On the Power of Number-Theoretic Operations with Respect to Counting

Ulrich Hertrampf[1]

Theoretische Informatik
Medizinische Universität Lübeck
D-23560 Lübeck, Germany
hertramp@informatik.mu-luebeck.de

Heribert Vollmer[2]

Department of Mathematics
University of California
Santa Barbara, CA 93106, USA
vollmer@math.ucsb.edu

Klaus W. Wagner[1]

Theoretische Informatik
Universität Würzburg
D-97072 Würzburg, Germany
wagner@informatik.uni-wuerzburg.de

Abstract

We investigate function classes $\langle\#\mathrm{P}\rangle_f$ *which are defined as the closure of* $\#\mathrm{P}$ *under the operation* f *and a set of known closure properties of* $\#\mathrm{P}$*, e.g. summation over an exponential range.*

First, we examine operations f *under which* $\#\mathrm{P}$ *is closed (i.e.,* $\langle\#\mathrm{P}\rangle_f = \#\mathrm{P}$*) in every relativization. We obtain the following complete characterization of these operations:* $\#\mathrm{P}$ *is closed under* f *in every relativization if and only if* f *is a finite sum of binomial coefficients over constants.*

Second, we characterize operations f *with respect to their power in the counting context in the unrelativized case. For closure properties* f *of* $\#\mathrm{P}$*, we have* $\langle\#\mathrm{P}\rangle_f = \#\mathrm{P}$*. The other end of the range is marked by operations* f *for which* $\langle\#\mathrm{P}\rangle_f$ *corresponds to the counting hierarchy. We call these operations* counting hard *and give general criteria for hardness. For many operations* f *we show that* $\langle\#\mathrm{P}\rangle_f$ *corresponds to some subclass* $\mathcal{C}$ *of the counting hierarchy. This will then imply that* $\#\mathrm{P}$ *is closed under* f *if and only if* $\mathrm{UP} = \mathcal{C}$*; and on the other hand,* f *is counting hard if and only if* $\mathcal{C}$ *contains the counting hierarchy.*

[1]Supported by Deutsche Forschungsgemeinschaft, Grant No. Wa 847/1-1, "k-wertige Schaltkreise".

[2]Supported by a Feodor Lynen Fellowship from the Alexander von Humboldt Foundation and by NSF Grant No. CCR-93-02057. Permanent address: Theoretische Informatik, Universität Würzburg, Am Exerzierplatz 3, D-97072 Würzburg.

1 Introduction

In 1979, Valiant [36] introduced the class $\#\mathrm{P}$ of all functions counting the number of solutions of NP problems, or equivalently, of all accepting paths of nondeterministic polynomial-time Turing machines. This class has attracted the attention of theoreticians, both from an algorithmic as well as from a structural point of view.

In the last few years, the question of closure properties of $\#\mathrm{P}$ (and GapP [9], which is the class of all differences of $\#\mathrm{P}$ functions) has become an important subject in complexity theory [4, 9, 27, 30, 32, 41]. Let us first define what we mean by a *closure property.*

An operator Θ for this paper is a scheme, which specifies how to construct, from certain given functions $f_1, \ldots, f_r$, a new function g. We say that a class of functions $\mathcal{F}$ possesses closure property Θ (or, $\mathcal{F}$ *is closed under* Θ) if the following holds:

$$\forall i\colon f_i \in \mathcal{F} \Rightarrow g \in \mathcal{F}.$$

An operator is called a *functional operator* (or, an *operation*), if the scheme is defined by a function $h\colon \mathbb{N}^r \to \mathbb{N}$, such that

$$g(x) = h(f_1(x), \ldots, f_r(x));$$

in other words: the value of the generated function depends only on the values of the given functions. We will always identify the operator Θ with the number-theoretic function h.

The classes #P and GapP are known to possess several functional closure properties, e. g., addition and multiplication [30], and subtraction (in the case of GapP [9]); and non-functional properties, e. g., summation of exponentially many function values and product of polynomially many function values [4, 9]. These closure properties have been used to obtain a number of inclusions between complexity classes [4, 5, 9, 10, 12, 13, 14, 24, 25, 26, 33]: closure properties play an important role when we want to simulate one machine type by another. To mention only two examples, closure of #P under polynomials plays a crucial role for Toda's result that the polynomial-time hierarchy is Turing reducible to the class PP of probabilistic polynomial-time-decidable sets [33]; and more general closure properties have been used in [5, 10] to show that PP is closed under intersection and even under truth-table reductions. But to give evidence that certain inclusions between classes do not hold, it has become an important topic to identify closure properties that #P or GapP probably do not have [27, 32, 41].

In this paper, we will examine classes $\langle \#\mathrm{P} \rangle_f$, which are defined as the closure of #P under operation f as well as some other operators #P is known to be closed under. This means, that #P is closed under f if and only if $\#\mathrm{P} = \langle \#\mathrm{P} \rangle_f$.

In Section 3, we will identify all *relativizable* closure properties of #P, i. e., those operations f, for which $\#\mathrm{P} = \langle \#\mathrm{P} \rangle_f$ in all relativizations. Starting with observations such as "if f is a function which is not monotonic increasing, then f cannot be a relativizable closure property of #P," we give a complete characterization of all relativizable functional closure properties of #P. Our main result in Section 3 is: A function f is a relativizable closure property of #P if and only if f is a sum of constantly many binomial coefficients over constants.

This result is obtained by making use of very recent results concerning separations of the so-called classes with finite acceptance types (examined in [13, 14]). These classes are generalizations of the class US [6] (later often also called 1-NP). Hertrampf [17, 19] was able to completely answer the question of which of those classes are separable by oracles. For his result, he built on a refinement of techniques developed in [7, 8, 37], which are nowadays referred to as the *leaf language approach* to define complexity classes (see e.g. [19, 20, 21, 23] or the recent textbook [28]).

In Section 4 we start a detailed classification of number-theoretic operations with respect to their power in the counting context. Examinations of this kind were started in a paper by Ogiwara and Hemachandra [27], in which a closure property f was called #P*-hard* if #P being closed under f is equivalent to #P being closed under every operation computable deterministically in polynomial time. Now, if a function f is hard in this sense, then we can conclude that #P is closed under f if and only if the hierarchy $\#\mathrm{P} \cup \#\mathrm{P}^{\#\mathrm{P}} \cup \#\mathrm{P}^{\#\mathrm{P}^{\#\mathrm{P}}} \cup \ldots$ (the so-called *hierarchy of counting functions* [42]) collapses to #P (which in turn is equivalent to a collapse of the counting hierarchy of sets, i. e., the hierarchy CH, defined as $\mathrm{PP} \cup \mathrm{PP}^{\mathrm{PP}} \cup \mathrm{PP}^{\mathrm{PP}^{\mathrm{PP}}} \cup \ldots$, to UP). This notion of hardness is non-constructive in nature; and as pointed out in [27], if #P is *not* closed under every polynomial time operation, then trivially every such operation that is not a closure property of #P is hard in the sense above. But, as argued in [27], their notion is nevertheless well suited to give evidence of the complexity of an operation "within the vocabulary of evidence available today to complexity theorists" [27].

Our definition of the classes $\langle \#\mathrm{P} \rangle_f$ of course directly suggests the following notion of constructive hardness and of reducibility between operations. We say that f is *counting hard* if and only if $\langle \#\mathrm{P} \rangle_f$ contains the above-defined hierarchy of counting functions; and we say that f reduces to f', if, essentially, an application of f can be simulated by finitely many applications of f'. Every property f that is counting hard is also hard for #P in the sense of Ogiwara and Hemachandra (but not necessarily vice versa), and therefore, #P is closed under f if and only if the hierarchy of counting functions collapses to #P. If f is counting hard and f reduces to f', then also f' is counting hard.

We characterize operations f by identifying those classes $\mathcal{C}$, in the range between UP and the counting hierarchy CH, to which $\langle \#\mathrm{P} \rangle_f$ corresponds (for a formal statement, see Section 5). If such a correspondence holds, then we are able to conclude the following:

(1) #P is closed under f if and only if $\mathrm{UP} = \mathcal{C}$.

(2) f is counting hard if and only if $\mathcal{C} = \mathrm{CH}$.

At the low end we find of course operations f for which $\#\mathrm{P} = \langle \#\mathrm{P} \rangle_f$, i. e., operations that are closure properties of #P. At the high end, we have the counting hard operations. For f counting hard, $\langle \#\mathrm{P} \rangle_f$ corresponds to $\mathcal{C} = \mathrm{CH}$, and thus #P is closed under f if and only if UP = CH. For the class GapP, we will make a similar definition (gap hard) and we will obtain similar results.

We obtain a number of general criteria which allow us to identify large classes of functions that are counting hard or gap hard. An interesting consequence is that if f is a functional closure property of GapP, and we define f' to be different from f but equal to f on an infinite interval, then f' cannot be a closure property of GapP unless the counting hierarchy collapses. We will then make convenient use of our reducibility notion to show that a number of natural arithmetic operations are counting hard.

We will demonstrate how our approach allows us to obtain simple and unified proofs for a number of operations examined in the literature, and how our approach allows us to prove a number of interesting consequences for other operations, e. g., rational powers. We show that if GapP is closed under the function $f(x) = \lfloor x^{3/2} \rfloor$ (or any function f such that $f(y^2) = y^3$ for all $y \in \mathbb{N}$), then the counting hierarchy collapes. We find this example especially interesting because of the following: Collapsing results of the form mentioned above are characterized by so-called *witness reduction* [15]. The operations up to now shown to be hard have this reduction property more or less inherent in their definition; think e. g. of integer division or modified subtraction. However, the function $f(x) = \lfloor x^{3/2} \rfloor$ is a monotonic increasing function, greater than the identity, and there is no form of a witness reduction present in its definition. Nevertheless, we show that closure under f can be characterized by a collapse of the hierarchy of counting functions.

We not only give general criteria for hardness, but we also show, for operations f from several large classes of functions, that $\langle \#\mathrm{P} \rangle_f$ corresponds to a probably strict subclass of CH. Our results in this direction concern among others those operations considered by Ogiwara and Hemachandra [27] as so-called *intermediate closure properties*, i. e., operations that are neither known to be closure properties of #P nor proven to be hard in their sense. What we will show here is that the closure of #P under decrement corresponds to the class NP, and thus decrement cannot be counting hard unless the counting hierarchy collapses to NP; and that the closure of #P under division by any prime number k corresponds to $\mathrm{MOD}_k\,\mathrm{P}$, and thus division by k cannot be counting hard unless the counting hierarchy collapses to $\mathrm{MOD}_k\,\mathrm{P}$. Since modified subtraction and integer division are counting hard, this shows that in this context, modified subtraction is really more powerful than decrement, and general integer division is more powerful than division by any fixed prime number. We prove similar results for minimum and maximum, and generalize this to all periodic functions and even to all functions that are finitely invertible (i. e. functions f for which for every y there are only finitely many x such that $f(x) = y$). One consequence is that while, as we mentioned above, $f(x) = \lfloor x^{3/2} \rfloor$ is gap hard, it cannot be counting hard (unless, as we will see, the counting hierarchy collapses to the polynomial time hierarchy).

