The Chain Method to Separate Counting Classes

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Abstract

We introduce a new method to separate counting classes of a special type by oracles. Among the classes, for which this method is applicable, are NP, coNP, US (also called 1-NP), $\oplus$P and all other MOD-classes, PP and $\mathrm{C_{=}P}$, classes of Boolean Hierarchies over the named classes, classes of finite acceptance type, and many more. As an important special case, we completely characterize all relativizable inclusions between classes NP(k) from the Boolean Hierarchy over NP and other classes defined by what we will call bounded counting.

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1 Introduction

A probabilistic Turing machine is essentially a nondeterministic Turing machine M, where we view M's nondeterministic choices as outcomes of a fair coin toss. Thus, every input is accepted with a certain probability. We say that the language accepted by M consists of all of its inputs which are accepted with probability greater than one half. The class PP, introduced by Gill [8] in 1977, is defined to consist of all languages accepted by such machines running in polynomial time.

It has been observed (see e.g. [20]) that PP can also be characterized in another way using Valiant's class #P [23]. This class consists of those functions *counting* the number of accepting paths of nondeterministic polynomial time Turing machines. Then, a language L belongs to PP if and only if there is a #P-function which exactly on inputs from L is greater than a given threshold (a constant ratio of the number of all paths of the underlying Turing machine). Consequently, PP has later been referred to as a *counting class*, and the oracle hierarchy built using PP as elementary class, defined by Wagner [25, 26] in 1986, has been called the *counting hierarchy* [21, 22]. Since then, a lot of other classes which also have some counting process inherent in their definition, appeared; for an overview see [10]. The most famous example is maybe the class $\mathrm{C}_{=}\mathrm{P}$ [20, 26]. All these classes can be defined almost exclusively by putting some restrictions on the outcomes of #P-functions. We use the following general framework: Let V be a (finite or infinite) set of integers. Then the class $(V)\mathrm{P}$ is the class of all languages L for which there exists a function $f \in \#\mathrm{P}$ such that $x \in L \iff f(x) \in V$. In other words: L is a language from $(V)\mathrm{P}$ if there is a nondeterministic Turing machine M which has a number of accepting paths from the set V exactly for inputs which are words in L. A similar approach has been suggested in [7]. We will also consider sets V of vectors of natural numbers and vectors of #P functions; the correspondence to Turing machines is then that given some input, we not only count the number of accepting paths, but for some alphabet Σ, we count the number of outputs of all symbols from Σ in M's accepting computations. An input is accepted iff the vector of numbers formed in this way belongs to V. For a precise definition, see Section 3.

Structural complexity theory deals with complexity classes as the above and their inclusion relations. While a detailed inclusion structure between the examined classes is known, almost none of the inclusions is known to be proper. Therefore, researchers have started to study "relativized complexity classes" and their inclusion structure, which in a lot of cases is much easier. While recent results have made this approach somewhat questionable, we believe that there is still enough reason to study relativizations (see also the vehement defense in [1]). Especially, when we are dealing with complexity classes defined in a uniform way as introduced above, we think that to investigate, whether inclusion relations hold under all relativizations, will shed some light on the unrelativized case. As stated in [4], if there is an oracle that separates a class $\mathcal{C}_1$ from class $\mathcal{C}_2$, then no direct simulation technique can be applied to prove $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Moreover, oracle separations for polynomial time classes directly reflect lower bounds for (unrelativized) constant depth circuits. Unfortunately, for counting classes as introduced above, even their separation by oracles turned out to be a very difficult task. While Torán in 1988 [21] was the first to develop a somewhat general technique which he used to separate the lower levels of the counting hierarchy, not much more progress in that direction has been made since then. The question whether there is an oracle relative to which the counting hierarchy is strict, is still an open problem, which is strongly connected to a famous open problem in circuit complexity (the $\mathrm{TC}^0 \stackrel{?}{=} \mathrm{NC}^1$-question, so to speak the $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$-problem for circuit complexity; see [2]).

However, for the special case of counting classes $(V)\mathrm{P}$, where the set V is of bounded significance, some progress has been made very recently. These are classes where the vector set V is such that the relevant range for every component is bounded by some fixed constant. Special cases for these classes are those with finite or co-finite V, i.e. the classes which have been called finite acceptance type classes (see [10]). The most important examples are NP, coNP, and the classes of the Boolean Hierarchy over NP. Another prominent class with finite acceptance type is the class US [3], later also called 1-NP [10], where by definition a word is accepted by some machine M iff M has exactly one accepting path in the computation on that word as an input. Thus, 1-NP in our terminology can be defined as $(\{1\})\mathrm{P}$. Classes with finite acceptance type and some combinations of them were studied intensively in [9], but the question of their separability remained open in a lot of cases.

In [5] a uniform way to define complexity classes was introduced (nowadays referred to as the *leaf language approach,* see [14, 15, 17] and the recent textbook [19]), and a general and sufficient criterion for two classes defined in such a way to be separable by an oracle was given. (See also [24], where this has been proven independently.) Though this result considerably simplified questions whether oracles with certain properties exist by reducing them to combinatorial questions, the argumentations still were sometimes a bit clumsy. Hertrampf in [11], however, demonstrated that the criterion from [5] is much easier applicable in the case of counting classes of bounded significance. In [13], he pushed his techniques a bit further and was able to present an algorithm, that given two explicit counting classes with bounded significance decides whether they can be separated by an oracle. However, his algorithm has the drawback that it does not give explicit answers when the classes are given in a parameterized form (as is typically the case for hierarchies). Thus, his algorithm could not solve the question, whether certain hierarchies of counting classes are strict, or whether classes given in a more general form are relativizably closed under complement.

