B^* \to A^+$. Now, relate to any language $L \subseteq A^+$ (called in this situation a leaf language) the language $M^{-1}(L) \subseteq B^*$. Denote by $\mathrm{Leaf}_{\mathrm{b}}(L)$ the set of languages $M^{-1}(L)$, for all machines M specified above which have balanced computation trees, and denote by $\mathrm{Leaf}_{\mathrm{u}}(L)$ the set of languages $M^{-1}(L)$, for all machines M specified above (which may have unbalanced computation trees). For a set of languages $\mathcal{C}$, let $\mathrm{Leaf}_{\mathrm{b}}(\mathcal{C})$ be the union of $\mathrm{Leaf}_{\mathrm{b}}(L)$, for all $L \in \mathcal{C}$, analogously define $\mathrm{Leaf}_{\mathrm{u}}(\mathcal{C})$.

Obviously, we have $\mathrm{Leaf}_{\mathrm{b}}(L) \subseteq \mathrm{Leaf}_{\mathrm{u}}(L)$ for every language L, and there exist languages L where $\mathrm{Leaf}_{\mathrm{b}}(L) = \mathrm{Leaf}_{\mathrm{u}}(L)$ is unlikely. (For example, $\mathrm{Leaf}_{\mathrm{b}}((01)^*) = \mathrm{P}$ and $\mathrm{Leaf}_{\mathrm{u}}((01)^*) = \oplus\mathrm{P}$.) However, $\mathrm{Leaf}_{\mathrm{b}}(L) = \mathrm{Leaf}_{\mathrm{u}}(L)$ is always true for *paddable languages* (we call a language $L \subseteq A^*$ paddable iff there exists an $a \in A$ such that $xay \in L \leftrightarrow xy \in L$ for all $x, y \in A^*$.

It turns out that many inportant complexity classes have natural and useful descriptions in terms of leaf languages (see e.g. [BCS92, Ve93, HLSVW93, HVW95, HVW96, JMT96]). In particular, a close relationship between some classes of regular leaf languages and complexity classes within PSPACE was established in [HLSVW93]. In [HLSVW93] and [BV98], a close relationship between the DDH and the polynomial hierarchy $\{\Sigma_n^p\}$ was established: $\mathrm{Leaf}_{\mathrm{b}}(\mathrm{BC}(\Sigma_n)) = \mathrm{Leaf}_{\mathrm{u}}(\mathrm{BC}(\Sigma_n)) = \mathrm{BC}(\Sigma_n^p)$ (where $\mathrm{BC}(\mathcal{C})$ is the boolean closure of the class $\mathcal{C}$) and $\mathrm{Leaf}_{\mathrm{b}}(\Sigma_n) = \mathrm{Leaf}_{\mathrm{u}}(\Sigma_n) = \Sigma_n^p$, for any $n > 0$. In [SW98, Se02] this was generalized to all levels of the typed boolean hierarchies over the DDH and over the polynomial hierarchy. Let $\mathcal{C}$ be a level of the typed boolean hierarchy over the DDH and $\mathcal{C}^l$ be the corresponding level of the typed boolean hierarchy over the polynomial hierarchy. Then $\mathcal{C}^l = \mathrm{Leaf}_{\mathrm{b}}(\mathcal{C}) = \mathrm{Leaf}_{\mathrm{u}}(\mathcal{C})$. It would be interesting to know which classes $\mathcal{C}$ have the property that if a regular language L is not in $\mathcal{C}$ then $\mathrm{Leaf}_{\mathrm{u}}(\mathcal{C}) \not\subseteq \mathcal{C}^l$ (unless the polynomial hierarchy collapses)? The paper [BKS99] shows that Σ_1 is such a class. It seems that notions and results of our paper maybe of use in finding new such natural classes $\mathcal{C}$.

We now consider a special class of leaf languages namely the languages of finite counting type (cf. [GW87, GNW90, CHVW98]). For a set $V \subseteq \mathbb{N}^k$ we define

$$L(V) =_{\mathrm{def}} \{x \in \{0, 1, \ldots, k\}^+ : (\#_1(x), \ldots, \#_k(x)) \in V\}$$

where $\#_i(x)$ is the number of occurences of i in x. The set V and the language $L(V)$ are said to be *of finite counting type* iff there exists an $m \geq 0$ such that

$$(n_1, \ldots, n_k) \in V \Leftrightarrow (\min(n_1, m), \ldots, \min(n_k, m)) \in V.$$

It is obvious that the languages of finite counting type are aperiodic and hence starfree. It is easy to see that in fact any language of finite counting type is in $BC(\Sigma_1)$.

