Complexity Classes of Optimization Functions[1]

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Abstract

In this paper, complexity classes of functions defined via taking maxima or minima (cf. the work of Krentel) or taking middle elements (cf. the work of Toda) are examined. A number of axioms for a class to be a so called *p-founded class of optimization functions* are given. It is shown that many natural function classes fulfill these axioms. Then these classes are examined concerning their relationship to complexity classes of sets. To this end, complexity preserving operators S and F for encoding function classes into set classes and vice versa are introduced. It is shown, how these operators translate closure properties from one class to another, how they relate operators on classes of functions and classes of sets, and how they encode classes of maximum, minimum, or median functions into well-studied classes of sets. The fixpoints of the compositional operator $\mathrm{F}\cdot\mathrm{S}$ are examined and shown to be exactly those function classes "closed under binary search." Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two such fixpoints, and $\mathcal{K}_1$ and $\mathcal{K}_2$ be their corresponding classes of sets (i.e. their images under the operator S). Then $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if and only if $\mathcal{K}_1 \subseteq \mathcal{K}_2$, and $\mathcal{F}_1 = \mathcal{F}_2$ if and only if $\mathcal{K}_1 = \mathcal{K}_2$. A number of natural classes of functions are shown to be such fixpoints. Thus we build hierachies of function classes with the same inclusional relationships as the polynomial time hierarchy of sets and the counting hierarchy of sets, and we prove many interesting structural properties of these hierarchies.

[1]Appeared in *Information & Computation* **120** (1995), pp. 198–219.

1. Introduction

Structural complexity theory has mainly been a theory of the complexity of sets. There are only a few classes of functions that attracted attention in the last years. Among them are "classical" classes like

- the class $\#\mathrm{P}$ from Valiant (1979) of all functions counting the number of accepting paths of a nondeterministic polynomial time Turing machine,
- the class $\mathrm{Opt}\,\mathrm{P}$ from Krentel (1988) of all functions yielding the maximum (or minimum) output over all paths of a nondeterministic polynomial time Turing machine, and generalizations of that class from Krentel (1992), and
- the class $\mathrm{Max}\,\mathrm{P}$ of maximization functions from $\mathrm{Opt}\,\mathrm{P}$, and the class $\mathrm{Min}\,\mathrm{P}$ of minimization functions from $\mathrm{Opt}\,\mathrm{P}$, see Köbler (1989); i.e. $\mathrm{Opt}\,\mathrm{P} = \mathrm{Max}\,\mathrm{P} \cup \mathrm{Min}\,\mathrm{P}$.

Recently some other classes of functions appeared:

- the class spanP studied by Köbler, Schöning, and Torán (1989) of all functions counting the number of different outputs over all paths of a nondeterministic polynomial time Turing machine,
- the class $\mathrm{Mid}\,\mathrm{P}$ studied by Toda (1990) of all functions yielding the median of the outputs (without multiplicities) over all paths of a nondeterministic polynomial time Turing machine,
- the class GapP studied by Fenner, Fortnow, and Kurtz (1991) of all functions counting the number of accepting paths minus the number of rejecting paths of a nondeterministic Turing machine, and
- the class $\mathrm{Med}\,\mathrm{P}$ studied by Vollmer and Wagner (1993) and Vollmer (1994b), which is defined similar to $\mathrm{Mid}\,\mathrm{P}$, but now, multiplicities are taken into account.

These classes can be divided into two groups: One group consists of classes of functions that are defined via a process of counting the number of paths of a Turing machine with some property. To this group belong $\#\mathrm{P}$ directly by definition, spanP which can be shown to be equal (in our terminology, see Section 2) to the class $\#\mathrm{NP}$, and GapP which was shown in Fenner, Fortnow, and Kurtz (1991) to be equal to $\#\mathrm{P} - \#\mathrm{P}$. The second group consists of the classes $\mathrm{Opt}\,\mathrm{P}$, $\mathrm{Max}\,\mathrm{P}$, $\mathrm{Min}\,\mathrm{P}$, $\mathrm{Med}\,\mathrm{P}$, and $\mathrm{Mid}\,\mathrm{P}$. These classes are defined via some selection process on the set of outputs over all paths of a Turing machine; and the selection process is in all cases some kind of optimization (also the definition of a median requires a sort of optimization: it is the maximal element of a set such that the number of greater elements is not greater than half the cardinality of the set).

In this paper we examine classes of the second kind, i.e. classes of optimization functions. Our starting point is the following striking list of analogous results. Let (for a function f) $\mathrm{proj}\,(f) =_{\mathrm{def}} \{\, (x, k) \mid f(x) \geq k \,\}$. The following ist known from Krentel (1988), Köbler (1989), Vollmer and Wagner (1993), Toda (1990):

$$\begin{array}{lcl} f \in \mathrm{Max}\,\mathrm{P} & \Longleftrightarrow & \mathrm{proj}\,(f) \in \mathrm{NP} \\ f \in \mathrm{Min}\,\mathrm{P} & \Longleftrightarrow & \mathrm{proj}\,(f) \in \mathrm{coNP} \\ f \in \mathrm{Med}\,\mathrm{P} & \Longleftrightarrow & \mathrm{proj}\,(f) \in \mathrm{PP} \\ f \in \mathrm{Mid}\,\mathrm{P} & \Longleftrightarrow & \mathrm{proj}\,(f) \in \mathrm{PP}^{\mathrm{NP}} \end{array}$$

In Toda (1990), it was shown that Mid P allows the following characterization of $\mathrm{FP}^{\mathrm{NP}}$: A function f belongs to $\mathrm{FP}^{\mathrm{PP}}$ if and only if it can be written as the composition of an easy function (from FP) and a function from Mid P, in symbols: $\mathrm{FP}^{\mathrm{PP}} = \mathrm{FP}^{\mathrm{Mid\,P}}[1]$. In Krentel (1988), Köbler (1989), Vollmer and Wagner (1993), the following analogous characterizations of $\mathrm{FP}^{\mathrm{NP}} = \mathrm{FP}^{\mathrm{coNP}}$ and $\mathrm{FP}^{\mathrm{PP}}$ were given:

$$\begin{aligned} \mathrm{FP}^{\mathrm{NP}} &= \mathrm{FP}^{\mathrm{Max\,P}}[1] \\ \mathrm{FP}^{\mathrm{coNP}} &= \mathrm{FP}^{\mathrm{Min\,P}}[1] \\ \mathrm{FP}^{\mathrm{PP}} &= \mathrm{FP}^{\mathrm{Med\,P}}[1] \end{aligned}$$

In Toda (1990), the question, whether similar characterizations can be given for other classes, was left open.

We now try to capture the essential properties of function classes as the above by seven axioms, and we call classes which fulfill these axioms *p-founded classes of optimization functions.* We show that FP (the class of functions computable in polynomial time on a deterministic Turing machine) is such a class, and that if $\mathcal{F}$ fulfills the axioms, then also function classes which are constructed from $\mathcal{F}$ via one of the operators Max, Min, or Med are p-founded classes of optimization functions. The just mentioned operators are defined via a maximization process, a minimization process, and a process of taking middle elements (with multiplicities[2]), respectively, over an exponential range of values. Thus, every class constructed from FP by finitely many applications of any of the above operators fulfills the optimization axioms.

In order to compare classes of functions and classes of sets as in the above listed results we introduce the operators S (which converts a function class into a set class) and F (which converts a set class into a function class). The operator S, applied to a class of functions $\mathcal{F}$, yields exactly the closure of $\mathrm{proj}\,(\mathcal{F}) =_{\mathrm{def}} \{\,\mathrm{proj}\,(f) \mid f \in \mathcal{F}\,\}$ under many-one reducibility. The operator F is chosen to be an inverse to S as good as possible. We decided to choose these operators because they fulfill the following requirements:

1. They are easy to compute and thus don't alter the complexity too much when we go from one class to another. S can be computed by asking only one query to a function from the class under consideration, that is it has the property $\mathrm{S}\mathcal{F} \subseteq \mathrm{P}^{\mathcal{F}}[1]$; thus the complexities of $\mathcal{F}$ and $\mathrm{S}\cdot\mathcal{F}$ are intimately related. F can be computed in polynomial time asking questions to an oracle from the set class under consideration, that is: $\mathrm{F}\mathcal{K} \subseteq \mathrm{FP}^{\mathcal{K}}$; thus the complexities of $\mathcal{K}$ and $\mathrm{F}\cdot\mathcal{K}$ don't differ too much, though in general $\mathrm{F}\mathcal{K}$ seems to be of slightly higher complexity than $\mathcal{K}$.

2. They translate "natural" classes of functions into "natural" classes of sets and vice versa, i.e. they establish a strong relationship between the above operators Max, Min, and Med on function classes and well studied operators on set classes, namely $\exists$, $\forall$, and C (the polynomially length-bounded existential, universal, and counting quantifier).

3. They translate "natural" closure properties of set classes into related closure properties of function classes and vice versa.

The just mentioned point of translation of closure properties is of special interest. Call a class of sets $\mathcal{K}$ an *abstract polynomial complexity class* if and only if it contains the class P and is closed under union, intersection, and polynomial time disjunctive and conjunctive reducibilities. Then it can

[2] In this paper we will leave Mid P aside and concentrate more on Med P, which is a "pure" median class, while in Mid P the median selection process is combined with an existential quantifier, as can be seen from the list of results above. For a detailed examination of the relationship between these two classes, see Vollmer and Wagner (1993).

be observed that a number of famous complexity theoretic results given in the past for special complexity classes are already valid for abstract polynomial complexity classes or the boolean closures of such classes, for instance Beigel, Reingold, and Spielman (1991), Fortnow and Reingold (1991), and many more. The notion *abstract family of languages (afl)* has a similar role in the theory of formal languages in the sense that a lot of results first given for special language classes are already valid for afl's, see Ginsburg, Greibach, and Hopcroft (1960). And it turns out that there is a strong connection between the above two notions. The set of all complexity classes corresponding to afl's (i.e. the set of many-one closures of afl's) is exactly the set of all abstract polynomial complexity classes closed under existential quantification. This extends the well-known fact that the class NP is the afl-closure of the class P.

If we now confront the closure properties required in the definition of an abstract polynomial complexity class and the closure properties required in the definition of p-founded classes of optimization functions, then we see that they correspond to each other under the operators S and F. A class $\mathcal{K}$ of sets is an abstract polynomial complexity class if and only if the corresponding class of functions $\mathcal{F}$ is a strongly p-founded class of optimization functions (we will see later what "strongly" here means). This shows that the notions we defined are really very natural.

The above given list of results from the literature can now be written as: $\mathrm{S}\cdot\mathrm{Max\,P} = \mathrm{NP}$, $\mathrm{S}\cdot\mathrm{Min\,P} = \mathrm{coNP}$, $\mathrm{S}\cdot\mathrm{Med\,P} = \mathrm{PP}$, and $\mathrm{S}\cdot\mathrm{Mid\,P} = \mathrm{PP}^{\mathrm{NP}}$. Additionally, one can show: $\mathrm{F}\cdot\mathrm{NP} = \mathrm{Max\,P}$, $\mathrm{F}\cdot\mathrm{coNP} = \mathrm{Min\,P}$, $\mathrm{F}\cdot\mathrm{PP} = \mathrm{F}\cdot\mathrm{S}\cdot\mathrm{Med\,P} \subseteq \mathrm{FP}^{\mathrm{Med\,P}}[1]$, and $\mathrm{F}\cdot\mathrm{PP}^{\mathrm{NP}} = \mathrm{F}\cdot\mathrm{S}\cdot\mathrm{Mid\,P} \subseteq \mathrm{FP}^{\mathrm{Mid\,P}}[1]$. Thus, our encoding of functions into sets and vice versa corresponds to a number of results from the literature. We then prove that generally for any class of optimization functions $\mathcal{F}$, we have $\mathrm{FP}^{\mathcal{F}} = \mathrm{FP}^{\mathrm{F}\cdot\mathrm{S}\cdot\mathcal{F}}[1]$, and similarly for any abstract polynomial complexity class $\mathcal{K}$, we have $\mathrm{FP}^{\mathcal{K}} = \mathrm{FP}^{\mathrm{F}\cdot\mathcal{K}}[1]$. Direct corollaries to this general results are then the above cited characterizations for $\mathcal{K}$ equal to NP, coNP, PP, or $\mathrm{PP}^{\mathrm{NP}}$ (for the last case, observe that $\mathrm{FP}^{\mathrm{PP}} = \mathrm{FP}^{\mathrm{PP}^{\mathrm{NP}}}$, see Toda and Watanabe (1991)). Thus, we generally answer Toda's question for any abstract polynomial class $\mathcal{K}$. A consequence of our general theorem is the following: To decide membership in a set from $\mathrm{P}^{\mathcal{K}}$, it is sufficient to evaluate a function from $\mathrm{F}\cdot\mathcal{K}$ for only one argument and then check whether the result is odd. This strengthens a result from Ogiwara (1991), where the same result is obtained under much stronger assumptions for $\mathcal{K}$.

Clearly, the classes (either of functions or of sets) which are fixpoints under one of the compositional operators $\mathrm{S}\cdot\mathrm{F}$ or $\mathrm{F}\cdot\mathrm{S}$ are most interesting: If $\mathcal{F}_1$ and $\mathcal{F}_2$ are classes of functions and $\mathcal{K}_1$ and $\mathcal{K}_2$ are classes of sets, and $\mathcal{F}_1$ and $\mathcal{K}_1$ are transformed into each other under S and F, and analogously for $\mathcal{F}_2$ and $\mathcal{K}_2$, then $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if and only if $\mathcal{K}_1 \subseteq \mathcal{K}_2$, and $\mathcal{F}_1 = \mathcal{F}_2$ if and only if $\mathcal{K}_1 = \mathcal{K}_2$. But the remarks above show that $\mathrm{F}\mathcal{K}$ in general is of slightly higher complexity than $\mathcal{K}$, and therefore we cannot hope to find many such fixpoints.

However, the situation is better than could be expected: If we apply to a set class $\mathcal{K}$ first the operator F to construct a function class and then to the result the operator S to construct a set class, then (under some weak assumptions to $\mathcal{K}$) the result is again $\mathcal{K}$: $\mathrm{S}\cdot\mathrm{F}\cdot\mathcal{K} = \mathcal{K}$. This means that the additional power of $\mathrm{F}\cdot\mathcal{K}$ compared to $\mathcal{K}$ cannot be used when we have only restricted access via S to functions from that class; that is all reasonable classes of sets are fixpoints of $\mathrm{S}\cdot\mathrm{F}$. On the other side, we will see that the class $\mathrm{F}\cdot\mathrm{S}\cdot\mathcal{F}$ captures a notion of binary search over the function class $\mathcal{F}$. Thus, a function class is a fixpoint of $\mathrm{F}\cdot\mathrm{S}$ if and only if it is "closed under binary search." We will characterize the power of $\mathrm{F}\cdot\mathrm{S}\cdot\mathcal{F}$ compared to $\mathcal{F}$ in several ways. We already mentioned that only one query to a function from $\mathrm{F}\cdot\mathrm{S}\cdot\mathcal{F}$ is sufficient to simulate an $\mathrm{FP}^{\mathcal{F}}$-computation: $\mathrm{FP}^{\mathcal{F}} = \mathrm{FP}^{\mathrm{F}\cdot\mathrm{S}\cdot\mathcal{F}}[1]$. Even more, we show that a polynomial time computation with an oracle is equal in power to binary search over oracle functions plus subtraction, i.e. $\mathrm{FP}^{\mathcal{F}} = \mathrm{FS}\mathcal{F} - \mathrm{FS}\mathcal{F}$.

And interestingly it will turn out that we can show that a lot of p-founded classes of optimization functions are fixpoints. For function classes $\mathcal{F}$ with certain properties, all classes which are constructed from $\mathcal{F}$ via finitely many applications of Max, Min, and $\mathrm{FP}^{(\cdot)}$ are closed under binary search. We define the min-max hierarchy over a function class $\mathcal{F}$ to consist of the classes $\Sigma_0^{\mathrm{p}}\mathcal{F} = \Pi_0^{\mathrm{p}}\mathcal{F} = \Delta_0^{\mathrm{p}}\mathcal{F} = \mathcal{F}$, $\Sigma_{k+1}^{\mathrm{p}}\mathcal{F} = \mathrm{Max}\,\Pi_k^{\mathrm{p}}\mathcal{F}$, $\Pi_{k+1}^{\mathrm{p}}\mathcal{F} = \mathrm{Min}\,\Sigma_k^{\mathrm{p}}\mathcal{F}$, $\Delta_{k+1}^{\mathrm{p}}\mathcal{F} = \mathrm{FP}^{\Sigma_k^{\mathrm{p}}\mathcal{F}}$. Under natural assumptions for $\mathcal{F}$, we can show that all classes of the min-max hierarchy over $\mathcal{F}$ are fixpoints of FS. Thus we can build a hierarchy of functions with the same inclusional structure as the polynomial time hierarchy of sets; and two classes in the function hierarchy are the same if and only if the two corresponding classes from the polynomial time hierarchy are the same. We prove a number of interesting structural properties of the min-max hierarchy, e.g. we show that functions from any Delta-Level can be written as a difference of two functions from the preceding Sigma-Level (or Pi-Level). Moreover, we show that functions from any Delta-level can be computed with only one query to a function from the preceding Sigma- or Pi-level. We give natural sufficient conditions for the Delta-levels to be included in the next Sigma- and Pi-levels. We show that functions from any Pi-level (and only those) can be written as a difference of an FP-function and a function from the corresponding Sigma-level; and that functions from any Sigma-level (and only those) can be written as a difference of an FP-function and a function from the corresponding Pi-level. We characterize the union over all classes of the min-max hierarchy over $\mathcal{F}$ as closure of $\mathcal{F}$ under Max and Min, or alternatively, under Max and subtraction, or Min and subtraction. This systematically extends results of Krentel (1992). There, so called alternating min-max machines were used to define a hierarchy of classes $\Sigma_0^{\mathrm{MM}} \subseteq \Sigma_1^{\mathrm{MM}} \subseteq \Sigma_2^{\mathrm{MM}} \subseteq \cdots$. A lot of natural complete problems (under metric reducibility) for different levels of that hierarchy were given; but from a structural point of view, not very much was known about that hierarchy up to now. Our just mentioned results on general min-max hierarchies imply directly (choosing $\mathcal{F} = \mathrm{FP}$) a number of very interesting structural results about Krentel's Σ^{MM} hierarchy.

Using Med instead of Max and Min, we can similarly build a hierarchy which corresponds to the counting hierarchy of sets. We prove a similar inclusion structure for this hierarchy as for the min-max hierarchy. Again, we prove among others for the Delta-levels a characterization as in the first case. Thus, we get results which extend Toda's characterization of P^{PP} from Toda (1990) to other levels of the counting hierarchy.

All in all, we see that extracting the important properties of well-known function classes and subsuming them under the name *p-founded class of optimization functions* allows us to generalize a number of important structural results from the literature. Thus, we first reveal the essence of the prior results and give unified proofs for a number of up to now separately examined cases. But second, we also get stronger results than those known previously, e. g. results known up to know to hold for Opt P or PP are now shown to hold for the whole Σ^{MM}-hierarchy or the whole counting hierarchy. And third, we even show that our framework can be applied far more generally in the context of the so called abstract polynomial complexity classes.

Finally, we want to point out, that our general results will allow us to give directly a number of very interesting corollaries. Using Toda's Theorem from Toda (1991), we get immediately that Krentel's generalization of Opt P to the polynomial time hierarchy is included in $\mathrm{FP}^{\mathrm{PP}}$, i. e. $\bigcup_{i\geq 0}\Sigma_i^{\mathrm{MM}} \subseteq \mathrm{FP}^{\mathrm{PP}}$. Using our structural results about the min-max hierarchy we see that Σ_1^{MM} is closed under subtraction if and only if NP = coNP, see Ogiwara and Hemachandra (1993). Similarly, we get that PP is closed under Turing reduction (a current topical open question) if and only if Med P is closed under subtraction.

2. Preliminaries

We assume the reader is familiar with the basic notions of complexity theory as for example oracle Turing machine, number of queries, many-one reducibility ($\leq_m^p$), truth table reducibility ($\leq_{tt}^p$), ..., see Balcázar, Díaz, and Gabarró (1988), Balcázar, Díaz, and Gabarró (1990). Our notation here is standard. Additionally, we use conjunctive reducibility ($\leq_{ctt}^p$), and disjunctive reducibility ($\leq_{dtt}^p$). A truth table reduction is called *conjunctive (disjunctive)* if the circuit computing from the answers of the oracle the result of the whole reduction consists of a single (unbounded fan-in) AND gate (OR gate, resp.).