In this paper, using a refinement of Hertrampf's main result from [13], we first observe that if there is a relativizable simulation of $(V)\mathrm{P}$ machines by $(U)\mathrm{P}$ machines (for vector sets V and U of bounded significance), then this simulation can be done by one single machine with certain convenient properties. Thus, we get that $(V)\mathrm{P}^A \subseteq (U)\mathrm{P}^A$ for all oracle sets A if and only if there exist positive linear combinations of multinomial coefficients $p_1, \dots, p_k$ (where k is the dimension of U) such that for all v, $v \in V \iff (p_1(v), \dots, p_k(v)) \in U$. Building on that, we develop two very easily applicable methods, called the First and Second Chain Method, which allow us for a lot of counting classes of bounded significance, to decide whether they can be separated by an oracle or not. Exploiting these methods, we completely resolve the question which relativizable inclusions between classes of the Boolean Hierarchy over NP and other classes of bounded significance exist. Moreover, we almost completely resolve the question which relativizable inclusions between classes of the Boolean Hierarchy over 1-NP exist (see Table 2), thus significantly improving results from [9, 10, 13]. Surprisingly, we obtain very easily that no counting class of bounded significance can be relativizably closed under complement. We show that no class of the form $(V)\mathrm{P}$, where V is not of bounded significance can be relativizably included in a class $(U)\mathrm{P}$, where U is of bounded significance.

2 Preliminaries

2.1 Notations

We will generally assume familiarity with basic concepts of complexity theory, like resource bounded Turing machines, random access input tapes, many-one reductions, etc. Refer to standard literature in complexity theory to get an overview.

Vectors of natural numbers will generally be written in the form $(a_1, \dots, a_k)$. A comparison of vectors is always meant componentwise, i.e. we write $(a_1, \dots, a_k) \leq (b_1, \dots, b_k)$, if and only if for all $j \in \{1, \dots, k\}$ we have $a_j \leq b_j$. If moreover the two vectors are not equal, then we write $(a_1, \dots, a_k) < (b_1, \dots, b_k)$. Note that this does not mean that every a_j has to be strictly less than b_j. Given a vector $v = (v_1, \dots, v_k)$ and a number m we define $\min(v, m) =_{\mathrm{def}} (\min(v_1, m), \dots, \min(v_k, m))$.

We will deal with the following complexity classes:

- The classes P, NP and coNP are well known.
- The class 1-NP consists of all languages L, such that there is a nondeterministic polynomial time Turing machine M, which on all inputs from L has exactly one accepting path, but on all inputs, which do not belong to L either has no accepting path or more than one accepting path. This class is also often denoted US (for "unique solution").
- The class $\oplus$P consists of all languages L, such that there is a nondeterministic polynomial time Turing machine M, which on all inputs from L has an odd number of accepting paths, but on all inputs, which do not belong to L has an even number of accepting paths.
- The class $\mathrm{MOD}_k\mathrm{P}$ (for $k \geq 2$) consists of all languages L, such that there is a nondeterministic polynomial time Turing machine M, which on all inputs from L has a number of accepting paths which is not divisible by k, but on all inputs, which do not belong to L has a number of accepting paths, which is a multiple of k (including 0).
- The class PP consists of all languages L, such that there is a nondeterministic polynomial time Turing machine M, which on all inputs from L has more accepting than rejecting paths, but on all inputs, which do not belong to L has at least as much rejecting as accepting paths.
- The class $\mathrm{C}_=\mathrm{P}$ consists of all languages L, such that there is a nondeterministic polynomial time Turing machine M, which on all inputs from L has as much accepting as rejecting paths, but on all inputs, which do not belong to L the number of rejecting paths is different from the number of accepting paths.
- If $\mathcal{C}$ is a complexity class, then the class co-$\mathcal{C}$ is defined as the class of all languages L, such that $\overline{L} \in \mathcal{C}$.
- If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two complexity classes, then the class $\mathcal{C}_1 \wedge \mathcal{C}_2$ is defined as the class of all languages L, such that there are languages $L_1 \in \mathcal{C}_1$ and $L_2 \in \mathcal{C}_2$ satisfying $L = L_1 \cap L_2$. $\mathcal{C}_1 \vee \mathcal{C}_2$ is analogously defined via $L = L_1 \cup L_2$; finally, $\mathcal{C}_1 \oplus \mathcal{C}_2$ is defined via $L = L_1 \Delta L_2$, where Δ denotes the symmetric difference.
- The classes of the Boolean Hierarchy over a class $\mathcal{C}$ are the classes, which can be obtained by iterated application of the operators $\wedge$, $\vee$, and co- to the class $\mathcal{C}$, e.g. $\mathcal{C} \wedge \text{co-}\mathcal{C}$ or $(\mathcal{C} \wedge \text{co-}\mathcal{C}) \vee \mathcal{C}$.
- The classes NP(k) and coNP(k) are the classes of the Boolean Hierarchy over NP, which are defined as follows: $\mathrm{NP}(1) = \mathrm{NP}$, for even k, $\mathrm{NP}(k) = \mathrm{NP}(k-1) \wedge \mathrm{coNP}$, and for odd k, $\mathrm{NP}(k) = \mathrm{NP}(k-1) \vee \mathrm{NP}$; moreover, $\mathrm{coNP}(k) = \text{co-NP}(k)$. It is a well known fact [27] that all nontrivial classes from the Boolean Hierarchy coincide with either NP(k) or coNP(k) for some $k > 0$. Moreover, in [18] it was shown that these classes can also be defined using the $\oplus$ operator: $\mathrm{NP}(k) = \mathrm{NP} \oplus \ldots \oplus \mathrm{NP}$, where the right hand side contains exactly k times the class NP.
- The class #P is the class of all functions f for which there exists a nondeterministic polynomial time bounded Turing machine M such that $f(x)$ is equal to the number of accepting computation paths of M run on input x.

2.2 Subsets of $\mathbb{N}^k$

In the subsequent sections we will define complexity classes via subsets of $\mathbb{N}^k$. We then will make use of the following definition:

Definition 2.1 Let $V \subseteq \mathbb{N}^k$.

1. We say V *is a set of bounded significance*, if there is a number m, such that for all vectors v

$$v \in V \iff \min(v, m) \in V.$$

We call m a bound for V.

2. We say, a finite or infinite sequence $v_1, v_2, v_3, \ldots \in \mathbb{N}^k$ is an *alternating chain with respect to* V, if $v_1 \leq v_2 \leq v_3 \leq \ldots$ and $v_i \in V \iff v_{i+1} \notin V$ for all $i = 1, 2, 3, \ldots$. We say that this chain has *positive signature* if $v_1 \in V$, otherwise, we say that it has *negative signature.*

Theorem 2.2 *Let $k \geq 1$ and $V \subseteq \mathbb{N}^k$.*

1. *If V is of bounded significance with bound m then there is no alternating chain w.r.t. V of length greater than $m \cdot k + 1$.*
2. *If V is not of bounded significance then there exists an infinite alternating chain w.r.t. V.*

Proof: Part 1. is an easy exercise.