2 Preliminaries

We assume familiarity with general notions of set theory and of complexity theory [1, 22].

The class FP_+ (resp., FP) is the class of all nonnegative integer functions (resp., all integer functions) computable in polynomial time by deterministic Turing machines.

The main complexity class we want to deal with in this paper is the function class #P, which was originally defined by Valiant [36]. #P consists of all functions $f\colon \Sigma^* \to \mathbb{N}$ for which there is a polynomial-time nondeterministic Turing machine M such that for all x, the number of accepting paths of M on input x is exactly $f(x)$.

The class GapP, defined by Fenner, Fortnow, and Kurtz [9], consists of all differences of #P functions, or equivalently, it is the closure of #P under subtraction.

The hierarchy of counting functions, defined by Wagner [42], is the hierarchy

$$\mathrm{FCH} =_{\mathrm{def}} \#\mathrm{P} \cup \#\mathrm{P}^{\#\mathrm{P}} \cup \#\mathrm{P}^{\#\mathrm{P}^{\#\mathrm{P}}} \cup \ldots$$

The hierarchy of gap functions, defined by Vollmer and Wagner [41], is the hierarchy

$$\mathrm{FCH}_{\mathbb{Z}} =_{\mathrm{def}} \mathrm{GapP} \cup \mathrm{GapP}^{\mathrm{GapP}} \cup \mathrm{GapP}^{\mathrm{GapP}^{\mathrm{GapP}}} \cup \ldots$$

It is easy to see that $\mathrm{FCH}_{\mathbb{Z}}$ consists exactly of all differences of FCH functions, or equivalently, it is the closure of FCH under subtraction.

We use the following operators on classes of functions $\mathcal{F}$.

- We say that $h \in \mathrm{Sum}\,\mathcal{F}$ iff there exist $f \in \mathcal{F}$ and a polynomial p such that
$$h(x) = \sum_{|y| \le p(|x|)} f(x, y).$$
- We say that $h \in \mathrm{WProd}\,\mathcal{F}$ iff there exist $f \in \mathcal{F}$ and a polynomial p such that
$$h(x) = \prod_{0 \le y \le p(|x|)} f(x, y).$$

  (WProd stands for "weak product," i. e., a product of only polynomially many values. This terminology is from [42], where the operators $\mathrm{Sum}, \mathrm{WSum}, \mathrm{Prod}, \mathrm{WProd}$ of summation of exponentially many values, summation of polynomially many values, product of exponentially many values, and product of polynomially many values were considered.)

- We say that $h \in \mathrm{Choose}\,\mathcal{F}$ iff there exist $f \in \mathcal{F}$ and a polynomially bounded function $g \in \mathrm{FP}$ (i. e., $g(x) \leq p(|x|)$ for some polynomial p and all x) such that

$$h(x) = \binom{f(x)}{g(x)}.$$

We also consider the simple arithmetic operations addition, multiplication, subtraction, integer division (denoted by :) as operators on functions, i. e. for any operator $\circ$ from the above list and two classes $\mathcal{F}_1, \mathcal{F}_2$ of functions, let

$$\mathcal{F}_1 \circ \mathcal{F}_2 =_{\mathrm{def}} \{\, f_1 \circ f_2 \mid f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2 \,\}.$$

Additionally, we consider modified subtraction: For integers x, y, let $x \dot{-} y =_{\mathrm{def}} \max\{0, x-y\}$. The operation of substitution (i. e. composition) is denoted by Sub.

It is well known [4, 9, 30] that #P and GapP are closed under $+, \cdot, \mathrm{Sum}, \mathrm{WProd}$, Choose, and that GapP is additionally closed under subtraction. Observe, that $+, -, \cdot$ are functional closure properties, and that $\mathrm{Sum}, \mathrm{WProd}, \mathrm{Choose}$ are *not*. Of course, one could consider the *functional* operations $f_k(n) = \binom{n}{k}$ (which is a closure property of #P) and $f(n,m) = \binom{n}{m}$. However, in the definition of Choose, compared to the definition of f, there is an additional requirement present which refers not only to function values but also to the original input x.

We will later consider any number theoretic function f as a functional operator and thus, it should lead to no confusion if we speak of the closure of a class of functions $\mathcal{F}$ under function f.

If $\mathcal{O}_1, \ldots, \mathcal{O}_k$ are operators as above, and $\mathcal{F}$ is a class of functions, then $[\mathcal{F}]_{\mathcal{O}_1,\ldots,\mathcal{O}_k}$ denotes the algebraic closure of $\mathcal{F}$ under the operations $\mathcal{O}_1, \ldots, \mathcal{O}_k$ and the operations of identification of variables, restriction (i. e. substitution of constants), and introduction of variables (i. e. composition with one of the identity functions i_k^n defined by $i_k^n(x_1, \ldots, x_n) =_{\mathrm{def}} x_k$). These latter operations are included for technical reasons to obtain "smoother" classes.

UP [35] is the class of all sets whose characteristic functions are in #P. SPP [9] is the class of all sets whose characteristic functions are in GapP. We will also refer to Gill's class PP [11], and Papadimitriou and Zachos's class $\oplus\mathrm{P}$ [29] and its generalization to the classes $\mathrm{MOD}_k\,\mathrm{P}$ [4]. The counting hierarchy (of sets) [42] is the hierarchy

$$\mathrm{CH} =_{\mathrm{def}} \mathrm{PP} \cup \mathrm{PP}^{\mathrm{PP}} \cup \mathrm{PP}^{\mathrm{PP}^{\mathrm{PP}}} \cup \ldots$$

We consider the following general definition of counting classes beyond #P. For any class $\mathcal{C}$, let $\# \cdot \mathcal{C}$ denote the set of all functions f such that there exists a set $A \in \mathcal{C}$ and a polynomial p such that

$$f(x) = \bigl|\{\, y \mid |y| \leq p(|x|) \wedge (x,y) \in A \,\}\bigr|.$$

We use three operators transforming classes of functions into classes of sets, defined as follows (see [40, 43]): For any class of functions $\mathcal{F}$, let $\mathrm{U} \cdot \mathcal{F}$ be the class of sets whose characteristic functions are in $\mathcal{F}$, $\mathrm{X} \cdot \mathcal{F}$ be the class of sets whose characteristic functions are in $\mathcal{F} - \mathcal{F}$, and $\exists \cdot \mathcal{F}$ be the class of all sets A for which there exists a function $f \in \mathcal{F}$ such that for all x, $x \in A \iff f(x) > 0$. We will generally abbreviate $\mathrm{U} \cdot \# \cdot \mathcal{C}$ by $\mathrm{U} \cdot \mathcal{C}$. Obviously, we have $\exists \cdot \#\mathrm{P} = \mathrm{NP}$ and, as already mentioned, $\mathrm{U} \cdot \#\mathrm{P} = \mathrm{UP}$, $\mathrm{X} \cdot \#\mathrm{P} = \mathrm{U} \cdot \mathrm{GapP} = \mathrm{SPP}$; moreover, it is known that $\exists \cdot \mathrm{GapP} = \mathrm{PP}$ [9] and $\mathrm{U} \cdot \mathrm{FCH} = \mathrm{X} \cdot \mathrm{FCH} = \exists \cdot \mathrm{FCH} = \mathrm{CH}$ [42]. Going the other way round, it is not hard to see that $\# \cdot \mathrm{UP} = \#\mathrm{P}$ and it is known that $\# \cdot \mathrm{CH} = \mathrm{FCH}$ [42].

We also need the so-called complexity classes with finite acceptance types, which were considered in [13, 14, 17, 19]. Using the class #P, these classes can be defined as follows.

Let A be a finite set of nonnegative integers. Then the class $(A)\mathrm{P}$ is defined as the class of all sets L, such that there is a function $f \in \#\mathrm{P}$, satisfying

$$x \in L \Leftrightarrow f(x) \in A.$$

The classes $(A)\mathrm{P}$ are called *classes with finite acceptance types*. For example, the class US (or 1-NP) in this notation is $\mathrm{US} = 1\text{-}\mathrm{NP} = (\{1\})\mathrm{P}$.

This definition can be generalized as follows: Let A and R be sets of nonnegative integers, and let $A \cap R = \emptyset$. Then the class $(A,R)\mathrm{P}$ is defined as the class of all sets L such that there is a function $f \in \#\mathrm{P}$ satisfying

$$x \in L \Rightarrow f(x) \in A \text{ and } x \notin L \Rightarrow f(x) \in R.$$

For example, UP equals $(\{1\},\{0\})\mathrm{P}$.

3 Relativizable closure properties of #P

Now we shall investigate relativizable functional closure properties of #P. First we consider only functions from $\mathbb{N}$ into $\mathbb{N}$. Let f be a function, $f\colon \mathbb{N} \to \mathbb{N}$.

We want to find out whether f is a relativizable closure property of $\#\mathrm{P}$. If it is, then for all functions $g\colon \Sigma^* \to \mathbb{N}$ we have:

$$g \in \#\mathrm{P} \implies f \circ g \in \#\mathrm{P}.$$

We eliminate step by step many functions that are not relativizable closure properties of $\#\mathrm{P}$.

3.1 Lemma. *Suppose $f\colon \mathbb{N} \to \mathbb{N}$, and for some number $m \in \mathbb{N}$,*

$$f(m) > f(m+1).$$

Then f is not a relativizable closure property of $\#\mathrm{P}$.

Proof. Assume, to the contrary, that f is a relativizable closure property of $\#\mathrm{P}$.

Along with f, the function $f'\colon \mathbb{N} \to \mathbb{N}$ defined by $f'(n) = f(m+n) + f(m) \cdot \binom{n}{2}$ is a relativizable closure property of $\#\mathrm{P}$, because $\#\mathrm{P}$ is (relativizably) closed under addition, multiplication, and binomial coefficient with constants. Now $f'(1) = f(m+1)$ and $f'(n) > f(m+1)$ for all $n \neq 1$. Let L be a set in 1-NP. Then there is a function $g \in \#\mathrm{P}$ with $x \in L \iff g(x) = 1$. But also the function h defined by $h(x) = f'(g(x))$ is in $\#\mathrm{P}$, and we have $x \in L \iff g(x) = 1 \iff h(x) \leq f(m+1)$. Thus, $L \in \mathrm{coNP}$. But it is well known that there is an oracle separating 1-NP from coNP, and consequently f' (and thus also f) cannot be a relativizable closure property of $\#\mathrm{P}$. □

3.2 Lemma. *Suppose $f\colon \mathbb{N} \to \mathbb{N}$, and for some number $m \in \mathbb{N}$,*

$$f(m) < f(m+1) = f(m+2).$$

Then f is not a relativizable closure property of $\#\mathrm{P}$.

Proof. Assume that f is a relativizable closure property of $\#\mathrm{P}$.