For a reducibility notion $\leq_y^x$ and a set B, let $\mathcal{R}_y^x(B)$ denote the set of all languages reducible to B via $\leq_y^x$, i.e. $\mathcal{R}_y^x(B) = \{ A \mid A \leq_y^x B \}$. If $\mathcal{K}$ is a class of sets, then $\mathcal{R}_y^x(\mathcal{K}) = \bigcup_{B\in\mathcal{K}} \mathcal{R}_y^x(B)$. We say that $\mathcal{K}$ is closed under $\leq_y^x$ if and only if $\mathcal{K} = \mathcal{R}_y^x(\mathcal{K})$.

For a language A, the characteristic function of A is denoted by c_A. Moreover, for a class of languages $\mathcal{K}$, let $\mathrm{char}(\mathcal{K})$ denote the class of characteristic functions of the members of $\mathcal{K}$, i.e. $\mathrm{char}(\mathcal{K}) =_{\mathrm{def}} \{ c_A \mid A \in \mathcal{K} \}$.

The set of natural numbers is denoted by $\mathbb{N}$. To be able to carry out ordinary subtraction we consider functions over the set $\mathbb{Z}$ of integers. For $y \in \mathbb{N}$, let $[y] =_{\mathrm{def}} \{-y+1, -y+2, \dots, y\}$.

For a rational number x, let $\lfloor x \rfloor =_{\mathrm{def}} \max\{n \in \mathbb{Z} \mid n \leq x\}$, and let $\lceil x \rceil =_{\mathrm{def}} \min\{n \in \mathbb{Z} \mid n \geq x\}$. For $z \in \mathbb{Z}$, $\mathrm{sgn}(z)$ denotes the sign of z. By $\dot{-}$ we denote modified subtraction, i. e. $a \dot{-} b = \max\{0, a-b\}$. For functions f and g, $f \circ g$ denotes the composition of f and g, i.e. $(f \circ g)(x) = f(g(x))$. Integer division is denoted by $\div$. By $x \bmod y$, the unique number r such that $a = b \cdot (a \div b) + r$ is denoted. For a finite set M, the number of elements of M is written as $\# M$. For a string w the length of w is written as $|w|$. For a set M, a set Ω of operations on M, and an $X \subseteq M$, we denote by $\Gamma_\Omega(X)$ the closure of X under operations from Ω. For sets M and M', the marked union of M and M' is the set $0 \cdot M \cup 1 \cdot M'$.

FP denotes the class of functions that are computable by a deterministic Turing machine in polynomial time. An FP-computation with oracle X (either a class of functions or a class of sets) if denoted by FP^X. If the number of queries of the oracle machine is bounded by some constant k, the resulting class is denoted by $\mathrm{FP}^X[k]$. Analogously, we use the notation $\mathrm{P}^X[k]$ for a polynomial time decision procedure with a bounded number of oracle queries to X. FP_+ denotes the set of functions from FP, which are everywhere non-negative. We assume that integers are one-one encoded into words over $\{0,1\}$ via some easily computable function, e.g. in binary or dyadic.

In the sequel of the paper it will turn out that we we have very often the need to encode a sequence of integers in one single integer. For this aim, for $m, y \in \mathbb{N}$ and $u_1, u_2, \dots, u_m \in \mathbb{Z}$, let

$$\langle u_1, u_2, \dots, u_m \rangle_y =_{\mathrm{def}} \sum_{i=1}^{m} (u_i + y) \cdot (2y)^{i-1} - 1.$$

Note that for given $U, y \in \mathbb{N}$, there is exactly one decomposition $m \in \mathbb{N}$, $u_1, \dots, u_m \in [y]$, such that $\langle u_1, \dots, u_m \rangle_y = U$. In the rest of the paper we will sometimes omit the subscript to $\langle \cdots \rangle$, if no confusion can arise. For $U, y, i \in \mathbb{N}$, let $[U, y, i]$ be the i-th number in the decomposition of U according to bound y and let $m[U, y]$ be the number of integers encoded in U, i.e. if $U = \langle u_1, \dots, u_m \rangle_y$, $u_1, \dots, u_m \in [y]$, then

$$[U, y, i] =_{\mathrm{def}} \begin{cases} u_i, & \text{if } i \in \{1, \dots, m\} \\ 0, & \text{otherwise} \end{cases}$$

and $m[U,y] =_{\text{def}} m$.

Obviously, encoding via $\langle\cdots\rangle$ and decomposition are computable in polynomial time.

Additionally, we will make use of an encoding scheme with two bounds, defined as follows: For $m, y, z \in \mathbb{N}$ and $u_1, u_2, \ldots, u_m \in \mathbb{Z}$, let

$$\langle u_1, \ldots, u_{2m} \rangle_{y,z} =_{\text{def}} \sum_{i=1}^{m} [(u_{2i-1} + y) \cdot (2y)^{i-1} (2z)^{i-1} + (u_{2i} + z) \cdot (2y)^{i} (2z)^{i-1}] - 1$$

Again, for given $U, y \in \mathbb{N}$, there is exactly one decomposition $m \in \mathbb{N}$, $u_1, u_2, \ldots, u_{2m}$ where $u_{2i-1} \in [y]$ and $u_{2i} \in [z]$ for $i \in \{1, \ldots, m\}$, such that $\langle u_1, u_2, \ldots, u_{2m} \rangle_{y,z} = U$. Again, we feel sometimes free to omit the subscripts to $\langle\cdots\rangle$.

We use the following operators on classes of sets. Let $\mathcal{K}$ be any class of sets.

$$\begin{array}{rcl}
f \in \#\mathcal{K} & \iff & \text{there exist } B \in \mathcal{K},\ g \in \mathrm{FP} \text{ such that} \\
 & & f(x) = \#\{\, y \mid 0 \le y \le g(x) \wedge (x,y) \in B \,\}. \\
A \in \exists\mathcal{K} & \iff & \text{there exist } B \in \mathcal{K},\ g \in \mathrm{FP} \text{ such that} \\
 & & x \in A \iff (\exists y)(0 \le y \le g(x) \wedge (x,y) \in B). \\
A \in \forall\mathcal{K} & \iff & \text{there exist } B \in \mathcal{K},\ g \in \mathrm{FP} \text{ such that} \\
 & & x \in A \iff (\forall y)(0 \le y \le g(x) \Rightarrow (x,y) \in B). \\
A \in \mathrm{C}\mathcal{K} & \iff & \text{there exist } f \in \#\mathcal{K},\ g \in \mathrm{FP} \text{ such that} \\
 & & x \in A \iff f(x) \ge g(x).
\end{array}$$

The classes of the polynomial hierarchy over $\mathcal{K}$ are defined by $\Sigma_0^{\mathrm{p}} \cdot \mathcal{K} = \Pi_0^{\mathrm{p}} \cdot \mathcal{K} = \Delta_0^{\mathrm{p}} \cdot \mathcal{K} = \mathcal{K}$, $\Sigma_{k+1}^{\mathrm{p}} \cdot \mathcal{K} = \exists \Pi_k^{\mathrm{p}} \cdot \mathcal{K}$, $\Pi_{k+1}^{\mathrm{p}} \cdot \mathcal{K} = \forall \Sigma_k^{\mathrm{p}} \cdot \mathcal{K}$, and $\Delta_{k+1}^{\mathrm{p}} \cdot \mathcal{K} = \mathrm{P}^{\Sigma_k^{\mathrm{p}} \cdot \mathcal{K} \cup \Pi_k^{\mathrm{p}} \cdot \mathcal{K}}$. Furthermore define $\mathrm{PH} \cdot \mathcal{K} = \bigcup_{k \ge 0} \Delta_k^{\mathrm{p}} \mathcal{K}$, $\Sigma_k^{\mathrm{p}} = \Sigma_k^{\mathrm{p}} \cdot \mathrm{P}$, $\Pi_k^{\mathrm{p}} = \Pi_k^{\mathrm{p}} \cdot \mathrm{P}$, and $\mathrm{PH} = \mathrm{PH} \cdot \mathrm{P}$.

The counting hierarchy over $\mathcal{K}$ is the smallest family of classes containing $\mathcal{K}$ and being closed under $\exists$, $\forall$, C. Let $\mathrm{CH} \cdot \mathcal{K}$ be the union of all classes of the counting hierarchy over $\mathcal{K}$, and let $\mathrm{CH} = \mathrm{CH} \cdot \mathrm{P}$. Since it is known from Wagner (1986a), Torán (1991) that for every $\mathcal{K}$, $\exists\mathcal{K} \cup \forall\mathcal{K} \subseteq \mathrm{C}\mathcal{K}$, $\mathrm{CH} \cdot \mathcal{K}$ can be obtained as the union of all $\mathrm{C}^k\mathcal{K}$, for $k \ge 0$.

We use the following operators on classes of functions. Let $\mathcal{F}$ be any class of functions.

$$\begin{array}{rcll}
h \in \mathrm{Max}\,\mathcal{F} & \iff & \text{there exist } f \in \mathcal{F}, g \in \mathrm{FP} & \text{s.t. } h(x) = \max_{0 \le y \le g(x)} f(x,y) \\
h \in \mathrm{Min}\,\mathcal{F} & \iff & \text{there exist } f \in \mathcal{F}, g \in \mathrm{FP} & \text{s.t. } h(x) = \min_{0 \le y \le g(x)} f(x,y) \\
h \in \text{r-Med}\,\mathcal{F} & \iff & \text{there exist } f \in \mathcal{F}, g \in \mathrm{FP} & \text{s.t. } h(x) = \text{r-med}_{0 \le y \le g(x)} f(x,y) \\
h \in \text{l-Med}\,\mathcal{F} & \iff & \text{there exist } f \in \mathcal{F}, g \in \mathrm{FP} & \text{s.t. } h(x) = \text{l-med}_{0 \le y \le g(x)} f(x,y)
\end{array}$$

Here r-med M for a multiset M denotes the "right-middle" element of M, i.e. if $M = \{m_1, m_2, \ldots, m_k\}$ (as a multiset, possibly $k > \#M$), $m_1 \le m_2 \le \cdots \le m_k$, then r-med $M = m_{\lceil (k+1)/2 \rceil}$. Similarly, l-med $M = m_{\lceil k/2 \rceil}$. The class $\mathrm{Med}\,\mathcal{F}$ denotes the union of r-Med $\mathcal{F}$ and l-Med $\mathcal{F}$.

The operator Mid from Toda (1990) yields in contrast to ours the middle element of a set without taking into account that elements may appear with multiplicity greater than 1. For more on the relation between Med and Mid see Vollmer and Wagner (1993). There, it was shown that for instance Med FP and Mid FP are incomparable to each other, unless the counting hierarchy collapses. Moreover, there exist relativized worlds, in which both classes are incomparable. As a second remark, we observe that the union of our classes Max FP and Min FP is identical to what Krentel denotes by Opt P.

As an abbreviation, we will often only write "$y \leq g(x)$" as subscript of the operators above and mean "$0 \leq y \leq g(x)$". We remark that we might as well use two functions $g, g' \in \mathrm{FP}$ and let y range over $g'(x) \leq y \leq g(x)$; but this change would not have any consequences upon the results that we obtain.

For an arithmetic operation op, e.g. $\mathrm{op} \in \{-, \dot{-}, +, \cdot\}$, and for $\mathrm{op} = \circ$, we denote for functions f, f' by f op f' the function $\lambda x.f(x)$ op $f'(x)$, and for classes of functions $\mathcal{F}, \mathcal{F}'$ by $\mathcal{F}$ op F' the class $\{ f \text{ op } f' \mid f \in \mathcal{F}, f' \in \mathcal{F}' \}$.

The following relationships between the operators are easily seen to hold:

2.1 Proposition. *If $\mathcal{F}$ is closed under addition with FP_+-functions then*

1. $\mathrm{Max}(\mathcal{F} - \mathrm{FP}) = \mathrm{Max}\,\mathcal{F} - \mathrm{FP}$.
2. $\mathrm{Min}(\mathcal{F} - \mathrm{FP}) = \mathrm{Min}\,\mathcal{F} - \mathrm{FP}$.
3. $\mathrm{Med}(\mathcal{F} - \mathrm{FP}) = \mathrm{Med}\,\mathcal{F} - \mathrm{FP}$.

Proof. Let $h \in \mathrm{Max}(\mathcal{F} - \mathrm{FP})$, $h(x) = \max_{y \leq g'(x)}(f(x,y) - g(x,y))$ for some $f \in \mathcal{F}$, $g' \in \mathrm{FP}$, $g \in \mathrm{FP}$. We have to replace g by a function which depends only on x. Choose $g'' \in \mathrm{FP}$ monotonic increasing such that $g''(x,y) \geq g(x,y)$ and let $g_1(x) = g''(x, g'(x))$. Choose $g_2 \in \mathrm{FP}_+$ such that $g(x,y) = g_1(x) - g_2(x,y)$. Then $h(x) = \max_{y \leq g'(x)}(f(x,y) - g(x,y)) = \max_{y \leq g'(x)}(f(x,y) + g_2(x,y)) - g_1(x)$, so $h \in \mathrm{Max}\,\mathcal{F} - \mathrm{FP}$.

If $h(x) = \max_{y \leq g'(x)} f(x,y) - g(x)$, then clearly $h(x) = \max_{y \leq g'(x)}(f(x,y) - g(x))$.

This completes the proof for Max, the proofs for Min and Med proceed completely analogously. □

2.2 Proposition. *If $\mathcal{F}$ is closed under addition with FP_+-functions then*

1. $\mathrm{Max}(-\mathcal{F}) = -\,\mathrm{Min}\,\mathcal{F}$ *and* $\mathrm{Max}(\mathrm{FP} - \mathcal{F}) = \mathrm{FP} - \mathrm{Min}\,\mathcal{F}$.
2. $\mathrm{Min}(-\mathcal{F}) = -\,\mathrm{Max}\,\mathcal{F}$ *and* $\mathrm{Min}(\mathrm{FP} - \mathcal{F}) = \mathrm{FP} - \mathrm{Max}\,\mathcal{F}$.
3. $\mathrm{Med}(-\mathcal{F}) = -\,\mathrm{Med}\,\mathcal{F}$ *and* $\mathrm{Med}(\mathrm{FP} - \mathcal{F}) = \mathrm{FP} - \mathrm{Med}\,\mathcal{F}$.

Proof. Similar to the proof of 2.1. We give the proof for the second part of Statement 1. If $h \in \mathrm{Max}(\mathrm{FP} - \mathcal{F})$, $h(x) = \max_{y \leq g'(x)}(g(x,y) - f(x,y))$, $f \in \mathcal{F}$, $g' \in \mathrm{FP}$, $g \in \mathrm{FP}$, choose as in 2.1 $g_1 \in \mathrm{FP}, g_2 \in \mathrm{FP}_+$ such that $g(x,y) = g_1(x) - g_2(x,y)$. Then $h(x) = \max_{y \leq g'(x)}(g_1(x) - (f(x,y) + g_2(x,y))) = g_1(x) - \min_{y \leq g'(x)}(f(x,y) + g_2(x,y))$. □

Moreover, it is easy to see the following relationships:

2.3 Proposition. *If $\mathcal{F} \circ \mathrm{FP} \subseteq \mathcal{F}$, then*

1. $\mathrm{Max}\,\mathrm{Max}\,\mathcal{F} = \max(\mathrm{Max}\,\mathcal{F}, \mathrm{Max}\,\mathcal{F}) = \mathrm{Max}\,\mathcal{F}$.
2. $\mathrm{Min}\,\mathrm{Min}\,\mathcal{F} = \min(\mathrm{Min}\,\mathcal{F}, \mathrm{Min}\,\mathcal{F}) = \mathrm{Min}\,\mathcal{F}$.
3. $\min(\mathrm{Max}\,\mathcal{F}, \mathrm{Max}\,\mathcal{F}) \subseteq \mathrm{Max}\,\min(\mathcal{F}, \mathcal{F})$.
4. $\max(\mathrm{Min}\,\mathcal{F}, \mathrm{Min}\,\mathcal{F}) \subseteq \mathrm{Min}\,\max(\mathcal{F}, \mathcal{F})$.

3. P-founded Classes of Optimization Functions

We now make clear, what classes of functions we are going to examine in this paper. We call them *p-founded classes of optimization functions.* We mainly think of classes like Krentel's Opt P or our modification of Toda's class Mid P to be prototypical examples of such classes of optimization functions. Of course, we want results similar to those listed in the introduction to hold generally for optimization classes. Therefore, our definition should be wide enough to include as many classes as possible, but on the other side it should capture those properties of the just mentioned classes which distinguish them from other kinds of function classes like e.g. Valiant's class # P.

Thus, we are led to the following definition. The hardest part is clearly requirement 7. We would like our classes to be able to simulate the parallel function evaluation of several arguments given as a tuple, by one evaluation of another function of only one value (or in other terms, we want the ability to simulate parallel oracle queries by only one query). We will later (in Definition 5.2) call this property *closure under strong tuple formation.* However, it seems that the Med-classes don't have this strong property, but it turns out that they fulfill the slightly weaker requirement 7 below. We will come back to this point in the explanation following the definition.

3.1 Definition. A class $\mathcal{F}$ of functions is a *p-founded class of optimization functions (pfo)*, if and only if the following holds:

1. For every $f \in \mathcal{F}$, there exist functions $g_1, g_2 \in \mathrm{FP}$ such that for all x, $g_1(x) < f(x) < g_2(x)$. ($\mathcal{F}$ is *polynomially length bounded*)

2. $\mathrm{id} \in \mathcal{F}$, where $\mathrm{id}(x) =_{\mathrm{def}} x$ for all x.

3. $\mathcal{F} \circ \mathrm{FP} \subseteq \mathcal{F}$
(*closure under input transformations*)

4. If $f \in \mathcal{F}$, $g \in \mathrm{FP}_+$, then $f + g, f \cdot g \in \mathcal{F}$
(*closure under addition and multiplication with* FP_+*-functions*)

5. If $f \in \mathcal{F}$, $g \in \mathrm{FP}$, then $\min(f, g), \max(f, g) \in \mathcal{F}$
(*closure under min-max-operations with* FP*-functions*)

6. If $f_1, f_2 \in \mathcal{F}$, $A \in \mathrm{P}$, then the function h defined as follows is in $\mathcal{F}$:

$$h(x) =_{\mathrm{def}} \begin{cases} f_1(x), & \text{if } x \in A; \\ f_2(x), & \text{otherwise.} \end{cases}$$

(*closure under definition by cases*)

7. For every $f \in \mathcal{F}$, $g, g' \in \mathrm{FP}_+$, such that $f(x, u) \in [g'(x)]$ for $u \in [g(x)]$, there exists $h \in \mathcal{F}$, $g'' \in \mathrm{FP}$, and r such that $r(x, u) \in [g''(x)]$ for $u \in [g(x)]$, and for every $m \in \mathbb{N}$, $u_1, \ldots, u_m \in [g(x)]$,

$$h(x, \langle u_1, \ldots, u_m \rangle_{g(x)}) = \langle r(x, u_1), f(x, u_1), r(x, u_2), f(x, u_2), \ldots, f(x, u_m) \rangle_{g''(x), g'(x)}.$$

(*closure under tuple formation*)

We will see (3.4) that Opt P, Med FP, and generalizations of them are really pfo's. And on the other hand, it is known that typical "classes of counting functions" like # P are not pfo's, since they are not closed under polynomial time min-max-operations (unless the polynomial time hierarchy collapses).

In order to explain our notion of a pfo, let us discuss the cases of the classes Max FP, Min FP, and Med FP. It is relatively easy to see, that these classes can be alternatively defined to consist of all functions that yield the maximal (minimal, median, resp.) element in the sequences of outputs over all paths of nondeterministic polynomial time Turing machines.

This alternative characterization of the classes shows directly, that all values of functions from any of them are bounded in length by a polynomial in the length of the input, and that the identity functions is included in all of these classes. Moreover, since we can, instead of printing some value z on any branch of a machine M, as well print z plus some positive value computable deterministically in polynomial time (or z multiplied by such a value) without changing the relative order of different values from different paths, all three classes are closed under addition (multiplication, resp.) with FP_+ functions. Similar simple arguments also show closure under min-max-operations with FP functions and closure under definition by cases.