In [CHVW98] a very useful characterization of relativizable inclusiond between complexity classes defined by leaf languages of finite counting type is proved. For $z = (z_1, \ldots, z_k) \in \mathbb{N}$ and $v = (v_1, \ldots, v_k) \in \mathbb{N}^k$ let $z \leq v$ iff $z_i \leq v_i$ for $i = 1, \ldots, k$, and let $\binom{v}{z} =_{\mathrm{def}} \prod_{i=1}^{k} \binom{v_i}{v_k}$.

Theorem 5.1 *[CHVW98] If $V \subseteq \mathbb{N}^k$ and $W \subseteq \mathbb{N}^l$ are of finite counting type then the following statements are equivalent:*

(1) $\mathrm{Leaf}_{\mathrm{b}}(L(V))^{\mathcal{O}} \subseteq \mathrm{Leaf}_{\mathrm{b}}(L(W))^{\mathcal{O}}$ *for every oracle* $\mathcal{O}$.

(2) *there exist functions* $\alpha_1, \ldots, \alpha_l : \mathbb{N}^k \to \mathbb{N}$ *such that*

$$v \in V \Leftrightarrow \left(\sum_{z \leq v} \alpha_1(z)\binom{v}{z}, \ldots, \sum_{z \leq v} \alpha_l(z)\binom{v}{z} \right) \in W.$$

At this point another notion of reducibility comes into the play. A language $L \subseteq A^*$ is *polylogtime reducible* to $K \subseteq B^*$, for short $L \leq_{\mathrm{m}}^{\mathrm{plt}} K$, iff there exist functions $f : A^* \times \mathbb{N} \to B$ and $g : A^* \to \mathbb{N}$, computable in polylogarithmic time on a deterministic Turing machine with random access to the input bits, such that $x \in L \leftrightarrow f(x,1)f(x,2)\ldots f(x,g(x)) \in K$ for every $x \in A^*$. In the early 1990s, Bovet, Crescenzi, Silvestri and Vereshchagin proved the following ingenious result.

Theorem 5.2 *[BCS92, Ve93] The following are equivalent for languages L and K:*
(1) $L \leq_{\mathrm{m}}^{\mathrm{plt}} K$
(2) $\mathrm{Leaf_b}(L)^{\mathcal{O}} \subseteq \mathrm{Leaf_b}(K)^{\mathcal{O}}$ *for every oracle* $\mathcal{O}$.

Generally, $L \leq_{\mathrm{m}}^{\mathrm{qf}} K$ does not seem to imply $L \leq_{\mathrm{m}}^{\mathrm{plt}} K$, but for special cases this implication is valid.

Theorem 5.3 *Let $L \subseteq A^+$ and $V \subseteq \mathbb{N}^k$. Then $L \leq_{\mathrm{m}}^{\mathrm{qf}} L(V)$ implies $L \leq_{\mathrm{m}}^{\mathrm{plt}} L(V)$.*

Proof. Let $L \leq_{\mathrm{m}}^{\mathrm{qf}} L(V)$ via the interpretation

$$I = (\phi_U(\bar{x}), \phi_<(\bar{x},\bar{y}), \phi_b(\bar{x}), \ldots, \phi_\perp(\bar{x}), \phi_\top(\bar{x}), \phi_S(\bar{x},\bar{y})),\ \bar{x} = (x_1, \ldots, x_n).$$

For a set $K \in \mathrm{Leaf_b}(L)$ there exist polynomial time computable functions g, h such that

$$x \in K \leftrightarrow f(x) =_{\mathrm{def}} g(x,1)g(x,2)\ldots g(x,h(x)) \in L.$$

For $x \in L$ and $i_1, \ldots, i_n \in \{0, 1, \ldots, h(x)\}$ define

$$g'(x, i_1, \ldots, i_n) =_{\mathrm{def}} \begin{cases} b \text{ if } f(x) \models \phi_b(i_1, \ldots, i_n) \text{ and } f(x) \models \phi_U(i_1, \ldots, i_n) \\ 0 \text{ otherwise,} \end{cases}$$

and

$$f'(x) =_{\mathrm{def}} g'(x, 0, \ldots, 0, 0)g'(x, 0, \ldots, 0, 1)\ldots g'(x, h(x), \ldots, h(x), h(x)).$$

In general we have $f(x)^I \neq f'(x)$, but there holds $\#_i(f(x)^I) = \#_i(f'(x))$ for $i = 1, \ldots, k$. Consequently,

$$\begin{aligned} x \in K &\Leftrightarrow f(x) \in L \Leftrightarrow f(x)^I \in L(V) \Leftrightarrow (\#_1(f(x)^I), \ldots, \#_k(f(x)^I) \in V \\ &\Leftrightarrow (\#_1(f'(x)), \ldots, \#_k(f'(x)) \in V \Leftrightarrow f'(x) \in L(V). \end{aligned}$$

This means $K \in \mathrm{Leaf_b}(L(V)))$. Since this proof of $\mathrm{Leaf_b}(L)) \subseteq \mathrm{Leaf_b}(L(V)))$ is relativizable, we obtain $L \leq_{\mathrm{m}}^{\mathrm{plt}} L(V)$ by Theorem 5.2. □

The main theorem of this section gives two characterizations of the qf-reducibility between languages of finite counting type.

