Classifying ω-Regular Partitions

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Abstract. We try to develop a theory of ω-regular partitions in parallel with the theory around the Wagner hierarchy of regular ω-languages. In particular, we generalize a theorem of L. Staiger and K. Wagner to the case of partitions, prove decidability of all levels of the Boolean hierarchy of regular partitions over open sets, establish coincidence of reducibilities by continuous functions and by functions computed by finite automata on the class of regular $\mathbf{\Delta}^0_2$-partitions, and show undecidability of the first-order theory of the structure of Wadge degrees of regular k-partitions for all $k \geq 3$.
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1 Introduction

This paper is devoted to the theory of infinite behavior of computing devices that is of primary importance for theoretical and practical computer science. More exactly, we consider topological aspects of this theory in the simplest case of finite automata. A series of papers in this direction culminated with the paper [Wag79] giving in a sense the finest possible topological classification of regular ω-languages (i.e., of subsets of X^ω for a finite alphabet X recognized by finite automata) known as the Wagner hierarchy. Here we shall try to develop a similar theory for partitions.

Let M be a set, $P(M)$ the class of subsets of M, and for each $k \geq 2$ let k^M be the set of all functions $A : M \to k$ (we identify a natural number $k \in \omega$ with the set $\{0, \ldots, k-1\}$). We call maps $A \in k^M$ *k-partitions* of M because they are in a natural bijective correspondence with the tuples $(A_0, \ldots, A_{k-1})$ of pairwise disjoint sets satisfying $A_0 \cup \cdots \cup A_{k-1} = M$. For any class $\mathcal{C} \subseteq P(M)$, let $\mathcal{C}_k$ denote the set of $\mathcal{C}$-partitions, i.e. partitions $A \in k^M$ such that $A^{-1}(i) \in \mathcal{C}$ for each $i < k$.

The aim of this paper is to generalize the theory around the Wagner hierarchy from the case of regular ω-regular languages to the case of regular k-partitions of X^ω. Note that the ω-languages are in a bijective correspondence with 2-partitions of X^ω. In particular, we generalize a theorem of L. Staiger and K.

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Wagner from [SW74] to the case of partitions, prove decidability of all levels of the Boolean hierarchy of regular partitions over open sets, establish coincidence of reducibilities $\leq_{CA}$ by continuous functions and $\leq_{DA}$ by functions computed by finite automata on the class of regular $\mathbf{\Delta}^0_2$-partitions of X^ω, and show undecidability of the first-order theory of the structure $(\mathcal{R}_k; \leq_{CA})$ of CA-degrees of regular partitions for each $k \geq 3$. Our results show that, though the case of partitions is certainly more complicated than the case of sets, there is a hope to develop a full analog of the Wagner hierarchy for partitions.

The rest of the paper is organized as follows. In Section 2 we collect notions and basic tools we will rely upon. In Section 3 we generalize the Staiger-Wagner theorem to the case of partitions. In Section 4 we develop a full analog of the Wagner hierarchy for regular $\mathbf{\Delta}^0_2$-partitions. In Section 5 we characterize some structures of CA-degrees of partitions, show the coincidence of CA- and DA-reducibilities on regular $\mathbf{\Delta}^0_2$-partitions and prove that for any $k \geq 3$ the first-order theory of the structure $(\mathcal{R}_k; \leq_{CA})$ is undecidable (from the description of this structure for $k = 2$ in [Wag79] it follows that the theory is decidable in this case).

2 Notions and Tools

We use some standard notation and terminology on posets which may be found in any book on the subject, see e.g. [DP94]. We will not be very cautious when applying notions about posets also to quasiorders (known also as preorders); in such cases we mean the corresponding quotient-poset of the quasiorder.

A poset $(P; \leq)$ will be often shorter denoted just by P. Any subset of P may be considered as a poset with the induced partial ordering. In particular, this applies to the "cones" $\check{x} = \{y \in P \mid x \leq y\}$ and $\hat{x} = \{y \in P \mid y \leq x\}$ defined by any $x \in P$. A *well partial order* is a poset P that has neither infinite descending chains nor infinite antichains; for such posets there is a canonical rank function rk assigning ordinals to the elements of P.

By *a forest* we mean a finite poset in which every upper cone $\check{x}$ is a chain. *A tree* is a forest having the biggest element (called *the root* of the tree). Note that any forest is uniquely representable as a disjoint union of trees, the roots of the trees being the maximal elements of the forest.