Now, the most difficult property to show is closure under tuple formation. Our proofs will rely on the following observations: Consider a Max FP function h defined by a nondeterministic polynomial time Turing machine M. If we want to compute the pair of values $\langle h(v), h(w) \rangle$ for some v, w, we first start M on input v; and in every halting configuration we then start M on input w. If M then stops, we simply print the concatenation of the two values obtained (normalized in some way concerning the lengths of the values). The maximal value over all these concatenations is exactly the pair we are interested in.

This strong form of closure under tuple formation can be proved in the same way for Min, but unfortunately not for Med. If as above, we concatenate M's computations on v and w then paths with the same value in the first computation can have the consequence that the median value of the global computation is not the concatenation of the median values of the single computations. As an example, let the first computation have three paths with values 0, 1, 1, resp. (median 1), and let the second computation have three paths with values 0, 1, 2, resp. (again median 1). The concatenation of these medians is 11, but the concatenated computation has 9 paths with values 00, 01, 02, 10, 11, 12, 10, 11, 12. If we order this values in such a way that all numbers that have their first path in common appear as a coherent subsequence (as we did in the example), then the middle element of that sequence will be exactly the value we are interested in. But if we take their natural order, then paths will be mixed up and we get the median 10. To overcome this difficulty we have to take care that in every level each value appears at most once. One can for example modify the machines (and the functions computed) in such a way that, on path y, instead of value z the value zy is produced. That ensures different values on different paths, and the proof used for Min and Max works also for Med. The price is that we do not obtain the function we really want, because all values have a "garbage tail". This is the reason for the form of the tuple formation property we have in the definition of pfo's. There, the function r is the formal counterpart of the garbage tail. However, this weak form is strong enough to ensure the properties we want to have for pfo's.

Our next goal will be to show that classes constructed from pfo's by application of the operators Max, Min, or Med, are again pfo's. But before we go on, we first need two technical results. We will sometimes have to combine (for example via addition, case distinction or so on) two functions from a class with different domains. The following easy to prove lemma states that sometimes we may extend domains of functions without changing relevant values.

3.2 Lemma. *(Domain extension lemma)*
Let $\mathcal{F}$ be a pfo. For every $f \in \mathcal{F}$, $g, g' \in \mathrm{FP}$ such that $g' > g$ there exists an $f' \in \mathcal{F}$ such that $\max_{y \leq g(x)} f(x,y) = \max_{y \leq g'(x)} f'(x,y)$ ($\min_{y \leq g(x)} f(x,y) = \min_{y \leq g'(x)} f'(x,y)$, $\mathrm{med}_{y \leq g(x)} f(x,y) = \mathrm{med}_{y \leq g'(x)} f'(x,y)$, resp.)

Proof. We give the proof for Max, the other cases proceed in the same spirit. Let $g'' \in \mathrm{FP}$ such that $g''(x) \leq f(x,y)$ for all $y \leq g(x)$. Define

$$f'(x,y) = \begin{cases} f(x,y), & \text{if } y \leq g(x); \\ g''(x), & \text{otherwise.} \end{cases}$$

□

We will sometimes make implicit use of Lemma 3.2. I.e. if we have for example two functions $h, h' \in \operatorname{Max} \mathcal{F}$ for some pfo $\mathcal{F}$, and they are defined via $f, f' \in \mathcal{F}$, then we assume tacitly that the maximization process over f and f' is carried out over the same domain.

The second technical result we need is the following:

3.3 Lemma. *If $\mathcal{F}$ is a pfo, then*

1. $\mathcal{F} \subseteq \operatorname{Min} \mathcal{F} \cap \operatorname{Max} \mathcal{F}$.
2. $\operatorname{Min} \mathcal{F} \cup \operatorname{Max} \mathcal{F} \subseteq \operatorname{Med} \mathcal{F}$.

Proof. Statement 1 is obvious. We give the proof for 2: Let $h \in \operatorname{Min} \mathcal{F}$, $h(x) = \min_{y \leq g(x)} f(x,y)$ for some $f \in \mathcal{F}$, $g \in \mathrm{FP}$. Since f is polynomially length-bounded we can easily find $g' \in \mathrm{FP}$ which fulfills $g'(x) < f(x,y)$ for all $y \leq g(x)$. Defining

$$f'(x,y) = \begin{cases} f(x,y), & \text{if } y \leq g(x) \\ g'(x), & \text{otherwise.} \end{cases}$$

we clearly have $h \in \mathcal{F}$ (definition by cases) and $h(x) = \mathrm{med}_{y \leq 2g(x)} f'(x,y)$.

The case for Max proceeds analogously. □

The next theorem shows that the notion of a pfo is really suitable for our goals. It states that if we are given a pfo (e.g. FP) and we apply the function operators that we are mainly interested in, what we get is again a pfo.

3.4 Theorem.

1. FP *is a pfo.*
2. *If $\mathcal{F}$ is a pfo, then so are* $\operatorname{Max} \mathcal{F}$, $\operatorname{Min} \mathcal{F}$, *and* $\operatorname{Med} \mathcal{F}$.

Proof. The first statement is obvious. We prove the second. As we will see, the proofs for requirements 1 to 6 from Definition 3.1 are more or less straightforward, but the proof of closure under tuple formation is quite involved.

Suppose that $\mathcal{F}$ is a pfo.

If $h \in \operatorname{Max} \mathcal{F}$ ($\operatorname{Min} \mathcal{F}$, $\operatorname{Med} \mathcal{F}$, resp.) via $f \in \mathcal{F}$ and f is polynomially length bounded, then so is h.

Since $\mathrm{id} \in \mathcal{F}$, is is easy to see $\mathrm{id} \in \operatorname{Max} \mathcal{F}$ ($\operatorname{Min} \mathcal{F}$, $\operatorname{Med} \mathcal{F}$, resp.).

Now suppose $h \in \operatorname{Max} \mathcal{F}$, $h(x) = \max_{y \leq g(x)} f(x,y)$ for $f \in \mathcal{F}, g \in \mathrm{FP}$, and suppose $g' \in \mathrm{FP}$. Then

$$h \circ g'(x) = \max_{y \leq g(g'(x))} f(g'(x), y)$$

and $h \circ g' \in \operatorname{Max} \mathcal{F}$, since $\lambda xy.f(g'(x),y) \in \mathcal{F}$ and $g \circ g' \in \mathrm{FP}$. The same argument holds for $\operatorname{Min} \mathcal{F}$ and $\operatorname{Med} \mathcal{F}$.

Furthermore,

$$h(x) + g'(x) = \max_{y \leq g(x)} f(x,y) + g'(x) = \max_{y \leq g(x)} (f(x,y) + g'(x))$$

and $h + g' \in \operatorname{Max} \mathcal{F}$, since $\lambda xy.f(x,y) + g'(x) \in \mathcal{F}$. The same argument holds for $\operatorname{Min} \mathcal{F}$ and $\operatorname{Med} \mathcal{F}$, and a similar argument holds (in all three cases) for multiplication with positive FP-functions.

Moreover,

$$\max\{h(x), g'(x)\} = \max_{y \leq g(x)} \max\{f(x,y), g'(x)\}$$

and

$$\min\{h(x), g'(x)\} = \max_{y \leq g(x)} \min\{f(x,y), g'(x)\}$$

and therefore $\max(f, g') \in \operatorname{Max} \mathcal{F}$, $\min(f, g') \in \operatorname{Max} \mathcal{F}$, since $\lambda xy. \max\{f(x,y), g'(x)\} \in \mathcal{F}$ and $\lambda xy. \min\{f(x,y), g'(x)\} \in \mathcal{F}$. The same argument holds for $\operatorname{Min} \mathcal{F}$ and $\operatorname{Med} \mathcal{F}$.

Let $h_1, h_2 \in \operatorname{Max} \mathcal{F}$, $h_i(x) = \max_{y \leq g_i(x)} f_i(x,y)$, where $f_i \in \mathcal{F}, g_i \in \mathrm{FP}$ ($i \in \{1,2\}$), and let $A \in \mathrm{P}$. By Lemma 3.2, we may suppose $g_1 = g_2$. Let

$$h(x) =_{\mathrm{def}} \begin{cases} h_1(x), \text{if } x \in A; \\ h_2(x), \text{otherwise.} \end{cases}$$

Then

$$h(x) = \max_{y \leq g_1(x)} \begin{cases} f_1(x,y), & \text{if } x \in A; \\ f_2(x,y), & \text{otherwise.} \end{cases}$$

Therefore $h \in \operatorname{Max} \mathcal{F}$; and similar arguments hold for $\operatorname{Min} \mathcal{F}$ and $\operatorname{Med} \mathcal{F}$.

We now come to closure under tuple formation. Our proof follows the ideas presented in the explanation following Definition 3.1. First, let's turn to the easier cases of the operators Max and Min.

Let $h \in \operatorname{Max} \mathcal{F}$, $g, g' \in \mathrm{FP}_+$ such that $h(x,u) \in [g'(x)]$ for $u \in [g(x)]$. Let $h(x,u) = \max_{y \leq g_1(x,u)} f(x,u,y)$ for $f \in \mathcal{F}$, $g_1 \in \mathrm{FP}$.

Choose $g_2 \in \mathrm{FP}$ such that $g_1(x,u) \leq g_(x)$ for $u \in [g(x)]$. By the domain extension lemma (Lemma 3.2), there exists an $f_1 \in \mathcal{F}$ such that $h(x,u) = \max_{y < g_2(x)} f_1(x,u,y)$ for $u \in [g(x)]$.

Defining $f_2(x,u,y) =_{\rm def} \max\{-g'(x)+1, \min\{f_1(x,u,y), g'(x)\}\}$, we have $h(x,u) = \max_{y<g_2(x)} f_2(x,u,y)$ and $f_2(x,u,y) \in [g'(x)]$ for all y and $u \in [g(x)]$.

Define $f_3(x,z) =_{\rm def} f_2(x, z \div g_2(x), z \bmod g_2(x))$; surely $f_3 \in \mathcal{F}$. Then we have that for $u \in [g(x)]$ and $y < g_2(x)$ that $f_3(x, u \cdot g_2(x) + y) = f_2(x,u,y)$. For $z \in [g(x) \cdot g_2(x)]$, surely $z \div g_2(x) \in [g(x)]$ and therefore also $f_3(x,z) \in [g'(x)]$.

By the assumption, $\mathcal{F}$ is closed under tuple formation. Therefore, there exist $f_4 \in \mathcal{F}$, $\hat{g} \in \mathrm{FP}$, and r, such that $r(x,z) \in [\hat{g}(x)]$ for all y and $u \in [g(x)]$, and

$$\begin{aligned} &f_4(x, \langle u_1 g_2(x) + y_1, \ldots, u_m g_2(x) + y_m \rangle_{g(x) \cdot g_2(x)}) \\ &\quad = \langle r(x, u_1 g_2(x) + y_1), f_3(x, u_1 g_2(x) + y_1), \ldots, f_3(x, u_m g_2(x) + y_m) \rangle_{\hat{g}(x), g'(x)} \cdot \end{aligned}$$

for $u_1, \ldots, u_m \in [g(x)]$ and $y_1, \ldots, y_m < g_2(x)$.

Define

$$f_5(x,y,U) =_{\rm def} f_4(x, \langle [U, g(x), 1] g_2(x) + y_1, \ldots, [U, g(x), m[U, g(x)]] g_2(x) + y_m \rangle),$$

where $y_1, y_2, \ldots$ are the digits of y in it's $g_2(x)$-ary representation. Then we have for $u_1, \ldots, u_m \in [g(x)]$ that

$$f_5(x, y, \langle u_1, \ldots, u_m \rangle_{g(x)}) = f_4(x, \langle u_1 g_2(x) + y_1, \ldots, u_m g_2(x) + y_m \rangle),$$

Finally, define $h'(x,U) =_{\rm def} \max_{y < g_2(x)^{m[U,g(x)]}} f_5(x,y,U)$.

Then we have for $u_1, \ldots, u_m \in [g(x)]$:

$$\begin{aligned} h'(x, \langle u_1, \ldots, u_m \rangle_{g(x)}) &= \max_{y_1, \ldots, y_m < g_2(x)} \langle r(x, u_1 g_2(x) + y_1), f_2(x, u_1, y_1), \ldots, f_2(x, u_m, y_m) \rangle_{\hat{g}(x), g'(x)} \\ &= \langle s(x, u_1), \max_{y < g_2(x)} f_2(x, u_1, y), \ldots, \max_{y < g_2(x)} f_2(x, u_m, y) \rangle_{\hat{g}(x), g'(x)} \end{aligned}$$

where $s(x,u) =_{\rm def} \max\{\, r(x, u g_2(x) + y) \mid y < g_2(x) \wedge f_2(x,u,y) = \max_{z < g_2(x)} f_2(x,u,z) \,\}$; obviously $s(x,u) \in [\hat{g}(x)]$ for $u \in [g(x)]$.

Thus,

$$h'(x, \langle u_1, \ldots, u_m \rangle_{g(x)}) = \langle s(x, u_1), h(x, u_1), \ldots, h(x, u_m) \rangle_{\hat{g}(x), g'(x)}$$

for $u_1, \ldots, u_m \in [g(x)]$.

A similar argument holds for $\mathrm{Min}\,\mathcal{F}$.

We remark for later use (Section 5) that, if r is the constant zero function, then s is also constant zero.

The case for Med is a bit more involved. The argumentation will be similar to the above, but we have to ensure, following the remarks preceeding this proof, that the arising Med-function is one-one. If we think in terms of machines, as in the case of Med FP in the preceeding explanation, we have to insert a "garbage tail" coding a computation path of the machine.

Let $h \in \mathrm{Med}\,\mathcal{F}$, $g, g' \in \mathrm{FP}_+$ such that $h(x,u) \in [g'(x)]$ for $u \in [g(x)]$. Let $h(x,u) = \mathrm{med}_{y \leq g_1(x,u)} f(x,u,y)$ for $f \in \mathcal{F}$, $g_1 \in \mathrm{FP}$.

Choose $g_2 \in \mathrm{FP}$ such that $g_1(x,u) \leq g_2(x)$ for $u \in [g(x)]$. By the domain extension lemma (Lemma 3.2), there exists an $f_1 \in \mathcal{F}$ such that $h(x,u) = \max_{y < g_2(x)} f_1(x,u,y)$ for $u \in [g(x)]$.

Defining $f_2(x,u,y) =_{\rm def} \max\{-g'(x)+1, \min\{f_1(x,u,y), g'(x)\}\}$, we have $h(x,u) = \max_{y<g_2(x)} f_2(x,u,y)$ and $f_2(x,u,y) \in [g'(x)]$ for all y and $u \in [g(x)]$.

Define $f_2'(x,u,y) =_{\text{def}} f_2(x,u,y)\cdot g_2(x)+y$. Hence $f_2' \in \mathcal{F}$, $f_2'(x,u,\cdot)$ is one-one on $[0,g_2(x)-1]$, and $f_2'(x,u,y) \in [g'(x)\cdot g_2(x)]$ for $u \in [g(x)]$. Set $g''(x) =_{\text{def}} g'(x)\cdot g_2(x)$.

Define $f_3(x,z) =_{\text{def}} f_2'(x, z \div g_2(x), z \bmod g_2(x))$; surely $f_3 \in \mathcal{F}$. Then we have that for $u \in [g(x)]$ and $y < g_2(x)$ that $f_3(x, u\cdot g_2(x)+y) = f_2'(x,u,y)$. For $z \in [g(x)\cdot g_2(x)]$, surely $z \div g_2(x) \in [g(x)]$ and therfore also $f_3(x,z) \in [g''(x)]$.

By the assumption, $\mathcal{F}$ is closed under tuple formation. Therefore, there exist $f_4 \in \mathcal{F}$, $\hat{g} \in \text{FP}$, and r, such that $r(x,z) \in [\hat{g}(x)]$ for all y and $u \in [g(x)]$, and

$$\begin{aligned} &f_4(x, \langle u_1 g_2(x)+y_1,\ldots,u_m g_2(x)+y_m\rangle_{g(x)\cdot g_2(x)}) \\ &\quad = \langle r(x,u_1g_2(x)+y_1), f_3(x,u_1g_2(x)+y_1),\ldots,f_3(x,u_mg_2(x)+y_m)\rangle_{\hat{g}(x),g''(x)}\cdot \end{aligned}$$

for $u_1,\ldots,u_m \in [g(x)]$ and $y_1,\ldots,y_m < g_2(x)$.

Define

$$f_5(x,y,U) =_{\text{def}} f_4(x, \langle [U,g(x),1]g_2(x)+y_1,\ldots,[U,g(x),m[U,g(x)]]g_2(x)+y_m\rangle),$$

where $y_1,y_2,\ldots$ are the digits of y in it's $g_2(x)$-ary representation. Then we have for $u_1,\ldots,u_m \in [g(x)]$ that

$$f_5(x,y,\langle u_1,\ldots,u_m\rangle_{g(x)}) = f_4(x,\langle u_1g_2(x)+y_1,\ldots,u_mg_2(x)+y_m\rangle),$$

Finally, define $h'(x,U) =_{\text{def}} \max_{y<g_2(x)^{m[U,g(x)]}} f_5(x,y,U)$.

Then we have for $u_1,\ldots,u_m \in [g(x)]$:

$$\begin{aligned} h'(x,\langle u_1,\ldots,u_m\rangle_{g(x)}) &= \operatorname*{med}_{y_1,\ldots,y_m<g_2(x)} \langle r(x,u_1g_2(x)+y_1), f_2'(x,u_1,y_1),\ldots,f_2'(x,u_m,y_m)\rangle_{\hat{g}(x),g''(x)} \\ &= \langle s(x,u_1), \operatorname*{med}_{y<g_2(x)} f_2'(x,u_1,y),\ldots, \operatorname*{med}_{y<g_2(x)} f_2'(x,u_m,y)\rangle_{\hat{g}(x),g''(x)} \end{aligned}$$

where $s(x,u) =_{\text{def}} r(x,ug_2(x)+t(x,u))$ and $t(x,u) =_{\text{def}}$ the unique y such that $f_2'(x,u,y) = \text{med}_{z<g_2(x)} f_2'(x,u,z)$. The last equality and the uniqueness condition hold because of the one-oneness of f_2'.

Thus,

$$\begin{aligned} h'(x,\langle u_1,\ldots,u_m\rangle_{g(x)}) &= \langle s(x,u_1), (\text{med}_{y<g_2(x)} f_2(x,u_1,y))\cdot g_2(x)+t(x,u_1), \\ &\qquad \ldots, (\text{med}_{y<g_2(x)} f_2(x,u_m,y))\cdot g_2(x)+t(x,u_m)\rangle_{\hat{g}(x),g''(x)} \\ &= \langle s(x,u_1), h(x,u_1)\cdot g_2(x)+t(x,u_1), \\ &\qquad \ldots, h(x,u_m)\cdot g_2(x)+t(x,u_m)\rangle_{\hat{g}(x),g''(x)} \\ &= \langle s'(x,u_1), h(x,u_1), \\ &\qquad \ldots, h(x,u_m)\rangle_{\hat{g}(x)g_2(x),g'(x)}, \end{aligned}$$

for $u_1,\ldots,u_m \in [g(x)]$ and a suitable chosen function s'; which finishes the proof. $\square$

4. Converting Functions into Sets and vice versa

We now come to the comparison between function classes and set classes. To that end we introduce two operators, F and S, which convert a set class (in this paper a set is always a subset of $\mathbb{Z}^k$ for some $k \in \mathbb{N}$) into a function class, and a function class into a set class, respectively, where it is intended that these conversions preserve the complexities of the objects as well as possible.

4.1 Definition. Let $\mathcal{F}$ be a class of functions and $\mathcal{K}$ be a class of sets. Then we say that

1. $A \in \text{S}\cdot\mathcal{F}$ if and only if there exist $f \in \mathcal{F}$, $g \in \text{FP}$ such that

$$x \in A \iff f(x) \geq g(x).$$

2. $f \in \mathrm{F}\cdot\mathcal{K}$ if and only if there exist $A \in \mathcal{K}$, $g_1, g_2 \in \mathrm{FP}$ such that $(x, y+1) \in A \Rightarrow (x,y) \in A$ and

$$f(x) = \max\{\, y \mid g_1(x) \le y \le g_2(x) \wedge (x,y) \in A \,\},$$

if this set is not empty, and $f(x) = g_1(x)$, otherwise.

