On Different Reducibility Notions for Function Classes*

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Abstract. This paper continues research of Toda (The Complexity of Finding Medians, 31^{st} *Symposium on Foundations of Computer Science* (1990), pp. 778–787) on problems complete for function classes like $\mathrm{FP}^{\#\mathrm{P}}$ and $\mathrm{Mid\,P}$ under Krentel's metric reductions. We first show that metric reductions wipe out the difference between $\mathrm{Mid\,P}$ and other related classes of functions which are probably different from $\mathrm{Mid\,P}$. In order to obtain a more detailed classification of naturally arising functional problems we then define and examine a stricter notion of reducibility and show that a number of problems, among them those proved by Toda to be hard for $\mathrm{Mid\,P}$ under metric reductions, are complete for different classes of median functions related to $\mathrm{Mid\,P}$ under our stricter reducibility. Finally, we use these results to exhibit new natural complete sets for the well studied classes of sets PP, $\mathrm{PP}^{\mathrm{NP}}$, and P^{PP}.

1 Introduction

In [9] Toda introduced the class of functions $\mathrm{Mid\,P}$. It consists of those functions that yield the middle element in the set of output values of nondeterministic polynomial time Turing machines. This class is contained in $\mathrm{FP}^{\#\mathrm{P}}$, the class of all functions computable by a deterministic polynomial time Turing machine with oracle queries to Valiant's class $\#\mathrm{P}$ [11]. Toda compared functions from these classes under metric reductions, first introduced by Krentel [6] in the context of the function class $\mathrm{Opt\,P}$, and a characterization theorem for $\mathrm{FP}^{\#\mathrm{P}}$ allowed him to exhibit a number of natural complete problems for that class.

However, the power of a Turing machine performing a metric reduction is rather high. The class $\mathrm{Mid\,P}$ for instance is probably not closed under metric reductions but its closure is equal to $\mathrm{FP}^{\#\mathrm{P}}$. We consider two other classes of median functions: A class $\mathrm{Med\,P}$ which is defined to consist of those functions that yield the middle element in the ordered *sequence* (not *set*, as for $\mathrm{Mid\,P}$) of output values of nondeterministic polynomial time Turing machines; this class is (under reasonable complexity theoretic assumptions) incomparable to $\mathrm{Mid\,P}$ (and there are oracles separating them) but it is also contained in $\mathrm{FP}^{\#\mathrm{P}}$; and a class $\mathrm{M\overline{ed}\,P}$ defined to yield the middle accepting path itself (in any reasonable encoding) of machines as above, a (probably strict) subclass of both other median

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classes. We then show that the closure of these classes under metric reductions is also equal to $\mathrm{FP}^{\#\mathrm{P}}$; and that all four classes of functions share the same hard problems under metric reductions. Thus this kind of reducibility wipes out the differences between the considered classes.

This is an unsatisfactory situation. For instance, the function yielding the middle satisfying assignment of a propositional formula seems to be dissimilar to the function yielding the middle weight of a knapsack. The first one is a typical element of $\mathrm{M\overline{ed}\,P}$, since by Cook's Theorem [3] a satisfying assignment corresponds directly to an accepting path of a Turing machine. The second one however is of another kind, since the contents of the knapsack corresponds to a path of a Turing machine while its weight is similar to the output on that path; thus this function is an element of $\mathrm{Mid\,P}$ (or $\mathrm{Med\,P}$, depending on whether we are willing to count multiplicities), and we will see that it is not in $\mathrm{M\overline{ed}\,P}$ (unless the classes are equal). But *both* functions are complete for $\mathrm{FP}^{\#\mathrm{P}}$ under metric reductions and therefore also hard or complete for all three of the median classes. The equivalence class of these problems seems to be rather large and contains functions with differing properties. So what we have to look for when we want to classify these functions better is a different reducibility notion.

We propose a stricter notion of reducibility which we call functional many-one reducibility. This notion copies one of the strictest reducibilities between sets: the intuitively well-understood many-one reducibility. We observe that all three median classes are closed under this reducibility and possess complete problems. Thus, we conclude that—in contrast to metric reducibility—if the classes share the same hard problems under functional many-one reducibility then they are equal (contradicting relativized worlds). Then we present a number of natural complete problems for the median classes and we show that the problems considered by Toda or slight modifications of them are complete in different median classes under our stricter reducibility.