Theorem 5.4 *For languages $V \subseteq \mathbb{N}^k$ and $W \subseteq \mathbb{N}^l$ of finite counting type the following are equivalent:*
(1) $L(V) \leq_{\mathrm{m}}^{\mathrm{qf}} L(W)$
(2) $L(V) \leq_{\mathrm{m}}^{\mathrm{plt}} L(W)$
(3) *there exist an $r \in \mathbb{N}^k$ and functions $\alpha_1, \ldots, \alpha_l : \mathbb{N}^k \to \mathbb{N}$ such that*

$$v \in V \Leftrightarrow \left(\sum_{z \leq r} \alpha_1(z)\binom{v}{z}, \ldots, \sum_{z \leq r} \alpha_l(z)\binom{v}{z} \right) \in W.$$

Proof. (1) $\Longrightarrow$ (2) follows from Theorem 5.3.

(2) $\Longrightarrow$ (3). If $L(V) \leq_{\mathrm{m}}^{\mathrm{plt}} L(W)$ then, by Theorem 5.1 and Theorem 5.2, there exist functions $\alpha_1, \ldots, \alpha_l : \mathbb{N}^k \to \mathbb{N}$ such that

$$v \in V \Leftrightarrow \left(\sum_{z \leq v} \alpha_1(z)\binom{v}{z}, \ldots, \sum_{z \leq v} \alpha_l(z)\binom{v}{z} \right) \in W.$$

Since W is of finite counting type there exists an $m \in \mathbb{N}$ such that

$$(n_1, \ldots, n_k) \in W \Leftrightarrow (\min(n_1, m), \ldots, \min(n_k, m)) \in W.$$

Defining for $i = 1, \ldots, l$

$$\beta_i(v) =_{\mathrm{def}} \begin{cases} \alpha_i(v) \text{ if } \sum_{z \leq v} \alpha_i(z) < m \\ 0 \text{ otherwise} \end{cases}$$

we obtain $\sum_{z \leq v} \alpha_i(z)\binom{v}{z} = \sum_{z \leq v} \beta_i(z)\binom{v}{z}$ or $m \leq \sum_{z \leq v} \alpha_i(z)\binom{v}{z} < \sum_{z \leq v} \beta_i(z)\binom{v}{z}$. Consequently,

$$v \in V \Leftrightarrow \left(\sum_{z \leq v} \beta_1(z)\binom{v}{z}, \ldots, \sum_{z \leq v} \beta_l(z)\binom{v}{z} \right) \in W.$$

Since there are only finitely many v such that $\beta_i(v) > 0$ there exists an $r \in \mathbb{N}^k$ such that $v \leq r$ for all v such that $\beta_i(v) > 0$. Hence

$$v \in V \Leftrightarrow \left(\sum_{z \leq r} \beta_1(z)\binom{v}{z}, \ldots, \sum_{z \leq r} \beta_l(z)\binom{v}{z} \right) \in W.$$

(3) $\Longrightarrow$ (1). It is not hard to see that there exists a qf-function f such that $\#_i(f(x)) = \sum_{z \leq r} \alpha_i(z)\binom{\#(x)}{z}$ for $i = 1, \ldots, k$ where $\#(x) = (\#_1(x), \ldots, \#_k(x))$. Consequently,

$$\begin{array}{rcl} x \in L(V) & \Leftrightarrow & \#(x) \in V \Leftrightarrow \left(\sum_{z \leq r} \alpha_1(z)\binom{\#(x)}{z}, \ldots, \sum_{z \leq r} \alpha_l(z)\binom{\#(x)}{z} \right) \\ & \Leftrightarrow & (\#_1(f(x)), \ldots, \#_k(f(x))) \in W \Leftrightarrow f(x) \in L(W) \end{array}.$$

This means $L(V) \leq_{\mathrm{m}}^{\mathrm{qf}} L(W)$. □

Examples of applications of the preceding theorem are given in the next section.

6 Degrees between $\overline{\mathbf{d}}_1$ and $\mathbf{d}_2$

In this section we will show that the degree structure between $\overline{\mathbf{d}}_1$ and $\mathbf{d}_2$ is sufficiently rich (we use the notation $D_k, \mathbf{d}_k$ from Section 4). For this reason we define for every $m \geq k \geq 0$ the language

$$U_{k,m} =_{\mathrm{def}} \{x \in \{0,1\}^+ : \#_1(x) \in \{k, k+1, \ldots, m\}\}.$$

Furthermore, denote by $\mathbf{u}_{k,m}$ the qf-degree of $U_{k,m}$, for $k \geq m \geq 0$, and set $\mathbf{u} = \mathbf{u}_{1,1}$. The next theorem gives some information on the structure of the set $\{\mathbf{u}_{k,m} : m \geq k \geq 0\}$ of qf-degrees (see also Figure 2). Note that some of the results in this theorem can also be derived from results in [GW87] and [GNW90] via Theorem 5.2 and the equivalence of 1 and 2 in Theorem 5.4. However, our proof via the equivalence of 1 and 3 in Theorem 5.4 is easier.