We will sometimes include a $\cdot$ between operators and arguments or between different operators. But this is only to improve readability by structuring the term. Therefore, when we write $\mathrm{S}\cdot\mathrm{F}\cdot\mathcal{K}$ or $\mathrm{S}\cdot\mathrm{F}\mathcal{K}$ or $\mathrm{SF}\cdot\mathcal{K}$ or $\mathrm{SF}\mathcal{K}$, we always mean the same class; we choose one of the possibilities depending on what we want to stress.

A seemingly simpler way to encode a function f into a set is to use the set $\mathrm{proj}\,(f) =_{\mathrm{def}} \{\,(x,k) \mid f(x) \ge k\,\}$ (the *projection* of f). However, we want the encoding $\mathrm{proj}\,(\mathcal{F}) =_{\mathrm{def}} \{\,\mathrm{proj}\,(f) \mid f \in \mathcal{F}\,\}$ of an interesting function class $\mathcal{F}$ to be closed under polynomial time many-one reducibility, but classes of the form $\mathrm{proj}\,(\mathcal{F})$ generally do not posess this property. But a simple proposition shows that in all interesting cases (i.e. always if $\mathcal{F} \circ \mathrm{FP} \subseteq \mathcal{F}$), the many-one closure of $\mathrm{proj}\,(\mathcal{F})$ is exactly $\mathrm{S}\cdot\mathcal{F}$. Thus, in a sense, S is the simplest operator we can choose.

4.2 Proposition. *If* $\mathcal{F} \circ \mathrm{FP} \subseteq \mathcal{F}$, *then* $\mathrm{S}\cdot\mathcal{F} = \mathcal{R}^{\mathrm{p}}_{m}(\mathrm{proj}\,(\mathcal{F}))$.

Note that operator S is the one which is used to convert the function class $\#\,\mathcal{K}$ into the set class $\mathrm{C}\cdot\mathcal{K}$, see Wagner (1986a), Wagner (1986b), Torán (1988), Torán (1991) (i.e. for all $\mathcal{K}$ we have $\mathrm{S}\,\#\,\mathcal{K} = \mathrm{C}\mathcal{K}$); and the operator F is a generalization of the exponentially bounded "monotonic" maximization operator which often appears in the literature, see Buss and Hay (1991), Wagner (1990), Allender and Wilson (1990), Ogiwara (1991).

The reasons for choosing these operators for the conversion are threefold: the operators are easy to compute (see Proposition 4.3); one operator is "almost" the inverse of the other (see Proposition 4.4); and they establish a very neat relationship between natural operators on classes of sets like $\exists$, $\forall$ and C and the operators Max, Min, and Med on classes of functions (as we shall see in Section 6).

4.3 Proposition. *Let* $\mathcal{F}$ *be a class of polynomially length-bounded functions closed under input transformations, and let* $\mathcal{K}$ *be a class of sets. Then the following hold:*

1. $\mathrm{S}\cdot\mathcal{F} \subseteq \mathrm{P}^{\mathcal{F}}[1]$.
2. $\mathrm{F}\cdot\mathcal{K} \subseteq \mathrm{FP}^{\mathcal{K}}$.
3. $\mathrm{P}^{\mathrm{S}\cdot\mathcal{F}} = \mathrm{P}^{\mathcal{F}}$ *and* $\mathrm{FP}^{\mathrm{S}\cdot\mathcal{F}} = \mathrm{FP}^{\mathcal{F}}$
4. $\mathrm{P}^{\mathrm{F}\cdot\mathcal{K}} = \mathrm{P}^{\mathcal{K}}$ *and* $\mathrm{FP}^{\mathrm{F}\cdot\mathcal{K}} = \mathrm{FP}^{\mathcal{K}}$
5. $\mathrm{F}\cdot\mathrm{P} = \mathrm{FP}$
6. $\mathrm{S}\cdot\mathrm{FP} = \mathrm{P}$

Proof. Obvious, see the remarks above and below. $\square$

So we see that while going from functions to sets we will definitely not alter the complexity too much (S can very easily be computed by asking one query to a function from the class under consideration). When going from sets to functions, the situation is a bit different: F seems to require a binary search to find the maximum; therefore the function class obtained from a set class may be a bit more complex, but not too much since binary search can be carried out in polynomial time. Thus we cannot expect the operators F and S to be inverse to each other on all reasonable classes. Nevertheless, we can prove surprisingly, that almost all reasonable classes of sets are fixpoints under the composition operator $\mathrm{S} \cdot \mathrm{F}$.

4.4 Proposition.

1. *If $\mathcal{K}$ is closed under $\leq_m^{\mathrm{p}}$ and $\mathrm{P} \subseteq \mathcal{K}$, then $\mathrm{S} \cdot \mathrm{F} \cdot \mathcal{K} = \mathcal{K}$.*

2. *If $\mathcal{F} \circ \mathrm{FP} \subseteq \mathcal{F}$, then $\mathrm{FS} \cdot \mathcal{F} = \mathrm{FS} \cdot \mathcal{F} - \mathrm{FP}$.*
   For all $\mathcal{F}$, $\mathcal{F} - \mathrm{FP} \subseteq \mathrm{FS} \cdot \mathcal{F} = \mathrm{FS} \cdot (\mathcal{F} - \mathrm{FP})$.

3. *FS is a closure operator on all pfo's $\mathcal{F}$.*

Proof. 1. Let $A \in \mathcal{K}$, $A \subseteq \mathbb{Z}^k$ for some $k \in \mathbb{N}$. We have to find a set $A' \in \mathcal{K}$ such that membership in A can be expressed as a maximization process over A'. This is achieved by defining $A' = A \times \{1\} \cup \mathbb{Z}^k \times \{0\}$. Because of the assumptions about $\mathcal{K}$, we have $A' \in \mathcal{K}$. Obviously $c_A(x) = \max\{\, y \mid 0 \leq y \leq 1 \wedge (x, y) \in A' \,\}$. (Here, x is a variable for an element from $\mathbb{Z}^k$.) Hence $c_A \in \mathrm{F} \cdot \mathcal{K}$. But since $x \in A \iff c_A(x) \geq 1$, we get $A \in \mathrm{SF} \cdot \mathcal{K}$.

Let $A \in \mathrm{SF} \cdot \mathcal{K}$, $x \in A \iff f(x) \geq g(x)$, where $f \in \mathrm{F} \cdot \mathcal{K}$, $g \in \mathrm{FP}$. Then $x \in A \iff \max\{\, y \mid g_1(x) \leq y \leq g_2(x) \wedge (x, y) \in B \,\} \geq g(x)$ for some $g_1, g_2 \in \mathrm{FP}$, $B \in \mathcal{K}$, $B \subseteq \mathbb{Z}^k$ for some $k \in \mathbb{N}$; where now x ranges over $\mathbb{Z}^{k-1}$. Consequently, $x \in A \iff g_1(x) \leq g(x) \leq g_2(x) \wedge (x, g(x)) \in B$.

Consider two cases:

- $B = \mathbb{Z}^k$. Hence $x \in A \iff g_1(x) \leq g(x) \leq g_2(x)$. Thus $A \in \mathrm{P}$. Because of the assumptions we have $\mathrm{P} \subseteq \mathcal{K}$.

- $B \neq \mathbb{Z}^k$. Hence we have $(x_0, y_0) \notin B$. Define

$$g''(x) = \begin{cases} (x, g(x)), & \text{if } g_1(x) \leq g(x) \leq g_2(x) \\ (x_0, y_0), & \text{otherwise.} \end{cases}$$

  Obviously, $g'' \in \mathrm{FP}$ and $x \in A \iff g''(x) \in B$. Consequently, $A \in \mathcal{K}$.

2. The proofs for both statements are similar. We give the second.

Let $f \in \mathcal{F}$, $g \in \mathrm{FP}$. Let $g', g'' \in \mathrm{FP}$ such that $g' \leq f \leq g''$.

$$\begin{aligned} f(x) - g(x) &= \max\{\, y \mid g'(x) - g(x) \leq y \leq g''(x) - g(x) \wedge f(x) - g(x) \geq y \,\} \\ &= \max\{\, y \mid g'(x) - g(x) \leq y \leq g''(x) - g(x) \wedge f(x) \geq g(x) + y \,\} \end{aligned}$$

Hence $f - g \in \mathrm{FS} \cdot \mathcal{F}$; and for all $\mathcal{F}$, surely $\mathrm{S}\mathcal{F} = \mathrm{S}(\mathcal{F} - \mathrm{FP})$.

3. Consequence of 1. and 2. □

Concerning the operator $\mathrm{F}\cdot\mathrm{S}$, we don't have a relationship as strong as the one given in the first statement of 4.4 for $\mathrm{S}\cdot\mathrm{F}$. We can only observe that

$$\mathcal{F} \subseteq \mathcal{F} - \mathrm{FP} \subseteq \mathrm{FS}\cdot\mathcal{F} \subseteq \mathrm{FP}^{\mathcal{F}}$$

and we will later see that the equality $\mathrm{FS}\cdot\mathcal{F} = \mathrm{FP}^{\mathcal{F}}$ is very unlikely for many pfo's. (E.g. if it holds for $\mathcal{F} = \mathrm{Max}\,\mathrm{FP}$, then the polynomial time hierarchy will collapse to NP.) On the other hand, we will see that surprisingly many natural classes of functions $\mathcal{F}$ have $\mathcal{F} - \mathrm{FP}$ as fixpoint of $\mathrm{F}\cdot\mathrm{S}$ (cf. Sections 6 and 7).

It is obvious that the operator FS can be computed by application of binary search. The next result shows that in fact the application of FS is nothing else than binary search over the values of a monotonic decreasing function.

Let $\mathcal{F}$ be a class of functions.

$$h \in \mathrm{BS}\cdot\mathcal{F} =_{\mathrm{def}} \text{ there exist } f \in \mathcal{F},\ g, g_1, g_2 \in \mathrm{FP} \text{ such that } f(x,y) \geq f(x,y+1) \text{ and}$$
$$h(x) = \begin{cases} \max\{\, y \mid g_1(x) \leq y \leq g_2(x) \wedge f(x,y) \geq g(x) \,\}, & \text{if } \{\ldots\} \neq \emptyset \\ g_1(x), & \text{otherwise} \end{cases}$$

4.5 Theorem. *If $\mathcal{F}$ is a pfo, then* $\mathrm{FS}\cdot\mathcal{F} = \mathrm{BS}\cdot\mathcal{F}$.

Proof. $\mathrm{BS}\cdot\mathcal{F} \subseteq \mathrm{FS}\cdot\mathcal{F}$ is obvious.

For the other direction, let $h \in \mathrm{FS}\cdot\mathcal{F}$, and let $f \in \mathcal{F}$, $g, g_1, g_2 \in \mathrm{FP}$ such that $f(x,y+1) \geq g(x,y+1)$ implies $f(x,y) \geq g(x,y)$ and $h(x) = \max\{\, y \mid g_1(x) \leq y \leq g_2(x) \wedge f(x,y) \geq g(x,y) \,\}$ if this set is not empty, and $h(x) = g_1(x)$, otherwise. To express this as a simple binary search, we have to get rid of g depending not only on x but also on y. Since $\mathcal{F}$ is closed under multiplication with FP_{+}-functions we can assume that the above set is not empty. Defining $f'(x,y) =_{\mathrm{def}} \max\{g(x,y) - 1, \min\{f(x,y), g(x,y)\}\}$ we obtain

$$f'(x,y) = \begin{cases} g(x,y), & \text{if } f(x,y) \geq g(x,y) \\ g(x,y) - 1, & \text{if } f(x,y) < g(x,y) \end{cases}$$

and $f' \in \mathcal{F}$. Let $g' \in \mathrm{FP}$ be such that $g(x,y) < g'(x)$ for $g_1(x) \leq y \leq g_2(x)$. Defining $f''(x,y) =_{\mathrm{def}} f'(x,y) + (g'(x) - g(x,y))$, we obtain

$$f''(x,y) = \begin{cases} g'(x), & \text{if } f(x,y) \geq g(x,y) \\ g'(x) - 1, & \text{if } f(x,y) < g(x,y) \end{cases}$$

and $f'' \in \mathcal{F}$.

Moreover, $f''(x,y) \geq f''(x,y+1)$ and $h(x) = \max\{\, y \mid g_1(x) \leq g_2(x) \wedge f''(x,y) \geq g'(x) \,\}$. □

Coming back to the inclusion chain

$$\mathcal{F} \subseteq \mathcal{F} - \mathrm{FP} \subseteq \mathrm{FS}\cdot\mathcal{F} \subseteq \mathrm{FP}^{\mathcal{F}}$$

which holds for all pfo's, we characterized in the previous result the power of $\mathrm{FS}\cdot\mathcal{F}$, compared to $\mathcal{F}$, as closure of $\mathcal{F}$ under binary search. Thus, the first two inclusions become equality if and only if $\mathcal{F}$ is closed under binary search.

We still have to compare the complexities of $\mathrm{FS}\cdot\mathcal{F}$ and $\mathrm{FP}^{\mathcal{F}}$. This will be done in the next theorem and it's corollary. We will see that surprisingly we only have to add the power of subtraction to $\mathrm{FS}\cdot\mathcal{F}$ and we already arrive at the full power of $\mathrm{FP}^{\mathcal{F}}$. It turns out that a polynomial number of adaptive queries to a function from $\mathcal{F}$ can be simulated by only one query to a slightly more powerful function (one from the class $\mathrm{FS}\cdot\mathcal{F}$). This will lead to many interesting corollaries in Sections 5 and 7.

4.6 Theorem. *If $\mathcal{F}$ is a pfo, then*

$$\mathrm{FP}^{\mathcal{F}} = (\mathrm{FS}\cdot\mathcal{F} \bmod 2\mathrm{FP}) - \mathrm{FP} = \mathrm{FS}\cdot\mathcal{F} - \mathrm{FS}\cdot\mathcal{F}.$$

Proof. Obviously, $\mathrm{FP}^{\mathcal{F}}$ is a superclass of the other two classes. For the other direction, we will have to simulate the sequence of adaptive queries using a tricky maximization process.

Let $f \in \mathcal{F}$ and $h \in \mathrm{FP}^f$ via the Turing machine M. Suppose that there exist $s, g, g' \in \mathrm{FP}$ such that $h(x) \in [g(x)]$, M asks on input x exactly $s(x)$ queries from $[g(x)]$, and $f(y) \in g'(x)$ for all $y \in [g(x)]$. Since $\mathcal{F}$ is closed under tuple formation there exist $f' \in \mathcal{F}$, $\hat{g} \in \mathrm{FP}$, and r such that

$$f'(x, \langle u_1, \ldots, u_m\rangle_{g(x)}) = \langle r(x,u_1), f(u_1), \ldots, r(x,u_m), f(u_m)\rangle_{\hat{g}(x),g'(x)}.$$

For $v_1, \ldots, v_{s(x)} \in [g'(x)]$ we write $\bar{v} = \langle v_1, \ldots, v_{s(x)}\rangle_{g'(x)}$. Now consider a computation of M on input x where $v_1, \ldots, v_{s(x)}$ are used instead of oracle answers. Let $\mathrm{output}(x, \bar{v})$ be the output of M in this computation and define $\mathrm{queries}(x,\bar{v}) =_{\mathrm{def}} \langle q_{s(x)}, \ldots, q_1\rangle_{g(x)}$ where q_i is the i-th query in this computation. Obviously, $\mathrm{output}, \mathrm{queries} \in \mathrm{FP}$.

Defining

$$\mathrm{path}(x,\bar{v}) =_{\mathrm{def}} \langle \mathrm{output}(x,\bar{v}), f'(x, \mathrm{queries}(x,\bar{v}))\rangle_{g(x)}$$

we have $\mathrm{path} \in \mathcal{F}$. Now define a two-placed function $\mathrm{answer} \in \mathrm{FP}$ such that

$$\mathrm{answer}(x, \langle u, \langle r_{s(x)}, v_{s(x)}, \ldots, r_1, v_1\rangle_{\hat{g}(x),g'(x)}\rangle_{g(x)} = \bar{v}$$

for all $v_1, \ldots, v_{s(x)} \in [g'(x)]$, $r_1, \ldots, r_{s(x)} \in [\hat{g}(x)]$, and $u \in [g(x)]$. Finally, define

$$\mathrm{better\_path}(x,z) =_{\mathrm{def}} \mathrm{path}(x, \mathrm{answers}(x,z)).$$

Surely, $\mathrm{better\_path} \in \mathcal{F}$.

Now let $q_1, \ldots, q_{s(x)}$ and $v_1, \ldots, v_{s(x)}$ be the sequences of correct queries and answers, resp., during M's work on input x using oracle f; i.e. $f(q_i) = v_i$, $\mathrm{queries}(x,\bar{v}) = \langle q_{s(x)}, \ldots, q_1\rangle_{g(x)}$, and $\mathrm{output}(x,\bar{v}) = h(x)$. Putting $r_i =_{\mathrm{def}} r(x, q_i)$, we obtain

$$z =_{\mathrm{def}} \mathrm{path}(x,\bar{v}) = \langle h(x), \langle r_{s(x)}, v_{s(x)}, \ldots, r_1, v_1\rangle_{\hat{g}(x),g'(x)}\rangle_{g(x)},$$

$\mathrm{answers}(x,z) = \bar{v}$ and $\mathrm{better\_path}(x,z) = z$. Define $\mathrm{correct\_path}(x) =_{\mathrm{def}} z$.

Now take arbitrary $v'_1, \ldots, v'_{s(x)} \in [g'(x)]$, $r'_1, \ldots, r'_{s(x)} \in [\hat{g}(x)]$, and $u' \in [g(x)]$. Define $\bar{v}' =_{\mathrm{def}} \langle v'_1, \ldots, v'_{s(x)}\rangle_{g'(x)}$ and $z' =_{\mathrm{def}} \langle u', \langle r'_{s(x)}, v'_{s(x)}, \ldots, r'_1, v'_1\rangle_{\hat{g}(x),g'(x)}\rangle_{g(x)}$. Define $q''_1, \ldots, q''_{s(x)}$ such that $\mathrm{queries}(x,\bar{v}') = \langle q''_{s(x)}, \ldots, q''_1\rangle_{g(x)}$, and define $u'' =_{\mathrm{def}} \mathrm{output}(x,\bar{v}')$, $v''_i =_{\mathrm{def}} f(q''_i)$, and $r''_i =_{\mathrm{def}} r(x, q''_i)$. Obviously, $\mathrm{answers}(x,z') = \bar{v}'$, and

$$z'' =_{\mathrm{def}} \mathrm{better\_path}(x,z') = \langle u'', \langle r''_{s(x)}, v''_{s(x)}, \ldots, r''_1, v''_1\rangle_{\hat{g}(x),g'(x)}\rangle_{g(x)}.$$

In the case $z' < z$ there exists a largest $i \in \{0, 1, \ldots, s(x)\}$ such that $v'_j = v_j$ for $j = 1, \ldots, i$, and $r'_j = r_j$ for $j = 1, \ldots, i-1$.

Case 1: $i < s(x)$ Because of $v'_j = v_j$ for $j = 1, \ldots, i$ we have $q''_j = q_j$ for $j = 1, \ldots, i+1$. Hence $v''_j = f(q''_j) = f(q_j) = v_j$ for $j = 1, \ldots, i+1$. Because of the choice of i and $z' < z$ we have $r'_i < r_i = r''_i$ or ($r'_i = r_i = r''_i$ and $v'_{i+1} < v_{i+1} = v''_{i+1}$). Hence $z' < z''$.

Case 2: $i = s(x)$ Here we have $u'' = \text{output}(x, \bar{v}') = \text{output}(x, \bar{v}) = h(x)$ and $v''_j = f(q''_j) = f(q_j) = v_j$ for $j = i, \ldots, s(x)$. Because of $z' < z$ we have $r'_{s(x)} < r_{s(x)} = r''_{s(x)}$ or $(r'_{s(x)} = r_{s(x)} = r''_{s(x)}$ and $u' < h(x) = u'')$. Hence $z' < z''$.