Along with f, the function $f'\colon \mathbb{N} \to \mathbb{N}$ defined by $f'(n) = f(m+n) + f(m+1) \cdot \binom{n}{3} \cdot 2$ is a relativizable closure property of $\#\mathrm{P}$. But $f'(0) < f'(1) = f'(2) < f'(i)$ for all $i > 2$. This implies the inclusion $(\{1,2\})\mathrm{P} \subseteq$ 1-NP. But that cannot be true under all relativizations, which can be easily shown using the algorithm from [19, Corollary 18]. □

Thus, we know that for non-constant relativizable closure properties $f\colon \mathbb{N} \to \mathbb{N}$ of $\#\mathrm{P}$,

$$f(0) = f(1) = \ldots = f(m) < f(m{+}1) < f(m{+}2) < \ldots$$

for some $m \geq 0$. We further prove:

3.3 Lemma. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a non-constant relativizable closure property of $\#\mathrm{P}$, and let m be the uniquely defined number with the property*

$$f(0) = f(1) = \ldots = f(m) < f(m{+}1) < f(m{+}2) < \ldots$$

Then for all $i \geq 0$ we have the relativizable inclusion $(A_i, \overline{A_i})\mathrm{P} \subseteq (B_i, R_i)\mathrm{P}$, where

$$\begin{aligned} A_i &= \{0, 1, \ldots, m-1, m, m+2, \ldots, m+2i\} \\ B_i &= \{f(m), f(m+2), \ldots, f(m+2i)\} \\ R_i &= \{f(m+1), f(m+3), \ldots, f(m+2i-1)\} \\ &\quad\ \cup \{x \mid x > f(m+2i)\} \end{aligned}$$

Proof. Suppose $L \in (A_i, \overline{A_i})\mathrm{P}$, so that there is a function $g \in \#\mathrm{P}$ with the property that $x \in L \iff g(x) \in A_i$. Then by assumption the function h with $h(x) = f(g(x))$ belongs to $\#\mathrm{P}$, and we have $x \in L \implies g(x) \in A_i \implies h(x) \in B_i$, and $x \notin L \implies g(x) \notin A_i \implies h(x) \in R_i$. Thus, also $L \in (B_i, R_i)\mathrm{P}$. All steps are relativizable, and thus the resulting inclusion of the two classes is relativizable. □

Now, using Lemma 3.3, we can exploit the hypergraph technique, which was introduced in [17] and extended in [19], to obtain the following necessary condition for relativizable closure properties.

3.4 Lemma. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a relativizable closure property of $\#\mathrm{P}$. Then there are nonnegative integers $\alpha_0, \alpha_1, \alpha_2, \ldots$, such that for all $n \in \mathbb{N}$ we have*

$$f(n) = \sum_{k \in \mathbb{N}} \alpha_k \binom{n}{k}.$$

To prove this lemma, we first have to make the reader familiar with oracle separations using combinatorial arguments on hypergraphs. We start by introducing some terminology.

By $\mathbb{N}$, we always denote the set of nonnegative integers, thus including the number 0. On the other hand, for a natural number n, we denote by $[n]$ the set $\{1, \ldots, n\}$, thus excluding the number 0.

Hypergraphs, (r,s)-hypergraphs, and $((r,s)$-$)$ hypergraph sequences are defined in [19] as follows.

3.5 Definition. A *hypergraph* H on n vertices is a mapping $H\colon 2^{[n]} \to \mathbb{N}$. We call the elements of $[n]$ vertices of the hypergraph H, and the subsets $I \subseteq [n]$ satisfying $H(I) = k > 0$ are called hyperedges of multiplicity k.

For each hypergraph H on n vertices, we define another mapping $\overline{H}\colon 2^{[n]} \to \mathbb{N}$ by $\overline{H}(I) = \sum_{J \subseteq I} H(J)$.

3.6 Definition. A hypergraph sequence (which we will in the following also call an *H-sequence*) is an infinite sequence $H_1, H_2, H_3, \ldots$ where H_n is a hypergraph on n vertices.

3.7 Definition. An (r,s)-hypergraph H on $n \cdot r$ vertices and an (r,s)-H-sequence $H_1, H_2, \ldots$ are defined analogously to the above definitions, but using r different sorts of vertices and s different sorts of edges (thus, $H\colon 2^{[n]\times[r]} \to \mathbb{N}^s$ is an (r,s)-hypergraph on $n \cdot r$ vertices).

We will call a hypergraph H on n vertices *uniform* if for all sets $I, J \subseteq [n]$ such that $|I| = |J|$ we have $H(I) = H(J)$. A hypergraph sequence is called uniform if all the hypergraphs in the sequence are uniform.

From [19, Theorem 11 and Corollary 14] we obtain immediately:

3.8 Theorem. *Let A, B, S, R be sets of nonnegative integers, let A and B be finite, S and R cofinite, and let $A \cap S = \emptyset$ and $B \cap R = \emptyset$. Further let $(A,S)\mathrm{P} \subseteq (B,R)\mathrm{P}$ in all relativizations. Then there is a uniform hypergraph sequence $H_1, H_2, \ldots$ such that for large enough n, and for all $J \subseteq [n]$ we have*

$$\begin{aligned} |J| \in A &\Longrightarrow \overline{H_n}(|J|) \in B \\ |J| \in S &\Longrightarrow \overline{H_n}(|J|) \in R \end{aligned}$$

Now we are prepared to give the proof for Lemma 3.4.

Proof. (of Lemma 3.4) Let $f : \mathbb{N} \to \mathbb{N}$ be a nonconstant relativizable closure property of #P, and let m be the unique value satisfying $f(0) = f(1) = \ldots = f(m) < f(m+1)$. Define a sequence $(\alpha_n)_{n\geq 0}$ of integer numbers by

$$\alpha_n = f(n) - \sum_{k<n} \alpha_k \binom{n}{k}.$$

Obviously, we then have

$$f(n) = \sum_{k\leq n} \alpha_k \binom{n}{k} = \sum_{k\in\mathbb{N}} \alpha_k \binom{n}{k},$$

the latter equality following from the fact that $\binom{n}{k} = 0$, whenever $k > n$. It remains to show that all coefficients α_k are nonnegative.

Clearly, $\alpha_0 = f(0)$, and for all $j \in \{1, \ldots, m\}$ we have $\alpha_j = 0$. Moreover, $\alpha_{m+1} = f(m+1) - f(0)$. Now let $n > m+1$ be such that $n - m$ is a positive and even number, and assume that $\alpha_i \geq 0$ for all $i \in \{0, \ldots, n-2\}$. We want to prove that $\alpha_{n-1} \geq 0$ and $\alpha_n \geq 0$. From Lemma 3.3 we know that there is a relativizable inclusion $(A, \overline{A})\mathrm{P} \subseteq (B,R)\mathrm{P}$, where $A = A_i$, $B = B_i$, and $R = R_i$ for $i = \frac{n-m}{2}$. Thus from Theorem 3.8 we know that there is a uniform hypergraph H of size n such that $|I| \in A \Rightarrow \overline{H}(I) \in B$ and $|I| \notin A \Rightarrow \overline{H}(I) \in R$. Now H can be represented by a function h, where $H(I) = h(|I|)$ for all subsets $I \subseteq [n]$, and similarly we can define a function $\overline{h}$ in such a way that $\overline{H}(I) = \overline{h}(|I|)$. Then obviously we have

$$\overline{h}(j) = \sum_{k\leq j} h(k) \binom{j}{k}$$

for all $j \leq n$. Clearly the set $\{\overline{h}(0), \overline{h}(1), \ldots, \overline{h}(m-1), \overline{h}(m), \overline{h}(m+2), \ldots, \overline{h}(n-2), \overline{h}(n)\}$ has to be a subset of B, and similarly $\{\overline{h}(m+1), \overline{h}(m+3), \ldots, \overline{h}(n-3), \overline{h}(n-1)\}$ has to be a subset of R. From the monotonicity of the function $\overline{h}$ and the cardinalities of the respective sets we can immediately conclude that $\overline{h}(j) = f(j)$ for all $j \in \{m, m+1, \ldots, n-1, n\}$, and so

$$\sum_{k\leq j} \alpha_k \binom{j}{k} = f(j) = \overline{h}(j) = \sum_{k\leq j} h(k) \binom{j}{k} \qquad (j \leq n)$$

which means $\alpha_k = h(k)$ for all $k \leq n$. But h is a nonnegative function, consequently all α_k's, especially α_{n-1} and α_n cannot be negative. This completes the proof. □

But we can further prove that the infinite sum in fact also globally has only finitely many nonzero summands:

3.9 Lemma. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a relativizable closure property of #P. Then there are nonnegative integers m and $\alpha_0, \alpha_1, \ldots, \alpha_m$ such that for all $n \in \mathbb{N}$ we have:*

$$f(n) = \sum_{0\leq k\leq m} \alpha_k \binom{n}{k}.$$

Proof. We already know that f can be represented by an infinite sum of this kind. Now assume that there are really infinitely many coefficients $\alpha_k > 0$. We know that the function $g(x) = 2^{|x|}$ belongs to #P. Thus also the function $h(x) = f(2^{|x|})$ has to belong to #P. But for all $k > 1$ such that $\alpha_k > 0$ we know that $h(0^{\lceil \log 2k \rceil}) = f(2^{\lceil \log 2k \rceil}) \geq f(2k) \geq \binom{2k}{k} > 2^k$ holds. Now the value 2^k is doubly exponential in the length of the input of h, and we can obtain this inequality for arbitrarily large k. That means, that for arbitrarily large input lengths there is a value under h, which is doubly exponential. But obviously the values of a function from #P are always exponentially bounded (in the length of the input). Thus h cannot be a function from #P. So the assumption that there are infinitely many k with α_k nonzero has led us to a contradiction. $\square$

Thus, we obtain:

3.10 Theorem. *A function $f: \mathbb{N} \to \mathbb{N}$ is a relativizable closure property of #P if and only if there are nonnegative integers m and $\alpha_0, \alpha_1, \ldots, \alpha_m$ such that for all $n \in \mathbb{N}$:*

$$f(n) = \sum_{0 \le k \le m} \alpha_k \binom{n}{k}.$$

Proof. Clearly, if f is of the given form, then f is a relativizable closure property of #P, since #P is relativizably closed under binomial coefficient with constants and under addition. This proves the "if" part. The "only if" part is the claim of Lemma 3.9. $\square$

Now we turn to the case of r-ary functions. Here we will obtain a sum ranging over all vectors below some given vector v, and the summands are given by multinomial coefficients rather than binomial coefficients. Recall, however, that a multinomial coefficient equals the product of the binomial coefficients of the respective components in the two vectors. This will lead to the second representation given in Theorem 3.13 below.

We first generalize the notion of a complexity class (A, R)P to the case where A and R are disjoint subsets of $\mathbb{N}^r$. In this case, we say that L belongs to (A, R)P, if there are functions $f_1, \ldots, f_r \in$ #P such that

$$\begin{aligned} x \in L &\Longrightarrow (f_1(x), \ldots, f_r(x)) \in A \\ x \notin L &\Longrightarrow (f_1(x), \ldots, f_r(x)) \in R \end{aligned}$$

Now we would like to proceed as above in the proof of Lemma 3.4. But this proof exploits an ordering of the function values under f such that $x < y$ implies $f(x) \le f(y)$, and for all x and y, we have either $x \le y$ or $y \le x$. This is certainly not true for r-ary functions. In order to overcome this difficulty, we further generalize the above complexity classes of the form (A, R)P to classes of the form $(A_1, \ldots, A_s)$P, where $s \ge 1$, A_i is a subset of $\mathbb{N}^r$ $(1 \le i \le s)$, and for $i \ne j$, we have $A_i \cap A_j = \emptyset$. These classes are no longer classes of ordinary sets, but instead they are classes of s-sets, where an s-set is an ordered collection of s pairwise disjoint sets.