These results show that our reducibility is better suited for the examination of function classes in $\mathrm{FP}^{\#\mathrm{P}}$. We can show that the two functions from above, which in our intuition have dissimilar properties, are indeed of different complexity by proving them complete for different median classes under functional many-one reducibility. Our paper develops a means for the classification of functions and the complexity classes they define which in this context is superior to the comparison via metric reductions.

Eventually, we prove that a completeness result for a function class under functional many-one reducibility directly yields completeness results for corresponding classes of sets under usual many-one reducibility. Thus, we exhibit more natural complete problems for PP, P^{PP}, and $\mathrm{PP}^{\mathrm{NP}}$. Wagner in [13] already presented problems complete for the last of these classes, but all of his problems were defined using so called *input languages*; i.e. problems of lower complexity become $\mathrm{PP}^{\mathrm{NP}}$-complete when its instances are presented in a succinct representation. Our problems are the (to our knowledge) first natural problems which are $\mathrm{PP}^{\mathrm{NP}}$-complete not because of the input representation but because of their intrinsic complexity.

2 Preliminaries

We assume the reader to be familiar with basic complexity theory notions. Our notation here is standard; see e.g. [1].

We start by defining the classes of median-functions we consider. Let FP denote the class of functions computable in deterministic polynomial time.

The following definition is equivalent to Toda's definition of the class Mid P [9]:

Definition 1. The class Mid P consists of those functions h, for which there exist $f, g \in \mathrm{FP}$, such that if the set of all values $f(x,y)$ for $0 \leq y \leq g(x)$ is

$$\{z_0, z_1, \ldots, z_k\}, \quad z_0 < z_1 < \cdots < z_k,$$

then $h(x) =_{\mathrm{def}} z_{\lfloor k/2 \rfloor}$.

Our version of a median class where multiplicities are taken into account is made precise in the following definition (which is from [12]):

Definition 2. The class Med P consists of those functions h, for which there exist $f, g \in \mathrm{FP}$, such that if the ordered sequence of all values $f(x,y)$ for $0 \leq y \leq g(x)$ (with multiplicities) is

$$z_0 \leq z_1 \leq \cdots \leq z_{g(x)},$$

then $h(x) =_{\mathrm{def}} z_{\lfloor g(x)/2 \rfloor}$.

It is obvious, that Med P (Mid P, resp.) is the class of functions that yield the middle element in the sequence (set, resp.) of the outputs over all paths of a nondeterministic polynomial time Turing machine. If this relation holds between such a machine M and a function f from Med P or Mid P, we say that f is computed by M.

The third kind of median class we consider is the following, again introduced in [12]:

Definition 3. Let $\mathcal{K}$ be a class of sets. Then the class $\mathrm{M\overline{ed}}\,\mathcal{K}$ consists of those functions h, for which there exist $A \in \mathcal{K}$, $g_1, g_2 \in \mathrm{FP}$ such that $h(x)$ is equal to the middle element of the set

$$\{\, y \mid g_1(x) \leq y \leq g_2(x) \wedge (x,y) \in A \,\}.$$

By nature of definiton, in this case only elements with multiplicity one can occur; and this makes it possible to show the following inclusionships [12]:

Proposition 4. $\mathrm{M\overline{ed}}\,\mathrm{P} \subseteq \mathrm{Med}\,\mathrm{P}$ *and* $\mathrm{M\overline{ed}}\,\mathrm{P} \subseteq \mathrm{Mid}\,\mathrm{P}$.

It was shown in [12] that under reasonable widely accepted complexity theoretic assumptions these inclusionships are strict: If $\overline{\mathrm{Med}}\,\mathrm{P} = \mathrm{Mid}\,\mathrm{P}$, then the second level of the polynomial time hierarchy is in PP (contradicting some relativized worlds [2]); and if $\overline{\mathrm{Med}}\,\mathrm{P} = \mathrm{Med}\,\mathrm{P}$, then the counting hierarchy [13, 10] collapses. Moreover, it was shown that under the same assumptions, $\mathrm{Med}\,\mathrm{P}$ and $\mathrm{Mid}\,\mathrm{P}$ are uncomparable: If $\mathrm{Med}\,\mathrm{P} \subseteq \mathrm{Mid}\,\mathrm{P}$, then the counting hierarchy collapses; and if $\mathrm{Mid}\,\mathrm{P} \subseteq \mathrm{Med}\,\mathrm{P}$, then PP contains the second level of the polynomial time hierarchy.