Theorem 6.1 *1.* $\overline{\mathbf{d}}_1 \leq \mathbf{u}_{k,m} \leq \mathbf{d}_2$ *for all* $m \geq k \geq 0$.

2. $\mathbf{u}_{0,m} = \overline{\mathbf{d}}_1$ *for all* $m \geq 0$.

3. $\mathbf{u}_{m,m} = \mathbf{u}$ *for all* $m \geq 1$.

4. $\overline{\mathbf{d}}_1 < \mathbf{u}$

5. $\mathbf{u}_{k,m} < \mathbf{u}_{k+1,m+1}$ *for all* $m > k \geq 1$.

6. $\mathbf{u}_{k,m} < \mathbf{u}_{k,m+1}$ *for all* $m \geq k \geq 1$.

7. $\mathbf{u}_{m-1,m} < \mathbf{u}_{1,m}$ *for all* $m \geq 3$.

8. $\mathbf{d}_1 \not\leq \mathbf{u}_{k,m}$ *for all* $m \geq k \geq 0$.

9. $\mathbf{u}_{k,m} \not\leq \overline{\mathbf{d}}_2$ *for all* $m \geq k \geq 1$.

10. $\mathbf{u}_{k,m} \not\leq \mathbf{u}_{l,m}$ *for all* $k, l \in \{1, 2, \ldots, m-2\}$ *such that* $k \neq l$ *and* $m \geq 4$.

Proof. Defining $\mathbb{N}^+ =_{\mathrm{def}} \mathbb{N} \setminus \{0\}$ we observe $D_1 = L(\mathbb{N}^+)$, $\overline{D_1} = L(\{0\})$, $D_2 = L(\mathbb{N}^+ \times \{0\})$, $\overline{D_2} = L(\{(0,0)\} \cup (\mathbb{N} \times \mathbb{N}^+))$, and $U_{k,m} = L(\{k, \ldots, m\})$ for $m \geq k \geq 0$. Obviously, these languages are of finite counting type. In all our proofs we will use Theorem 5.4, i.e., we will use the fact that, for languages $V \subseteq \mathbb{N}$ and $W \subseteq \mathbb{N}^l$ of finite counting type, $L(V) \leq_{\mathrm{m}}^{\mathrm{qf}} L(W)$ if and only if there exist an $r \in \mathbb{N}$ and functions $\alpha_1, \ldots, \alpha_l : \mathbb{N} \to \mathbb{N}$ such that

$$n \in V \Leftrightarrow \left(\textstyle\sum_{i \leq r} \alpha_1(i) \binom{n}{i}, \ldots, \sum_{i \leq r} \alpha_l(i) \binom{n}{i} \right) \in W.$$

1. It suffices to check $n \in \{0\} \Leftrightarrow n + m \in \{k, \ldots, m\}$ and $n \in \{k, \ldots, m\} \Leftrightarrow \left(\binom{n}{k}, \binom{n}{m+1}\right) \in \mathbb{N}^+ \times \{0\}$ for all $n \geq 0$.

2. It suffices to check $n \in \{0, \ldots, m\} \Leftrightarrow \binom{n}{m+1} \in \{0\}$ for all $n \geq 0$.

3. It suffices to check $n \in \{1\} \Leftrightarrow n \cdot m \in \{m\}$ and $n \in \{m\} \Leftrightarrow \binom{n}{m} \in \{1\}$ for all $n \geq 0$.

4. The relation $\overline{\mathbf{d}}_1 \leq \mathbf{u}$ is clear by Statement 1. Assume that $\mathbf{u} \leq \overline{\mathbf{d}}_1$, i.e., there exists an $\alpha : \mathbb{N} \to \mathbb{N}$ such that

$$n \in \{1\} \Leftrightarrow f(n) =_{\mathrm{def}} \sum_{i \leq n} \alpha(i) \binom{n}{i} \in \{0\}.$$

Since $f(1) = \alpha(0) + \alpha(1) = 0$ we obtain $f(0) = \alpha(0) = 0$, a contadiction.