We can generalize Theorem 3.8 as follows.

3.11 Theorem. *Let $A_1, \ldots, A_s$ be pairwise disjoint finite subsets of $\mathbb{N}^r$, and let $B_1, \ldots, B_s$ be pairwise disjoint finite subsets of $\mathbb{N}$. Further let $(A_1, \ldots, A_s)$P $\subseteq$ $(B_1, \ldots, B_s)$P in all relativizations. Then there is a uniform $(r, 1)$-hypergraph sequence $H_1, H_2, \ldots$ such that for large enough n, and for all $J \subseteq [n] \times [r]$, for all i, $1 \le i \le s$, we have*

$$|J| \in A_i \Longrightarrow \overline{H_n}(J) \in B_i.$$

Here, $|J|$ is defined to be the unique vector $(j_1, \ldots, j_r)$ satisfying $j_k = |J \cap ([n] \times \{k\})|$ $(1 \le k \le r)$.

Let $f: \mathbb{N}^r \to \mathbb{N}$. We have to investigate whether for all r-tuples of functions $(g_1, \ldots, g_r)$ with $g_i \in$ #P for all i, we have in all relativized worlds $f \circ (g_1, \ldots, g_r) \in$ #P, too. Here by $f \circ (g_1, \ldots, g_r)$ we mean the function h defined by $h(x) = f(g_1(x), \ldots, g_r(x))$.

In analogy to Lemma 3.1 we prove:

3.12 Lemma. *Let $f: \mathbb{N}^r \to \mathbb{N}$, and let $m_1, \ldots, m_r$ be natural numbers satisfying the inequality*

$$\begin{aligned} &f(m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_r) > \\ &\qquad f(m_1, \ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots, m_r) \end{aligned}$$

Then f is not a relativizable closure property of #P.

Proof. This follows from Lemma 3.1, since for fixed numbers $m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r$ and for a relativizable closure property f also the mapping $g \mapsto h$ is a relativizable closure property, where h is defined by $h(x) = f(m_1, \ldots, m_{i-1}, g(x), m_{i+1}, \ldots, m_r)$. $\square$

Now, we can state and prove the analogue of Theorem 3.10 for the general r-ary case.

3.13 Theorem. *Let $f: \mathbb{N}^r \to \mathbb{N}$ be a function. Then f is a relativizable closure property of #P if and only if there is a vector $v \in \mathbb{N}^r$ and there are coefficients α_u for all vectors $u \le v$ such that for all $m = (m_1, \ldots, m_r) \in \mathbb{N}^r$ the following holds:*

$$\begin{aligned} f(m_1, \ldots, m_r) &= \sum_{u \le v} \alpha_u \binom{m}{u} \\ &= \sum_{\substack{u = (u_1, \ldots, u_r) \\ u \le v}} \alpha_u \prod_{i=1}^{r} \binom{m_i}{u_i}. \end{aligned}$$

Proof. Lemma 3.12 showed that also r-ary relativizable functional closure properties of #P are always monotonic. Thus, f can be written in the form

$$f(x) = \sum_{y \in \mathbb{N}^r} \alpha_y \binom{x}{y}$$

for coefficients $\alpha_y \in \mathbb{Z}$. We first want to show that all α_y are nonnegative. Thus, given $y = (y_1, \ldots, y_r)$, we define m to be the maximum value in the set

$\{y_1, \dots, y_r\}$. If #P is closed under f, then #P is also closed under f', where

$$f'(x) = f(x) + \sum_{i=1}^{r} \binom{x}{b_i} \cdot (f(m, \dots, m) + 1).$$

Here, the vectors b_i are defined as $b_i = (0, \dots, 0, m+1, 0, \dots, 0)$, the value $m+1$ being the i-th component of b_i. f' can be represented in the form

$$f'(x) = \sum_{z \in \mathbb{N}^r} \alpha'_z \binom{x}{z},$$

and then obviously $\alpha_z = \alpha'_z$ for all $z \leq (m, \dots, m)$, especially $\alpha_y = \alpha'_y$. Now it suffices to prove that all α'_z with $z \leq (m, \dots, m)$ are nonnegative. For that we consider the set $V = \{f'(y) \mid y \in \mathbb{N}^r\}$. We divide V in two parts $V_1 = \{v \in V \mid v \leq f'(m, \dots, m)\}$ and $V_2 = V \setminus V_1$, such that from the definition and monotonicity of f' it is clear that $f'(z) \in V_1 \iff z \leq (m, \dots, m)$. V_1 is a finite set, so let $V_1 = \{v_1, \dots, v_s\}$. We define subsets $A_1, \dots, A_s$ of $\mathbb{N}^r$ by $A_i = \{z \mid f'(z) = v_i\}$ $(1 \leq i \leq s)$. Then certainly $(A_1, \dots, A_s)\mathrm{P} \subseteq (\{v_1\}, \dots, \{v_s\})\mathrm{P}$ in all relativizations, and thus by Theorem 3.11 we obtain a uniform hypergraph representing this inclusion. The uniformity of this hypergraph means that it is given by a function $h: \mathbb{N}^r \to \mathbb{N}$, and again, as in the proof of Lemma 3.4, we can associate a function $\overline{h}$ with it, such that f' and $\overline{h}$ have to agree at least on all inputs $z \in \mathbb{N}^r$ satisfying $z \leq (m, \dots, m)$, which yields again that the coefficients α'_z coincide with $h(z)$, and thus cannot be negative. Now we can conclude that α_y is nonnegative, and moreover the same argument as in the unary case shows that the number of nonzero coefficients cannot be infinite. This completes the proof of the theorem. $\square$

4 Hardness of operations and reducibilities

We now address the question of whether the unrelativized classes #P and GapP are closed under certain operations. In what follows, an operation is either a function $f: \mathbb{N}^r \to \mathbb{N}$ or a function $f: \mathbb{Z}^r \to \mathbb{Z}$, for some $r \geq 1$. We start by giving a formal definition of "hardness" of a finite set of operations $\{f_1, \dots, f_k\}$. For simplicity, we define

$$\langle \mathcal{F} \rangle_{f_1,\dots,f_k} =_{\mathrm{def}} [\mathcal{F}]_{f_1,\dots,f_k,+,\cdot,\mathrm{Sum},\mathrm{WProd},\mathrm{Choose}},$$

that is, $\langle \mathcal{F} \rangle_{f_1,\dots,f_k}$ is the closure of the class $\mathcal{F}$ under the operations $f_1, \dots, f_k$ and some other basic operators which are known to be closure properties of both #P and GapP.

4.1 Definition. A set $\{f_1, \dots, f_k\}$ is *counting hard* if $\mathrm{FCH} \subseteq \langle \#\mathrm{P} \rangle_{f_1,\dots,f_k}$.

Since #P and GapP are closed under $+, \cdot, \mathrm{Sum}, \mathrm{WProd}, \mathrm{Choose}$, and on the other hand FCH is closed under every FP_+ operation and $\mathrm{FCH}_{\mathbb{Z}}$ is closed under every FP operation, we immediately obtain:

4.2 Proposition.

1. *If* $\{f_1, \dots, f_k\} \subseteq \mathrm{FP}_+$ *is counting hard, then* $\langle \#\mathrm{P} \rangle_{f_1,\dots,f_k} = \mathrm{FCH}$*, and moreover:* #P *is closed under* $f_1, \dots, f_k \iff \#\mathrm{P} = \mathrm{FCH} \iff \mathrm{UP} = \mathrm{CH}$.
2. *If* $\{f_1, \dots, f_k, -\} \subseteq \mathrm{FP}$ *is counting hard, then* $\langle \#\mathrm{P} \rangle_{f_1,\dots,f_k,-} = \mathrm{FCH}_{\mathbb{Z}}$*, and moreover:* GapP *is closed under* $f_1, \dots, f_k \iff \mathrm{GapP} = \mathrm{FCH}_{\mathbb{Z}} \iff \mathrm{SPP} = \mathrm{CH}$.

Because of statement 2 of the previous proposition, in the following section we will also say that $\{f_1, \dots, f_k\}$ is *gap hard* if $\{f_1, \dots, f_k, -\}$ is counting hard.

Obviously, if f is counting hard, then f is #P-hard in the sense of Ogiwara and Hemachandra. However, as pointed out in [27], if #P is not closed under every FP_+ operation, then every FP_+ operation that is not a closure property of #P is trivially hard in their sense, but not necessarily in ours.

We can make use of the notion of constructive hardness best, when combined with the following form of reducibility. We say that $\{f_1, \dots, f_k\}$ reduces to $\{h_1, \dots, h_m\}$ if every application of a function f_i can be replaced by several applications of $h_1, \dots, h_m, +, \cdot, \binom{x}{2}, \binom{x}{3}, \binom{x}{4}, \dots$ The motivation for this is that $[+, \cdot, \binom{x}{2}, \binom{x}{3}, \binom{x}{4}, \dots]_{\mathrm{Sub}}$ is the set of all functional closure properties we currently know of (and it is exactly the set of all relativizable functional closure properties, see Theorem 3.13), and composing several functional closure properties of course yields again a functional closure property.

4.3 Definition. For operations $f_1, \dots, f_k$ and $h_1, \dots, h_m$ we say, $\{f_1, \dots, f_k\}$ *counting reduces* to $\{h_1, \dots, h_m\}$, in symbols

$$\{f_1, \dots, f_k\} \leq_{\mathrm{cnt}} \{h_1, \dots, h_m\},$$

if $\{f_1, \dots, f_k\} \subseteq [h_1, \dots, h_m, +, \cdot, \binom{x}{2}, \binom{x}{3}, \binom{x}{4}, \dots]_{\mathrm{Sub}}$.

Obviously, $\leq_{\text{cnt}}$ is a reflexive and transitive relation.

4.4 Proposition. *For operations $f_1, \dots, f_k$ and $h_1, \dots, h_m$, let*

$$\{f_1, \dots, f_k\} \leq_{\text{cnt}} \{h_1, \dots, h_m\}.$$

Then we have:

1. *For any class of functions $\mathcal{F}$,*
$$\langle \mathcal{F} \rangle_{f_1,\dots,f_k} \subseteq \langle \mathcal{F} \rangle_{h_1,\dots,h_m}.$$
2. *If $\{f_1, \dots, f_k\}$ is counting hard, then*
$$\{h_1, \dots, h_m\} \text{ is counting hard.}$$

It should be noted that the above two definitions (as well as the hardness notion from [27]) reflect our current knowledge about closure properties of #P and GapP. If someday, it will unexpectedly be proved that #P is closed under some operator $g \notin [+, \cdot, \mathrm{Sum}, \mathrm{WProd}, \mathrm{Choose}]_{\mathrm{Sub}}$, then obviously, adding g to the operators allowed in Definition 4.1 would possibly make more problems constructively hard for #P, but it would not effect the truth of Propositions 4.2 and 4.4. Similarly, it might be possible to add any other (today unknown) functional closure property of #P in Definition 4.3. However, even restricting ourselves according to our current knowledge to the operators mentioned above, we can prove that a lot of operations are constructively hard for one of the classes we are interested in.