Additionally, it was shown that the operator $\overline{\mathrm{Med}}$ allows a characterization of Toda's class: $\overline{\mathrm{Med}}\,\mathrm{NP} = \mathrm{Mid}\,\mathrm{P}$ [12].

Vollmer and Wagner proved an analogue to Toda's main theorem [9] also for the classes $\overline{\mathrm{Med}}\,\mathrm{P}$ and $\mathrm{Med}\,\mathrm{P}$. For a function f and a class of functions $\mathcal{F}$, $\mathrm{FP}^f[k]$ ($\mathrm{FP}^{\mathcal{F}}[k]$, resp.) denotes the class of functions computable by a deterministic Turing machine in polynomial time with k queries to f (a function from $\mathcal{F}$, resp.).

Proposition 5.

$$\begin{aligned} \mathrm{FP}^{\mathrm{PP}} &= \mathrm{FP}^{\mathrm{Mid}\,\mathrm{P}} = \mathrm{FP}^{\mathrm{Mid}\,\mathrm{P}}[1] \\ &= \mathrm{FP}^{\mathrm{Med}\,\mathrm{P}} = \mathrm{FP}^{\mathrm{Med}\,\mathrm{P}}[1] \\ &= \mathrm{FP}^{\overline{\mathrm{Med}}\,\mathrm{P}} = \mathrm{FP}^{\overline{\mathrm{Med}}\,\mathrm{P}}[1] \end{aligned}$$

(The first two equalities of this proposition were already proved by Toda [9].)

It was even shown in [12] that the $\mathrm{P}^{\mathrm{Mid}\,\mathrm{P}}[1]$, $\mathrm{P}^{\mathrm{Med}\,\mathrm{P}}[1]$, and $\mathrm{P}^{\overline{\mathrm{Med}}\,\mathrm{P}}[1]$ computations can be further normalized. They can be arranged to consist only of asking for one value of the oracle function class and then giving as output the rest of the integer division of this function value by 2. That is, for every $A \in \mathrm{P}^{\mathrm{PP}}$, there exists some $h \in \mathrm{Mid}\,\mathrm{P}$ (or $\mathrm{Med}\,\mathrm{P}$, or $\overline{\mathrm{Med}}\,\mathrm{P}$) such that for all x, $c_A(x) \equiv h(x) \pmod 2$.

Thus, although the remarks above show that the three median classes are probably different, there are contexts in which all three classes have the same power. This duality will again come up when we talk about reducibilities in the next sections.

3 Metric Reductions

The reducibility notion we first examine is a well known one, introduced by Krentel [6].

Definition 6. A function f is metrically reducible to function h, in symbols: $f \leq_{met}^{\mathrm{FP}} h$, if and only if there exist two functions $g, g' \in \mathrm{FP}$ such that for all x, $f(x) = g'(x, h(g(x)))$.

Metric reductions were examined by Toda [9] in the context of the class $\mathrm{Mid}\,\mathrm{P}$. We now show that they wipe out the differences between our three median classes.

Lemma 7. *Let f and h be functions. Then $f \leq_{met}^{\mathrm{FP}} h$ if and only if $f \in \mathrm{FP}^h[1]$. Thus, for a class of functions $\mathcal{F}$, the closure of $\mathcal{F}$ under metric reductions is equal to the class* $\mathrm{FP}^{\mathcal{F}}[1]$.

Proof. obvious. $\square$

Corollary 8. *The closure of each of the classes* $\mathrm{M\overline{ed}\,P}$, $\mathrm{Med\,P}$, *and* $\mathrm{Mid\,P}$ *is equal to* $\mathrm{FP}^{\#\mathrm{P}}$.

Proof. Consequence of the previous lemma and Proposition 5. $\square$

Corollary 9. *The classes* $\mathrm{M\overline{ed}\,P}$, $\mathrm{Med\,P}$, $\mathrm{Mid\,P}$, *and* $\mathrm{FP}^{\#\mathrm{P}}$ *have the same hard functions under* $\leq_m^{\mathrm{FP}}$*-reducibility.*

Thus we get a number of natural problems to be hard under metric reductions for all three median classes, namely those shown to be complete for $\mathrm{FP}^{\#\mathrm{P}}$ by Toda [9], e.g. Lexical middle satisfying assignment of a boolean formula, Lexical middle Hamiltonian circuit of an undirected graph, and many others. We will later come back to these problems and give exact definitions of them.