A *k-labeled poset* (or just a k-poset) is an object $(P; \leq, c)$ consisting of a finite poset $(P; \leq)$ and a labeling $c : P \to k$. Sometimes we simplify notation of a k-poset to (P, c) or even to P. A *morphism* $f : (P; \leq, c) \to (P'; \leq', c')$ between k-posets is a monotone function $f : (P; \leq) \to (P'; \leq')$ respecting the labelings, i.e. satisfying $c = c' \circ f$.

Let $\mathcal{F}_k$ and $\mathcal{T}_k$ be the sets of all finite k-forests and finite k-trees, respectively. Define a preorder $\leq$ on $\mathcal{F}_k$ as follows: $(P, c) \leq (P', c')$, if there is a morphism from (P, c) to (P', c') (this preorder maybe of course considered also on k-posets). By $\equiv$ we denote the equivalence relation on $\mathcal{F}_k$ induced by $\leq$. The quotient structure of $(\mathcal{F}_k; \leq)$ plays an important role in this paper because it is intimately related to the Boolean hierarchy of k-partitions, see [Se04,Se06].

For arbitrary finite k-trees $T_0, \ldots, T_n$, let $F = T_0 \sqcup \cdots \sqcup T_n$ be their join, i.e. the disjoint union. Then F is a k-forest whose trees are exactly $T_0, \ldots, T_n$. Of course, every k-forest is (equivalent to) the join of its trees. Note that the join operation applies also to k-forests, and the join of any two k-forests is clearly their supremum under $\leq$. Hence, $(\mathcal{F}_k; \sqcup)$ is an upper semilattice.

Fix a finite alphabet X containing more than one symbol (for simplicity we may assume that $X = m = \{x \mid x < m\}$ for a natural number $m > 1$, so $0, 1 \in X$). Let X^* and X^ω denote respectively the sets of all words and of all ω-words (i.e. sequences $\alpha : \omega \to X$) over X. The empty word is denoted by ε. Let $X^{\leq\omega} = X^* \cup X^\omega$. We use some almost standard notation concerning words and ω-words, so we are not too casual in reminding it here. For $w \in X^*$ and $\xi \in X^{\leq\omega}$, $w \sqsubseteq \xi$ means that w is a substring of ξ, $w \cdot \xi = w\xi$ denote the concatenation, $l = |w|$ is the length of $w = w(0) \cdots w(l-1)$. For $w \in X^*, W \subseteq X^*$ and $A \subseteq X^{\leq\omega}$, let $w \cdot A = \{w\xi : \xi \in A\}$ and $W \cdot A = \{w\xi : w \in W, \xi \in A\}$. For $k, l < \omega$ and $\xi \in X^{\leq\omega}$, let $\xi[k,l) = \xi(k) \cdots \xi(l-1)$ and $\xi[k] = \xi[0,k)$.

The set X^ω carries the Cantor topology with the open sets $W \cdot X^\omega$, where $W \subseteq X^*$. Continuous functions in this topology are called also *CA-functions*. A *CS-function* is a function $f : X^\omega \to X^\omega$ satisfying $f(\xi)(n) = \phi(\xi[n+1])$ for some $\phi : X^* \to X$. Every CS-function is a CA-function. In descriptive set theory CS-functions are known as Lipschitz functions. Both classes of functions are closed under composition. We relate to any pair $\xi, \eta \in X^\omega$ its "code" $\zeta = \langle \xi, \eta \rangle$ defined by $\zeta(2n) = \xi(n)$ and $\zeta(2n+1) = \eta(n)$, where $n < \omega$. It is well-known that $(\xi, \eta) \mapsto \langle \xi, \eta \rangle$ is a homeomorphism of spaces $X^\omega \times X^\omega$ and X^ω.