5 Characterizing the power of operations

It will be our aim in this section to derive, for operations f, equations of the form $\langle \#\mathrm{P} \rangle_f = \# \cdot \mathcal{C}$ for some complexity class $\mathcal{C}$, or at least inclusions as $\langle \#\mathrm{P} \rangle_f \subseteq \# \cdot \mathcal{C} \subseteq \langle \#\mathrm{P} \rangle_f - \mathrm{FP}$. This will then allow us to directly give consequences like "#P is closed under f iff UP $=$ U $\cdot\, \mathcal{C}$" or "f is counting hard iff U $\cdot\, \mathcal{C} =$ CH."

The starting point for our investigations is the following results from [41], which state that modified subtraction and integer division are not only hard in the sense of Ogiwara and Hemachandra [27], but even counting hard, and that decrement is gap hard.

5.1 Proposition.

1. $\langle \#\mathrm{P} \rangle_{\dot{-}} = \langle \#\mathrm{P} \rangle_{:} = \mathrm{FCH}$.
2. $\langle \mathrm{GapP} \rangle_{\dot{-}1,-} = \langle \#\mathrm{P} \rangle_{\dot{-}1,-} = \langle \#\mathrm{P} \rangle_{:,-} = \mathrm{FCH}_{\mathbb{Z}}$.

We now start, similar to what we did in Section 3, by considering functions with certain simple properties. In connection with results for only nonnegative numbers and the class #P, we tacitly assume that every f we mention is a function from $\mathbb{N}$ to $\mathbb{N}$, while when talking about GapP and the integers, f is from $\mathbb{Z}$ to $\mathbb{Z}$. We first consider functions $f\colon \mathbb{N} \to \mathbb{N}$ that are *constant almost everywhere* (a.e.), that is functions f for which there exists some $c \in \mathbb{N}$ such that $f(x) = f(c)$ for every $x > c$. We distinguish two cases (Theorem 5.2 and Theorem 5.4).

In this section, we will use the following notation. For a boolean expression α, let $[\![\alpha]\!]$ be 1 if α is true, and 0 otherwise. Let

$$\mathrm{sgn}(x) =_{\mathrm{def}} \begin{cases} 1, & \text{if } x \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

To avoid possible misunderstandings we remark that by "monotonic" we will in the following always mean "nondecreasing."

5.2 Theorem. *If f is monotonic and non-constant but constant a.e., then*

1. $\langle \#\mathrm{P} \rangle_f = \# \cdot \mathrm{NP}$.
2. *Any extension of f to the domain of the integers is gap hard.*

Proof. For part 1, we first show that $\langle \#\mathrm{P} \rangle_{\mathrm{sgn}} = \# \cdot \mathrm{NP}$.

For the inclusion from left to right, observe that $\#\mathrm{P} \subseteq \# \cdot \mathrm{NP}$ and that $\# \cdot \mathrm{NP}$ is closed under $+, \cdot, \mathrm{Sum}, \mathrm{WProd}, \mathrm{Choose}$ [38]; thus it remains to show that $\# \cdot \mathrm{NP}$ is closed under sgn. Consider $f \in \# \cdot \mathrm{NP}$,

$$f(x) = \big|\{\, y \mid |y| \leq p(|x|) \wedge (x, y) \in A \,\}\big|$$

for some polynomial p and a set $A \in \mathrm{NP}$. Define

$$A' =_{\mathrm{def}} \{\, (x, 0) \mid (\exists y, |y| \leq p(|x|))(x, y) \in A \,\}.$$

Then, $A' \in \mathrm{NP}$, and obviously,

$$\mathrm{sgn}\, f(x) = \big|\{\, y \mid |y| \leq p(|x|) \wedge (x, y) \in A' \,\}\big|;$$

thus, $\mathrm{sgn}\, f \in \# \cdot \mathrm{NP}$.

For the inclusion from right to left, consider $f \in \# \cdot \mathrm{NP}$,

$$f(x) = \big|\{\, y \mid |y| \leq p(|x|) \wedge (x, y) \in A \,\}\big|$$

for some polynomial p and a set $A \in \mathrm{NP}$. Let $(x, y) \in A \iff h(x, y) > 0$ for some $h \in \#\mathrm{P}$. Then, obviously,

$$f(x) = \sum_{y \leq p(|x|)} \mathrm{sgn}\, h(x, y),$$

and thus, $f \in \langle \#\mathrm{P}\rangle_{\mathrm{sgn}}$.

Next, we show that any f which fulfills the assumptions of the theorem can be reduced to the sgn function, and vice versa, which implies $\langle \#\mathrm{P}\rangle_f = \langle \#\mathrm{P}\rangle_{\mathrm{sgn}} = \# \cdot \mathrm{NP}$. So, let f be monotonic, non-constant, but fulfilling $f(x) = d$ for all $x > c$.

Then,

$$f(x) = \sum_{i=0}^{c} [\![x \geq i]\!] \cdot a_i = \sum_{i=0}^{c} a_i \cdot \mathrm{sgn}\binom{x}{i},$$

where $a_0 = f(0)$ and $a_i = f(i) - f(i-1)$ for $i > 0$. Observe, that all a_i's are positive numbers. Thus, we showed that $f \leq_{\mathrm{cnt}} \mathrm{sgn}$. On the other hand,

$$\mathrm{sgn}(x) = \binom{f(x+c)}{d}$$

shows $\mathrm{sgn} \leq_{\mathrm{cnt}} f$.

For part 2, recall that $\{-, \dot{-}1\}$ is counting hard (Proposition 5.1). Now, it only remains to observe that $\{-, \dot{-}1\} \leq_{\mathrm{cnt}} \{-, \mathrm{sgn}\}$ by

$$x \dot{-} 1 = (x-1) \cdot \mathrm{sgn}(x).$$

□

5.3 Corollary. *If f is monotonic and non-constant but constant a.e., then*

1. $\#\mathrm{P}$ *is closed under* $f \iff \#\mathrm{P} = \# \cdot \mathrm{NP} \iff \mathrm{UP} = \mathrm{NP}$.
2. f *is counting hard* $\iff \# \cdot \mathrm{NP} = \mathrm{FCH} \iff \mathrm{NP} = \mathrm{CH}$.
3. *For every polynomial-time-computable extension* f' *of* f, GapP *is closed under* $f' \iff \mathrm{GapP} = \mathrm{FCH}_{\mathbb{Z}} \iff \mathrm{SPP} = \mathrm{CH}$.

Proof. This corollary as well as Corollaries 5.5, 5.12, 5.14, 5.16, 5.18, 5.20 state consequences obtained from equations among complexity classes (Theorems 5.2, 5.4, 5.11, 5.13, 5.15, 5.17, and 5.19, resp.) The results of the corollaries are always obtained by applying one of our function-to-set operators to the classes involved in the equations. All proofs are very similar. We only give the proof for Corollary 5.3 in detail.

Statement 1: If $\#\mathrm{P}$ is closed under f, then by Theorem 5.2, we have $\#\mathrm{P} = \# \cdot \mathrm{NP}$. Applying the operator U now yields $\mathrm{UP} = \mathrm{NP}$. On the other hand, if $\mathrm{UP} = \mathrm{NP}$, applying the operator $\#$, we obtain $\#\mathrm{P} = \# \cdot \mathrm{NP}$; thus by Theorem 5.2, $\#\mathrm{P}$ is closed under f.

Statement 2: If f is counting hard, then $\# \cdot \mathrm{NP} = \mathrm{FCH}$ by Theorem 5.2. This implies, applying the operator U, that $\mathrm{NP} = \mathrm{CH}$. On the other hand, if $\mathrm{NP} = \mathrm{CH}$, then applying $\#$ shows that $\# \cdot \mathrm{NP} = \mathrm{FCH}$, which implies that f is counting hard by Theorem 5.2.

Statement 3: Follows immediately from Proposition 4.2.

□

5.4 Theorem. *If f is non-monotonic and constant a.e., then*

1. $\langle \#\mathrm{P}\rangle_f = \# \cdot \mathrm{PH}$.
2. *Any extension of f to the domain of the integers is gap hard.*

Proof. For part 1, we first show that $\langle \#\mathrm{P}\rangle_{1\dot{-}} = \# \cdot \mathrm{PH}$.

For the inclusion from left to right, observe that $\#\mathrm{P} \subseteq \# \cdot \mathrm{PH}$ and that $\# \cdot \mathrm{PH}$ is closed under $+, \cdot, \mathrm{Sum}, \mathrm{WProd}, \mathrm{Choose}$ [38]; thus it remains to show that $\# \cdot \mathrm{PH}$ is closed under $1\dot{-}$. Consider $f \in \# \cdot \mathrm{PH}$,

$$f(x) = \left|\{\, y \mid |y| \leq p(|x|) \wedge (x,y) \in A \,\}\right|$$

for some polynomial p and a set $A \in \mathrm{PH}$. Define

$$A' =_{\mathrm{def}} \{\, (x,0) \mid (\forall y, |y| \leq p(|x|))(x,y) \notin A \,\}.$$

Then, $A' \in \mathrm{PH}$, and obviously,

$$1 \dot{-} f(x) = \left|\{\, y \mid |y| \leq p(|x|) \wedge (x,y) \in A' \,\}\right|;$$

thus, $1 \dot{-} f \in \# \cdot \mathrm{PH}$.

The inclusion from right to left follows by an inductive argument, using the quantifier structure of PH, from the equation

$$\# \cdot \forall \cdot \text{co-}\mathcal{C} = \mathrm{Sum}(1 \dot{-} \# \cdot \mathcal{C})$$

for $\mathcal{C}$ any class from the polynomial time hierarchy. This equation can be proved as follows: f is in $\mathrm{Sum}(1 \dot{-} \# \cdot C)$ iff there exist $A \in \mathcal{C}$ and polynomials p, p' such that

$$\begin{aligned} f(x) &= \textstyle\sum_{|z| \leq p'(|x|)} \big(1 \dot{-} \#\{\, y \mid |y| \leq p(|(x,z)|) \\ &\qquad \wedge (x,y,z) \in A \,\}\big) \\ &= \#\{\, z \mid |z| \leq p'(|x|) \\ &\qquad \wedge (\forall y, |y| \leq p(|(x,z)|))(x,y,z) \notin A \,\} \end{aligned}$$

iff $f \in \# \cdot \forall \cdot \text{co-}\mathcal{C}$.

Next, we show that any f satisfying the assumptions of the theorem can be reduced to the $1\dot{-}$ function, and vice versa, which implies $\langle \#\mathrm{P}\rangle_f = \langle \#\mathrm{P}\rangle_{1\dot{-}} = \# \cdot \mathrm{PH}$. So, let f be non-monotonic, non-constant, and fulfilling $f(x) = d$ for all $x > c$.