5. The relation $\mathbf{u}_{k,m} \leq \mathbf{u}_{k+1,m+1}$ follows from $n \in \{k, \ldots, m\} \Leftrightarrow n+1 \in \{k+1, \ldots, m+1\}$. On the other hand, assume that $\mathbf{u}_{k+1,m+1} \leq \mathbf{u}_{k,m}$, i.e., there exists an $\alpha : \mathbb{N} \to \mathbb{N}$ such that

$$n \in \{k+1, \ldots, m+1\} \Leftrightarrow f(n) =_{\mathrm{def}} \sum_{i \leq n} \alpha(i) \binom{n}{i} \in \{k, \ldots, m\}.$$

Since $f(m+1) = \sum_{i \leq m+1} \alpha(i) \binom{m+1}{i} \leq m$ we obtain $\alpha(1) = \alpha(2) = \cdots = \alpha(m) = 0$ and $\alpha(0) + \alpha(m+1) \leq m$. Consequently, $f(0) = \alpha(0) < k$ and $f(k+1) = \alpha(0) \geq k$, a contradiction.

6. The relation $\mathbf{u}_{k,m} \leq \mathbf{u}_{k,m+1}$ follows from

$$n \in \{k, \ldots, m\} \Leftrightarrow n + 2 \cdot \binom{n}{m+1} \in \{k, \ldots, m+1\}.$$

Now assume $\mathbf{u}_{k,m+1} \leq \mathbf{u}_{k,m}$, i.e., there exists an $\alpha : \mathbb{N} \to \mathbb{N}$ such that

$$n \in \{k, \ldots, m+1\} \Leftrightarrow f(n) =_{\mathrm{def}} \sum_{i \leq n} \alpha(i) \binom{n}{i} \in \{k, \ldots, m\}.$$

Since $f(m+1) = \sum_{i \leq m+1} \alpha(i) \binom{m+1}{i} \leq m$ we obtain $\alpha(1) = \alpha(2) = \cdots = \alpha(m) = 0$ and $\alpha(0) + \alpha(m+1) \leq m$. Consequently, $f(0) = \alpha(0) < k$ and $f(k) = \alpha(0) \geq k$, a contradiction.

7. The relation $\mathbf{u}_{m-1,m} \leq \mathbf{u}_{1,m}$ follows from

$$n \in \{m-1, m\} \Leftrightarrow \binom{n}{m-1} \in \{1, \ldots, m\}.$$

Now assume $\mathbf{u}_{1,m} \leq \mathbf{u}_{m-1,m}$, i.e., there exists an $\alpha : \mathbb{N} \to \mathbb{N}$ such that

$$n \in \{1, \ldots, m\} \Leftrightarrow f(n) =_{\text{def}} \sum_{i \leq n} \alpha(i) \binom{n}{i} \in \{m-1, m\}.$$

Since $f(m) = \sum_{i \leq m} \alpha(i) \binom{m}{i} \in \{m-1, m\}$ we obtain $\alpha(2) = \alpha(3) = \cdots = \alpha(m-2) = 0$ and $\alpha(0) + m \cdot \alpha(1) + m \cdot \alpha(m-1) + \alpha(m) \in \{m-1, m\}$. If $\alpha(1) = 0$ then $f(1) = \alpha(0) \geq m-1$ and $f(0) = \alpha(0) < m-1$, a contradiction. If $\alpha(1) = 1$ then $\alpha(0) = 0$ and $f(1) = m < m-1$, a contradiction.

8. Assume $\mathbf{d}_1 \leq \mathbf{u}_{k,m}$, i.e., there exists an $\alpha : \mathbb{N} \to \mathbb{N}$ such that

$$n \in \mathbb{N}^+ \Leftrightarrow f(n) =_{\text{def}} \sum_{i \leq n} \alpha(i) \binom{n}{i} \in \{k, \ldots, m\}.$$

For $r > m$ we obtain $f(s) = \sum_{i \leq s} \alpha(i) \binom{s}{i} \leq m$ and hence $\alpha(1) = \alpha(2) = \cdots = \alpha(s-1) = 0$. Consequently, $f(n) = \alpha(0)$, a contradiction.

9. Assume $\mathbf{u} \leq \overline{\mathbf{d}}_2$, i.e., there exist $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ such that

$$n = 1 \Leftrightarrow f(n) =_{\text{def}} \Big(\sum_{i \leq n} \alpha(i) \binom{n}{i}, \sum_{i \leq n} \beta(i) \binom{n}{i}\Big) \in \{(0,0)\} \cup (\mathbb{N} \times \mathbb{N}^+).$$

If $f(1) = (0,0)$ then $\alpha(0) = \beta(0) = 0$ and hence $f(0) = (0,0)$, a contradiction. If $f(1) \in \mathbb{N} \times \mathbb{N}^+$ then $\beta(0) + \beta(1) > 0$ and hence $f(2) \in \mathbb{N} \times \mathbb{N}^+$, a contradiction.