4 Functional Many-one Reductions

Having seen that metric reductions are not sufficient to distinguish between the classes $\mathrm{M\overline{ed}\,P}$, $\mathrm{Med\,P}$, and $\mathrm{Mid\,P}$, we now proceed by introducing a stricter notion of reducibility which is similar in spirit to many-one reducibility between sets (see also [13, 5]).

Definition 10. A function f is polynomial time functionally many-one reducible to function h, in symbols: $f \leq_m^{\mathrm{FP}} h$, if and only if there exists a function $g \in \mathrm{FP}$ such that for all x, $f(x) = h(g(x))$.

The following is easy to verify:

Proposition 11. $\mathrm{M\overline{ed}\,P}$, $\mathrm{Med\,P}$ *and* $\mathrm{Mid\,P}$ *are closed under* $\leq_m^{\mathrm{FP}}$.

Proof. Let h be a function from any of the considered classes and let h be computed by the Turing machine M, and let $f(x) =_{\mathrm{def}} h(g(x))$ for some $g \in \mathrm{FP}$. Then f is computed by the machine M' which on input x first computes $g(x)$ and then simulates the work of M on input $g(x)$. $\square$

Complete problems can be constructed artificially:

Theorem 12. $\mathrm{M\overline{ed}\,P}$, $\mathrm{Med\,P}$ *and* $\mathrm{Mid\,P}$ *have* $\leq_m^{\mathrm{FP}}$ *complete functions.*

Proof. The construction proceeds in a canonical way: Define

$\textsc{Univ}_{\mathrm{M\overline{ed}}}(x, M, 0^j, 0^k)$ to be the middle y, $|y| \leq j$, such that the Turing machine encoded (in any reasonable way) by M on input (x, y) run for k steps accepts;

$\textsc{Univ}_{\mathrm{Med}}(x, M, 0^k)$ to be the middle element in the sequence of outputs over all paths of the Turing machine encoded by M on input x run for k steps; and
$\textsc{Univ}_{\mathrm{Mid}}(x, M, 0^k)$ to be the middle element in the set of outputs over all paths of the Turing machine encoded by M on input x run for k steps.

$\textsc{Univ}_{\overline{\mathrm{Med}}}$, $\textsc{Univ}_{\mathrm{Med}}$, and $\textsc{Univ}_{\mathrm{Mid}}$ are $\leq_m^{\mathrm{FP}}$-complete for $\mathrm{M\overline{ed}\,P}$, $\mathrm{Med\,P}$ and $\mathrm{Mid\,P}$, resp. All proofs are completely analogously. We only sketch one: Let $f \in \mathrm{Mid\,P}$ be computed by some nondeterministic Turing machine M. Let $\langle M \rangle$ be the encoding of M, and let p be a polynomial bounding the running time of M. Then the following function g reduces f to $\textsc{Univ}_{\mathrm{Mid}}$:

$$g(x) =_{\mathrm{def}} (x, \langle M \rangle, 0^{p(|x|)}).$$

□

Corollary 13. *If a function hard for* $\mathrm{Mid\,P}$ *under* $\leq_m^{\mathrm{FP}}$ *is in* $\mathrm{Med\,P}$*, then the second level of the polynomial time hierarchy is in* PP*.*
If a function hard for $\mathrm{Med\,P}$ *under* $\leq_m^{\mathrm{FP}}$ *is in* $\mathrm{Mid\,P}$*, then the counting hierarchy collapses.*

Proof. Consequence of the closure of the considered classes under $\leq_m^{\mathrm{FP}}$ and the remarks following Proposition 4. □

Thus we have the following result which should be contrasted with Corollary 9:

Corollary 14. *If the classes* $\mathrm{M\overline{ed}\,P}$, $\mathrm{Med\,P}$, $\mathrm{Mid\,P}$*, and* $\mathrm{FP}^{\#\mathrm{P}}$ *have the same* $\leq_m^{\mathrm{FP}}$*-hard functions, then the counting hierarchy collapses and* PP *includes the second level of the polynomial time hierarchy.*

Proof. Follows from Theorem 12 and Corollary 13. □

So it turns out that the notion of functional many-one reducibility in contrast to metric reducibility is well-suited to elaborate the differences between the considered classes of median functions. We will do so in the following subsections by presenting a number of natural problems complete for these classes under $\leq_m^{\mathrm{FP}}$.