Then,

$$\begin{aligned} f(x) &= \textstyle\sum_{i=0}^{c} [\![x = i]\!] \cdot f(i) + [\![x > c]\!] \cdot d \\ &= \textstyle\sum_{i=0}^{c} [\![\binom{x}{i} \geq 1]\!] \cdot [\![\binom{x}{i+1} = 0]\!] \cdot f(i) \\ &\qquad + [\![\binom{x}{c+1} \geq 1]\!] \cdot d \\ &= \textstyle\sum_{i=0}^{c} (1 \dot{-} (1 \dot{-} \binom{x}{i})) \cdot (1 \dot{-} \binom{x}{i+1}) \cdot f(i) \\ &\qquad + (1 \dot{-} (1 \dot{-} \binom{x}{c+1})) \cdot d, \end{aligned}$$

which shows $f \leq_{\text{cnt}} 1 \dot{-}$.

To show $1 \dot{-} \leq_{\text{cnt}} f$, we distinguish two cases: If there exists $c' \leq c$ such that $f(c') > d = f(x)$ for all $x > c$, then

$$1 \dot{-} x = \binom{f(x \cdot (c+1) + c')}{f'(c)}.$$

Otherwise, there exist c', c'' such that $c' < c'' \leq c$ and $f(c'') < f(c') \leq d = f(x)$ for all $x > c$. Then

$$\mathrm{sgn}(x) = \binom{f(x \cdot (c+1) + c'')}{d}$$

and

$$1 \dot{-} x = \binom{f(\mathrm{sgn}(x) \cdot (c'' - c') + c')}{f(c')}.$$

For part 2, we remark that $\{-, \dot{-} 1\} \leq_{\text{cnt}} \{-, 1 \dot{-}\}$, since $x \dot{-} 1 = (x-1) \cdot \mathrm{sgn}(x)$. Thus, by the above, $\{-, \dot{-} 1\} \leq \{-, f\}$. The hardness now follows by Proposition 5.1. □

5.5 Corollary. *If f is non-monotonic and constant a.e., then*

1. #P *is closed under f* $\iff$ #P = # · PH $\iff$ UP = PH.
2. *f is counting hard* $\iff$ # · PH = FCH $\iff$ PH = CH.
3. *For every polynomial-time-computable extension f' of f,* GapP *is closed under f'* $\iff$ GapP = $\text{FCH}_{\mathbb{Z}}$ $\iff$ SPP = CH.

We now know that every function that is constant on an infinite interval is gap hard. An immediate consequence is the following corollary.

5.6 Corollary. *If f is a closure property of* GapP*, and $f' \neq f$ is any function such that $f(x) = f'(x)$ on an infinite interval, then $\{f, f'\}$ is gap hard, and thus:* GapP *is closed under f'* $\iff$ GapP = $\text{FCH}_{\mathbb{Z}}$ $\iff$ SPP = CH.

Proof. Under the assumptions of the corollary, $f - f'$ has an infinite constant interval, and is therefore gap hard by Theorems 5.2 and 5.4. However, trivially $f - f' \leq_{\text{cnt}} \{f, f', -\}$. □

These general results allow us now to directly conclude the following (since $\max\{-x, 0\}$ and $\min\{x, 0\}$ are constant a.e).

5.7 Corollary. *Maximum and minimum are gap hard.*

This in turn allows us to conclude (as already shown in [32]) that GapP is closed under maximum or minimum if and only if the counting hierarchy collapses to SPP. But we can also give a number of new results.

5.8 Corollary. *Taking absolute values is gap hard.*

Proof. $f(x) = x - \mathrm{abs}(x)$ has an infinite constant interval. □

This directly implies that taking square roots (or to be more precise, any function that behaves like the square-root function on all square numbers), is gap hard, since for sq any such function, $\mathrm{abs} \leq_{\text{cnt}} \mathrm{sq}$ by $\mathrm{abs}(x) = \mathrm{sq}(x^2)$ for all integers x. This can be generalized as follows:

5.9 Corollary. *If $p, q \in \mathbb{N}$ are such that either (1) p is odd and q is even, or (2) $p < q$, then every function f with $f(x) = \lfloor x^{p/q} \rfloor$ for $x \geq 0$ is gap hard.*

Proof. To prove part 1, we simply observe, that

$$f'(x) =_{\text{def}} x^p - (x^q)^{\frac{p}{q}} = x^p - \mathrm{abs}(x)^p$$

has an infinite constant interval, but is not constant; and $f' \leq_{\text{cnt}} f$.

For part 2, consider the functions

$$f_0(x) =_{\text{def}} x^p \text{ and } f_1(x) =_{\text{def}} \lfloor (x^q + 1)^{p/q} \rfloor.$$

If $x = 0$, then $f_0(x) = 0 \neq 1 = f_1(x)$. If $x \geq 1$, then $x^p < (x^q + 1)^{p/q} < x^p + 1$, since generally for $x > 0$ we have $x^r + 1 < (x+1)^r$ iff $r > 1$. Thus, for $x > 0$, we have $f_0(x) = f_1(x)$. Now, $f_0 - f_1$ has an infinite constant interval, but is not constant; and $f_0 - f_1 \leq_{\text{cnt}} f$. □

5.10 Remark. The proof of the previous result shows that for assumption (1), it actually does not matter whether we use the function $\lfloor x^{p/q} \rfloor$ or the function $\lceil x^{p/q} \rceil$, but for assumption (2) it is important.

Ogiwara and Hemachandra proved that #P cannot be closed under decrement unless NP $\subseteq$ SPP [27]. However, they could not prove that #P closed under decrement is equivalent to the collapse of the counting hierarchy to UP. Therefore, they considered decrement as a so-called *intermediate closure property.* We mentioned in the beginning that decrement is gap hard. Next, we show that it cannot be counting hard (unless the counting hierarchy collapses to NP).

5.11 Theorem. $\langle \#\text{P} \rangle_{\dot{-}1} \subseteq \langle \# \cdot \text{NP} \rangle_{\dot{-}1} = \# \cdot \text{NP} \subseteq \langle \#\text{P} \rangle_{\dot{-}1} - \text{FP}$.

Proof. The first inclusion is trivial, since $\langle\cdot\rangle$ is monotonic.

We next prove the equality. $\#\cdot\mathrm{NP}$ is known to be closed under addition, multiplication, summation, weak product, and selection (Choose) [38]. Thus we only have to show that $\#\cdot\mathrm{NP}$ is closed under decrement. Consider an arbitrary $f \in \#\cdot\mathrm{NP}$,

$$f(x) = \left|\left\{\, y \mid |y| \le p(|x|) \wedge (x,y) \in A \,\right\}\right|$$

for some polynomial p and some set $A \in \mathrm{NP}$. Define

$$\begin{aligned} A' =_{\mathrm{def}} \{\, (x,y) \mid{} & (x,y) \in A \\ & \wedge (\exists y', |y'| \le p(|x|))(x,y') \in A \,\}. \end{aligned}$$

Then, $A' \in \mathrm{NP}$, and obviously,

$$f(x) \dot{-} 1 = \left|\left\{\, y \mid |y| \le p(|x|) \wedge (x,y) \in A' \,\right\}\right|.$$

Second inclusion: We show that even $\#\cdot\text{co-NP} \subseteq \langle\#\mathrm{P}\rangle_{\dot{-}1}$, which suffices, because $\#\cdot\mathrm{NP} \subseteq \#\cdot\text{co-NP}$ ([34, Theorem 4.1.6], see also [40]). Let $f \in \#\cdot\text{co-NP}$,

$$f(x) = \left|\left\{\, z \mid 0 \le z \le 2^{p(|x|)} \wedge h(x,z) = 0 \,\right\}\right|$$

for some $h \in \#\mathrm{P}$ and some polynomial p. Then there exists an $h' \in \#\mathrm{P}$ and $g \in \mathrm{FP}$ such that $h(x,z) + h'(x,z) = g(x) + 1$ for $0 \le z \le 2^{p(|x|)}$. Thus,

$$\begin{aligned} f(x) &= \textstyle\sum_{z=0}^{2^{p(|x|)}} ((h(x,z) \dot{-} 1) + h'(x,z) - g(x)) \\ &= \textstyle\sum_{z=0}^{2^{p(|x|)}} ((h(x,z) \dot{-} 1) + h'(x,z)) \\ &\qquad - (2^{p(|x|)} + 1) \cdot g(x) \end{aligned}$$

□

5.12 Corollary.

1. $\#\mathrm{P}$ *is closed under decrement* $\Rightarrow$ $\mathrm{NP} \subseteq \mathrm{SPP}$, *and*
   $\mathrm{NP} = \mathrm{UP} \Rightarrow$ $\#\mathrm{P}$ *is closed under decrement*
2. *Decrement is counting hard* $\Rightarrow \mathrm{NP} = \mathrm{CH}$.

As already mentioned, statement 1 was already shown in [27]. Other operations considered intermediate by Ogiwara and Hemachandra are maximum and minimum. Above, we showed that both are gap hard. Next, we will show that they cannot be counting hard (unless the counting hierarchy collapses to NP).

5.13 Theorem.

1. $\#\cdot\mathrm{NP} \subseteq \langle\#\mathrm{P}\rangle_{\min} \subseteq \mathrm{FCH} \subseteq \langle\#\mathrm{P}\rangle_{\min} - \mathrm{FP}$ *and* $\exists\cdot\langle\#\mathrm{P}\rangle_{\min} = \mathrm{NP}$.
2. $\langle\#\mathrm{P}\rangle_{\max} \subseteq \mathrm{FCH} \subseteq \langle\#\mathrm{P}\rangle_{\max} - \mathrm{FP}$ *and* $\exists\cdot\langle\#\mathrm{P}\rangle_{\max} = \mathrm{NP}$.

5.14 Corollary.

1. $\#\mathrm{P}$ *is closed under* min $\Rightarrow \mathrm{SPP} = \mathrm{CH}$.
2. $\#\mathrm{P}$ *is closed under* max $\Rightarrow \mathrm{SPP} = \mathrm{CH}$.
3. min *is counting hard* $\Rightarrow \mathrm{NP} = \mathrm{CH}$.
4. max *is counting hard* $\Rightarrow \mathrm{NP} = \mathrm{CH}$.

The first two statements of the previous corollary were already proved in [27].

Next, we consider the operation of integer division by constants. It was shown in [32] that closure of GapP under this operation can be characterized by some collapse of certain MOD-classes to SPP, but for #P, the status of this operation remained open. We show:

5.15 Theorem. *Let k be prime.*

1. $\langle\#\mathrm{P}\rangle_{:k} \subseteq \#\cdot\mathrm{MOD}_k\mathrm{P} \subseteq \langle\#\mathrm{P}\rangle_{:k} - \mathrm{FP}$ *and* $\exists\cdot\langle\#\mathrm{P}\rangle_{:k} = \mathrm{NP}$.
2. $\langle\#\mathrm{P}\rangle_{:k,-} = \langle\#\mathrm{P}\rangle_{:k} - \mathrm{FP} = \#\cdot\mathrm{MOD}_k\mathrm{P} - \mathrm{FP}$.