Let $\{\mathbf{\Sigma}^0_n\}_{n>0}$ denote the Borel hierarchy in X^ω, i.e. $\mathbf{\Sigma}^0_1$ is the class of open sets, $\mathbf{\Sigma}^0_2$ is the class of countable unions of closed sets and so on. Let $\mathbf{\Pi}^0_n$ be the dual class (i.e., the class of all complements) for $\mathbf{\Sigma}^0_n$, and $\mathbf{\Delta}^0_n = \mathbf{\Sigma}^0_n \cap \mathbf{\Pi}^0_n$. For any $n > 0$, $\mathbf{\Sigma}^0_n$ contains $\emptyset, X^\omega$ and is closed under countable unions and finite intersections, while $\mathbf{\Delta}^0_n$ is a Boolean algebra. For any $n > 0$, $\mathbf{\Sigma}^0_n \cup \mathbf{\Pi}^0_n \subseteq B(\mathbf{\Sigma}^0_n) \subseteq \mathbf{\Delta}^0_{n+1}$, and $\mathbf{\Sigma}^0_n \not\subseteq \mathbf{\Pi}^0_n$. For a class of sets $\mathcal{C}$, $B(\mathcal{C})$ denotes the Boolean closure of $\mathcal{C}$. For more information on descriptive set theory see e.g. [Ke94].

By *automaton* (over X) we mean a triple $\mathcal{A} = (Q, f, i)$ consisting of a finite non-empty set Q of states, a transition function $f : Q \times X \to Q$ and an initial state $i \in Q$. The transition function is naturally extended to the function $f : Q \times X^* \to Q$ defined by induction $f(q, \varepsilon) = q$ and $f(q, u \cdot x) = f(f(q,u), x)$, where $u \in X^*$ and $x \in X$. Similarly, we may define the function $f : Q \times X^\omega \to Q^\omega$ by $f(q, \xi)(n) = f(q, \xi[n])$. Relate to any automaton $\mathcal{A}$ the set of cycles $C_\mathcal{A} = \{f_\mathcal{A}(\xi) \mid \xi \in X^\omega\}$ where $f_\mathcal{A}(\xi)$ is the set of states which occur infinitely often in the sequence $f(i, \xi) \in Q^\omega$. Note that in this paper we consider only deterministic finite automata.

Automata equipped with appropriate additional structures are used as acceptors (devises accepting words or ω-words) and transducers (devices computing functions on words or ω-words). For the case of ω-words, there are several notions of acceptors of which we will use only one. A *Muller acceptor* has the form $(\mathcal{A}, \mathcal{F})$ where $\mathcal{A}$ is an automaton and $\mathcal{F} \subseteq C_\mathcal{A}$; it recognizes the set $L(\mathcal{A}, \mathcal{F}) = \{\xi \in X^\omega \mid f_\mathcal{A}(\xi) \in \mathcal{F}\}$. It is well known that Muller acceptors recognize exactly the *regular ω-languages* called also regular sets in this paper. The class $\mathcal{R}$ of all regular ω-languages is a proper subclass of $B(\mathbf{\Sigma}^0_2)$ that in turn is a proper subclass of $\mathbf{\Delta}^0_3$.

The notion of Muller acceptor is generalized to the notion of Muller k-acceptor (a devise that recognizes k-partitions of X^ω) in a straightforward way. Namely, it is a pair $(\mathcal{A}, c)$ where $\mathcal{A}$ is an automaton and $c : C_\mathcal{A} \to k$ is a k-partition of $C_\mathcal{A}$. Such a k-acceptor recognizes the k-partition $L(\mathcal{A}, c) = c \circ f_\mathcal{A}$ where $f_\mathcal{A} : X^\omega \to C_\mathcal{A}$ is the map defined above. We have the following characterization of the ω-regular partitions.

Proposition 1. *A partition $L : X^\omega \to k$ is regular iff it is recognized by a Muller k-acceptor.*

Proof. For $k = 2$ the assertion is trivial, so we assume $k > 2$. If $L = L(\mathcal{A}, c)$ for a Muller k-acceptor $(\mathcal{A}, c)$ then $L^{-1}(l) = L(\mathcal{A}, c^{-1}(l))$ for all $l < k$. Therefore, L is a regular partition.