4.1 Problems Complete for $\mathrm{M\overline{ed}\,P}$

We start with a problem concerning boolean circuits.

Problem:	LexMidCVal
Input:	a boolean circuit C with n inputs
Output:	the lexicographic middle element of all strings of length n that given as input to C produce the output 1.

Theorem 15. LexMidCVal *is* $\leq_m^{\mathrm{FP}}$*-complete for* $\mathrm{M\overline{ed}\,P}$*.*

Proof. Obviously, $\textsc{LexMidCVal} \in \overline{\mathrm{Med}}\,\mathrm{P}$. We now have to show that $\textsc{Univ}_{\overline{\mathrm{Med}}} \leq_m^{\mathrm{FP}} \textsc{LexMidCVal}$:

If we want to evaluate $\textsc{Univ}_{\overline{\mathrm{Med}}}(x, \langle M \rangle, 0^j, 0^k)$, we produce the circuit C with j inputs that simulates the computation of M on input (x, y), were x is coded directly into the circuit and y corresponds to C's inputs. This construction can be performed in polynomial time (cf. e.g. [1]). Then the middle y of length at most j that is accepted by M if run for k steps is the middle input y that forces C to output 1. $\square$

In the context of polynomial time complexity of sets, a problem defined using the notion of a circuit can most often be changed to a problem defined using the notion of a boolean formula without changing the complexity. For instance, given a circuit, to decide whether it has an input that produces output 1 is equivalent to the question whether a boolean formula has a satisfying assignment. In our context, this seems not to be the case because of the following:

Consider the proof of Cook's Theorem [3] saying that SAT is NP-complete: There, given a Turing machine M and an input x a boolean formula F is constructed such that there is a one-one relation between accepting paths of M and satisfying assignments for F. There are certain variables in F that correspond directly to the nondeterministic guesses of M and given an assignment of these variables the assignment of the other ones in a satisfying assignment can be computed in deterministic polynomial time.

If we want to describe the circuit from a $\textsc{LexMidCVal}$-instance by a boolean formula, then the same phenomenon arises: We encode the circuit by a conjunction of small formulae describing local connectivity properties of the circuit. Each input and each gate of the circuit corresponds to a boolen variable. If we are given the values of the inputs of the circuits, then the values of the internal gates can be computed deterministically. Thus, if we know the values of the boolean variables corresponding to the inputs, then we can compute deterministically the values of the remaining variables in a satisfying assignment in an obvious step by step way, reflecting directly the level by level evaluation of the circuit.

We now make the following definition:

Definition 16. Given a boolean formula F with variables V then we say that a subset $V_0 \subseteq V$ is a set of *external variables* if for every satisfying assignment of F the assignment of the variables in $V_1 =_{\mathrm{def}} V \setminus V_0$ (the so called *internal variables*) can be computed by the above described completion algorithm from the assignment of the variables V_0. An assignment of F with respect to V_0 is a mapping $b: V_0 \to \{\mathrm{true}, \mathrm{false}\}$. To determine the value of F under b, we first extend b to a mapping of all variables to truth values and then evaluate F as usual.

Thus, in the proof of Cook's theorem the given nondeterministic Turing machine M can be described by a boolen formula in such a way that the variables corresponding to the nondeterminism of M are *external variables,* the others are *internal.* This allows us to prove:

Problem: LexMidSat

Input: a propositional formula F and a sequence V of distinguished variables of F

Output: the lexicographic middle satisfying assignment of F w.r.t. V, if V is external; and any assignment else

Theorem 17. LexMidSat *is* $\leq_m^{\mathrm{FP}}$*-complete for* $\mathrm{M\overline{ed}\,P}$.

Proof. As already mentioned, an argumentation following Cook's Theorem would do the proof. But it is simpler to observe LexMidCVal $\leq_m^{\mathrm{FP}}$ LexMidSat: Given a cicuit C, build a boolean formula F with one variable for each gate of C as described above. The variables corresponding to C's inputs are chosen to be the set of distinguished variables. □

An analogous problem LexMid3Sat defined similar to the previous one, but only for conjunctive formulae with three literals per clause can by standard techniques be shown to be $\leq_m^{\mathrm{FP}}$-complete for $\mathrm{M\overline{ed}\,P}$, too.