Proof. Statement 1, first inclusion: Since $\#\mathrm{P} \subseteq \#\cdot\mathrm{MOD}_k\mathrm{P}$ and since for k prime, $\#\cdot\mathrm{MOD}_k\mathrm{P}$ is known to be closed under addition, multiplication, summation, weak product, and selection (Choose) [38], we only have to show that $\#\cdot\mathrm{MOD}_k\mathrm{P}$ is closed under division by k. Consider a function $f \in \#\cdot\mathrm{MOD}_k\mathrm{P}$,

$$f(x) = \left|\left\{\, y \mid |y| \le p(|x|) \wedge (x,y) \in A \,\right\}\right|$$

for some polynomial p and a set $A \in \mathrm{MOD}_k\mathrm{P}$. Define

$$\begin{aligned} A' =_{\mathrm{def}} \{\, (x,y) \mid{} & (x,y) \in A \\ & \wedge \left|\left\{\, y' \mid y' \le y \wedge (x,y') \in A \,\right\}\right| \\ & \equiv 0 \pmod{k} \,\} \end{aligned}$$

Then $A' \in \mathrm{MOD}_k\mathrm{P}$ by the closure properties of $\mathrm{MOD}_k\mathrm{P}$ proved in [4], and obviously,

$$f(x) : k = \left|\left\{\, y \mid |y| \le p(|x|) \wedge (x,y) \in A' \,\right\}\right|.$$

Statement 1, second inclusion: Let $f \in \#\cdot\mathrm{MOD}_k\mathrm{P}$,

$$f(x) = \left|\left\{\, z \mid 0 \le z \le 2^{p(|x|)} \wedge h(x,z) \equiv 0 \pmod{k} \,\right\}\right|$$

for some $h \in \#\mathrm{P}$. Since k is prime, then we may suppose without loss of generality [4] that $h(x,z) \in \{0,1\} \pmod{k}$. Moreover, there exists some $h' \in \#\mathrm{P}$ and $g \in \mathrm{FP}$ such that $h(x,z) + h'(x,z) = g(x) + 1$ for all $0 \le z \le 2^{p(|x|)}$. Then,

$$\begin{aligned} f(x) &= \left|\left\{\, z \mid 0 \le z \le 2^{p(|x|)} \right.\right. \\ &\qquad \left.\left. \wedge (h(x,z) : k) \cdot k = h(x,z) \,\right\}\right| \\ &= \left|\left\{\, z \mid 0 \le z \le 2^{p(|x|)} \right.\right. \\ &\qquad \left.\left. \wedge (h(x,z) : k) \cdot k + h'(x,z) = g(x) + 1 \,\right\}\right| \end{aligned}$$

Abbreviating $(h(x,z):k)\cdot k+h'(x,z)$ by $h''(x,z)$, we then have

$$\begin{aligned} f(x) &= \sum_{z=0}^{2^{p(|x|)}}(h''(x,z)-g(x)) \\ &= \sum_{z=0}^{2^{p(|x|)}} h''(x,z)-(2^{p(|x|)}+1)\cdot g(x) \end{aligned}$$

Statement 2 then follows easily, since $\#\cdot\mathrm{MOD}_k\,\mathrm{P}-\mathrm{FP}=\mathrm{GapP}^{\mathrm{MOD}_k\,\mathrm{P}}$, which is closed under subtraction. The last equality of statement 1 can be shown by induction on the number of steps in the generation of functions in $\langle\#\mathrm{P}\rangle_{:k}$. This will be done in the full paper.

□

5.16 Corollary. *Let k be prime.*

1. $\#\mathrm{P}$ *is closed under integer division by* $k \Rightarrow \mathrm{SPP}=\mathrm{MOD}_k\,\mathrm{P}$, *and* $\mathrm{UP}=\mathrm{MOD}_k\,\mathrm{P}\Rightarrow\#\mathrm{P}$ *is closed under integer division by* k.
2. *Integer division by k is counting hard* $\Rightarrow \mathrm{NP}=\mathrm{MOD}_k\,\mathrm{P}=\mathrm{CH}$.
3. GapP *is closed under integer division by* $k \iff \mathrm{SPP}=\mathrm{MOD}_k\,\mathrm{P}$.
4. *Integer division by k is gap hard* $\iff \mathrm{MOD}_k\,\mathrm{P}=\mathrm{CH}$.

The first implication of statement 1 was already shown in [27]. Statement 3 was already shown in [32].

It turns out that integer division by the constant m has similar power as any non-constant *m-periodic* function f (that is, a function f, for which $f(x)=f(x+m)$ for all x). Thus, we get:

5.17 Theorem. *Let f be non-constant and m-periodic for a prime number m.*

1. $\#\cdot\mathrm{MOD}_m\,\mathrm{P}\subseteq\langle\#\mathrm{P}\rangle_f$.
2. $\langle\#\mathrm{P}\rangle_{f,-}=\langle\#\mathrm{P}\rangle_f-\mathrm{FP}=\#\cdot\mathrm{MOD}_m\,\mathrm{P}-\mathrm{FP}$.

Proof. The first statement can be proved very similarly to the corresponding proof of Theorem 5.15. The key argument again is that the function $\bmod m$ can in essence be simulated by f as follows: Let d_1,d_2 be such that $f(d_1)<f(d_2)$. Then,

$$\begin{aligned} x\equiv 0 \pmod m \quad &\Rightarrow \tbinom{f(d_1+x)}{f(d_2)}=0 \\ x\equiv d_2-d_1 \pmod m &\Rightarrow \tbinom{f(d_1+x)}{f(d_2)}=1 \end{aligned}$$

It follows from [4] that it suffices to restrict oneself to these two cases (for m prime). Thus, $\#\cdot\mathrm{MOD}_m\,\mathrm{P}\subseteq\langle\#\mathrm{P}\rangle_f$.

For statement 2, we show that $f\le_{\mathrm{cnt}}\{\,:m,-\}$. The other inclusions then follow as in the proof of Theorem 5.15. So let f be as in the statement of the theorem. Then obviously,

$$f(x)=f(x-m\cdot(x:m)),$$

that is, the function f is already determined by the finitely many values $f(0),\ldots,f(m-1)$. This finite part of the function can be expressed as a finite sum over weighted binomial coefficients, very similar to the proof of Theorem 5.2. To be more precise, let $a_0=_{\mathrm{def}} f(0)$ and let $a_i=_{\mathrm{def}} f(i)-\sum_{j=0}^{i-1}a_j\cdot\binom{i}{j}$ for $0<i<m$. Observe, that all a_i's are integer constants. Then we have, for $0\le x\le m-1$,

$$f(x)=\sum_{i=0}^{m-1}a_i\cdot\binom{x}{i}.$$

□

5.18 Corollary. *Let f be non-constant and m-periodic for a prime number m.*

1. $\#\mathrm{P}$ *is closed under* $f \Rightarrow \mathrm{UP}=\mathrm{MOD}_m\,\mathrm{P}$.
2. f *is counting hard* $\Rightarrow \mathrm{MOD}_m\,\mathrm{P}=\mathrm{CH}$.
3. GapP *is closed under* $f \iff \mathrm{SPP}=\mathrm{MOD}_m\,\mathrm{P}$.
4. f *is gap hard* $\iff \mathrm{MOD}_m\,\mathrm{P}=\mathrm{CH}$.

An interesting combination of Theorem 5.4 and Theorem 5.17 is the following result concerning functions which we call *m-periodic a.e.*, i. e. functions f for which there exists $c\in\mathbb{N}$, such that $f(x)=f(x+m)$ for all $x>c$. Let $\{\mathrm{MOD}_m,\exists,\forall\}^*\mathrm{P}$ be the oracle hierarchy defined by the classes NP and $\mathrm{MOD}_m\,\mathrm{P}$.

5.19 Theorem. *Let f be non-periodic, not constant a.e., but m-periodic a.e. for a prime m.*

1. $\langle\#\mathrm{P}\rangle_f=\#\cdot\{\mathrm{MOD}_m,\exists,\forall\}^*\mathrm{P}$.
2. *Any extension of f to the domain of the integers is gap hard.*

Proof. Statement 1 is proved as follows:

To prove the inclusion from left to right, we show that $f\le_{\mathrm{cnt}}\{:m,1\dot{-},\mathrm{sgn}\}$. Let c and m be as in the definition of periodic a.e. Then

$$\begin{aligned} f(x) = {} & [\![x<c]\!]\cdot g_1(x) \\ & +[\![x\ge c]\!]\cdot g_2(x-c-((x-c):m)\cdot m) \end{aligned}$$

g_1 and g_2 are finite functions which can be expressed as follows:

$$g_1(x)=\sum_{i=0}^{c-1}[\![x=i]\!]\cdot a_i \quad\text{and}\quad g_2(x)=\sum_{i=0}^{m-1}[\![x=i]\!]\cdot b_i,$$

where $a_i = g_1(i)$ $(0 \leq i < c)$ and $b_i = g_2(i)$ $(0 \leq i < m)$. $[\![x \geq u]\!]$ can be expressed as $\operatorname{sgn}\binom{x}{u}$, $[\![x < u]\!]$ as $1 \dot{-} [x \geq u]$, and $[\![x = u]\!]$ as $[\![x \geq u]\!] \cdot [\![x < u+1]\!]$. Now it remains to observe that $\langle \#\mathrm{P} \rangle_{:m, 1\dot{-}, \mathrm{sgn}} \subseteq \# \cdot \{\mathrm{MOD}_m, \exists, \forall\}^*\mathrm{P}$, which follows, since the latter class is closed under $+, \cdot, \mathrm{Sum}, \mathrm{WProd}, \mathrm{Choose}$ [38] as well as $1 \dot{-}, \mathrm{sgn}$ by a relativization of Theorem 5.4 and $: k$ by a relativization of Theorem 5.17.

To prove the inclusion from right to left, we show that $\# \cdot \{\mathrm{MOD}_m, \exists, \forall\}^*\mathrm{P} \subseteq \langle \#\mathrm{P} \rangle_f$. First, we note that every class from the oracle hierarchy defined by $\mathrm{MOD}_m\,\mathrm{P}$ and NP can be characterized by a corresponding sequence of MOD_m, $\exists$, and $\forall$ quantifiers applied to the class P. We then proceed inductively as follows: $\# \cdot \mathrm{MOD}_m \cdot \mathcal{C} \subseteq \langle \# \cdot \mathcal{C} \rangle_f$ as in the proof of Theorem 5.17, first statement. $\# \cdot \exists \cdot \mathcal{C} \subseteq \langle \# \cdot \mathcal{C} \rangle_f$ and $\# \cdot \forall \cdot \mathcal{C} \subseteq \langle \# \cdot \mathcal{C} \rangle_f$ as in the proof of Theorem 5.4, since $\{\mathrm{sgn}, 1 \dot{-}\} \leq_{\mathrm{cnt}} f$ by the considerations below: If there is $x_0 \leq c$ such that $f(x_0) > f(x_0 + m)$, then

$$1 \dot{-} x = \binom{f(x_0 + x \cdot m)}{f(x_0)},$$

and $\mathrm{sgn}(x) = 1 \dot{-} (1 \dot{-} x)$. In the other case, there must be an $x_0 \leq c$ such that $f(x_0) < f(x_0 + m)$. Then

$$\mathrm{sgn}(x) = \binom{f(x_0 + x \cdot m)}{f(x_0 + m)}.$$

Moreover, since f is periodic for $x > c$, there must exist $x_1 < x_2$ such that $f(x_1) > f(x_2)$. Then,

$$1 \dot{-} x = \binom{f(x_1 + \mathrm{sgn}(x) \cdot (x_2 - x_1))}{f(x_1)}.$$

For statement 2, it suffices to notice that $f(x) - f(x+m)$ has an infinite constant interval and is not constant. □

5.20 Corollary. *Let f be non-periodic, not constant a.e., but m-periodic a.e. for a prime m.*

1. $\#\mathrm{P}$ *is closed under* $f \iff \mathrm{UP} = \{\mathrm{MOD}_m, \exists, \forall\}^*\mathrm{P}$.
2. f *is counting hard* $\iff \{\mathrm{MOD}_m, \exists, \forall\}^*\mathrm{P} = \mathrm{CH}$.
3. *For every polynomial-time-computable extension f' of f,* GapP *is closed under* $f' \iff \mathrm{GapP} = \mathrm{FCH}_{\mathbb{Z}} \iff \mathrm{SPP} = \mathrm{CH}$.