For part 2 we proceed by induction on k, the case $k = 1$ being trivial. So let $k > 1$ and assume that the claimed equivalence holds for all dimensions less than k. Let further V be a set, which is not of bounded significance. We have to show that there is an infinite chain of the type described above.

We distinguish two cases: In case 1, we have for every $m > 0$ two vectors a and b, such that $a \geq (m, \ldots, m)$ and $b \geq (m, \ldots, m)$, and $a \in V$, but $b \notin V$. This means that whenever we have any finite alternating chain $(v_1, \ldots, v_r)$ w.r.t. V, we can extend it by choosing the maximum component of v_r as the value of m, and then adding the vector a or b. Thus we can extend the chain infinitely often to obtain an infinite chain as desired.

Now in case 2, we assume that there is an m, such that either all vectors greater than or equal to $(m, \ldots, m)$ are in V, or none is. Let U denote the set of all vectors greater than or equal to $(m, \ldots, m)$. For every $\nu \in \{1, \ldots, k\}$ and every $\mu \in \{0, \ldots, m-1\}$, let U_ν^μ be the set $\{v \in \mathbb{N}^k \mid \text{the } \nu\text{-th component of } v \text{ has value } \mu\}$. Then each U_ν^μ is isomorphic to $\mathbb{N}^{k-1}$, and we have

$$\mathbb{N}^k = U \cup \bigcup_{\nu=1}^{k} \bigcup_{\mu=0}^{m-1} U_\nu^\mu.$$

Note that the above union in general is not a disjoint union, since every two U_ν^μ for different values of ν overlap.

Now let V_ν^μ be $V \cap U_\nu^\mu$. Then each of these sets can be viewed as a subset of $\mathbb{N}^{k-1}$, and if all of these $k \cdot m$ sets were of bounded significance, then clearly also V was. Thus there is at least one pair (ν, μ), such that V_ν^μ is not of bounded significance. By the assumption of our induction, there exists an infinite chain $(v_1, v_2, v_3, \ldots)$, where all v_i are elements of U_ν^μ and satisfy $v_i \in V_\nu^\mu \iff i \equiv 0 \bmod 2$. The same chain then can be taken as the infinite chain in $\mathbb{N}^k$ for V, because U_ν^μ is a subset of $\mathbb{N}^k$. □

3 Complexity Classes Defined by Counting

Many well-known complexity classes can be defined by counting as done in the following definition.

Definition 3.1 For $V \subseteq \mathbb{N}^k$ the complexity class $(V)\mathrm{P}$ is defined as the class of all languages L such that there exist functions $f_1, \ldots, f_k \in \#\mathrm{P}$ satisfying

$$x \in L \iff (f_1(x), \ldots, f_k(x)) \in V.$$

Using the class $\#\mathrm{P}^{\mathrm{A}}$ ($\#\mathrm{P}$ relative to oracle A) instead of $\#\mathrm{P}$, we obtain the relativized complexity class $(V)\mathrm{P}^A$.

Immediately by definition, $\mathrm{NP} = (\mathbb{N}\setminus\{0\})\mathrm{P}$, $\mathrm{coNP} = (\{0\})\mathrm{P}$, $1\text{-NP} = (\{1\})\mathrm{P}$, $\oplus\mathrm{P} = (2\cdot\mathbb{N}+1)\mathrm{P}$, and $\text{co-MOD}_k\mathrm{P} = (k\cdot\mathbb{N})\mathrm{P}$ for $k \geq 2$. It is known (see e.g. [20]) that PP and $\mathrm{C}_=\mathrm{P}$ can be characterized in a way which we could express in the above framework as $\mathrm{PP} = (\{(m,n) \mid m,n \in \mathbb{N} \wedge m > n\})\mathrm{P}$ and $\mathrm{C}_=\mathrm{P} = (\{(n,n) \mid n \in \mathbb{N}\})\mathrm{P}$.

In what follows, we are particularly interested in complexity classes $(V)\mathrm{P}$ where $V \subseteq \mathbb{N}^k$ is of bounded significance. For these classes, the following is easy to see:

Proposition 3.2 *If $V \subseteq \mathbb{N}^k$ is of bounded significance, then there is some $j \geq 1$ such that $(V)\mathrm{P} \subseteq \mathrm{NP}(j)$.*

The main interest of structural complexity focuses on the inclusion relationships between differently defined complexity classes. In the above terms: Under which conditions do we have $(V)\mathrm{P} \subseteq (U)\mathrm{P}$ for given $V \subseteq \mathbb{N}^k$ and $U \subseteq \mathbb{N}^{k'}$? The following theorem gives a necessary and sufficient condition for such an inclusion to hold under all relativizations, if U and V are sets of bounded significance.

If $u, v \in \mathbb{N}^k$, $u = (u_1, \ldots, u_k)$, $v = (v_1, \ldots, v_k)$, then the *multinomial coefficient* $\binom{u}{v}$ is defined as

$$\binom{u}{v} =_{\mathrm{def}} \prod_{i=1}^{k} \binom{u_i}{v_i}.$$

If for $p\colon \mathbb{N}^k \to \mathbb{N}$ there are $z' \in \mathbb{N}^k$ and natural numbers α_z for all $z \leq z'$ such that for all $v \in \mathbb{N}^k$ we have $p(v) = \sum_{z \leq z'} \alpha_z \binom{v}{z}$, then we say p is a *positive linear combination of multinomial coefficients.*

Theorem 3.3 *Let $V \subseteq \mathbb{N}^k$ and $U \subseteq \mathbb{N}^{k'}$ for some k, k' be sets of bounded significance. Then, we have: $(V)\mathrm{P} \subseteq (U)\mathrm{P}$ under all relativizations, if and only if there exist functions $p_1, \ldots, p_{k'}\colon \mathbb{N}^k \to \mathbb{N}$, all of which are positive linear combinations of multinomial coefficients, such that for all $v \in \mathbb{N}^k$,*

$$v \in V \iff (p_1(v), \ldots, p_{k'}(v)) \in U.$$

Proof: "only if": Since V and U are of bounded significance, there are numbers m, m' such that $v \in V \iff \min(v, m) \in V$ and $u \in U \iff \min(u, m') \in U$. Thus, the finite sets $V \cap \{0, \ldots, m\}^k$ and $U \cap \{0, \ldots, m'\}^{k'}$ are bounded counting types in the sense of [13].