Next consider the problem of finding a Hamiltonian circuit in a given undirected graph G. Again, we make a distinction between *external edges* and *internal edges*, if G is such that a possible Hamiltonian circuit is already completely determined by the external edges. We describe a Hamiltonian circuit by ordering the external edges and then transforming the circuit into a sequence of bits, one for each external edge showing whether it is in or is not in the circuit. The ordering of Hamiltonian circuits is then defined to be the lexicographic ordering of their descriptions.

Problem: LexMidHamC

Input: an undirected graph G and a set E of distinguished edges

Output: the lexicographic middle Hamiltonian circuit, if E is external; any sequence of edges else

Theorem 18. LexMidHamC *is* $\leq_m^{\mathrm{FP}}$*-complete for* $\mathrm{M\overline{ed}\,P}$.

Proof. Obviously, LexMidHamC $\in \mathrm{M\overline{ed}\,P}$. We now reduce LexMid3Sat to LexMidHamC following a proof concerning a similar problem given in [9]. The details require a long explanation which can be found in [8]. We only remark that the absence or non-absence of certain edges in the Hamiltonian circuit corresponds directly to the assignment of certain variables in the given formula. The set of external edges can be determined directly from the given set of external variables of the formula. □

4.2 Problems Complete for Med P

Our first problem is related to a problem proven Opt P-complete by Krentel [6].

Problem: MedWeightSat

Input: a propositional formula F with weights on its variables

Output: the middle number in the sequence of the sums of the weights of variables assigned "true" over all satisfying assignments of F

Theorem 19. MEDWEIGHTSAT *is* $\leq_m^{\mathrm{FP}}$*-complete for* Med P.

Proof. Obviously, MEDWEIGHTSAT $\in$ Med P. We now prove that UNIV$_{\mathrm{Med}}$ $\leq_m^{\mathrm{FP}}$ MEDWEIGHTSAT:

Given a Turing machine M, define a formula F such that there is a one-one correspondence between accepting paths of M and satisfying assignments for F. Give weight 2^{i-1} to the variable describing the i-th last bit of the output of M; and give weight 0 to all other variables. Then the weight of a satisfying assignment is equal to the value printed by M on the corresponding path. □

Problem: MEDKNAPSACK

Input: integers $x_1, \ldots, x_n, M, N$

Output: the middle sum $\sum_{i \in S} x_i - M$ in the sequence of sums over all $S \subseteq \{1, \ldots, n\}$ that sum up to a value between 0 and $N - M$.

Problem: MEDTSPTOUR

Input: weighted undirected graph G

Output: the length of the middle Travelling Salesperson Tour over the sequence of all possible round-trips in G.

Theorem 20. MEDKNAPSACK *and* MEDTSPTOUR *are complete for* Med P *under* $\leq_m^{\mathrm{FP}}$*-reductions.*

Proof.

1. MEDWEIGHTSAT $\leq_m^{\mathrm{FP}}$ MEDKNAPSACK can be shown similar to a proof found in [6] reducing Maximum Satisfying Assignment to Knapsack, using standard encoding tricks from other knapsack-style reductions (see [4]).
2. MEDWEIGHTSAT $\leq_m^{\mathrm{FP}}$ MEDTSPTOUR can be shown using a reduction similar to Krentel's from Weighted Sat to TSP [6, 7].

□

4.3 Problems Complete for Mid P

In this subsection we will first consider again a problem from propositional logic, now concerning Σ_1-formulae, and then turn to problems which are defined similar to those examined in the previous subsection.

We define a restriction of *quantified boolean formulae* (see [1]) as follows: A Σ_1*-formula* is a boolean formula F where some of the variables appear within the scope of an $\exists$-quantifier. An assignment for F is a mapping of the variables not within the scope of any quantifier, the so called *free* variables, to boolean values. An assignment is satisfying if the mapping of the free variables can be extended in some way to a mapping of all variables which satisfies the formula $\hat{F}$ which is equal to F but with all quantifiers deleted.

Problem:	LEXMIDΣ_1SAT
Input:	a boolean Σ_1 formula F
Output:	the lexicographic middle satisfying assignment of F.

Theorem 21. LEXMIDΣ_1SAT *is* $\leq_m^{\text{FP}}$*-complete for* Mid P.

Proof. LEXMIDΣ_1SAT $\in$ Mid P, since we can use the nondeterminism of the Turing machine to guess the assignment of the free variables *and* a value for the quantified variables. If such a guess leads to a satisfying assignment, then the machine prints the assignment of the free variables.