In the case $z' > z$ we obtain in a completely analogous manner $z' > z''$. Altogether, we obtain

$$\text{better\_path}(x, z') \begin{cases} = z', & \text{if } z' = \text{correct\_path}(x) \\ > z', & \text{if } z' < \text{correct\_path}(x) \\ < z', & \text{if } z' > \text{correct\_path}(x) \end{cases}$$

Defining $A =_{\text{def}} \{ (x, z') \mid \text{better\_path}(x, z') \geq z' \} = \{ (x, z') \mid z' \leq \text{correct\_path}(x) \}$, we obtain $A \in \mathrm{S}\mathcal{F}$, $(x, z' + 1) \in A \Rightarrow (x, z') \in A$, and

$$\text{correct\_path}(x) = \max\{ z' \mid 0 \leq z' \leq \bar{g}(x) \wedge (x, z') \in A \}$$

for some suitable chosen function $\bar{g} \in \mathrm{FP}$. Hence, correct_path $\in \mathrm{FS} \cdot \mathcal{F}$.

For some $t \geq 0$ we obtain $\text{correct\_path}(x) = \langle h(x), t \rangle_{g(x)} = h(x) + g(x) + 2 \cdot g(x)(t + g(x)) - 1$ and therefore $h(x) = \text{correct\_path}(x) \bmod 2g(x) - g(x) + 1$.

Finally, defining correct_path$'$ in the same manner as correct_path, but now using as ouput function the constant zero function, we see that

$$h(x) = \text{correct\_path}(x) - \text{correct\_path}'(x).$$

□

4.7 Corollary. *For any pfo* $\mathcal{F}$, $\mathrm{FP}^{\mathcal{F}} = \mathrm{FP}^{\mathrm{F} \cdot \mathrm{S} \cdot \mathcal{F}}[1]$.

Colloquially, 4.6 can be interpreted as

Polynomial Time Oracle Computation = Binary Search over Oracle Functions + Subtraction

Let us summarize what we have seen up to now. We know that

$$\mathcal{F} \subseteq \mathcal{F} - \mathrm{FP} \subseteq \mathrm{FS} \cdot \mathcal{F} \subseteq \mathrm{FP}^{\mathcal{F}}$$

for all pfo's. We have seen what operators we have to apply to go in the inclusion chain from one class to it's right neighbour. We argued earlier that application of S preserves the complexity of the class more or less; and that application of F yields a slightly more powerful class. We know that set classes $\mathcal{K}$ are fixpoints of SF already under very weak assumptions (Proposition 4.4). The additional power of F$\mathcal{K}$ compared to $\mathcal{K}$ cannot be used when we go back again to the set side, i.e. $\mathrm{S} \cdot \mathrm{F}\mathcal{K} = \mathcal{K}$. But if we start with a function class $\mathcal{F}$, go to the set side to the class $\mathrm{S}\mathcal{F}$, and then back to the function side, the class $\mathrm{FS}\mathcal{F}$ we obtain in general is more powerful than the class $\mathcal{F}$ from which we started; it is the closure of $\mathcal{F}$ under binary search (Corollary 4.5). The number of classes on the function side is richer than on the set side.

Nevertheless, it is worthwhile to identify as many fixpoints of FS as possible besides FP (see 4.3). If $\mathcal{F}_1$ and $\mathcal{F}_2$ are such fixpoints, and their corresponding set classes are $\mathcal{K}_1 = \mathrm{S}\mathcal{F}_1$ and $\mathcal{K}_2 = \mathrm{S}\mathcal{F}_2$, then we know that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if and only if $\mathcal{K}_1 \subseteq \mathcal{K}_2$, and $\mathcal{F}_1 = \mathcal{F}_2$ if and only if $\mathcal{K}_1 = \mathcal{K}_2$. It will be our goal to find fixpoints in Sections 6 and 7. But first, we will compare closure properties of a function class $\mathcal{F}$ and the corresponding set class $\mathrm{S}\mathcal{F}$.

5. Classes of Functions versus Classes of Sets: Closure Properties

In this section, we want to examine, which properties of classes of sets imply which properties of associated classes of functions, and vice versa.

We start by subsuming several pleasant properties of classes of sets under the name *abstract polynomial complexity class.*

5.1 Definition. A class $\mathcal{K}$ of sets is an *abstract polynomial complexity class (apc)* if and only if $\mathcal{K}$ includes the class P and is closed under union, intersection, $\leq_{ctt}^{\mathrm{p}}$, and $\leq_{dtt}^{\mathrm{p}}$.

These requirements for set classes are not too strict, and a large number of reasonable classes fulfill them. The following classes are apc's: P, NP, Σ_k^{p}, Π_k^{p}, Δ_k^{p}, and Θ_k^{p} for all k, $\oplus$P, PP, PH, CH, PSPACE.

Many interesting and important results on special complexity classes are already valid under the apc-assumptions, for instance, equivalence of an NP-computation with an oracle and an existential quantifier and similarly coNP vs. universal quantifier and PP vs. counting quantifier, cf. Torán (1991), closure of counting classes under intersection, cf. Beigel, Reingold, and Spielman (1991), Fortnow and Reingold (1991).

In the theory of formal languages a set of assumptions known under the notion *abstract family of languages (afl)* has a similar role in the sense that a number of important results on special language classes are already valid for afl's, see Ginsburg, Greibach, and Hopcroft (1960). A class of languages over the alphabet Σ is an afl if and only if it contains the set Σ^* and is closed under ε-free homomorphisms, inverse homomorphisms, intersection with regular languages, union, concatenation, and Kleene star. Interestingly, there are strong connections between the notion of an afl and our notion of apc. It was shown in Vollmer and Wagner (1994), Vollmer (1994a) that for any class $\mathcal{K}$ of sets, the following are equivalent:

1. $\mathcal{K}$ is an abstract polynomial complexity class closed under existential quantification.
2. $\mathcal{K}$ is an abstract family of languages closed under many-one reducibility.

We want to confront abstract polynomial complexity classes with a special form of p-founded classes of optimization functions.

5.2 Definition. A class of functions $\mathcal{F}$ is a *strongly p-founded class of optimization functions (spfo)*, if and only if

1. $\mathcal{F}$ is a pfo.
2. For every $f \in \mathcal{F}$, $g, g' \in$ FP such that $f(x,u) \in [g'(x)]$ for all $u \in [g(x)]$, there exists $h \in \mathcal{F}$ such that for all $m \in \mathbb{N}$, $u_1, \ldots, u_m \in [g(x)]$,

$$h(x, \langle u_1, \ldots, u_m \rangle_{g(x)}) = \langle f(x,u_1), \ldots, f(x,u_m) \rangle_{g'(x)}.$$

(*closure under strong tuple formation*)

Condition 2 above is a strong form of closure under tuple formation, required in the definition of a pfo. This requirement is fulfilled by Max and Min classes, as can be seen from the discussion following Definition 3.1 and the proof of 3.4. So we have the following result:

5.3 Theorem.

1. FP *is a spfo.*
2. *If* $\mathcal{F}$ *is a spfo, then so are* $\mathrm{Max}\,\mathcal{F}$ *and* $\mathrm{Min}\,\mathcal{F}$.

The discussion following Definition 3.1 shows why we cannot include Med classes in the statement of Theorem 5.3. But we will see later (Section 7) that if $\mathcal{F}$ is a pfo fulfilling some additional reasonable assumptions, then $\mathrm{FS\,Max}\,\mathcal{F}$, $\mathrm{FS\,Min}\,\mathcal{F}$, and also $\mathrm{FS\,Med}\,\mathcal{F}$ are spfo's.

Before we give a general result relating closure properties of function classes and those of set classes in Theorem 5.7, we first turn to the special point of closure under subtraction by a function in FP and closure under complementation.

5.4 Theorem. *Let* $\mathcal{K}$ *be closed under* $\leq_m^{\mathrm{p}}$. *Then*

1. $\mathrm{F}\cdot\mathcal{K} = \mathrm{F}\cdot\mathcal{K} - \mathrm{FP}$.
2. $\mathrm{F}\cdot\mathrm{co}\mathcal{K} = -\mathrm{F}\mathcal{K}$.

Proof. 1. Let $f(x) = \max\{\, y \mid g_1(x) \leq y \leq g_2(x) \wedge (x,y) \in A\,\}$, $A \in \mathcal{K}$, $g_1, g_2 \in \mathrm{FP}$, where $(x, y+1) \in A$ implies $(x,y) \in A$. Let $g \in \mathrm{FP}$. Then, $f(x) - g(x) = \max\{\, y \mid g_1(x) - g(x) \leq y \leq g_2(x) - g(x) \wedge (x, y + g(x)) \in A\,\}$. Defining $A' \leq_m^{\mathrm{p}} A$ via $(x,y) \in A' \iff (x, y+g(x)) \in A$ then shows $f - g \in \mathrm{F}\mathcal{K}$.

2. Let $f \in \mathrm{F}\cdot\mathrm{co}\mathcal{K}$, $f(x) = \max\{\, y \mid g_1(x) < y \leq g_2(x) \wedge (x,y) \in A\,\}$, $A \in \mathrm{co}\mathcal{K}$, $g_1, g_2 \in \mathrm{FP}$, where $(x, y+1) \in A$ implies $(x,y) \in A$. Define $f'(x) =_{\mathrm{def}} \max\{\, y \mid 0 \leq y < g_2(x) - g_1(x) \wedge (x, g_2(x) - y) \in \overline{A}\,\}$. Then $(x,y) \in \overline{A} \Rightarrow (x, y+1) \in \overline{A}$. Defining $A' \leq_m^{\mathrm{p}} A$ via $(x,y) \in A' \iff (x, g_2(x) - y) \in \overline{A}$ then shows $f' \in \mathrm{F}\mathcal{K}$. But $f(x) + f'(x) + 1 = g_2(x)$. Hence $\mathrm{F}\cdot\mathrm{co}\mathcal{K} \subseteq -\mathrm{F}\mathcal{K} + \mathrm{FP} = -\mathrm{F}\mathcal{K}$. Analogously, one shows $\mathrm{F}\mathcal{K} \subseteq -\mathrm{F}\cdot\mathrm{co}\mathcal{K}$. □

5.5 Theorem. *For all* $\mathcal{F}$, $\mathrm{S}(-\mathcal{F}) = \mathrm{coS}\mathcal{F}$.

Proof. Let $A \in \mathrm{S}(-\mathcal{F})$, i.e. $x \in A \iff -f(x) \geq g(x)$ for $f \in \mathcal{F}$ and $g \in \mathrm{FP}$. Then, $x \in \overline{A} \iff -f(x) < g(x) \iff f(x) > -g(x) \iff f(x) \geq -g(x) + 1$. Thus, $\overline{A} \in \mathrm{S}\mathcal{F}$ and therefore $A \in \mathrm{coS}\mathcal{F}$, which proves $\mathrm{S}(-\mathcal{F}) \subseteq \mathrm{coS}\mathcal{F}$. On the other hand, $\mathrm{coS}\mathcal{F} \subseteq \mathrm{coS}(-(-\mathcal{F})) \subseteq \mathrm{cocoS}(-\mathcal{F}) \subseteq \mathrm{S}(-\mathcal{F})$. □

5.6 Corollary. *If* $\mathcal{F} \circ \mathrm{FP} \subseteq \mathcal{F}$, *then* $\mathrm{FS}(-\mathcal{F}) = -\mathrm{FS}\cdot\mathcal{F}$.

Now, the next theorem gives relationships between closure properties of function classes $\mathcal{F}$ and set classes $\mathrm{S}\cdot\mathcal{F}$ as well as between set classes $\mathcal{K}$ and function classes $\mathrm{F}\cdot\mathcal{K}$. A very nice summary of these single results then is Corollary 5.8.

5.7 Theorem. *Row k of Table 1 has to be interpreted as follows:*

- *If* $\mathcal{K} = \mathrm{S}\mathcal{F}$ *and* $\boxed{\mathrm{A}}$ *then* $\boxed{\mathrm{B}}$.

Table 1: Translation of closure properties

	$\boxed{\text{A}}$	$\boxed{\text{B}}$
1	$\mathrm{FP} \subseteq \mathcal{F}$	$\mathrm{P} \subseteq \mathcal{K}$
2	$\mathcal{F} \circ \mathrm{FP} \subseteq \mathcal{F}$	$\mathcal{K}$ closed under $\leq_m^{\mathrm{p}}$
3	$\mathcal{F} \circ \mathrm{FP} \subseteq \mathcal{F}$ $\mathcal{F} + \mathrm{FP}_+ \subseteq \mathcal{F}, \mathcal{F} \cdot \mathrm{FP}_+ \subseteq \mathcal{F}$ $\mathcal{F}$ closed under definition by cases	$\mathcal{K}$ closed under $\leq_m^{\mathrm{p}}$ $\mathcal{K}$ closed under marked union
4	$\mathcal{F} \circ \mathrm{FP} \subseteq \mathcal{F}, \mathcal{F} + \mathrm{FP}_+ \subseteq \mathcal{F}$ $\min(\mathcal{F}, \mathcal{F}) \subseteq \mathcal{F}$	$\mathcal{K}$ closed under $\leq_m^{\mathrm{p}}$ $\mathcal{K}$ closed under intersection
5	$\mathcal{F} \circ \mathrm{FP} \subseteq \mathcal{F}, \mathcal{F} + \mathrm{FP}_+ \subseteq \mathcal{F}$ $\max(\mathcal{F}, \mathcal{F}) \subseteq \mathcal{F}$	$\mathcal{K}$ closed under $\leq_m^{\mathrm{p}}$ $\mathcal{K}$ closed under union
6	$\mathrm{FP} \subseteq \mathcal{F}$ $\min(\mathcal{F}, \mathrm{FP}) \subseteq \mathcal{F}, \max(\mathcal{F}, \mathrm{FP}) \subseteq \mathcal{F}$ $\mathcal{F}$ closed under strong tuple formation	$\mathrm{P} \subseteq \mathcal{K}$ $\mathcal{K}$ closed under $\leq_{ctt}^{\mathrm{p}}$ and $\leq_{dtt}^{\mathrm{p}}$
7	$\mathcal{F} = -\mathcal{F}$	$\mathcal{K}$ closed under complement

- *If* $\mathcal{F} = \mathrm{F}\mathcal{K}$ *and* $\boxed{\mathrm{B}}$ *then* $\boxed{\mathrm{A}}$.

Proof. We give only those proofs that are not straightforward.

3. $\boxed{\mathrm{A}} \Rightarrow \boxed{\mathrm{B}}$: Let $A_1, A_2 \in \mathcal{K}$, $x \in A_i \iff f_i(x) \geq g_i(x)$ for $f_i \in \mathcal{F}$, $g_i \in \mathrm{FP}$ $(i = 1, 2)$. Let $A = 0A_1 \cup 1A_2$. Define

$$f(x) = \begin{cases} f_1(z), & \text{if } x = 0z; \\ f_2(z), & \text{if } x = 1z. \end{cases}$$

$$g(x) = \begin{cases} g_1(z), & \text{if } x = 0z; \\ g_2(z), & \text{if } x = 1z. \end{cases}$$

Surely, $f \in \mathcal{F}$, $g \in \mathrm{FP}$, and

$$x \in A \iff f(x) \geq g(x).$$

This shows $A \in \mathcal{K}$.

$\boxed{\mathrm{B}} \Rightarrow \boxed{\mathrm{A}}$: Let $f_1, f_2 \in \mathcal{F}$, $f_i(x) = \max\{\, y \mid g_{i,1}(x) \leq y \leq g_{i,2}(x) \wedge (x, y) \in B_i \,\}$ for $B_i \in \mathcal{K}$, $g_{i,1}, g_{i,2} \in \mathrm{FP}$ $(i = 1, 2)$. Let $A \in \mathrm{P}$. Suppose

$$f(x) =_{\mathrm{def}} \begin{cases} f_1(x), & \text{if } x \in A; \\ f_2(x), & \text{otherwise.} \end{cases}$$

Define

$$B = \{\, (x, y) \mid (x \in A \wedge (x, y) \in B_1) \vee (x \notin A \wedge (x, y) \in B_2) \,\}.$$

Surely, $B \leq_m^{\mathrm{p}} 0B_1 \cup 1B_2$, therefore $B \in \mathcal{K}$. Define

$$g_j(x) = \begin{cases} g_{1,j}, & \text{if } x \in A; \\ g_{2,j}, & \text{otherwise.} \end{cases}, \qquad (j = 1, 2)$$

Surely, $g_1, g_2 \in \mathrm{FP}$. Then

$$f(x) = \max\{\, y \mid g_1(x) \leq y \leq g_2(x) \wedge (x, y) \in B \,\}.$$

6. $\boxed{\mathrm{A}} \Rightarrow \boxed{\mathrm{B}}$: Let $A \in \mathcal{K}$, and $x \in A \iff f(x) \geq g(x)$ for $f \in \mathcal{F}$, $g \in \mathrm{FP}$.

Suppose $x \in B \iff x_1 \in A \wedge x_2 \in A \wedge \cdots \wedge x_m \in A$ where the $x_1, \ldots, x_m$ are computable from x in polynomial time and $x_1, \ldots, x_m \in [g_1(x)]$ for some $g_1 \in \mathrm{FP}$. Let $g_2 \in \mathrm{FP}$ be such that $f(z), g(z) \in [g_2(x)]$ for all $z \in [g_1(x)]$. Then

$$\begin{aligned} x \in B &\iff f(x_1) \geq g(x_1) \wedge \cdots \wedge f(x_m) \geq g(x_m) \\ &\iff \min\{f(x_1), g(x_1)\} \geq g(x_1) \wedge \cdots \wedge \min\{f(x_m), g(x_m)\} \geq g(x_m) \\ &\iff \langle \min\{f(x_1), g(x_1)\}, \ldots, \min\{f(x_m), g(x_m)\} \rangle_{g_2(x)} \geq \langle g(x_1), \ldots, g(x_m) \rangle_{g_2(x)}. \end{aligned}$$

Now suppose $x \in B \iff x_1 \in A \vee x_2 \in A \vee \cdots \vee x_m \in A$ where the $x_1, \ldots, x_m$ are computable from x in polynomial time. Let g_1, g_2 be as above. Then

$$\begin{aligned} x \in B \quad &\iff \quad f(x_1) \geq g(x_1) \vee \cdots \vee f(x_m) \geq g(x_m) \\ &\iff \quad \max\{f(x_1), g(x_1) - 1\} > g(x_1) - 1 \vee \cdots \\ &\qquad\qquad \vee \max\{f(x_m), g(x_m) - 1\} > g(x_m) - 1 \\ &\iff \quad \langle \max\{f(x_1), g(x_1) - 1\}, \ldots, \max\{f(x_m), g(x_m) - 1\} \rangle_{g_2(x)} \\ &\qquad\qquad > \langle g(x_1) - 1, \ldots, g(x_m) - 1 \rangle_{g_2(x)} \end{aligned}$$

$\boxed{\mathrm{B}} \Rightarrow \boxed{\mathrm{A}}$: Let $f \in \mathcal{F}$, and $f(x) = \max\{\, y \mid g_1(x) \leq y \leq g_2(x) \wedge (x,y) \in A \,\}$ for $A \in \mathcal{K}$, $g_1, g_2 \in \mathrm{FP}$. Let $\hat{f}(x, \langle u_1, \ldots, u_m \rangle_{g_3(x)}) = \langle f(u_1), \ldots, f(u_m) \rangle_{g_4(x)}$ for $g_3, g_4 \in \mathrm{FP}$. We have to show $\hat{f} \in \mathcal{F}$.

Define a set $\hat{A}$ via the following condition:

$$(x, \langle u_1, \ldots, u_m \rangle_{g_3(x)}, \langle z_1, \ldots, z_m \rangle_{g_4(x)}) \in \hat{A}$$

if and only if *either*

(i) for all $i = 1, \ldots, m$ we have $(u_i, z_i) \in A$;
or

(ii) there is some $i \in \{1, \ldots, m\}$ such that $(u_i, z_i + 1) \in A$ and for all $j = i+1, \ldots, m$ we have $(u_j, z_j) \in A$.

Is is clear that $\hat{A}$ is in the closure of $\mathcal{K}$ under conjunctive and disjunctive reductions. But since $\hat{f}(x,u)$ is equal to the maximal sequence z in appropriate bounds such that $(x,u,z) \in \hat{A}$, we see that $\hat{f} \in \mathcal{F}$. $\square$

5.8 Corollary.

1. *If $\mathcal{F}$ is a spfo, then $\mathrm{S} \cdot \mathcal{F}$ is an apc.*

2. *If $\mathcal{K}$ is an apc, then $\mathrm{F} \cdot \mathcal{K}$ is a spfo.*

Using the above relationships between classes of functions and classes of sets we can translate results from the "function side" to the "set side" and vice versa. As an example, take Corollary 4.6 which translates to

5.9 Corollary. *If $\mathcal{K}$ is an apc, then*

$$\mathrm{FP}^{\mathcal{K}} = \mathrm{F}\mathcal{K} - \mathrm{F}\mathcal{K} = (\mathrm{F}\mathcal{K} \bmod 2\mathrm{FP}) - \mathrm{FP}.$$

For a class of functions $\mathcal{F}$, let $\oplus \mathcal{F}$ denote the class of all sets A for which there exists some $f \in \mathcal{F}$ such that for all x, $x \in A \iff f(x) \equiv 1 \pmod 2$.