The proof now is an easy consequence of [13], Theorem 17. Hertrampf proves there that under the above assumptions, there is a function $f\colon \{0, \ldots, m\}^k \to \{0, \ldots, m'\}^{k'}$ such that for all $v \in \{0, \ldots, m\}^k$, we have

$$v \in V \iff \min\left(\sum_{z \leq v} \binom{v}{z} \cdot f(z), m'\right) \in U.$$

Defining $F\colon \mathbb{N}^k \to \mathbb{N}^{k'}$ as $F(v) =_{\mathrm{def}} \sum_{z \leq (m,\ldots,m)} \binom{v}{z} \cdot f(z)$, we thus have for all $v \in \{0, \ldots, m\}^k$ that $v \in V \iff \min(F(v), m') \in U \iff F(v) \in U$. Moreover it is shown in [13] that f can always be chosen such that for every $v \in \mathbb{N}^k$, we have $F(v) \in U \iff F(\min(v, m)) \in U$, which implies that for all $v \in \mathbb{N}^k$, we have $v \in V \iff F(v) \in U$. Thus, if we define the sequence of functions $p_1, \ldots, p_{k'}$ as

$$(p_1(v), \ldots, p_{k'}(v)) =_{\mathrm{def}} F(v)$$

for all $v \in \mathbb{N}^k$, then the $p_1, \ldots, p_{k'}$ are as required.

"if": If there are $p_1, \ldots, p_{k'}$ as assumed, then these functions immediately define a relativizable simulation of $(V)\mathrm{P}$ machines by $(U)\mathrm{P}$ machines. $\square$

The following direct consequence of Theorem 3.3 provides a very convenient criterion, which can be used to prove chain theorems like in Sections 4 and 5. We have to introduce some notation.

For $r \in \{1, \ldots, k\}$ let $e_r \in \mathbb{N}^k$ be the vector having 1 at place r and 0 everywhere else. For $V \subseteq \mathbb{N}^k$, $u \in \mathbb{N}^k$ and $r \in \{1, \ldots, k\}$, we say that u is r-proper for V if there exists a $u' \leq u$ such that $u' \in V \iff u' + e_r \notin V$.

Theorem 3.4 *Let $V \subseteq \mathbb{N}^k$ and $U \subseteq \mathbb{N}^{k'}$ be of bounded significance, and suppose that $(V)\mathrm{P} \subseteq (U)\mathrm{P}$ under all relativizations. Then we have that for every set $\{v_1, \ldots, v_s\} \subseteq \mathbb{N}^k$ there exists a multi-set $\{u_1, \ldots, u_s\} \subseteq \mathbb{N}^{k'}$ such that for all $i, j \in \{1, \ldots, s\}$,*

1. $v_i \in V \iff u_i \in U$
2. $v_i \leq v_j \iff u_i \leq u_j$
3. *If for some r, v_i is r-proper for V and $v_i + e_r \leq v_j$, then $u_i < u_j$.*

To demonstrate the power of the previous theorem, we give the following application:

Corollary 3.5 *Let $V, U \subseteq \mathbb{N}$, $0 \notin V$, and $|U| < |V| < \infty$. Then there is an oracle A, such that*

$$(V)\mathrm{P}^A \not\subseteq (U)\mathrm{P}^A.$$

Proof: V and U are one-dimensional sets of bounded significance, the minimal element in V is nonzero and thus all elements of V are 1-proper. Now the set V itself would by the previous theorem enforce a set $U' \subseteq U$, which is of the same cardinality as V. This is a contradiction to the assumption $|U| < |V|$. □

4 The First Chain Theorem and its Applications

4.1 The Theorem

In this section, we will see that in the case that $(V)\mathrm{P} \subseteq (U)\mathrm{P}$ under all relativizations, some structural properties of the set V are transferred to U. The following consequence of Theorem 3.3 was originally proven in [6]:

Theorem 4.1 (First Chain Theorem) *Let $V \subseteq \mathbb{N}^k$ and $U \subseteq \mathbb{N}^{k'}$ be of bounded significance, and suppose that $(V)\mathrm{P} \subseteq (U)\mathrm{P}$ under all relativizations. If there is an alternating chain with respect to V, then there is an alternating chain of the same length and the same signature with respect to U.*

4.2 Applications to Boolean Hierarchies over NP and 1-NP

In this subsection we will apply Theorem 4.1 to obtain separation results for classes of the Boolean Hierarchies over NP and 1-NP. We have already mentioned that every class of the Boolean Hierarchy over NP (i.e., every class which can be built from NP by the operations $\wedge$, $\vee$, and co-) coincides with one of the classes

$$\mathrm{NP}(k) = \underbrace{\mathrm{NP} \oplus \cdots \oplus \mathrm{NP}}_{k \text{ times}} \quad \text{or} \quad \text{co-NP}(k) = \underbrace{\mathrm{NP} \oplus \cdots \oplus \mathrm{NP}}_{k \text{ times}} \oplus 1.$$