Conversely, let L be a regular k-partition. Then $L^{-1}(l) \in \mathcal{R}$ for all $l < k$, hence $L^{-1}(l) = L(\mathcal{A}_l, \mathcal{F}_l)$ for some Muller acceptors $(\mathcal{A}_l, \mathcal{F}_l)$, $l < k$. Let $\mathcal{A} = (Q, f, i)$ be the product of automata $\mathcal{A}_l = (Q_l, f_l, i_l)$ for $l < k - 1$, so $Q = Q_0 \times \cdots \times Q_{k-2}$, $f((q_0, \ldots, q_{k-2}), x) = (f_0(q_0, x), \ldots, f_{k-2}(q_{k-2}, x))$ and $i = (i_0, \ldots, i_{k-2})$. As is well-known and easy to see, $pr_l(f_\mathcal{A}(\xi)) = f_{\mathcal{A}_l}(\xi)$ for all $l < k - 1$ and $\xi \in X^\omega$, where $pr_l : Q \to Q_l$ is the projection to the l-th coordinate. Since the sets $L^{-1}(l)$, $l < k - 1$, are pairwise disjoint, so are also the sets $pr_l^{-1}(\mathcal{F}_l)$. Let $c : C_\mathcal{A} \to k$ be the unique partition of $C_\mathcal{A}$ satisfying $c^{-1}(l) = pr_l^{-1}(\mathcal{F}_l)$ for all $l < k-1$. Then the Muller k-acceptor $(\mathcal{A}, c)$ recognizes L, completing the proof.

An important role in this paper is played by a preorder $\leq_0$ and a partial order $\leq_1$ on $C_\mathcal{A}$ introduced in [Wag79] and defined as follows: $D \leq_0 E$ iff some (equivalently, each) state in E is reachable in the graph of the automaton $\mathcal{A}$ from some (equivalently, each) state in D; $D \leq_1 E$ iff $D \supseteq E$.

A *synchronous transducer* (over X) is a tuple $\mathcal{T} = (Q, f, g, i)$ consisting of an automaton (Q, f, i) and an output function $g : Q \times X \to X$. The output function is naturally extended to a function $g : Q \times X^\omega \to X^\omega$ denoted by the same letter. The transducer $\mathcal{T}$ *computes* the function $g_\mathcal{T} : X^\omega \to X^\omega$ defined by $g_\mathcal{T}(\xi) = g(i, \xi)$. An *asynchronous transducer* (over X) is defined as a synchronous transducer with only one exception: this time the output function g maps $Q \times X$ into X^*. As a result, the value $g(q, \xi)$ is in $X^{\leq\omega}$, and the function $g_\mathcal{T}$ maps X^ω into $X^{\leq\omega}$. Nevertheless, we usually consider the case when $g_\mathcal{T}$ maps X^ω into X^ω; this condition is easily described in terms of $\mathcal{T}$. Functions computed by synchronous (asynchronous) transducers are called DS-functions (respectively DA-functions). As is well known, both of these classes of functions are closed under composition, and every DS-function (DA-function) is a CS-function (resp., CA-function). We will use the following well-known deep fact established in [BL69]: for every regular set A, if $\forall\xi(\langle\xi, g(\xi)\rangle \in A)$ for some CS-function g on X^ω then $\forall\xi(\langle\xi, h(\xi)\rangle \in A)$ for some DS-function h on X^ω. For more information on automata on infinite words see e.g. [Th90,PP04].

We will study several reducibilities on subsets of X^ω and, more generally, on k-partitions of X^ω. A k-partition A is said to be *CA-reducible* to a k-partition B (in symbols $A \leq_{CA} B$), if $A = B \circ g$ for some CA-function $g : X^\omega \to X^\omega$. The relations $\leq_{DA}$, $\leq_{CS}$ and $\leq_{DS}$ on k^{X^ω} are defined in the same way but using the other three classes of functions. The introduced relations on k^{X^ω} are preorderings. The CA-reducibility is widely known in descriptive set theory as Wadge reducibility, and CS-reducibility as Lipschitz reducibility. The other two reducibilities are effective automatic versions of these. By $\equiv_{CA}$ we denote the induced equivalence relation which gives rise to the corresponding quotient partial ordering. Following a well established jargon, we call this ordering the structure of CA-degrees. For a set A and a class of sets $\mathcal{C}$, $\mathcal{C} \leq_{CA} A$ means that any set from $\mathcal{C}$ is CA-reducible to A. The same applies to the other reducibilities and to partitions in place of sets. Note that the class $\mathcal{R}_k$ of regular k-partitions is closed downwards under DA- and DS-reducibilities but is not closed under CA- and CS-reducibilities. Any level of the Borel hierarchy is closed under CA-reducibility (and thus under all four reducibilities). Every $\mathbf{\Sigma}$-level $\mathcal{C}$ (and also every $\mathbf{\Pi}$-level) of the Borel hierarchy has a CA-complete set C which means that $\mathcal{C} = \{A : A \leq_{CA} C\}$.