5.10 Corollary. *If $\mathcal{K}$ is an apc, then $\mathrm{P}^{\mathcal{K}} = \oplus \mathrm{F}\mathcal{K}$.*

Corollary 5.10 is a generalization of a result from Ogiwara (1991). He needed the additional assumption that $\exists\mathcal{K} = \mathcal{K}$ or $\forall\mathcal{K} = \mathcal{K}$.

6. Classes of Functions versus Classes of Sets: Operators

In this section we state a strong connection between the set operators $\exists$, $\forall$, and C, and the function operators Max, Min, and Med. This will allow us to identify surprisingly many fixpoints of FS in Section 7. We start with the operators Max and Min, for which we have the strongest results. Given a function class $\mathcal{F}$, applying the operator Max and then going to the set-side yields the same result as going at first to the set side and then applying the set operator $\exists$. Analogous results hold when we start with a class of sets $\mathcal{K}$. The strong relationships between Max and Min given earlier allow us then to directly deduce analogous results for the Min operator.

6.1 Theorem. *For all $\mathcal{F}$ such that $\mathcal{F} \circ \mathrm{FP} \subseteq \mathcal{F}$ and $\mathcal{F} + \mathrm{FP}_+ \subseteq \mathcal{F}$, and for all $\mathcal{K} \supseteq \mathrm{P}$ closed under $\leq_m^{\mathrm{p}}$,*

1. $\mathrm{S} \cdot \mathrm{Max} \cdot \mathcal{F} = \exists \cdot \mathrm{S}\mathcal{F}$ *and* $\mathrm{F} \cdot \exists \cdot \mathcal{K} = \mathrm{Max} \cdot \mathrm{F}\mathcal{K}$.
2. $\mathrm{S} \cdot \mathrm{Min} \cdot \mathcal{F} = \forall \cdot \mathrm{S}\mathcal{F}$ *and* $\mathrm{F} \cdot \forall \cdot \mathcal{K} = \mathrm{Min} \cdot \mathrm{F}\mathcal{K}$.

Proof. 1. If $A \in \mathrm{S} \cdot \mathrm{Max}\,\mathcal{F}$, then there exist $h \in \mathrm{Max}\,\mathcal{F}$, $g \in \mathrm{FP}$ such that $x \in A \iff h(x) \geq g(x)$. Hence there exist $f \in \mathcal{F}$, $g' \in \mathrm{FP}$ such that $x \in A \iff \max_{y \leq g'(x)} f(x,y) \geq g(x) \iff (\exists y)(y \leq g'(x) \wedge f(x,y) \geq g(x))$. Therefore, $A \in \exists \mathrm{S}\mathcal{F}$.

On the other hand, let $A \in \exists \mathrm{S}\mathcal{F}$, then there exist $f \in \mathcal{F}$, $g, g' \in \mathrm{FP}$ such that $x \in A \iff (\exists y \leq g'(x))(f(x,y) \geq g(x,y))$. Now choose an FP-function $g_1 \geq g$ monotonic increasing in the second argument and define $g_2(x) = g_1(x, g'(x))$. Now defining $f'(x,y) = f(x,y) + (g_2(x) - g(x,y)) \cdot c_{\{0,1,\ldots,g'(x)\}}(y)$ we get $x \in A \iff (\exists y \leq g'(x))(f'(x,y) \geq g_2(x)) \iff \max_{y \leq g'(x)} f'(x,y) \geq g_2(x)$, which shows $A \in \mathrm{S}\,\mathrm{Max}\,\mathcal{F}$.

If $f \in \mathrm{Max}\,\mathrm{F}\mathcal{K}$, then $f(x) = \max_{y \leq g(x)} \max\{\, z \mid g'(x) \leq z \leq g''(x) \wedge (x,y,z) \in A \,\}$ for $A \in \mathcal{K}$ and $g, g', g'' \in \mathrm{FP}$; but then $f(x) = \max\{\, z \mid g'(x) \leq z \leq g''(x) \wedge (\exists y)(y \leq g(x) \wedge (x,y,z) \in A) \,\}$, which shows $f \in \mathrm{F}\exists\mathcal{K}$.

If $h \in \mathrm{F}\exists\mathcal{K}$, $h(x) = \max\{\, z \mid g'(x) \leq z \leq g''(x) \wedge (\exists y)(y \leq g(x) \wedge (x,y,z) \in A) \,\}$ for $A \in \mathcal{K}$, $g, g', g'' \in \mathrm{FP}$, then $h(x)$ is the maximal value of the function

$$f(x,y,z) = \begin{cases} z, & \text{if } (x,y,z) \in A \\ g'(x), & \text{otherwise} \end{cases}$$

in the range $y \leq g(x)$, $g'(x) \leq z \leq g''(x)$. But $f \in \mathrm{F}\mathcal{K}$, since

$$\begin{aligned} f(x,y,z) &= \max\{\, u \mid g'(x) \leq u \leq z \wedge (x,y,z) \in A \,\} \\ &= \max\{\, u \mid g'(x) \leq u \leq z \wedge (x,y,z,u) \in A' \,\} \end{aligned}$$

for some $A' \in \mathcal{K}$.

2. By 2.2, 5.4, and 5.5, we conclude: $\mathrm{S}\,\mathrm{Min}\,\mathcal{F} = \mathrm{S}(-\mathrm{Max}(-\mathcal{F})) = \mathrm{coS}\,\mathrm{Max}(-\mathcal{F}) = \mathrm{co}\exists\mathrm{S}(-\mathcal{F}) = \forall\mathrm{cocoS}\mathcal{F} = \forall\mathrm{S}\mathcal{F}$ and $\mathrm{F}\forall\mathcal{K} = \mathrm{Fco}\exists\mathrm{co}\mathcal{K} = -\mathrm{F}\exists\mathrm{co}\mathcal{K} = -\mathrm{Max}\,\mathrm{Fco}\mathcal{K} = \mathrm{Min}(-(-\mathrm{F}\mathcal{K})) = \mathrm{Min}\,\mathrm{F}\mathcal{K}$. □

6.2 Corollary.

1. $f \in \mathrm{NP}$ *if and only if* $\mathrm{proj}\,(f) \in \mathrm{Max}\,\mathrm{FP}$.
2. $f \in \mathrm{coNP}$ *if and only if* $\mathrm{proj}\,(f) \in \mathrm{Min}\,\mathrm{FP}$.
3. $f \in \mathrm{FP}^{\mathrm{NP}}$ *if and only if there exist* $g \in \mathrm{FP}$ *and* $h \in \mathrm{Max}\,\mathrm{FP}$ *such that* $f = g \circ h$ *if and only if there exist* $g \in \mathrm{FP}$ *and* $h \in \mathrm{Min}\,\mathrm{FP}$ *such that* $f = g \circ h$.

For Med, the situation is not so comfortable.

6.3 Theorem. *For all $\mathcal{F}$ such that $\mathcal{F} \circ \mathrm{FP} \subseteq \mathcal{F}$ and $\mathcal{F} + \mathrm{FP}_+ \subseteq \mathcal{F}$ and for all $\mathcal{K} \supseteq \mathrm{P}$ closed under $\leq_m^{\mathrm{p}}$,*

$$\mathrm{S} \cdot \mathrm{Med} \cdot \mathcal{F} = \mathrm{C} \cdot \mathrm{S}\mathcal{F} \textit{ and } \mathrm{Med} \cdot \mathrm{F}\mathcal{K} \subseteq \mathrm{F} \cdot \mathrm{C} \cdot \mathcal{K} = \mathrm{FS} \cdot \mathrm{Med} \cdot \mathrm{F}\mathcal{K}$$

Proof.

If $A \in \mathrm{S\,Med}\,\mathcal{F}$, then there exist $h \in \mathrm{Med}\,\mathcal{F}$, $g \in \mathrm{FP}$ such that $x \in A \iff h(x) \geq g(x) \iff \mathrm{med}_{y \leq g'(x)} f(x,y) \geq g(x)$ for some $g' \in \mathrm{FP}$ and $f \in \mathcal{F}$. Therefore $x \in A \iff \#\{\, y \leq g'(x) \mid f(x,y) \geq g(x) \,\} \geq \frac{g'(x)}{2}$ and we get $A \in \mathrm{CS}\mathcal{F}$.

If $A \in \mathrm{CS}\mathcal{F}$, then for some $f \in \mathcal{F}$, $g, g_1, f_2 \in \mathrm{FP}$: $x \in A \iff \#\{\, y \leq g_2(x) \mid f(x,y) \geq g(x,y) \,\} \geq g_1(x)$. Now we can expand the range of f to yield a function $f' \in \mathcal{F}$ for which the following holds for some $g_3 \in \mathrm{FP}$: $x \in A \iff \#\{\, y \leq g_3(x) \mid f'(x,y) \geq g(x,y) \,\} \geq \frac{g_3(x)}{2}$. Then, using methods of the proof of Proposition 2.1, we get $f'' \in \mathcal{F}$ and $g_4 \in \mathrm{FP}$ such that $x \in A \iff \#\{\, y \leq g_3(x) \mid f''(x,y) \geq g_4(x) \,\} \geq \frac{g_3(x)}{2}$. Therefore, $x \in A \iff \mathrm{med}_{y \leq g_3(x)} f''(x,y) \geq g_4(x)$, which shows $A \in \mathrm{S\,Med}\,\mathcal{F}$.

Since $\mathcal{K}$ is closed under $\leq_m^{\mathrm{p}}$, the class $\mathrm{F}\mathcal{K}$ includes $\mathrm{F}\mathcal{K} \circ \mathrm{FP}$ and $\mathrm{F}\mathcal{K} + \mathrm{FP}_{+}$. Hence $\mathrm{FS} \cdot \mathrm{Med}\,\mathrm{F}\mathcal{K} = \mathrm{F} \cdot \mathrm{C} \cdot \mathrm{SF}\mathcal{K} = \mathrm{FC}\mathcal{K}$ by 4.4. □

Although we don't get equality between $\mathrm{F} \cdot \mathrm{C}\mathcal{K}$ and $\mathrm{Med}\,\mathrm{F}\mathcal{K}$ in 6.3, we can at least state that $\mathrm{Med} \cdot \mathrm{F}\mathcal{K} \subseteq \mathrm{F} \cdot \mathrm{C}\mathcal{K} = \mathrm{FS} \cdot \mathrm{Med} \cdot \mathrm{F}\mathcal{K} \subseteq \mathrm{FP}^{\mathrm{Med}\,\mathrm{F}\mathcal{K}}[1]$. We first prove the following lemma:

6.4 Lemma. *If $\mathcal{F}$ is a pfo, then*

$$\mathrm{FS} \cdot \mathrm{Med}\,\mathcal{F} \subseteq \mathrm{Med} \cdot \mathrm{FS}\mathcal{F} - \mathrm{Med} \cdot \mathrm{FS}\mathcal{F}.$$

Proof. The proof combines the maximization method from the proof of Theorem 4.6 with the proof idea of the main result from Toda (1990) in a non-straightforward way.

Let $h \in \mathrm{FS} \cdot \mathrm{Med}\,\mathcal{F}$, and $h(x) = \max\{\, z \mid g_1(x) \leq z < g_2(x) \wedge \mathrm{med}_{y < g_3(x)} f(x,y,z) \geq g(x) \,\}$ for $f \in \mathcal{F}$, $g, g_1, g_2, g_3 \in \mathrm{FP}$, where $\mathrm{med}_{y < g_3(x)} f(x,y,z+1) \geq g(x)$ implies $\mathrm{med}_{y \leq g_3(x)} f(x,y,z) \geq g(x)$. That g and g_3 can be assumed to be a one-place function without loss of generality is a consequence of the domain extension lemma. Suppose for simplicity that $g_2(x) - g_1(x)$ is a power of two and let $p(x) =_{\mathrm{def}} \log(g_2(x) - g_1(x))$. In what follows, we put $p = p(x)$, $g_i = g_i(x)$, $g = g(x)$, and we omit the subscripts to $\langle \cdots \rangle$.

Since $\mathcal{F}$ is a pfo, there exists $\hat{f} \in \mathcal{F}$, and r, such that for all z such that $g_1(x) \leq z \leq g_2(x)$ and all y such that $0 \leq y < g_3(x)$,

$$\hat{f}(x, \langle y_p, z_p, \ldots, y_1, z_1 \rangle) = \langle r(x, y_p, z_p), \langle y_p, f(x, y_p, z_p) \rangle, \ldots r(x, y_1, z_1), \langle y_1, f(x, y_1, z_1) \rangle \rangle.$$

Define $\langle z_1, \ldots, z_{p+1} \rangle = Z(c_1, \ldots, c_p)$ inductively as follows:

$$\begin{array}{lcl} z_1 & = & \frac{g_1+g_2}{2} \\ z_{i+1} & = & \left\{ \begin{array}{ll} z_i + \frac{g_2-g_1}{2^{i+1}}, & \text{if } c_i \geq g \\ z_i - \frac{g_2-g_1}{2^{i+1}}, & \text{if } c_i < g \end{array} \right. , i = 1 \ldots p-1 \\ z_{p+1} & = & \left\{ \begin{array}{ll} z_p, & \text{if } c_p \geq g \\ z_p - 1, & \text{if } c_p < g \end{array} \right. \end{array}$$

Define a set A_1 such that for all $y_1, \ldots, y_p$, $0 \leq y_1, \ldots, y_p < g_3(x)$, and all $c_1, \ldots, c_p \in [g_0(x)]$,

$$(x, \langle y_1, \ldots, y_p \rangle, d) \in A_1 \iff \langle u', \hat{f}(x, \langle y_p, z_p, \ldots, y_1, z_1 \rangle) \rangle \geq d,$$

where $d = \langle u, \langle r_p, \langle y_p, c_p \rangle, \ldots, r_1, \langle y_1, c_1 \rangle \rangle \rangle$ and $u' = z_{p+1}$ in the above terminology. Clearly, $A_1 \in \mathrm{S}\mathcal{F}$. Define A_2 similar to A_1, but changing the condition for u' to $u' = 0$.

Define for $i = 1, 2$

$$f_i(x, \langle y_1, \dots, y_p \rangle) =_{\mathrm{def}} \max\{\, d \text{ in appropriate bounds} \mid (x, \langle y_1, \dots, y_p \rangle, d) \in A_i \,\}$$

Then,

$$\begin{aligned} f_1(x, \langle y_1, \dots, y_p \rangle) &= \langle\ z_{p+1},\ \langle r(x, y_p, z_p), \langle y_p, f(x, y_p, z_p)\rangle, \dots, r(x, y_1, z_1), \langle y_1, f(x, y_1, z_1)\rangle\rangle\rangle \\ f_2(x, \langle y_1, \dots, y_p \rangle) &= \langle\ \quad 0,\ \langle r(x, y_p, z_p), \langle y_p, f(x, y_p, z_p)\rangle, \dots, r(x, y_1, z_1), \langle y_1, f(x, y_1, z_1)\rangle\rangle\rangle \end{aligned}$$

where $\langle z_1, \dots, z_p \rangle = Z(f(x, y_1, z_1), \dots, f(x, y_p, z_p))$.

But now, observe that the function f_1 simulates by binary search the computation of $h(x)$, i.e. the search for the maximal value z as in the definition of h. Consider the middle value of $f_1(x, \langle y_1, \dots, y_p \rangle)$ over all $\langle y_1, \dots, y_p \rangle$. Here, we have that $f(x, y_1, z_1) \geq g(x)$ if and only if $h(x) \geq z_1$. The next "candidate" z_2 is z_1 incremented or decremented by some power of two. Again, we have $f(x, y_2, z_2) \geq g(x)$ if and only if $h(x) \geq z_2$, and so on. Thus the sequence $z_1, z_2, \dots$ converges to $z_{p+1} = h(x)$. (The y's have to be inserted into the function values to obtain uniqueness, cf. the discussion following Definition 3.1.) Therefore, we see that the first component of the middle value of $f_1(x, \langle y_1, \dots, y_p \rangle)$ over all $\langle y_1, \dots, y_p \rangle$ is the value $h(x)$. Thus, subtracting the middle value of $f_2(x, \langle y_1, \dots, y_p \rangle)$ from the middle value of $f_1(x, \langle y_1, \dots, y_p \rangle)$ yields exactly $h(x)$. Thus, $\mathrm{FS} \cdot \mathrm{Med}\, \mathcal{F} \subseteq \mathrm{Med} \cdot \mathrm{FS}\mathcal{F} - \mathrm{Med} \cdot \mathrm{FS}\mathcal{F}$. □

6.5 Corollary. *If $\mathcal{F}$ is a pfo, then* $\mathrm{FS} \cdot \mathrm{Med}\, \mathcal{F} \subseteq \mathrm{FP}^{\mathrm{Med} \cdot \mathrm{FS} \cdot \mathcal{F}}[1]$.

Proof. Follows directly from the proof of Lemma 6.4. □

The previous corollary implies directly (for $\mathcal{F} = \mathrm{F} \cdot \mathcal{K}$ for an apc $\mathcal{K}$) the above mentioned inclusion.

6.6 Corollary.

1. $f \in \mathrm{Med}\,\mathrm{FP}$ *if and only if* $\mathrm{proj}\,(f) \in \mathrm{PP}$.
2. $f \in \mathrm{FP}^{\mathrm{PP}}$ *if and only if there exist* $g \in \mathrm{FP}$ *and* $h \in \mathrm{Med}\,\mathrm{FP}$ *such that* $f = g \circ h$.

For later use, we give the following consequence of Lemma 6.4:

6.7 Corollary. *If $\mathcal{F}$ is a pfo, then*

$$\mathrm{Med}\,\mathrm{FP}^{\mathcal{F}} - \mathrm{Med}\,\mathrm{FP}^{\mathcal{F}} = \mathrm{FS} \cdot \mathrm{Med}\,\mathrm{FP}^{\mathcal{F}} - \mathrm{FS} \cdot \mathrm{Med}\,\mathrm{FP}^{\mathcal{F}} = \mathrm{FP}^{\mathrm{Med}\,\mathrm{FP}^{\mathcal{F}}}.$$

Proof. The last equality follows from 4.6. The inclusion $\subseteq$ of the first equality is then obvious. Using Lemma 6.4, we now have to show that $(\mathrm{Med}\,\mathrm{FP}^{\mathcal{F}} - \mathrm{Med}\,\mathrm{FP}^{\mathcal{F}}) - (\mathrm{Med}\,\mathrm{FP}^{\mathcal{F}} - \mathrm{Med}\,\mathrm{FP}^{\mathcal{F}}) \subseteq \mathrm{Med}\,\mathrm{FP}^{\mathcal{F}} - \mathrm{Med}\,\mathrm{FP}^{\mathcal{F}}$.

Suppose as a first step that $h(x) = h_1(x) + h_2(x)$, $h_i(x) = \mathrm{med}_{y < g(x)} f_i(x, y)$, $f_i \in \mathrm{FP}^{\mathcal{F}}$, $g \in \mathrm{FP}$ for $i \in \{1, 2\}$. Define

$$\begin{aligned} f(x, \langle x, y \rangle) &= \langle f_1(x, y_1) + f_2(x, y_1), f_2(x, y_2), y_1, f_1(x, y_1)\rangle \\ f'(x, \langle x, y \rangle) &= \langle 0, f_2(x, y_2), y_1, f_1(x, y_1)\rangle \end{aligned}$$

Obviously, $f, f' \in \mathrm{FP}^{\mathcal{F}}$, and

$$h(x) = \underset{y_1, y_2 < g(x)}{\mathrm{med}} f(x, \langle x, y \rangle) - \underset{y_1, y_2 < g(x)}{\mathrm{med}} f'(x, \langle x, y \rangle).$$

But now it is easy to modify the just given proof to deal with four different functions. □

We now identify fixpoints of FS among the considered classes of functions.

6.8 Theorem. *If $\mathcal{F}$ is a pfo, then* $\mathrm{FS} \cdot \mathcal{F} \subseteq \mathrm{Max}\,\mathcal{F} - \mathrm{FP}$.