The proof of this fact (see [27]) heavily depends on the property of NP being closed under union and intersection. Since 1-NP is not known to be closed under union, we do not know about a similar characterization result for the classes of the Boolean Hierarchy over 1-NP. However, because NP $\subseteq$ co-1-NP and 1-NP $\subseteq$ NP $\wedge$ coNP, we know that the boolean closure of NP (i.e., the union of all classes of the Boolean Hierarchy over NP) coincides with the boolean closure of 1-NP. Moreover, because NP $\subseteq$ 1-NP $\oplus$ 1 the boolean closure of NP coincides with the union of all classes of the form 1-NP $\oplus \cdots \oplus$ 1-NP. Thus, the classes of the type

$$\text{1-NP}(k) =_{\text{def}} \underbrace{\text{1-NP} \oplus \cdots \oplus \text{1-NP}}_{k \text{ times}} \quad \text{and} \quad \text{co-1-NP}(k) = \underbrace{\text{1-NP} \oplus \cdots \oplus \text{1-NP}}_{k \text{ times}} \oplus 1$$

are of particular interest. Since 1-NP is not known to be closed under union, we are also interested in classes of the type

$$\bigvee_k \text{1-NP} =_{\text{def}} \underbrace{\text{1-NP} \vee \cdots \vee \text{1-NP}}_{k \text{ times}} \quad \text{and} \quad \text{co-}\bigvee_k \text{1-NP} = \underbrace{\text{co-1-NP} \wedge \cdots \wedge \text{co-1-NP}}_{k \text{ times}},$$

which have already been studied in [10].

All these classes $\text{NP}(k), \text{co-NP}(k), \text{1-NP}(k), \text{co-1-NP}(k), \bigvee_k \text{1-NP}$, and co-$\bigvee_k$ 1-NP can be characterized as counting classes as follows:

Definition 4.2 For $k \geq 1$, we define the following six vector-sets:

$A_k = \{(n_1, \ldots, n_k) \mid \#\{i \mid n_i > 0\} \text{ is odd}\}$
$B_k = \{(n_1, \ldots, n_k) \mid \#\{i \mid n_i > 0\} \text{ is even}\}$
$C_k = \{(n_1, \ldots, n_k) \mid \#\{i \mid n_i = 1\} \text{ is odd}\}$
$D_k = \{(n_1, \ldots, n_k) \mid \#\{i \mid n_i = 1\} \text{ is even}\}$
$E_k = \{(n_1, \ldots, n_k) \mid \#\{i \mid n_i = 1\} > 0\}$
$F_k = \{(n_1, \ldots, n_k) \mid \#\{i \mid n_i = 1\} = 0\}$

Proposition 4.3 *For every $k \geq 1$, (A_k)P = NP(k), (B_k)P = co-NP(k), (C_k)P = 1-NP(k), (D_k)P = co-1-NP(k), (E_k)P = $\bigvee_k$ 1-NP, (F_k)P = co-$\bigvee_k$ 1-NP.*

A close inspection of the significant parts of the respective sets A_k, ..., F_k easily yields the lengths and signatures of maximum-length alternating chains w.r.t. each of these sets.

Theorem 4.4 *For $k \geq 1$, we have the following lengths and signatures of maximum alternating chains with respect to A_k, B_k, C_k, D_k, E_k, and F_k:*

A_k:	$k+1$, *negative*	B_k:	$k+1$, *positive*
C_k:	$2k+1$, *negative*	D_k:	$2k+1$, *positive*
E_k:	$2k+1$, *negative*	F_k:	$2k+1$, *positive*

From Theorem 4.4 we can obtain many separation results. The results concerning the Boolean Hierarchies over NP and over 1-NP are summarized in Table 1, which represents our knowledge about the inclusion relationships between the classes NP(j), co-NP(j), 1-NP(j), co-1-NP(j), $\bigvee_j$ 1-NP, and co-$\bigvee_j$ 1-NP so far. In the table entry corresponding to the classes $\mathcal{C}(j)$ and $\mathcal{C}'(k)$ we list for any valid pair (j, k) (i.e. $j, k \geq 1$) two conditions; here, `false` denotes the never fulfilled condition. If the first condition is fulfilled, then $\mathcal{C}(j) \subseteq \mathcal{C}'(k)$ under all relativizations (by Theorem 4.5 or other obvious relationships like coNP $\subseteq$ 1-NP $\subseteq$ NP(2)); but if the second one is fulfilled, then from the First Chain Theorem we obtain the existence of an oracle A such that $\mathcal{C}(j)^A \not\subseteq \mathcal{C}'(k)^A$. To avoid redundancy, we do not provide columns for the classes co-NP(k), co-1-NP(k), and co-$\bigvee_k$ 1-NP, because e.g. the relations between NP(j) and co-1-NP(k) clearly are the same as between co-NP(j) and 1-NP(k), etc.

In Section 5, we will close some of the gaps in this table by using a more sophisticated chain theorem.

	$\mathrm{NP}(k)$		$1\text{-}\mathrm{NP}(k)$		$\bigvee_k 1\text{-NP}$	
	$\subseteq$	$\not\subseteq$	$\subseteq$	$\not\subseteq$	$\subseteq$	$\not\subseteq$
$\mathrm{NP}(j)$	$j \leq k$	$j > k$	$j \leq 2 \cdot \left\lfloor \frac{k}{2} \right\rfloor$	$j > 2k$	`false`	$j > 2k$
co-$\mathrm{NP}(j)$	$j < k$	$j \geq k$	$j < 2 \cdot \left\lceil \frac{k}{2} \right\rceil$	$j \geq 2k$	$j = 1$	$j \geq 2k$
$1\text{-}\mathrm{NP}(j)$	$j \leq \frac{k}{2}$	$j > \frac{k}{2}$	$j \leq k$	$j > k$	$j = 1$	$j > k$
co-$1\text{-}\mathrm{NP}(j)$	$j < \frac{k}{2}$	$j \geq \frac{k}{2}$	$j < k$	$j \geq k$	`false`	$j \geq k$
$\bigvee_j 1\text{-NP}$	$j \leq \frac{k}{2}$	$j > \frac{k}{2}$	$j \leq \frac{k}{2}$	$j > k$	$j \leq k$	$j > k$
co-$\bigvee_j 1\text{-NP}$	$j < \frac{k}{2}$	$j \geq \frac{k}{2}$	$j < \frac{k}{2}$	$j \geq k$	`false`	$j \geq k$

Table 1: Inclusion relationships obtained by the First Chain Theorem

By Theorem 4.4 it is clear that it is not possible to separate all separable counting classes using the First Chain Theorem; e.g. A_{2k}, C_k, and E_k are not distinguishable w.r.t. their maximum-length alternating chains.