10. Assume $\mathbf{u}_{k,m} \leq \mathbf{u}_{l,m}$ for $k, l \in \{1, 2, \ldots, m-2\}$ such that $k \neq l$ and $m \geq 4$. Then there exists an $\alpha : \mathbb{N} \to \mathbb{N}$ such that

$$n \in \{k, \ldots, m\} \Leftrightarrow f(n) =_{\text{def}} \sum_{i \leq n} \alpha(i) \binom{n}{i} \in \{l, \ldots, m\}.$$

Since $f(m) = \sum_{i \leq m} \alpha(i) \binom{m}{i} \leq m$ we obtain $\alpha(2) = \alpha(3) = \cdots = \alpha(m-2) = 0$ and $\alpha(0) + m \cdot \alpha(1) + m \cdot \alpha(m-1) + \alpha(m) \leq m$. This implies $f(k-1) = \alpha(0) + (k-1) \cdot \alpha(1)$ and $f(k) = \alpha(0) + k \cdot \alpha(1)$. If $\alpha(1) = 0$ then $f(k-1) = \alpha(0) = f(k)$, a contradiction. If $\alpha(1) = 1$ then $\alpha(0) = \alpha(m-1) = \alpha(m) = 0$ and hence $f(k-1) = k-1 < l$ and $f(k) = k \geq l$. Consequently $k = l$, a contradiction. □

Corollary 6.2 *Between $\overline{\mathbf{d}}_1$ and $\mathbf{d}_2$ there exist upward infinite chains and finite antichains of arbitrary length.*

Proof. For $m \geq 2$, the chains $\mathbf{u}_{1,m} < \mathbf{u}_{2,m+1} < \mathbf{u}_{3,m+2} < \ldots$ and $\mathbf{u}_{m,m+2} < \mathbf{u}_{m,m+3} < \mathbf{u}_{m,m+4} < \ldots$ for $m \geq 1$ are infinite. For $m \geq 4$, the set $\{\mathbf{u}_{1,m}, \mathbf{u}_{2,m}, \ldots, \mathbf{u}_{m-2,m}\}$ is an antichain of size $m-2$. □

A part of the qf-degree structure between $\overline{\mathbf{d}}_1$ and $\mathbf{d}_2$ is shown in Figure 2.

7 The Principal Ideal Generated by $\mathbf{u} \cup \bar{\mathbf{u}}$

In this section we describe the principal ideal of $(\mathbf{D}_{\text{qf}}; \leq)$ generated by $\mathbf{u} \cup \bar{\mathbf{u}}$. We work here with the binary alphabet $B = \{0, 1\}$ and use notation from the last section.

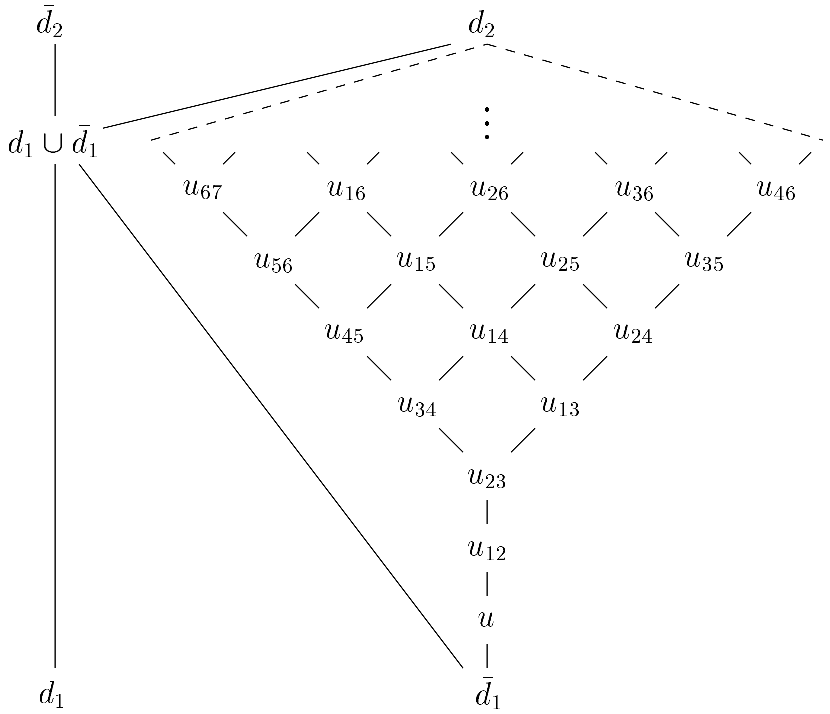

Figure 2. Part of the qf-degree structure between $\overline{\mathbf{d}}_1$ and $\mathbf{d}_2$.
All lines stand for $<$-relations, but not all known $<$-relations are shown, (e.g. $\mathbf{u}_{2,4} < \mathbf{u}_{1,6}$).