We now show UNIV$_{\text{Mid}}$ $\leq_m^{\text{FP}}$ LEXMIDΣ_1SAT: Given a machine M and an input x, build a Σ_1-formula F with free variables $y_1, \ldots, y_m$ saying "there is an accepting path of M on input x with output $y_1 \cdots y_m$." Then, the middle y printed by M is equal to the middle satisfying assignment of F. □

Problem:	MIDWEIGHTSAT
Input:	a propositional formula F with weights on its variables
Output:	the middle number in the set of the sums of the weights of variables assigned "true" over all satisfying assignments of F

Problem:	MIDKNAPSACK
Input:	integers $x_1, \ldots, x_n, M, N$
Output:	the middle sum $\sum_{i \in S} x_i - M$ in the set of sums over all $S \subseteq \{1, \ldots, n\}$ that sum up to a value between 0 and $N - M$.

Problem:	MIDTSPTOUR
Input:	weighted undirected graph G
Output:	the length of the middle Travelling Salesperson Tour over the set of all possible round-trips in G.

Theorem 22. MIDWEIGHTSAT, MIDKNAPSACK, *and* MIDTSPTOUR *are* $\leq_m^{\text{FP}}$*-complete for* Mid P.

Proof. All proofs are similar to the corresponding ones in the previous subsection.

□

5 Completeness Results for Classes of Sets

In this section we will prove that completeness results for a median class as just given allow one to directly deduce completeness results for PP, $\mathrm{PP}^{\mathrm{NP}}$ and P^{PP}. Thus using the results from the previous section we get a number of natural sets complete for these classes.

Theorem 23.

1. *If f is $\leq_m^{\mathrm{FP}}$-complete for* $\overline{\mathrm{Med}}\,\mathrm{P}$ *or* $\mathrm{Med}\,\mathrm{P}$*, then the set* $\{\, (x,k) \mid f(x) \geq k \,\}$ *is $\leq_m^{\mathrm{p}}$-complete for* PP.
2. *If f is $\leq_m^{\mathrm{FP}}$-complete for* $\mathrm{Mid}\,\mathrm{P}$*, then the set* $\{\, (x,k) \mid f(x) \geq k \,\}$ *is $\leq_m^{\mathrm{p}}$-complete for* $\mathrm{PP}^{\mathrm{NP}}$.

Proof. We give the proof for Statement 1. The proof for 2 is similar.

In [12] it was shown that the class PP is exactly equal to the class of sets A for which there exists an $f \in \overline{\mathrm{Med}}\,\mathrm{P}$ (or alternatively, $f \in \mathrm{Med}\,\mathrm{P}$) and a $g \in \mathrm{FP}$ such that for all x, $x \in A$ if and only if $f(x) \geq g(x)$. Thus $\{\, (x,k) \mid f(x) \geq k \,\}$ is in PP for $f \in \overline{\mathrm{Med}}\,\mathrm{P} \cup \mathrm{Med}\,\mathrm{P}$.

Let now f be complete for $\mathrm{Med}\,\mathrm{P}$ (the proof for $\overline{\mathrm{Med}}\,\mathrm{P}$ is completely analogous) and A be some set from the class PP. Then by the remarks above there exists $h \in \mathrm{Med}\,\mathrm{P}$ and a $g \in \mathrm{FP}$ such that for all x, $x \in A$ if and only if $h(x) \geq g(x)$. From the completeness of f we get the existence of some $g' \in \mathrm{FP}$ such that for all x, $h(x) = f(g'(x))$. Thus,

$$\begin{aligned} x \in A &\iff f(g'(x)) \geq g(x) \\ &\iff (g'(x), g(x)) \in \{\, (x,k) \mid f(x) \geq k \,\} \end{aligned}$$

Thus, the function $g'' \in \mathrm{FP}$, defined by $g''(x) =_{\mathrm{def}} (g'(x), g(x))$ reduces A to $\{\, (x,k) \mid f(x) \geq k \,\}$. □

Thus the problems from Subsections 4.1 and 4.2 yield a number of complete problems for PP and the problems from Subsection 4.3 yield natural complete probems for $\mathrm{PP}^{\mathrm{NP}}$; e.g. to determine whether the length of the middle element in the set of all TSP-tours of a graph G is greater than a given number k is complete for $\mathrm{PP}^{\mathrm{NP}}$.