Finally, we turn to the functions that are *finitely invertible,* i.e. those functions f such that for every y, there are only finitely many x such that $f(x) = y$.

5.21 Theorem. *Let f be finitely invertible, then* $\exists \cdot \langle \#\mathrm{P} \rangle_f \subseteq \mathrm{PH}$.

Proof. For this proof, we have to consider another function-to-set operator. For a function f, let $\mathrm{E}_k^f =_{\mathrm{def}} \{\, x \mid f(x) = k \,\}$, and for a class of functions $\mathcal{F}$, let

$$\mathrm{E}_k \cdot \mathcal{F} =_{\mathrm{def}} \{\, \mathrm{E}_k^f \mid f \in \mathcal{F} \,\}.$$

We then prove that $\mathrm{E}_k \cdot \langle \#\mathrm{P} \rangle_f \subseteq \mathrm{PH}$ for every $k \geq 0$ by a not too difficult, but a bit lengthy, inductive argument using the inductive generation of the concerned function classes. This will be done in the full paper. The statement of the theorem then easily follows, since obviously $\exists \cdot \langle \#\mathrm{P} \rangle_f \subseteq \text{co-}\mathrm{E}_k \cdot \langle \#\mathrm{P} \rangle_f$. □

5.22 Corollary. *Let f be finitely invertible. If f is counting hard, then the counting hierarchy collapses to the polynomial-time hierarchy.*

This result implies e.g. that while $f(x) = \lfloor x^{3/2} \rfloor$ is gap hard, it cannot be counting hard (unless $\mathrm{PH} = \mathrm{CH}$).

6 Conclusion

We presented results concerning closure of #P and GapP under number theoretic functions which are stronger than and subsume the results known up to now. We gave a complete characterization of all relativizable functional closure properties and examined structural consequences of some functions being closure properties of #P and GapP. Our results from Section 3 showed that if some function which is not a finite sum of binomial coefficients is a closure property of #P, then some collapses concerning US and other classes with finite acceptance types occur. Of course, this does not necessarily imply a larger collapse, e.g. of the counting hierarchy; and therefore, we could not hope to find restrictions as strict as in Section 3 also in Section 5. Nevertheless, we were able to completely classify large classes of functions with respect to their power as operations on counting functions.

In a preliminary version of this paper, we asked whether one can give relativization results similar to those presented here for the class GapP. This seems not to be possible using the present approach of separability of finite acceptance type classes. The reason for this is that all finite acceptance classes for GapP (i.e. those classes defined exactly as the classes (A)P in Section 2, but now requiring $f \in$ GapP) collapse to the class C=P (for a definition of this class, see [31, 42]). However, in the meantime Richard Beigel [3] showed, how to use the polynomial method [2], first to obtain an alternative proof of our Theorem 3.13, and second to prove that a function $f\colon \mathbb{Z} \to \mathbb{Z}$ is a relativizable closure property of GapP if and only if $f(n) = \sum_{0\leq k\leq m} \alpha_k \binom{n}{k}$ for some $m \in \mathbb{N}$ and integers $\alpha_0, \ldots, \alpha_m$.

Acknowledgments. The authors wish to thank Eric Allender, Rutgers University, New Brunswick, and Ronald V. Book and Celia Wrathall, University of California, Santa Barbara, for many valuable hints.

References

[1] J. L. Balcázar, J. Díaz, J. Gabarró, *Structural Complexity I* (Springer Verlag, Berlin – Heidelberg – New York, ²1994).

[2] R. Beigel, The polynomial method in circuit complexity; *8th Structure in Complexity Theory Conference* (1993), pp. 82–95.

[3] R. Beigel, Closure Properties of GapP and # P; Technical Report, Yale University, Department of Computer Science, Yale (1994).

[4] R. Beigel, J. Gill, U. Hertrampf, Counting classes: thresholds, parity, mods, and fewness; *7th Symposium on Theoretical Aspects of Computer Science* (1990), LNCS 415, pp. 49–57.

[5] R. Beigel, N. Reingold, D. Spielman, PP is closed under intersection; *23rd Symposium on Theory of Computing* (1991), pp. 1–9.

[6] A. Blass, Y. Gurevich, On the unique satisfiability problem; *Information and Control* **55** (1982), pp. 80–88.

[7] D. P. Bovet, P. Crescenzi, R. Silvestri, Complexity classes and sparse oracles; *6th Structure in Complexity Theory Conference* (1991), pp. 102–108.

[8] D. P. Bovet, P. Crescenzi, R. Silvestri, A uniform approach to define complexity classes; *Theoretical Computer Science* **104** (1992), pp. 263–283.

[9] S. Fenner, L. Fortnow, S. Kurtz, Gap-definable counting classes; *Journal of Computer and System Sciences* **48** (1994), pp. 116–148.

[10] L. Fortnow, N. Reingold, PP is closed under truth-table reductions; *6th Structure in Complexity Theory Conference* (1991), pp. 13–15.

[11] J. Gill, Computational complexity of probabilistic complexity classes; *SIAM Journal on Computing* **6** (1977), pp. 675–695.

[12] F. Green, J. Köbler, K. Regan, T. Schwentick, J. Torán, The power of the middle bit of a # P function; to appear in *Journal of Computer and System Sciences.*

[13] T. Gundermann, N. A. Nasser, G. Wechsung, A survey on counting classes; *5th Structure in Complexity Theory Conference* (1990), pp. 140–153.

[14] T. Gundermann, G. Wechsung, Counting classes of finite acceptance types, *Computers and Artificial Intelligence* **6** (1987), pp. 395–409.

[15] S. Gupta, The power of witness reduction; *6th Structure in Complexity Theory Conference* (1991), pp. 43–59.

[16] S. Gupta, On the closure of certain function classes under integer division by polynomial-bounded functions; *Information Processing Letters* **44** (1992), pp. 205–210.

[17] U. Hertrampf, Complexity classes with finite acceptance types; *11th Symposium on Theoretical Aspects of Computer Science* (1994), LNCS 775, pp. 543–553.

[18] U. Hertrampf, Complexity classes defined via k-valued functions; *9th Structure in Complexity Theory Conference* (1994), pp. 224–234.

[19] U. Hertrampf, Classes of bounded counting type and their inclusion relations; *12th Symposium on Theoretical Aspects of Computer Science* (1995), LNCS 900, pp. 60–70.

[20] U. Hertrampf, C. Lautemann, T. Schwentick, H. Vollmer, K. W. Wagner, On the power of polynomial time bit-reductions; *8th Structure in Complexity Theory Conference* (1993), pp. 200–207.

[21] U. Hertrampf, H. Vollmer, K. W. Wagner, On balanced vs. unbalanced computation trees; to appear in *Mathematical Systems Theory.*

[22] J. E. Hopcroft, J. D. Ullman, *Introduction to Automata Theory, Languages, and Computation* (Addison-Wesley, Reading, Mass., 1979).

[23] B. Jenner, P. McKenzie, D. Thérien, Logspace and logtime leaf languages; *9th Structure in Complexity Theory Conference* (1994), pp. 242–254.

[24] J. Köbler, U. Schöning, J. Torán, On counting and approximation; *Acta Informatica* **26** (1989), pp. 363–379.

[25] J. Köbler, U. Schöning, S. Toda, J. Torán, Turing machines with few accepting computations and low sets for PP; *Journal of Computer and System Sciences* **44** (1992), pp. 272–286.

[26] J. KÖBLER, S. TODA, On the power of generalized MOD-classes; *8th Structure in Complexity Theory Conference* (1993), pp. 147–155.

[27] M. OGIWARA, L. HEMACHANDRA, A complexity theory for feasible closure properties; *Journal of Computer and System Sciences* **46** (1993), pp. 295–325.

[28] C. H. PAPADIMITRIOU, *Computational Complexity* (Addison-Wesley, Reading, Mass., 1994).

[29] C. H. PAPADIMITRIOU, S. K. ZACHOS, Two remarks on the power of counting; *6th GI-Conference on Theoretical Computer Science* (1983), LNCS 145, pp. 269–275.

[30] K. REGAN, Enumeration problems; manuscript (1982).

[31] J. SIMON, On Some Central Problems in Computational Complexity; Dissertation, Cornell University, Ithaca (1975).

[32] T. THIERAUF, S. TODA, O. WATANABE, On closure properties of GapP; *Computational Complexity* **4** (1994), pp. 242–261.

[33] S. TODA, PP is as hard as the polynomial time hierarchy; *SIAM Journal on Computing* **20** (1991), pp. 865–877.

[34] S. TODA, Computational Complexity of Counting Complexity Classes; Dissertation, Tokyo Institute of Technology, Department of Computer Science, Tokyo (1991).

[35] L. G. VALIANT, Relative complexity of checking and evaluation, *Information Processing Letters* **5** (1976), pp. 20–23.

[36] L. G. VALIANT, The complexity of computing the permanent; *Theoretical Computer Science* **8** (1979), pp. 189–201.

[37] N. K. VERESHCHAGIN, Relativizable and nonrelativizable theorems in the polynomial theory of algorithms; *Izvestija Rossijskoj Akademii Nauk* **57** (1993), pp. 51–90. In Russian.

[38] H. VOLLMER, Komplexitätsklassen von Funktionen; Dissertation, Universität Würzburg, Institut für Informatik, Würzburg (1994).

[39] H. VOLLMER, K. W. WAGNER, Complexity classes of optimization functions; to appear in *Information and Computation*.

[40] H. VOLLMER, K. W. WAGNER, The complexity of finding middle elements; *International Journal of Foundations of Computer Science* **4** (1993), pp. 293–307.

[41] H. VOLLMER, K. W. WAGNER, Recursion theoretic characterizations of complexity classes of counting functions; Technical Report #91, Universität Würzburg, Institut für Informatik, 1994. Submitted for publication.

[42] K. W. WAGNER, Some observations on the connection between counting and recursion; *Theoretical Computer Science* **47** (1986), pp. 131–147.

[43] C. WRATHALL, Complete sets and the polynomial-time hierarchy; *Theoretical Computer Science* **3** (1977), pp. 23–33.