Proof. Let $h \in \mathrm{FS}\mathcal{F}$, $h(x) = \max\{\, y \mid g_1(x) \le y < g_2(x) \wedge f_1(x,y) \ge g(x,y) \,\}$ for $f_1 \in \mathcal{F}$ and $g, g_1, g_2 \in \mathrm{FP}$, such that $f_1(x,y+1) \ge g(x,y+1) \Rightarrow f_1(x,y) \ge g(x,y)$. Then, as in the proof of 2.1, we find $f_2 \in \mathcal{F}$, $g_3 \in \mathrm{FP}$ such that $h(x) = \max\{\, y \mid g_1(x) \le y < g_2(x) \wedge f_2(x,y) \ge g_3(x) \,\}$ where $f_2(x,y+1) \ge g_3(x) \Rightarrow f_2(x,y) \ge g_3(x)$. Define $f_3(x,y) = \max\{g_3(x) - 1, \min\{f_2(x,y), g_3(x)\}\}$; then $f_3 \in \mathcal{F}$, $f_3(x,y) \in \{g_3(x) - 1, g_3(x)\}$, and $f_3(x,y+1) \ge g_3(x) \Rightarrow f_3(x,y) \ge g_3(x)$; and

$$h(x) = \max\{\, y \mid g_1(x) \le y < g_2(x) \wedge f_3(x,y) = g_3(x) \,\}.$$

Let $g_4 \in \mathrm{FP}$ be such that $g_4 > 0$ and $g_1(x) \le y < g_2(x) \Rightarrow y \in [g_4(x) - 1]$.

$$\begin{aligned}
h(x) &= \max\{\, y \mid g_1(x) \le y < g_2(x) \wedge f_3(x,y) \cdot 2g_4(x) + g_4(x) = g_3(x) \cdot 2g_4(x) + g_4(x) \,\} \\
&= \max\{\, y \mid g_1(x) \le y < g_2(x) \wedge f_3(x,y) \cdot 2g_4(x) + g_4(x) + y \ge g_3(x) \cdot 2g_4(x) + g_4(x) \,\} \\
&= \max_{g_1(x) \le y < g_2(x)} (f_3(x,y) \cdot 2g_4(x) + g_4(x) + y) - g_3(x) \cdot 2g_4(x) - g_4(x) \\
&= \max_{y \le g_2(x) - g_1(x) - 1} (f_3(x, y + g_1(x)) \cdot 2g_4(x) + g_1(x) + g_4(x) + y) - g_3(x) \cdot 2g_4(x) - g_4(x)
\end{aligned}$$

□

From the previous theorem we can easily conclude that for a pfo $\mathcal{F}$ the classes $\mathrm{Max}\,\mathcal{F} - \mathrm{FP}$ and $\mathrm{Min}\,\mathcal{F} - \mathrm{FP}$ are fixpoints of FS.

6.9 Corollary. *Let $\mathcal{F}$ be a pfo. Then*

1. $\mathrm{FS}(\mathrm{Max}\,\mathcal{F} - \mathrm{FP}) = \mathrm{FS} \cdot \mathrm{Max}\,\mathcal{F} = \mathrm{Max} \cdot \mathrm{FS}\mathcal{F} = \mathrm{Max}\,\mathcal{F} - \mathrm{FP}$.
2. $\mathrm{FS}(\mathrm{Min}\,\mathcal{F} - \mathrm{FP}) = \mathrm{FS} \cdot \mathrm{Min}\,\mathcal{F} = \mathrm{Min} \cdot \mathrm{FS}\mathcal{F} = \mathrm{Min}\,\mathcal{F} - \mathrm{FP}$.

Proof. By 4.4 we have $\mathrm{FS}(\mathrm{Max}\,\mathcal{F} - \mathrm{FP}) = \mathrm{FS}\,\mathrm{Max}\,\mathcal{F}$ and $\mathrm{FS}(\mathrm{Min}\,\mathcal{F} - \mathrm{FP}) = \mathrm{FS}\,\mathrm{Min}\,\mathcal{F}$. By 6.8 and 2.3 we then conclude: $\mathrm{FS}\,\mathrm{Max}\,\mathcal{F} \subseteq \mathrm{Max}\,\mathrm{Max}\,\mathcal{F} - \mathrm{FP} \subseteq \mathrm{Max}\,\mathcal{F} - \mathrm{FP}$ which is again by 4.4 included in $\mathrm{FS}(\mathrm{Max}\,\mathcal{F} - \mathrm{FP})$. For Min, we then have by 5.6 and 2.3: $\mathrm{FS}\,\mathrm{Min}\,\mathcal{F} = -\mathrm{FS}\,\mathrm{Max}(-\mathcal{F}) = -(\mathrm{Max}(-\mathcal{F}) - \mathrm{FP}) = -\mathrm{Max}(-\mathcal{F}) - \mathrm{FP} = \mathrm{Min}\,\mathcal{F} - \mathrm{FP}$ which is included in $\mathrm{FS}(\mathrm{Min}\,\mathcal{F} - \mathrm{FP})$. □

As an immediate consequence of 6.9, 4.6, and 2.2 we obtain again an interesting query-tradeoff:

6.10 Corollary. *Let $\mathcal{F}$ be a pfo.*

1. $\mathrm{FP}^{\mathrm{Max}\,\mathcal{F}} = \mathrm{Max}\,\mathcal{F} - \mathrm{Max}\,\mathcal{F}$.
2. $\mathrm{FP}^{\mathrm{Min}\,\mathcal{F}} = \mathrm{Min}\,\mathcal{F} - \mathrm{Min}\,\mathcal{F}$.
3. *If $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$ then* $\mathrm{FP}^{\mathrm{Max}\,\mathcal{F}} = \mathrm{FP}^{\mathrm{Min}\,\mathcal{F}} = \mathrm{Max}\,\mathcal{F} + \mathrm{Min}\,\mathcal{F} - \mathrm{FP}$.

Since we do not have $\mathrm{FS}\,\mathrm{Med}\,\mathcal{F} = \mathrm{Med}\,\mathcal{F} - \mathrm{FP}$ for all pfo's $\mathcal{F}$, we cannot give a result similar to 6.10 for Med classes. Theorem 4.6 only yields $\mathrm{FP}^{\mathrm{Med}\,\mathcal{F}} = \mathrm{FS}\,\mathrm{Med}\,\mathcal{F} - \mathrm{FS}\,\mathrm{Med}\,\mathcal{F}$.

7. Classes of Functions versus Classes of Sets: Hierarchies

In analogy to the polynomial hierarchy of sets, we define the following hierarchy for functions.

7.1 Definition. Let $\mathcal{F}$ be a class of functions. Then the classes of the *min-max hierarchy over* $\mathcal{F}$ are defined by $\Sigma_0^{\mathrm{p}}\mathcal{F} = \Pi_0^{\mathrm{p}}\mathcal{F} = \Delta_0^{\mathrm{p}}\mathcal{F} = \mathcal{F}$, $\Sigma_{k+1}^{\mathrm{p}}\mathcal{F} = \mathrm{Max}\,\Pi_k^{\mathrm{p}}\mathcal{F}$, $\Pi_{k+1}^{\mathrm{p}}\mathcal{F} = \mathrm{Min}\,\Sigma_k^{\mathrm{p}}\mathcal{F}$, $\Delta_{k+1}^{\mathrm{p}}\mathcal{F} = \mathrm{FP}^{\Sigma_k^{\mathrm{p}}\mathcal{F}\cup\Pi_k^{\mathrm{p}}\mathcal{F}}$. Let $\mathrm{FPH}\cdot\mathcal{F} = \bigcup_{k\geq 0}\Delta_k^{\mathrm{p}}\mathcal{F}$.

In Krentel (1988), so called *polynomial k-alternating min-max Turing machines* were used to define the classes Σ_k^{MM}. The following observation is obvious:

7.2 Proposition. $\Sigma_k^{\mathrm{MM}} = \Sigma_k^{\mathrm{p}}\cdot\mathrm{FP}$.

7.3 Proposition. *For all pfo's $\mathcal{F}$, we have*

1. $\Sigma_k^{\mathrm{p}}(\mathcal{F}-\mathrm{FP}) = \Sigma_k^{\mathrm{p}}\mathcal{F}-\mathrm{FP}$.
2. $\Pi_k^{\mathrm{p}}(\mathcal{F}-\mathrm{FP}) = \Pi_k^{\mathrm{p}}\mathcal{F}-\mathrm{FP}$.
3. $\Sigma_k^{\mathrm{p}}(-\mathcal{F}) = -\Pi_k^{\mathrm{p}}\mathcal{F}$ *and* $\Sigma_k^{\mathrm{p}}(\mathrm{FP}-\mathcal{F}) = \mathrm{FP}-\Pi_k^{\mathrm{p}}\mathcal{F}$.
4. $\Pi_k^{\mathrm{p}}(-\mathcal{F}) = -\Sigma_k^{\mathrm{p}}\mathcal{F}$ *and* $\Pi_k^{\mathrm{p}}(\mathrm{FP}-\mathcal{F}) = \mathrm{FP}-\Sigma_k^{\mathrm{p}}\mathcal{F}$.

Proof. Straightforward inductive generalizations of 2.1, 2.2, and 2.3. □

The next theorem shows that min-max hierarchies of functions and polynomial hierarchies of sets correspond to each other.

7.4 Theorem. *For all pfo's $\mathcal{F}$ and apc's $\mathcal{K}$ and for all $k\geq 1$,*

1. $\mathrm{S}\cdot\Sigma_k^{\mathrm{p}}\mathcal{F} = \Sigma_k^{\mathrm{p}}(\mathrm{S}\mathcal{F})$ *and* $\mathrm{F}\cdot\Sigma_k^{\mathrm{p}}\mathcal{K} = \Sigma_k^{\mathrm{p}}(\mathrm{F}\mathcal{K})$.
2. $\mathrm{S}\cdot\Pi_k^{\mathrm{p}}\mathcal{F} = \Pi_k^{\mathrm{p}}(\mathrm{S}\mathcal{F})$ *and* $\mathrm{F}\cdot\Pi_k^{\mathrm{p}}\mathcal{K} = \Pi_k^{\mathrm{p}}(\mathrm{F}\mathcal{K})$.

Proof. Straightforward inductive generalization of 6.1. □

7.5 Corollary. $f\in\Sigma_k^{\mathrm{MM}}$ *if and only if* $\mathrm{proj}\,(f)\in\Sigma_k^{\mathrm{p}}$.

However, this does not mean that $\Sigma_k^{\mathrm{p}}\mathcal{F}$ and $\Pi_k^{\mathrm{p}}\mathcal{F}$ are fixpoints of FS. Theorem 7.4 yields only $\mathrm{FS}\cdot\Sigma_k^{\mathrm{p}}\mathcal{F} = \Sigma_k^{\mathrm{p}}(\mathrm{FS}\mathcal{F})$ and $\mathrm{FS}\cdot\Pi_k^{\mathrm{p}}\mathcal{F} = \Pi_k^{\mathrm{p}}(\mathrm{FS}\mathcal{F})$. The following theorem says that $\Sigma_k^{\mathrm{p}}\mathcal{F}-\mathrm{FP}$ and $\Pi_k^{\mathrm{p}}\mathcal{F}-\mathrm{FP}$ are fixpoints of FS.

7.6 Theorem. *For all pfo's $\mathcal{F}$ and for all $k\geq 1$,*

1. $\mathrm{FS}(\Sigma_k^{\mathrm{p}}\mathcal{F}-\mathrm{FP}) = \mathrm{FS}\cdot\Sigma_k^{\mathrm{p}}\mathcal{F} = \Sigma_k^{\mathrm{p}}(\mathrm{FS}\mathcal{F}) = \Sigma_k^{\mathrm{p}}\mathcal{F}-\mathrm{FP}$.
2. $\mathrm{FS}(\Pi_k^{\mathrm{p}}\mathcal{F}-\mathrm{FP}) = \mathrm{FS}\cdot\Pi_k^{\mathrm{p}}\mathcal{F} = \Pi_k^{\mathrm{p}}(\mathrm{FS}\mathcal{F}) = \Pi_k^{\mathrm{p}}\mathcal{F}-\mathrm{FP}$.

Proof. Immediate consequence of 6.9. □

Thus, all classes that are obtained by finitely many applications of the operators Min and Max to the class FP are fixpoints of $\mathrm{F}\cdot\mathrm{S}$; that is, any two of them coincide if and only if the two corresponding classes of the polynomial time hierarchy coincide:

7.7 Corollary. $\Sigma_k^{\mathrm{MM}} = \Sigma_{k+1}^{\mathrm{MM}}$ *if and only if* $\mathrm{PH} = \Sigma_k^{\mathrm{p}}$.

Using the previous theorem and 4.6, we obtain:

7.8 Theorem. *Let $\mathcal{F}$ be a pfo and let $k \geq 1$.*

1. $\mathrm{FP}^{\Sigma_k^{\mathrm{p}}\mathcal{F}} = \Sigma_k^{\mathrm{p}}\mathcal{F} - \Sigma_k^{\mathrm{p}}\mathcal{F} = \Sigma_k^{\mathrm{p}}\mathcal{F} + \Pi_k^{\mathrm{p}}(-\mathcal{F})$.
2. $\mathrm{FP}^{\Pi_k^{\mathrm{p}}\mathcal{F}} = \Pi_k^{\mathrm{p}}\mathcal{F} - \Pi_k^{\mathrm{p}}\mathcal{F} = \Pi_k^{\mathrm{p}}\mathcal{F} + \Sigma_k^{\mathrm{p}}(-\mathcal{F})$.

7.9 Corollary. $\mathrm{F}\Delta_{k+1}^{\mathrm{p}} = \Sigma_k^{\mathrm{MM}} - \Sigma_k^{\mathrm{MM}}$.

The next corollary was already proved by Krentel (1988).

7.10 Corollary. $\mathrm{F}\Delta_{k+1}^{\mathrm{p}} = \mathrm{FP}^{\Sigma_k^{\mathrm{MM}}}[1]$.

However, we cannot state generally that $\Delta_k^{\mathrm{p}}\mathcal{F}$ is included in the next Sigma- or Pi-level of the min-max hierarchy over $\mathcal{F}$ since this would imply that e.g. subtraction of $\mathcal{F}$-functions can be "simulated" by Min and Max operators which we do not believe for arbitrary pfo's $\mathcal{F}$. The situation becomes simpler if we make an additional assumption.

7.11 Theorem. *If $\mathcal{F}$ is a pfo, then*

1. $\mathrm{FP}^{\mathcal{F}} \subseteq \mathrm{Min}\,\mathrm{max}(\mathcal{F} - \mathrm{FP}, \mathrm{FP} - \mathcal{F}) \cap \mathrm{Max}\,\mathrm{min}(\mathcal{F} - \mathrm{FP}, \mathrm{FP} - \mathcal{F})$.
2. $\mathrm{Min}\,\mathrm{FP}^{\mathcal{F}} = \mathrm{Min}\,\mathrm{max}(\mathcal{F} - \mathrm{FP}, \mathrm{FP} - \mathcal{F})$ *and* $\mathrm{Max}\,\mathrm{FP}^{\mathcal{F}} = \mathrm{Max}\,\mathrm{min}(\mathcal{F} - \mathrm{FP}, \mathrm{FP} - \mathcal{F})$.

Proof. Let f be computed by a polynomial time machine M with an oracle function $h \in \mathcal{F}$. Without loss of generality assume that M asks exactly $p(|x|)$ queries to h on input x where p is a suitable polynomial. Assume further that on input x only queries z are asked where $h(z) \in [g(x)]$ for a suitable function $g \in \mathrm{FP}$. Using closure under tuple formation we may simulate the $p(|x|)$ queries with one query to another function h' from $\mathcal{F}$. h' will then answer with a sequence $y = \langle y'_{p(|x|)}, y_{p(|x|)} \ldots, y'_2, y_2, y'_1, y_1\rangle$, where $y_i \in [g(x)]$ and $y'_i \in [g''(x)]$ for $i = 1, 2, \ldots, p(|x|)$ and a some $g'' \in \mathrm{FP}$. In what follows, y will always denote such sequences with appropriate length.

Let $r_M(x, y, k)$ be the k-th query of M on input x, provided that $y_1, y_2, \ldots, y_{k-1}$ are the answers of the oracle function to the previous $k-1$ queries. Let $g_M(x, y)$ be the result of M on input x provided that y is as above and $y_1, y_2, \ldots, y_{p(|x|)}$ are the answers of the oracle function to the $p(|x|)$ queries of M. Let $g' \in \mathrm{FP}_+$ be such that $g_M(x, y) \in [g'(x) - 1]$ for all y as above.

Now it is evident that y is the answer of h' to the sequence of correct queries of M on input x if and only if

$$y = \hat{h}(x, y) =_{\mathrm{def}} h'(x, \langle r_M(x, y, 1), r_M(x, y, 2), \ldots, r_M(x, y, p(|x|))\rangle).$$

Since $h' \in \mathcal{F}$ and $\mathcal{F}$ is a pfo, also $\hat{h} \in \mathcal{F}$. If y_x encodes as above the correct oracle answers of h during M's work on x, i.e. if $y_x = \hat{h}(x, y_x)$, then

$$\begin{aligned} f(x) &= g_M(x, y_x) \\ &= \min_{y \text{ as above}} \max\{(\hat{h}(x,y) - y) \cdot g'(x) + g_M(x,y), \\ &\qquad\qquad (y - \hat{h}(x,y)) \cdot g'(x) + g_M(x,y)\} \end{aligned}$$

The minimal term over all appropriate y is that for $y = \hat{h}(x,y)$, therefore the minimal term is equal to $f(x)$.

The proofs for the other statements are similar. □

7.12 Corollary. *If $\mathcal{F}$ is a pfo and if $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$, then*

1. $\mathrm{FP}^{\mathcal{F}} \subseteq (\mathrm{Min}\max(\mathcal{F},\mathcal{F}) - \mathrm{FP}) \cap (\mathrm{Max}\min(\mathcal{F},\mathcal{F}) - \mathrm{FP})$.
2. $\mathrm{Min}\,\mathrm{FP}^{\mathcal{F}} = \mathrm{Min}\max(\mathcal{F},\mathcal{F}) - \mathrm{FP}$ *and* $\mathrm{Max}\,\mathrm{FP}^{\mathcal{F}} = \mathrm{Max}\min(\mathcal{F},\mathcal{F}) - \mathrm{FP}$.

If $\mathcal{F}$ is an spfo and if $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$, then $\mathrm{Min}\,\mathrm{FP}^{\mathcal{F}} = \mathrm{Min}\,\mathcal{F} - \mathrm{FP}$ *and* $\mathrm{Max}\,\mathrm{FP}^{\mathcal{F}} = \mathrm{Max}\,\mathcal{F} - \mathrm{FP}$.

Now we come back to the question of inclusionship of the Delta levels of the min-max hierarchy in the next Sigma- or Pi-level. The next theorem is a generalization of well-known facts for the case $\mathcal{F} = \mathrm{FP}$ to other classes.

7.13 Theorem. *If $\mathcal{F}$ is a pfo and if $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$, then for every $k \geq 1$,*

1. $\Delta^{\mathrm{p}}_{k+1}\mathcal{F} = \mathrm{FP}^{\Sigma^{\mathrm{p}}_k\mathcal{F}} = \mathrm{FP}^{\Pi^{\mathrm{p}}_k\mathcal{F}} \subseteq (\Sigma^{\mathrm{p}}_{k+1}\mathcal{F} - \mathrm{FP}) \cap (\Pi^{\mathrm{p}}_{k+1}\mathcal{F} - \mathrm{FP})$.
2. $\mathrm{Max}\,\Delta^{\mathrm{p}}_{k+1}\mathcal{F} = \Sigma^{\mathrm{p}}_{k+1}\mathcal{F} - \mathrm{FP}$.
3. $\mathrm{Min}\,\Delta^{\mathrm{p}}_{k+1}\mathcal{F} = \Pi^{\mathrm{p}}_{k+1}\mathcal{F} - \mathrm{FP}$.