Theorem 7.1 *The principal ideal* $\{\mathbf{x}|\mathbf{x} \leq \mathbf{u} \cup \bar{\mathbf{u}}\}$ *is shown in Figure 3.*

Before proving the theorem we formulate two auxiliary facts related to patterns for (the graphs of) finite automata.

Following [BKS99], we say that a (deterministic) finite automaton *contains the* co-NP*-pattern* (see Figure 4) if there are two reachable states p, q and words $v, w \in B^+$, $z \in B^*$ such that $p.v = p$, $p.w = q.v = q.w = q$, $p.z$ is an accepting state and $q.z$ is not an accepting set. Similar conditions are assumed for the co-1-NP*-pattern* shown in Figure 5.

The next result is an immediate consequence of some results in [BKS99] which rely heavily on a result in [PW97].

Proposition 7.2 *Let L be a star-free language and let $\mathcal{A}$ be the minimal automaton recognizing L. Then $L \in \Sigma_1$ iff $\mathcal{A}$ contains neither* co-NP*-pattern nor* co-1-NP*-pattern.*

Next we relate the introduced patterns to our reducibility.

Proposition 7.3 *Let $\mathcal{A}$ be a finite automaton recognizing a language L. If $\mathcal{A}$ contains the* co-NP*-pattern (*co-1-NP*-pattern) then $\overline{D}_1 \leq_{\mathrm{m}}^{\mathrm{qf}} L$ ($\overline{U} \leq_{\mathrm{m}}^{\mathrm{qf}} L$, respectively).*

Proof. Let $\mathcal{A}$ have the co-NP-pattern shown in Figure 4. Let the state p be reachable from the initial state by the word $y \in B^*$. Let $h : B^+ \to B^+$ be the semigroup morphism such that $h(0) = v$ and $h(1) = w$. Then the qf-function $u \mapsto yh(u)z$ reduces $\overline{D}_1$ to L. The same function works for the other pattern. □

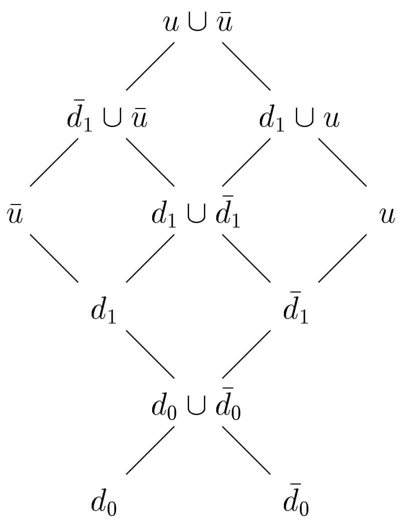

Figure 3. The principal ideal generated by $\mathbf{u} \cup \bar{\mathbf{u}}$.

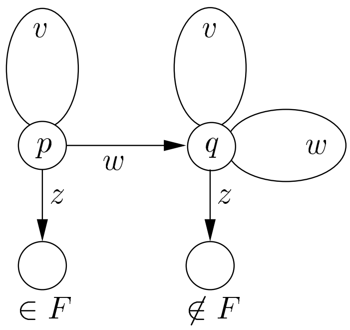

Figure 4. The co-NP-pattern.

Proof of Theorem 7.1. First we show that

$$\{\mathbf{x}|\mathbf{x} \leq \bar{\mathbf{u}}\} = \{\bar{\mathbf{u}}, \mathbf{d}_1, \mathbf{d}_0 \cup \bar{\mathbf{d}}_0, \mathbf{d}_0, \bar{\mathbf{d}}_0\}$$

(see Figure 3). It suffices to show that if $L <_{\mathrm{m}}^{\mathrm{qf}} \overline{U}$ then L is equivalent to one of $D_1, D_0 \oplus \overline{D}_0, D_0, \overline{D}_0$. Suppose that $L \not\leq_{\mathrm{m}}^{\mathrm{qf}} D_1$, i.e. $L \notin \Sigma_1$. By Propositions 7.2 and 7.3, $\overline{D}_1 \leq_{\mathrm{m}}^{\mathrm{qf}} L$ or $\overline{U} \leq_{\mathrm{m}}^{\mathrm{qf}} L$. This contradicts to $L <_{\mathrm{m}}^{\mathrm{qf}} U$ and Theorem 6.1.8.