However, all the relativizable inclusions $(V)\mathrm{P} \subseteq \mathrm{NP}(k)$ and $(V)\mathrm{P} \subseteq \text{co-NP}(k)$ for $V \subseteq \mathbb{N}^m$ with bounded significance can be completely characterized by the First Chain Theorem. We make this precise in the following theorem:

Theorem 4.5 *Let $k, m \geq 1$ and let $V \subseteq \mathbb{N}^m$ be of bounded significance.*

1. $(V)\mathrm{P}^A \subseteq \mathrm{NP}(k)^A$ *for all oracles A if and only if there is no alternating chain w.r.t. V which has length $k+1$ and positive signature.*
2. $(V)\mathrm{P}^A \subseteq \text{co-NP}(k)^A$ *for all oracles A if and only if there is no alternating chain w.r.t. V which has length $k+1$ and negative signature.*

Proof: We prove statement 1; statement 2 follows by complementation.

"only if": Assume that $(V)\mathrm{P}^A \subseteq \mathrm{NP}(k)^A$ for all oracles A. By Proposition 4.3 we have $(V)\mathrm{P}^A \subseteq (A_k)\mathrm{P}^A$ for all A, and by Theorem 4.1 and by Theorem 4.4, this means that the maximum alternating chain w.r.t. V which has positive signature has length no more than k.

"if": Assume that there is no alternating chain w.r.t. V which has length $k+1$ and positive signature. Then, the maximum alternating chain w.r.t. V which has negative signature has length at most $k+1$. For a language $L \in (V)\mathrm{P}^A$ there exist $f_1, \ldots, f_m \in \#\mathrm{P}^{\mathrm{A}}$ such that $x \in L \iff (f_1(x), \ldots, f_m(x)) \in V$. Defining for $i \geq 1$ the language

$$L_i =_{\mathrm{def}} \{x \mid \text{there exist } v_1, v_2 \ldots, v_i \in \mathbb{N}^m \text{ such that } v_1 < v_2 < \cdots < v_i \leq (f_1(x), \ldots, f_m(x)), \\ v_1, v_3, v_5 \ldots \in V \text{ and } v_2, v_4, v_6 \ldots \notin V\}$$

we obtain $L_i \in \mathrm{NP}^A$ (since V is of bounded significance), $L_i \supseteq L_{i+1}$, $L_{k+1} = \emptyset$, and $L = L_1 \Delta L_2 \Delta \cdots \Delta L_k \in \underbrace{\mathrm{NP}^A \oplus \cdots \oplus \mathrm{NP}^A}_{k \text{ times}}$. Thus, we have $L \in \mathrm{NP}(k)^A$. $\square$

4.3 Further Applications

We proceed with a separation of unbounded counting type classes (like $\oplus$P) from bounded counting type classes (like the classes of the Boolean Hierarchies over NP and 1-NP):

Theorem 4.6 *Let $V \subseteq \mathbb{N}^k$ not be of bounded significance, and let $U \subseteq \mathbb{N}^{k'}$ be of bounded significance. Then there exists an oracle A such that $(V)\mathrm{P}^A \not\subseteq (U)\mathrm{P}^A$.*

Proof: For $m \in \mathbb{N}$ let in the following $V^{\leq m} =_{\text{def}} \{v \in V \mid v \leq (\underbrace{m, \ldots, m}_{k \text{ times}})\}$.

Suppose $(V)\mathrm{P} \subseteq (U)\mathrm{P}$ under all relativizations. Since U is of bounded significance only the two following cases can happen: (i) For all u such that $u_0 \leq u$ we have $u \in U$, or (ii) for all u such that $u_0 \leq u$ we have $u \in \overline{U}$. Suppose first that case (ii) holds. Let m be arbitrary. By changing the simulation of a (V)P machine by a (U)P machine in such a way that all vectors greater than m in at least one component are "mapped to" (i.e., simulated by) vectors greater than u_0, we prove that $(V^{\leq m})\mathrm{P} \subseteq (U)\mathrm{P}$. Now, it suffices to notice that by considering large enough values of m, we can obtain alternating chains of any desired length in $V^{\leq m}$. However trivially every $V^{\leq m}$ is of bounded significance. Thus, by Theorem 4.1 the assumption $(V)\mathrm{P}^A \subseteq (U)\mathrm{P}^A$ for all A led us to a contradiction.

For case (i) we note that $(V)\mathrm{P} \subseteq (U)\mathrm{P}$ iff $(\overline{V})\mathrm{P} \subseteq (\overline{U})\mathrm{P}$, and moreover V is of bounded significance iff $\overline{V}$ is. Thus this case reduces to case (ii). □

All the classes $\oplus$P, MOD$_k$P, PP, C$_=$P can easily be defined as (V)P for some V which is not of bounded significance. On the other hand, we above represented all classes in the Boolean Hierarchies over NP and 1-NP as (U)P for some U of bounded significance. Thus we obtain the following corollary:

Corollary 4.7 *Let $\mathcal{C}_1$ be one of the classes* $\oplus$P, MOD$_k$P, PP, C$_=$P, *and let $\mathcal{C}_2$ be one of the classes* NP, coNP, 1-NP *or a class from the Boolean Hierarchies over these classes. Then there is an oracle A, such that ${\mathcal{C}_1}^A \not\subseteq {\mathcal{C}_2}^A$.*

Our next result is valid for all classes of the form (V)P for some V of bounded significance.

Theorem 4.8 *Let V be of bounded significance. Then there is an oracle A, such that $(V)\mathrm{P}^A \neq$ co-$(V)\mathrm{P}^A$.*

Proof: The maximum-length alternating chains of V and $\overline{V}$ have the same lengths but different signatures. By Theorem 4.1 we get an oracle such that $(V)\mathrm{P}^A \not\subseteq (\overline{V})\mathrm{P}^A =$ co-$(V)\mathrm{P}^A$. □

5 The Second Chain Theorem and its Applications

5.1 The Theorem

In this section we want to refine our chain technique, which is necessary to fill some of the gaps in Section 4. For example, the classes NP($2k$) and 1-NP(k) cannot be distinguished by looking only for the alternating chains we have studied in Section 4. Both of these classes are defined via sets B with maximum chain length $2k+1$ and negative signature. To overcome this difficulty, we introduce a new type of alternating chains as follows:

Definition 5.1 Let $V \subseteq \mathbb{N}^k$ be of bounded significance with minimum bound m. We say, a sequence $v_0, v_1, \ldots, v_s$ is a *type-2 alternating chain with respect to V of length $s+1$*, if

- $v_0 \leq v_1 \leq \cdots \leq v_s \leq (\underbrace{m, m, \ldots, m}_{k \text{ times}})$,
- $v_i \in V \iff v_{i+1} \notin V$ for $i = 0, 1, \ldots, s-1$, and
- if v_i and v_{i+1} differ in the j-th component, then the j-th component of v_{i+1} is equal to m $(i = 0, 1, \ldots, s-1,\ j = 1, \ldots, k)$.