Proof. By 7.3 we have $\mathrm{FP}^{\Sigma^{\mathrm{p}}_k\mathcal{F}} \subseteq \mathrm{FP}^{\Sigma^{\mathrm{p}}_k(\mathrm{FP}-\mathcal{F})} = \mathrm{FP}^{\mathrm{FP}-\Pi^{\mathrm{p}}_k\mathcal{F}} = \mathrm{FP}^{\Pi^{\mathrm{p}}_k\mathcal{F}}$. The other inclusion is analogous. Furthermore, we conclude

$$\begin{aligned} \mathrm{FP}^{\Sigma^{\mathrm{p}}_k\mathcal{F}} &\subseteq \mathrm{Max}\min(\Sigma^{\mathrm{p}}_k\mathcal{F} - \mathrm{FP}, \mathrm{FP} - \Sigma^{\mathrm{p}}_k\mathcal{F}) && \text{by 7.11} \\ &\subseteq \mathrm{Max}\min(\Sigma^{\mathrm{p}}_k\mathcal{F} - \mathrm{FP}, \Pi^{\mathrm{p}}_k\mathcal{F} - \mathrm{FP}) && \text{by 7.3} \\ &\subseteq \mathrm{Max}\min(\Sigma^{\mathrm{p}}_{k+1}\mathcal{F} - \mathrm{FP}, \Sigma^{\mathrm{p}}_{k+1}\mathcal{F} - \mathrm{FP}) && \text{by 3.3} \\ &\subseteq \mathrm{Max}\min(\Pi^{\mathrm{p}}_k\mathcal{F} - \mathrm{FP}, \Pi^{\mathrm{p}}_k\mathcal{F} - \mathrm{FP}) && \text{by 2.3} \\ &\subseteq \mathrm{Max}\,\mathrm{Min}\,\Pi^{\mathrm{p}}_k\mathcal{F} - \mathrm{FP} && \text{by 2.3} \\ &\subseteq \Sigma^{\mathrm{p}}_{k+1}\mathcal{F} - \mathrm{FP} && \text{by 2.3} \end{aligned}$$

The other statements are immediate consequences of Statement 1. □

The previous result yields two interesting corollaries. First, it shows that a function from any Delta-level of the min-max hierarchy can be written as a difference of two functions from the preceding Sigma- or Pi-level; and this allows us as a second consequence to characterize the union over all classes of the min-max hierarchy as closure of the base function class solely under Max and Min or Max and subtraction, or Min and subtraction. For the corresponding polynomial hierarchy of sets over $\mathrm{S}\mathcal{F}$, we get that the class $\mathrm{PH} \cdot \mathrm{S}\mathcal{F}$ is then equal to the closure of $\mathrm{S}\mathcal{F}$ under $\exists$ and $\forall$.

7.14 Corollary. *If $\mathcal{F}$ is a pfo and if $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$, then for every $k \geq 1$,*

$$\begin{aligned} \Delta^{\mathrm{p}}_{k+1}\mathcal{F} &= \Sigma^{\mathrm{p}}_k\mathcal{F} - \Sigma^{\mathrm{p}}_k\mathcal{F} = \Pi^{\mathrm{p}}_k\mathcal{F} - \Pi^{\mathrm{p}}_k\mathcal{F} \\ &= \mathrm{FP}^{\Sigma^{\mathrm{p}}_k\mathcal{F}}[1] = \mathrm{FP}^{\Pi^{\mathrm{p}}_k\mathcal{F}}[1] \end{aligned}$$

7.15 Corollary. *If $\mathcal{F}$ is a pfo and if $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$, then* $\mathrm{FPH} \cdot \mathcal{F} = \Gamma_{\mathrm{Max,Min}}(\mathcal{F}) - \mathrm{FP} = \Gamma_{\mathrm{Max},-}(\mathcal{F}) = \Gamma_{\mathrm{Min},-}(\mathcal{F})$ *is a fixpoint of* FS.

Abbreviating $\mathrm{FPH} \cdot \mathrm{FP}$ by FPH, we thus have $\mathrm{FPH} = \bigcup_{k \geq 0} \Sigma_k^{\mathrm{MM}}$.

In Ogiwara and Hemachandra (1993), is was shown that Max FP is closed under subtraction if and only if NP = coNP. Corollary 7.14 now allows a direct extension of their result:

7.16 Corollary. Σ_k^{MM} *is closed under subtraction if and only if* $\mathrm{FPH} = \Sigma_k^{\mathrm{MM}}$ *if and only if* $\Sigma_k^{\mathrm{p}} = \Pi_k^{\mathrm{p}}$.

7.17 Corollary. $\mathrm{FPH} \subseteq \mathrm{FP}^{\mathrm{PP}}$.

Proof. Follows from Toda (1991) and the monotonicity of the operator F. □

The relationship between classes of the min-max hierarchy over some $\mathcal{F}$ (under the assumptions of Corollary 7.14) are summarized in Fig. 1. All function classes shown there are fixpoints of FS, and classes of equal position in the left and right half of the diagram are transformed into each other via S or F. Two classes in one of the two halfs of the diagram are equal if and only if the two corresponding classes in the other half are equal.

The operator Med doesn't share the pleasant properties of Max and Min proved in 6.1. Therefore, we did not suceed in giving fixpoints for Med as in 6.9. But we will show, that a function hierarchy built solely by the operator Med has structural properties similar to those of the min-max hierarchy; we will show into which classes of sets the classes of the median hierarchy are transformed under application of operator S; and we will show how fixpoints of FS are located between classes of that hierarchy.

Let $\mathrm{Med}^k \cdot \mathcal{F}$ abbreviate $\underbrace{\mathrm{Med}\,\mathrm{Med} \cdots \mathrm{Med}}_{k \text{ times}} \cdot \mathcal{F}$.

7.18 Theorem. *If $\mathcal{F}$ is a pfo and $\mathcal{K} \supseteq \mathrm{P}$ is closed under $\leq_m^{\mathrm{p}}$, then*

$$\mathrm{S} \cdot \mathrm{Med}^k \mathcal{F} = \mathrm{C}^k \cdot \mathrm{S}\mathcal{F} \text{ and } \mathrm{Med}^k \, \mathrm{F}\mathcal{K} \subseteq \mathrm{F} \cdot \mathrm{C}^k \mathcal{K} = \mathrm{FS}\,\mathrm{Med}^k \cdot \mathrm{F}\mathcal{K}.$$

Proof. Inductive generalization of 6.3. □

7.19 Corollary. $f \in \mathrm{Med}^k \cdot \mathrm{FP}$ *if and only if* $\mathrm{proj}\,(f) \in \mathrm{C}^k \cdot \mathrm{P}$.

Inclusionship of Delta-levels in the next Sigma- or Pi-level in min-max hierarchies over pfo's $\mathcal{F}$ only holds under the additional assumption $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$. Also here, in the median hierarchy, we cannot expect classes of the form $\mathrm{FP}^{\mathrm{Med}^k \mathcal{F}}$ to be included in $\mathrm{Med}^{k+1} \mathcal{F}$ for all pfo's, since we don't think that Med can generally simulate subtraction. But again, under the additional assumption $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$, we get such a result.

7.20 Lemma. *If $\mathcal{F}$ is a pfo and if $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$, then*

1. $\mathrm{FP}^{\mathcal{F}} \subseteq \mathrm{Med}\,\mathcal{F} - \mathrm{FP}$.
2. $\mathrm{Med}\,\mathrm{FP}^{\mathcal{F}} = \mathrm{Med}\,\mathcal{F} - \mathrm{FP}$.

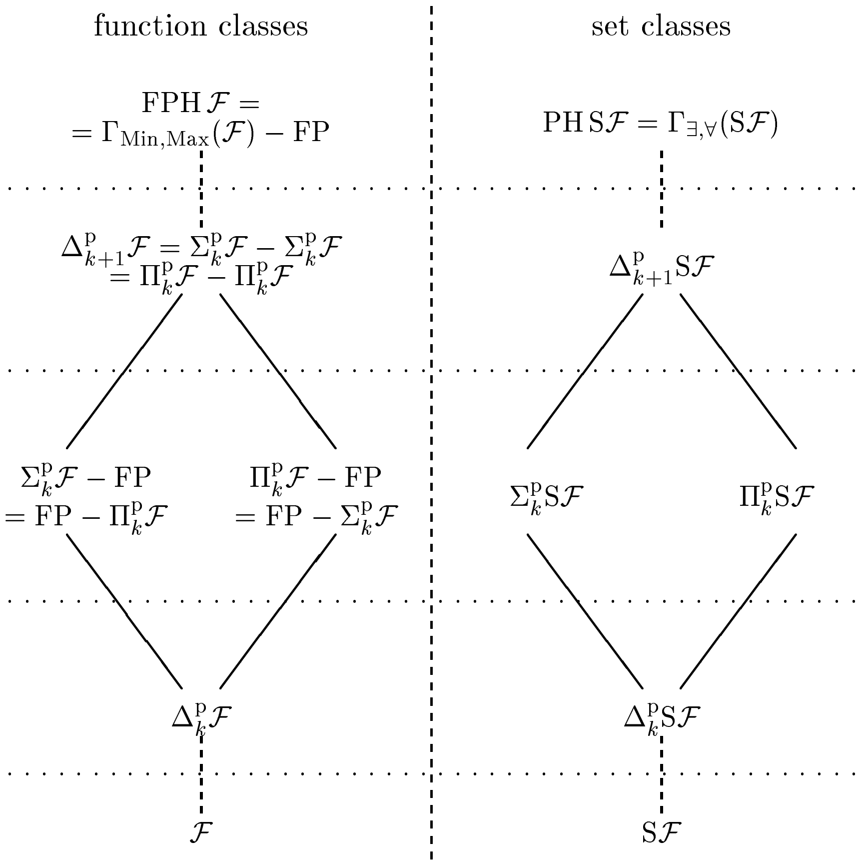

Figure 1: The min-max hierarchy

Proof. 1. As in the proof of Theorem 7.11, obtain functions g', g_M, and $\hat{h}$. Then, we continue as follows:

$$\begin{aligned} f(x) &= g_M(x, y_x) \\ &= \underset{\substack{y \text{ as above} \\ b \in \{-1,+1\}}}{\text{med}} \left[b \cdot (\hat{h}(x,y) - y) \cdot g'(x) + g_M(x,y) \right] \end{aligned}$$

Since $\mathcal{F}$ is a pfo and $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$, there exist $h_1 \in \mathcal{F}, g_1 \in \mathrm{FP}_+$ such that $\hat{h}(x,y) = g_1(x,y) - h_1(x,y)$, thus, $-(\hat{h}(x,y) - y) = h_1(x,y) - g_1(x,y) + y$. Now, define

$$h_2(x,y,b) =_{\text{def}} \begin{cases} (\hat{h}(x,y) - y) \cdot g'(x) + g_M(x,y), & \text{if } b \geq 1; \\ (h_1(x,y) - g_1(x,y) + y) \cdot g'(x) + g_M(x,y), & \text{if } b \leq 0. \end{cases}$$

Surely, $h_2 \in \mathcal{F} - \mathrm{FP}$ via definition by cases. Now,

$$f(x) = \underset{\substack{y \text{ as above} \\ 0 \leq b \leq 1}}{\text{med}} h_2(x,y,b),$$

thus $f \in \mathrm{Med}\,\mathcal{F} - \mathrm{FP}$.

2. The proof of Statement 1 showed that the value of the $\mathrm{FP}^{\mathcal{F}}$ computation under consideration appears in the middle of an exponentially long sequence of values (one for each possible encoding y of oracle answers). If we now have an exponential number of $\mathrm{FP}^{\mathcal{F}}$ computations and we are interested in the middle value over all outcomes of these, then we can simulate each single computation as above; and therefore all correct outcomes appear in the middle, and the middle element of these (which is the middle element of the whole sequence) is the value of the $\mathrm{Med}\,\mathrm{FP}^{\mathcal{F}}$ computation. □

Now we have the following result on Delta-levels in the median hierarchy:

7.21 Theorem. *If $\mathcal{F}$ is a pfo and if $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$, then for all $k \geq 1$,*

1. $\mathrm{FP}^{\mathrm{Med}^k \mathcal{F}} = \mathrm{Med}^k \mathcal{F} - \mathrm{Med}^k \mathcal{F} = \mathrm{Med}^k \mathcal{F} + \mathrm{Med}^k \mathcal{F} - \mathrm{FP}$.
2. $\mathrm{FP}^{\mathrm{Med}^k \mathcal{F}} \subseteq \mathrm{Med}\,\mathrm{FP}^{\mathrm{Med}^k \mathcal{F}} = \mathrm{Med}^{k+1} \mathcal{F} - \mathrm{FP}$.

Proof. 1.
$$\begin{aligned} \mathrm{FP}^{\mathrm{Med}^k \mathcal{F}} &\subseteq \mathrm{FS} \cdot \mathrm{Med}^k \mathcal{F} - \mathrm{FS} \cdot \mathrm{Med}^k \mathcal{F} && \text{by 4.6} \\ &\subseteq \mathrm{FS\,Med} \cdot \mathrm{Med}^{k-1} \mathcal{F} - \mathrm{FS\,Med} \cdot \mathrm{Med}^{k-1} \mathcal{F} \\ &\subseteq \mathrm{Med\,FP}^{\mathrm{Med}^{k-1} \mathcal{F}} - \mathrm{Med\,FP}^{\mathrm{Med}^{k-1} \mathcal{F}} && \text{by 6.7} \\ &\subseteq \mathrm{Med}^k \mathcal{F} - \mathrm{Med}^k \mathcal{F} && \text{by 7.21, 2.} \\ &\subseteq \mathrm{Med}^k \mathcal{F} + \mathrm{Med}^k \mathcal{F} - \mathrm{FP} && \text{by 2.2, 3.} \\ &\subseteq \mathrm{FP}^{\mathrm{Med}^k \mathcal{F}} \end{aligned}$$

2. Direct consequence of 7.20 and an inductive generalization of 2.2, part 3. □

7.22 Corollary. $\mathrm{FP}^{\mathrm{C}^k \cdot \mathrm{P}} = \mathrm{Med}^k \cdot \mathrm{FP} - \mathrm{Med}^k\,\mathrm{FP} = \mathrm{FP}^{\mathrm{Med}^k \cdot \mathrm{FP}[1]}$.

7.23 Corollary. *If* Med FP *is closed under subtraction, then* PP *is closed under Turing reductions.*

Concerning fixpoints in the area of the median hierarchy, we now know the following (under the assumptions of 7.21):

$$\mathrm{Med}^k \mathcal{F} \subseteq \mathrm{FS} \cdot \mathrm{Med}^k \mathcal{F} \subseteq \mathrm{FP}^{\mathrm{Med}^k \mathcal{F}} = \mathrm{FS} \cdot \mathrm{FP}^{\mathrm{Med}^k \mathcal{F}} \subseteq \mathrm{Med}^{k+1} \mathcal{F} - \mathrm{FP}$$

and

$$\mathrm{FS} \cdot \Gamma_{\mathrm{Med}}(\mathcal{F}) - \mathrm{FP} = \Gamma_{\mathrm{Med}}(\mathcal{F}) - \mathrm{FP}.$$

7.24 Corollary. *The counting hierarchy collapses if and only if the* Med *hierarchy collapses.*

The just mentioned fixpoint has also another characterization. The polynomial time hierarchy of counting functions FCH was defined in Wagner (1986b) to be the union of the classes $\#\mathrm{P}$, $\#\mathrm{P}^{\#\mathrm{P}}$, $\#\mathrm{P}^{\#\mathrm{P}^{\#\mathrm{P}}}$, ... We define $\mathrm{FCH}\cdot\mathcal{F}$ here to be the polynomial time hierarchy of counting functions relative to $\mathcal{F}$, i.e. let $0\#\mathrm{P}^{\mathcal{F}} =_{\mathrm{def}} \mathcal{F}$, $(k+1)\#\mathrm{P}^{F} =_{\mathrm{def}} \#\mathrm{P}^{k\#\mathrm{P}^{\mathcal{F}}}$, and $\mathrm{FCH}\cdot\mathcal{F} = \bigcup_{k\geq 0} k\#\mathrm{P}^{\mathcal{F}}$.

7.25 Theorem. *If $\mathcal{F}$ is a pfo and if $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$, then*

$$\mathrm{FCH}\cdot\mathcal{F} - \mathrm{FP} = \Gamma_{\mathrm{Med}}(\mathcal{F}) - \mathrm{FP}.$$

Proof. By an induction, using 6.3 and 7.20, one can show that for all $k \geq 0$, $k\#\mathrm{P}^{\mathcal{F}} - \mathrm{FP} \subseteq \mathrm{FS}\cdot\mathrm{Med}^k\mathcal{F} \subseteq \mathrm{FP}^{k\#\mathrm{P}^{\mathcal{F}}} \subseteq (k+1)\#\mathrm{P}^{\mathcal{F}} - \mathrm{FP}$. Therefore $\bigcup_{k\geq 0} k\#\mathrm{P}^{\mathcal{F}} = \bigcup_{k\geq 0}\mathrm{Med}^k\mathcal{F} - \mathrm{FP} = \Gamma_{\mathrm{Med}}(\mathcal{F}) - \mathrm{FP}$. □

7.26 Corollary. $\mathrm{FCH} = \Gamma_{\mathrm{Med}}(\mathrm{FP})$.

The results on the median hierarchy (under the assumptions of Theorem 7.21) are summarized in Fig. 2. Not all function classes shown there are fixpoints of FS, but all classes between two horizontal lines lead to the same class of sets (depicted in the right half of the diagram), and the corresponding fixpoint is then the maximal class of all classes between two such lines.

8. Conclusion

In this paper we systematically studied classes of optimization functions.

As we pointed out in the introduction and later after Definition 3.1, these classes are very different from the second group of function classes mentioned earlier, namely the counting classes. But also in the context of classes of counting functions, it is possible to exhibit a suitable set of axioms (analogously to those of Definition 3.1) and to examine in a spirit similar to that of the present paper classes that fulfill the axioms. This was done in Vollmer (1994a).

Finally, we want to mention the following: If we take not the mininum, median, or maximum, but the i-th largest element out of an exponential range of values, where i is an FP-computable function, then we get again the Med classes. Now it would be interesting to examine the variety of the classes that we get when i is not allowed to range between 0 and the number of elements but is restricted in any reasonable way.

Acknowledgement

Thanks to Johannes Köbler (Ulm), and Krzysztof Loryś (Wrocław) for many helpful discussions.

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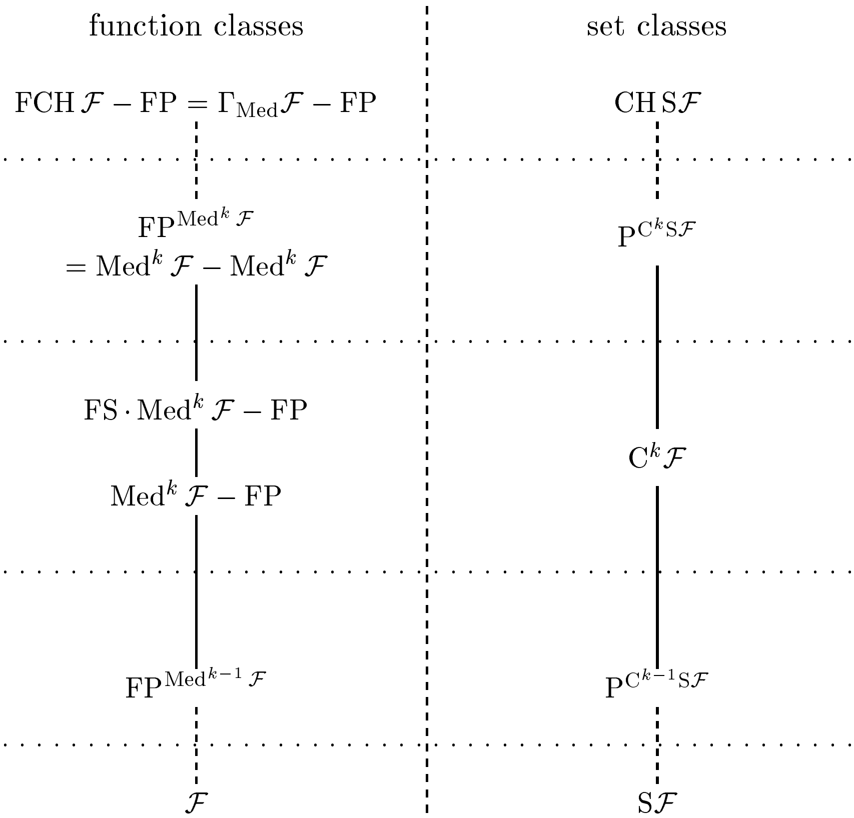

Figure 2: The median hierarchy

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