The operation $A \oplus B$ on k-partitions of X^ω, defined by $(A \oplus B)(0\xi) = A(\xi)$ and $(A \oplus B)(i\xi) = B(\xi)$ for all $0 < i < m$ and $\xi \in X^\omega$ (recall that $X = m$), induces the operation of supremum in the structures of degrees under all four reducibilities introduced above. The class of regular k-partitions is closed under $\oplus$.

We conclude this section with recalling some unary operations on k-forests and on k-partitions of X^ω introduced in [Se04,Se06]. For every k-forest F and every $i < k$, let $p_i(F)$ be the k-tree obtained from F by joining a new biggest element and assigning the label i to this element. It is clear that any k-forest is equivalent to a term of signature $\{\sqcup, p_0, \ldots, p_{k-1}, 0, \ldots, k-1\}$ without free variables (the constant symbol i in the signature is interpreted as the singleton tree carrying the label i). For all $i < k$ and k-partition A of X^ω, define a k-partition $p_i(A)$ by

$$[p_i(A)](\xi) = \begin{cases} i, & \text{if } \xi \not\in 0^*1X^\omega, \\ A(\eta), & \text{if } \xi = 0^n1\eta. \end{cases}$$

It is easy to see that the class of regular k-partitions is closed under $p_0, \ldots, p_{k-1}$.

In [Se04,Se06] it was shown that the structures $(\mathcal{F}_k; \leq, \sqcup, p_0, \ldots, p_{k-1})$, $(k^{X^\omega}; \leq_{CA}, \oplus, p_0, \ldots, p_{k-1})$ and $(\mathcal{R}_k; \leq_{CA}, \oplus, p_0, \ldots, p_{k-1})$ are dc-semilattices in terms of the following notion introduced in [Se82].

Definition 1. *By a semilattice with discrete closures (dc-semilattice for short) we mean a structure $(S; \leq, \sqcup, p_0, \ldots, p_{k-1})$ satisfying the following axioms:*

1) $(S; \leq, \cup)$ is an upper semilattice, i.e., for all $x, y \in S$, $x \cup y$ is a supremum of x, y in the preorder $(S; \leq)$.

2) Every p_i, $i < k$, is a closure operation on $(S; \leq)$, i.e. it satisfies $x \leq p_i(x)$, $x \leq y \to p_i(x) \leq p_i(y)$ and $p_i(p_i(x)) \leq p_i(x)$.

3) The operations p_i have the following discreteness property: for all distinct $i, j < k$, $p_i(x) \leq p_j(y) \rightarrow p_i(x) \leq y$.

4) Every $p_i(x)$ is join-irreducible, i.e. $p_i(x) \leq y \cup z \rightarrow (p_i(x) \leq y \vee p_i(x) \leq z)$.

3 Regular $B(\mathbf{\Sigma}^0_1)$-Partitions

In this section we extend to the case of partitions the following nice result due to L. Staiger and K. Wagner [SW74]: every regular $\mathbf{\Delta}^0_2$-set is a Boolean combination of open sets. We will generalize the following equivalent reformulation of this result: if a regular set L is not a Boolean combination of open sets then $\mathbf{\Sigma}^0_2 \leq_{CA} L$ or $\mathbf{\Pi}^0_2 \leq_{CA} L$.

For all distinct $i, j < k$, let $A_{i,j} : X^\omega \rightarrow k$ be the unique partition satisfying $A^{-1}(i) = B$, $A^{-1}(j) = \overline{B}$ and $A^{-1}(l) = \emptyset$ for all $l \in k \setminus \{i, j\}$ where B is a CA-complete set for $\mathbf{\Sigma}^0_2$. Observe that the partitions $A_{i,j}$ $(i, j < k, i \neq j)$ are pairwise CA-incomparable.

Let $(\mathcal{A}, c)$ be a Muller k-acceptor, $\leq_0$ the preorder on $C_\mathcal{A}$ from Section 2 and $\equiv_0$ the corresponding equivalence relation. We say that $\equiv_0$ respects the labeling c if $D \equiv_0 E$ implies $c(D) = c(E)$ for all $D, E \in C_\mathcal{A}$.

Theorem 1. *For any Muller k-acceptor $(\mathcal{A}, c)$ the following conditions are equivalent:*

(i) $L(\mathcal{A}, c) \notin (B(\mathbf{\Sigma}^0_1))_k$;

(ii) The relation $\equiv_0$ does not respect the labeling c;

(iii) $A_{l,j} \leq_{CA} L(\mathcal{A}, c)$ for some distinct $l, j < k$.