If v_0 in V, then we say that this chain has *positive signature*, otherwise it has *negative signature.*

Informally, in a type-2 alternating chain $v_0, v_1, \ldots, v_s$ with respect to some V (with minimum bound m), the vector v_i is obtained from v_{i-1} by increasing at least one of the components from the basic v_0-level to the bound m. This immediately yields the following:

Proposition 5.2 *If $V \subseteq \mathbb{N}^k$ is of bounded significance, then the length of every type-2 alternating chain w.r.t. V is bounded by $k+1$.*

Theorem 5.3 (Second Chain Theorem) *Let $V \subseteq \mathbb{N}^k$ and $U \subseteq \mathbb{N}^{k'}$ be of bounded significance, and suppose that $(V)\mathrm{P}^A \subseteq (U)\mathrm{P}^A$ for all oracles A. If there is a type-2 alternating chain w.r.t. V, then there exists a type-2 alternating chain of the same length and the same signature w.r.t. U.*

Proof: Because $(V)\mathrm{P}^A \subseteq (U)\mathrm{P}^A$ for all oracles A, we conclude from Theorem 3.3 that there exist positive linear combinations of multinomial coefficients $p_1, \ldots, p_{k'}\colon \mathbb{N}^k \to \mathbb{N}$, such that for all $v \in \mathbb{N}^k$, $v \in V \iff p(v) =_{\mathrm{def}} (p_1(v), \ldots, p_{k'}(v)) \in U$. Let m and m' be the minimum bounds of V and U, resp. Since $p_1, \ldots, p_{k'}$ are positive linear combinations of multinomial coefficients, there exists an $M \geq 1$ such that for all $v, x \in \mathbb{N}^k$ and $\ell = 1, \ldots, k'$, we have $p_\ell(v + M \cdot x) = p_\ell(v)$ or $p_\ell(v + M \cdot x) \geq m'$.

Let $v_0, v_1, \ldots, v_s$ be a type-2 alternating chain w.r.t. V. Define for $i = 0, 1, \ldots, s$,

- $v'_i =_{\mathrm{def}} v_i + (M-1) \cdot (v_i - v_0)$,
- $u'_i =_{\mathrm{def}} p(v'_i)$, and
- $u_i =_{\mathrm{def}} \min(u'_i, m')$.

We will prove that $u_0, u_1, \ldots, u_s$ is a type-2 alternating chain w.r.t. U, and $v_i \in V \iff u_i \in U$ for all $i = 0, 1, \ldots, s$.

1. Obviously, $v'_0 < v'_1 < \cdots < v'_s$, and thus $u'_0 \leq u'_1 \leq \cdots \leq u'_s$; and $u_0 \leq u_1 \leq \cdots \leq u_s \leq (\underbrace{m', \ldots, m'}_{k' \text{ times}})$.
2. Since for all $i = 0, 1, \ldots, s$, v_i and v'_i differ only at positions where v_i is equal to m, we have $v_i \in V \iff v'_i \in V$; thus $v_i \in V \iff v'_i \in V \iff u'_i \in U \iff u_i \in U$. Hence the u_i in fact form an alternating chain.
3. Assume that for some $i \in \{0, 1, \ldots, s-1\}$ and some $j \in \{1, \ldots, k'\}$, u_i and u_{i+1} differ in the j-th component. Then also $u'_i = p(v'_i)$ and $u'_{i+1} = p(v'_{i+1})$ differ in the j-th component, which means

$$\begin{aligned} p_j(v'_{i+1}) &= p_j\left(v_{i+1} + (M-1) \cdot (v_{i+1} - v_0)\right) \\ &= p_j\left(M \cdot v_{i+1} + M \cdot v_i - M \cdot v_i - (M-1) \cdot v_0\right) \\ &= p_j\left(v'_i + M \cdot (v_{i+1} - v_i)\right) \\ &> p_j(v'_i) \end{aligned}$$

   By the choice of M, we immediately obtain $p_j(v'_{i+1}) \geq m'$, thus the j-th component of u_i is m'.

□

5.2 Applications to Boolean Hierarchies, Cont'd

In order to apply the second chain theorem to the Boolean Hierarchies over NP and over 1-NP, we note the following properties of the vector sets A_k, B_k, C_k, D_k, E_k, and F_k:

Lemma 5.4 *For $k \geq 1$, we have the following lengths and signatures of maximum type-2 alternating chains with respect to A_k, B_k, C_k, D_k, E_k, and F_k:*

A_k:	$k+1$,	*negative*	B_k:	$k+1$,	*positive*
C_k:	$k+1$,	*positive (if k is odd) or negative (otherwise)*	D_k:	$k+1$,	*negative (if k is odd) or positive (otherwise)*
E_k:	2,	*positive*	F_k:	2,	*negative*

We present our improved knowledge about the relativizable inclusion relationships between the classes NP(k), co-NP(k), 1-NP(k), co-1-NP(k), $\bigvee_k$ 1-NP, and co-$\bigvee_k$ 1-NP in Table 2, which has to be interpreted as Table 1 in Section 4. The only questions which remain not completely resolved in this context concern the inclusions between $\bigvee_j$ 1-NP (or co-$\bigvee_j$ 1-NP) and 1-NP(k) (or co-1-NP(k)).