We have shown that $L \leq_{\mathrm{m}}^{\mathrm{qf}} D_1$. It remains to show that if $L <_{\mathrm{m}}^{\mathrm{qf}} D_1$ then L is equivalent to one of $D_0 \oplus \overline{D}_0, D_0, \overline{D}_0$. Suppose that $L \not\leq D_0 \oplus \overline{D}_0$, then $L \notin \Delta_1$ and consequently $L \notin \Sigma_1$ or $L \notin \Pi_1$. Since $L <_{\mathrm{m}}^{\mathrm{qf}} D_1$, we get $L \notin \Pi_1$. By assertions dual to Propositions 7.2 and 7.3, we get $D_1 \leq_{\mathrm{m}}^{\mathrm{qf}} L$ or $U \leq_{\mathrm{m}}^{\mathrm{qf}} L$. This contradicts $L <_{\mathrm{m}}^{\mathrm{qf}} D_1$ and Theorem 6.1.9. Therefore $\mathbf{x} \leq \mathbf{d}_0 \cup \bar{\mathbf{d}}_0$ and, by Lemma 2.2.2, $\mathbf{x} \in \{\mathbf{d}_0 \cup \bar{\mathbf{d}}_0, \mathbf{d}_0, \bar{\mathbf{d}}_0\}$.

We have verified the assertion about the ideal $\{\mathbf{x}|\mathbf{x} \leq \bar{\mathbf{u}}\}$. Now let $\mathbf{x} \leq \mathbf{u} \cup \bar{\mathbf{u}}$. By distributivity (see Theorem 2.5), $\mathbf{x} = \mathbf{y} \cup \mathbf{z}$ for some $\mathbf{y} \leq \mathbf{u}$ and $\mathbf{z} \leq \bar{\mathbf{u}}$. But such degrees $\mathbf{y} \cup \mathbf{z}$ are exhausted by the 11 degrees shown on Figure 3. That all these 11 degrees are distinct follows from the above and from Theorem 6.1.8 and 6.1.9. □

Theorem 7.1 shows that (at least near the bottom) the structure of qf-degrees is simpler than those of m-degrees and of polynomial-time m-degrees (though it is certainly more complicated

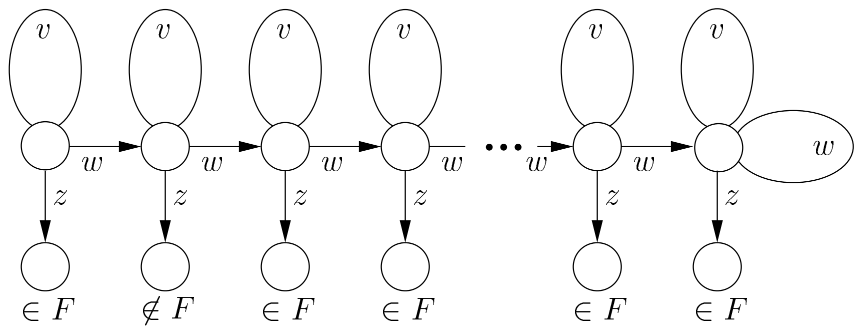

Figure 5. The co-1-NP-pattern.

that the structure of Wadge degrees). This support our belief that the qf-degrees are useful classifications of regular languages.

8 Future Work

There are many interesting open questions related to this paper. Some of them were already mentioned above. Here we mention some more, sometimes in the form of conjectures.

Conjecture 1. The relations $\leq_{\mathrm{m}}^{\mathrm{plt}}$ and $\leq_{\mathrm{m}}^{\mathrm{qf}}$ are decidable on the regular languages. This means that there exists an algorithm which computes from finite automata recognizing languages L, M whether or not $L \leq_{\mathrm{m}}^{\mathrm{qf}} M$.

This conjecture seems very hard because (together with Theorems 2.6 and 3.2) implies the decidability of all levels of the DDH which is a well-known open question of automata theory. One could weaken the Conjecture 1 in different ways to get less hard problems, e.g.

Conjecture 2. The relations $\leq_{\mathrm{m}}^{\mathrm{plt}}$ and $\leq_{\mathrm{m}}^{\mathrm{qf}}$ are decidable on $\mathrm{BC}(\Sigma_1)$.

The last conjecture seems rather plausible (though not easy to prove!) since in [GS01, Se01] the decidability of several natural problems related to $\mathrm{BC}(\Sigma_1)$ was established.

Many natural questions on the introduced degree structures also remain open, e.g.:

Question. Is there an infinite antichain or an infinite descending chain within $(\mathrm{BC}(\Sigma_1); \leq_{\mathrm{m}}^{\mathrm{qf}})$?

An interesting open problem related to Section 5 is the following: find a reducibility describing the relation $\forall \mathcal{O}(\mathrm{Leaf}_{\mathrm{u}}(L)^{\mathcal{O}} \subseteq \mathrm{Leaf}_{\mathrm{u}}(K)^{\mathcal{O}})$ in the same way as the polylogtime reducibility describes the corresponding relation for the balanced trees. We hope that our reducibility could be of use in solving this problem.

Several natural variants of the qf-reducibility are possible. E.g., one could consider the corresponding reducibility for the Straubing-Therien hierarchy by considering another signature in place of σ_A [Th82, Se01] (actually, in this context it is better to consider interpretations by Δ_1-formulas). This reducibility does not induce the upper semilattice and is probably not so closely relates to complexity theory as the one considered above.

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