Proof. (i)$\rightarrow$(ii). We prove contraposition. Let $\equiv_0$ respect c. Then the quotient-structure $(\widetilde{C}_\mathcal{A}, \leq_0, \tilde{c})$, where $\widetilde{C}_\mathcal{A} = C_\mathcal{A}/\equiv_0$ and $\tilde{c}(\widetilde{D}) = \widetilde{c(D)}$ for all $D \in C_\mathcal{A}$, is well-defined. Then for any $l < k$ the set $\tilde{c}^{-1}(l)$ is a Boolean combination of sets closed upwards in $(\widetilde{C}_\mathcal{A}, \leq_0)$, and hence the set $c^{-1}(l)$ is a Boolean combination of sets closed upwards in $(C_\mathcal{A}, \leq_0)$ By a well-known result (see e.g. [Wag79]), for any $l < k$ the set $\{\xi \mid f_\mathcal{A}(\xi) \in c^{-1}(l)\}$ is in $B(\mathbf{\Sigma}^0_1)$. Therefore, $L(\mathcal{A}, c) \in B(\mathbf{\Sigma}^0_1)_k$ which is a contradiction.

(ii)$\rightarrow$(iii). Let $D, E \in C_\mathcal{A}$ satisfy $D \equiv_0 E$ and $c(D) \neq c(E)$. Then $F \supseteq D, E$ for some $F \in C_\mathcal{A}$ and $c(F)$ is distinct from at least one of $c(D), c(E)$. Let e.g. $c(F) \neq c(D)$. Since $F \leq_1 D$, by a well-known result in [Wag79] we get a continuous function g on X^ω such that $f_\mathcal{A}(g(\xi)) \in F$ for $\xi \notin B$ and $f_\mathcal{A}(g(\xi)) \in D$ for $\xi \in B$. Taking $l = c(D)$ and $j = c(F)$ we obtain $A_{l,j} \leq_{CA} L(\mathcal{A}, c)$.

(iii)$\rightarrow$(i). Let $A_{l,j} \leq_{CA} L(\mathcal{A}, c) = L$, then $B \leq_{CA} L^{-1}(l)$. Therefore, $L^{-1}(l) \notin B(\mathbf{\Sigma}^0_1)$ and hence $L \notin (B(\mathbf{\Sigma}^0_1))_k$. This completes the proof of the theorem.

Remark. The proof above gives for the case $k = 2$ a new proof of the Staiger-Wagner theorem.

From the equivalence of (i) and (ii) above we immediately obtain

Corollary 1. *The relation "$L(\mathcal{A}, c) \in B(\mathbf{\Sigma}^0_1)_k$" is decidable.*

4 Boolean Hierarchy of Regular $B(\mathbf{\Sigma}^0_1)$-Partitions

In this section we develop for partitions a full analog of the Boolean hierarchy of regular sets inside $B(\mathbf{\Sigma}^0_1)$. For this we need the Boolean hierarchy of partitions developed in [Ko00,KW00] for the case of NP-partitions and simplified in [Se04] for the case of partitions over reducible bases.

We recall some necessary definitions from [Ko00,KW00,Se04]. Let M be a set and $\mathcal{L} \subseteq P(M)$ a class of subsets closed under $\cup, \cap$ and containing $\emptyset, M$; for the sake of brevity, we call such a class $\mathcal{L}$ *a base*. A base $\mathcal{L}$ is *reducible* if for all $C_0, C_1 \in \mathcal{L}$ there are disjoint $C'_0, C'_1 \in \mathcal{L}$ such that $C'_i \subseteq C_i$ for both $i < 2$ and $C_0 \cup C_1 = C'_0 \cup C'_1$. As is well known [Ke94], each level $\mathbf{\Sigma}^0_n$ of the Borel hierarchy is reducible. In [Se98] it was shown that bases $\mathcal{L}_n = \mathcal{R} \cap \mathbf{\Sigma}^0_{n+1}$, $n = 0, 1$, are reducible as well.