	NP(k)		1-NP(k)		$\bigvee_k$ 1-NP	
	$\subseteq$	$\not\subseteq$	$\subseteq$	$\not\subseteq$	$\subseteq$	$\not\subseteq$
NP(j)	$j \leq k$	$j > k$	$j \leq 2\cdot\left\lfloor\frac{k}{2}\right\rfloor$	$j > 2\cdot\left\lfloor\frac{k}{2}\right\rfloor$	`false`	`true`
co-NP(j)	$j < k$	$j \geq k$	$j < 2\cdot\left\lceil\frac{k}{2}\right\rceil$	$j \geq 2\cdot\left\lceil\frac{k}{2}\right\rceil$	$j = 1$	$j \geq 2$
1-NP(j)	$j \leq \frac{k}{2}$	$j > \frac{k}{2}$	$j \leq k$	$j > k$	$j = 1$	$j \geq 2$
co-1-NP(j)	$j < \frac{k}{2}$	$j \geq \frac{k}{2}$	$j < k$	$j \geq k$	`false`	`true`
$\bigvee_j$ 1-NP	$j \leq \frac{k}{2}$	$j > \frac{k}{2}$	$j \leq \frac{k}{2}$	$j > k$	$j \leq k$	$j > k$
co-$\bigvee_j$ 1-NP	$j < \frac{k}{2}$	$j \geq \frac{k}{2}$	$j < \frac{k}{2}$	$j \geq k$	`false`	`true`

Table 2: Inclusion relationships obtained by the First and Second Chain Theorems

In Section 4.2, we saw that the First Chain Theorem allows us to completely settle the question which inclusions $(V)\mathrm{P} \subseteq \mathrm{NP}(k)$ for some $k \geq 1$ and V of bounded significance relativizably hold (see Theorem 4.5). Next we will give a necessary and sufficient criterion for inclusions $\mathrm{NP}(k) \subseteq (V)\mathrm{P}$ to hold relativizably.

Theorem 5.5 *Let $k \geq 1$ and let $V \subseteq \mathbb{N}^{k'}$ be of bounded significance.*

1. $\mathrm{NP}(k) \subseteq (V)\mathrm{P}^A$ *for all oracles A if and only if there is a type-2 alternating chain w.r.t. V of length $k+1$ and negative signature.*
2. $\mathrm{coNP}(k) \subseteq (V)\mathrm{P}^A$ *for all oracles A if and only if there is a type-2 alternating chain w.r.t. V of length $k+1$ and positive signature.*

Proof: We prove statement 1; statement 2 follows by complementation.

Let $v_0, v_1, \ldots, v_k$ be a type-2 alternating chain w.r.t. V of negative signature. Let m be the minimum bound of V. For $L \in \mathrm{NP}(k)^A$, there exist languages $L_1, L_2, \ldots, L_k \in \mathrm{NP}^A$ such that $L_1 \supseteq L_2 \supseteq \cdots \supseteq L_k$ and $x \in L \iff \max\{i \mid x \in L_i\}$ is odd [27, 18]. Let $f_1, f_2, \ldots, f_k \in \#\mathrm{P}^\mathrm{A}$ be such that $x \in L_i \iff f_i(x) > 0$. Consequently, $f_i(x) > 0 \Longrightarrow f_{i-1}(x) > 0$ for $i = 2, \ldots, k$. Now, define $h_1, \ldots, h_{k'} \in \#\mathrm{P}^\mathrm{A}$ such that

$$h(x) = (h_1(x), \ldots, h_{k'}(x)) =_{\text{def}} v_0 + \sum_{i=1}^{k} (v_i - v_{i-1}) \cdot f_i(x).$$

We will show that $x \in L \iff h(x) \in V$. Let x be given and let $i_0 =_{\text{def}} \max\{i \mid x \in L_i\}$. Hence, $h(x) = v_0 + \sum_{i=1}^{i_0} (v_i - v_{i-1}) \cdot f_i(x)$ and $f_i(x) = 0$ for $i > i_0$. If the j-th component of v_{i_0} is equal to that of v_0, then this is true for all $i = 1, \ldots, i_0$. Hence, the j-th component of $h(x)$ is equal to that of v_{i_0}. If the j-th component of v_{i_0} is m, then because of $h(x) \geq v_{i_0}$ the j-th component of $h(x)$ is at least m. Hence, in both cases we obtain $h(x) \in V \iff v_{i_0} \in V$, and therefore $x \in L \iff \max\{i \mid x \in L_i\}$ is odd $\iff i_0$ is odd $\iff v_{i_0} \in V \iff h(x) \in V$, and thus $L \in (V)\mathrm{P}$.

This shows that $\mathrm{NP}(k)^A \subseteq (V)\mathrm{P}^A$ for all oracles A. On the other hand, if this is the case, then by the Second Chain Theorem and Lemma 5.4 there exists a type-2 alternating chain w.r.t. V of length $k+1$ and negative signature. □

6 Conclusion

We developed two main techniques for proving relativizable inclusions between certain bounded counting complexity classes, called the First and the Second Chain Theorem. Using both chain theorems, a comparison between classes of the Boolean Hierarchies over NP and over 1-NP could be obtained as easy corollaries. Theorems 4.5 and 5.5 completely characterize the inclusion relationships between the classes of the Boolean Hierarchy over NP – which are in our opinion the most important counting classes with bounded significance – and all other bounded significance classes. Also in many other cases, our methods yield sharp distinctions between the cases of relativizable inclusion and those of oracle separability.

These results should be contrasted to the result of [13], where an algorithm was constructed, which given two explicitly described vector sets V and U answers the question, whether $(V)\mathrm{P} \subseteq (U)\mathrm{P}$ under all relativizations, or not, thus solving the separability vs. inclusion problem. But this algorithm unfortunately is not suitable for a class with a parameter, like $\mathrm{NP}(k)$, so it could not directly be used to obtain general results for separations of whole hierarchies like in this paper.

A surprisingly easy to obtain consequence of our techniques is given in Theorem 4.8, where we showed that all classes, which allow a definition as counting classes of bounded significance are not relativizably closed under complement.

Some inclusion relations, however, remain unresolved, as already pointed out in Section 5. Also interesting of course is the question, whether there are generalizations of the chain theorems which could be suitable to obtain separations of two classes which are both described via vector-sets not of bounded significance.

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