The next theorem allows us to use the results from Section 4 to get completeness results for the class P^{PP}. Toda [9] proved a number of sets to be complete for this class, but he had to give an individual proof for each problem. Our result generalizes his work and it yields completeness results in a uniform more or less mechanical way. Toda showed that P^{PP} is exactly the class of sets A for which there exists an $f \in \mathrm{Mid}\,\mathrm{P}$ such that for all x, $x \in A \iff f(x) \equiv 1 \pmod 2$. Proposition 5 shows that in this characterization, we might as well choose $f \in \overline{\mathrm{Med}}\,\mathrm{P}$ or $f \in \mathrm{Med}\,\mathrm{P}$. We now prove that the set obtained from a complete function f is itself complete for P^{PP}:

Theorem 24. *If f is $\leq_m^{\mathrm{FP}}$-complete for* $\overline{\mathrm{Med}}\,\mathrm{P}$, $\mathrm{Med}\,\mathrm{P}$, *or* $\mathrm{Mid}\,\mathrm{P}$*, then the set* $\{\, x \mid f(x) \equiv 1 \pmod 2 \,\}$ *is $\leq_m^{\mathrm{p}}$-complete for* P^{PP}.

Proof. For f from any of the considered classes of median functions, the set $\{\, x \mid f(x) \equiv 1 \pmod 2 \,\}$ is in P^{PP}, since we can determine the middle element by binary search and then check if it is odd. Let now f be complete for $\mathrm{Med}\,\mathrm{P}$ (the proofs for $\overline{\mathrm{Med}}\,\mathrm{P}$ and $\mathrm{Mid}\,\mathrm{P}$ are completely analogous) and let A be some set from P^{PP}. Then the characteristic function from A is in $\mathrm{FP}^{\mathrm{PP}}$. As consequence of the remarks following Proposition 5 we get the existence of a function $h \in \mathrm{Med}\,\mathrm{P}$ such that $x \in A$ if and only if $h(x)$ is odd. From the completeness of f we get the existence of some $g' \in \mathrm{FP}$ such that for all x, $h(x) = f(g'(x))$. Thus,

$$\begin{aligned} x \in A &\iff f(g'(x)) \equiv 1 \pmod 2 \\ &\iff g'(x) \in \{\, x \mid f(x) \equiv 1 \pmod 2 \,\} \end{aligned}$$

Thus, g' reduces A to $\{\, x \mid f(x) \equiv 1 \pmod 2 \,\}$. □

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References

1. J. L. Balcázar, J. Díaz, J. Gabarró, *Structural Complexity I and II;* Springer (Berlin – Heidelberg – New York, 1988 and 1990).
2. R. Beigel, Perceptrons, PP, and the polynomial time hierarchy; 7^{th} *Structure in Complexity Theory Conference* (1992), pp. 14–19.
3. S. A. Cook, The complexity of theorem proving procedures; 3^{rd} *Symposium on the Theory of Computing* (1971), pp. 151–158.
4. M. R. Garey, D. S. Johnson, *Computers and Intractability: A Guide to the Theorey of* NP*-Completeness;* Freeman & Co. (New York, 1979).
5. J. Köbler, Strukturelle Komplexität von Anzahlproblemen, Dissertation, Fakultät für Informatik, Universität Stuttgart, 1989 .
6. M. W. Krentel, The complexity of optimization problems; *Journal of Computer and Systems Sciences* **36** (1988), pp. 490–509.
7. C. H. Papadimitriou, On the complexity of unique solutions; *Journal of the ACM* **31** (1982), pp. 392–400.
8. C. H. Papadimitriou, K. Steiglitz, *Combinatorial Optimization;* Prentice Hall (Englewood Cliffs, New Jersey, 1992).
9. S. Toda, The complexity of finding medians; 31^{st} *Symposium on Foundations of Computer Science* (1990), pp. 778–787.
10. J. Torán, Complexity classes defined by counting quantifiers; *Journal of the ACM* **38** (1991), pp. 753–774.
11. L. G. Valiant, The complexity of computing the permanent; *Theoretical Computer Science* **8** (1979), pp. 189–201.
12. H. Vollmer, K. W. Wagner, The complexity of finding middle elements; Technical report No. 53, University Würzburg; to appear in the *International Journal of Foundations of Computer Science.*
13. K. W. Wagner, The complexity of combinatorial problems with succinct input representation; *Acta Informatica* **23** (1986), pp. 325—356.

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