Relate to any base $\mathcal{L}$ and any $F = (F, \leq, c) \in \mathcal{F}_k$ a class $\mathcal{L}(F)$ of k-partitions as follows. We call a map $S : F \to \mathcal{L}$ *admissible* if it is monotone, $\bigcup_x S_x = M$ and $S_x \cap S_y = \emptyset$ for all incomparable $x, y \in F$. For any $x \in F$, let $\tilde{S}_x = S_x \setminus \bigcup\{S_y | y < x\}$. The sets $\tilde{S}_x$ are pairwise disjoint and exhaust M. Hence, we can relate to any admissible $S : F \to \mathcal{L}$ the k-partition $A = A_S \in k^M$ by $A(s) = c(x)$ where x is the unique element of F with $s \in \tilde{S}_x$. Finally, let $\mathcal{L}(F)$ be the set of all A_S for all admissible $S : F \to \mathcal{L}$. The family $\{\mathcal{L}(F)\}_{F \in \mathcal{F}_k}$ is called the *Boolean hierarchy of k-partitions over* $\mathcal{L}$. Let also $BH_k(\mathcal{L}) = \{\mathcal{L}(F) \mid F \in \mathcal{F}_k\}$ be the class of levels of this hierarchy. Note that in [Ko00,KW00] levels of the Boolean hierarchy of partitions were considered for arbitrary k-posets (with a similar definition), and in [Se04] it was shown that for reducible bases the levels defined above using only k-forests are sufficient; this simplifies considerably the Boolean hierarchy of partitions over reducible bases. This is important for this paper because here we consider only the reducible bases mentioned in the preceding paragraph. As is well-known and easy to see, $\bigcup\{\mathcal{L}(F) \mid F \in \mathcal{F}_k\} = B(\mathcal{L})_k$.

In [Se04] we have proved the following result. For a further reference, we give a short proofsketch.

Theorem 2. *For any $k \geq 2$, the structures $(BH_k(\mathcal{L}_0); \subseteq)$ and $(BH_k(\mathbf{\Sigma}^0_1); \subseteq)$ are isomorphic to the quotient structure of $(\mathcal{F}_k; \leq)$.*

Proofsketch. By [Ko00,KW00,Se04], the map $F \mapsto \mathbf{\Sigma}^0_1(F)$ induces a monotone function from $(\mathcal{F}_k; \leq)$ onto $(BH_k(\mathbf{\Sigma}^0_1); \subseteq)$. It remains to show that $F \not\leq G$ implies $\mathbf{\Sigma}^0_1(F) \not\subseteq \mathbf{\Sigma}^0_1(G)$. According to the end of Section 2, F is equivalent to a term t of signature $\{\sqcup, p_0, \ldots, p_{k-1}, 0, \ldots, k-1\}$ without free variables. Let B_F be the value of t in the structure $(k^{X^\omega}; \oplus, p_0, \ldots, p_{k-1}, 0, \ldots, k-1)$ where the constant symbols i, $i < k$, are interpreted as the constant partitions $A_i(\xi) = i$. As shown in [Se04], B_F is a greatest element in $(\mathbf{\Sigma}^0_1(F); \leq_{CA})$, and $B_F \in \mathcal{L}_0(F)$. As shown in [Se06], $F \leq G$ iff $B_F \leq_{CA} B_G$. Hence, $B_F \in \mathbf{\Sigma}^0_1(F) \setminus \mathbf{\Sigma}^0_1(G)$ whenever $F \not\leq G$. A similar proof works for the structure $(BH_k(\mathcal{L}_0); \subseteq)$. This completes the proof.

Our next goal is to show that, given a Muller k-acceptor $(\mathcal{A}, c)$ and a k-forest F, we can effectively decide whether $L(\mathcal{A}, c) \in \mathbf{\Sigma}^0_1(F)$. This is a generalization of an important property of the Wagner hierarchy. First we prove a lemma.

Lemma 1. *Let $(\mathcal{A}, c)$ be a Muller k-acceptor, G a k-forest, and $G \leq (C_{\mathcal{A}}, \leq_0, c)$. Then $\mathbf{\Sigma}^0_1(G) \leq_{CA} L(\mathcal{A}, c)$.*

Proof. Let $h : G \to (C_{\mathcal{A}}, \leq_0, c)$ be a morphism of k-posets. Let $A \in \mathbf{\Sigma}^0_1(G)$, we have to show $A \leq_{CA} L(\mathcal{A}, c)$. Let $S : G \to \mathbf{\Sigma}^0_1$ be an admissible map that defines A as explained in the beginning of this section. Generalizing a well-known property of Muller acceptors [Wag79], it is straightforward to construct a continuous function g on X^ω such that $f_{\mathcal{A}}(\xi) = h(x)$ whenever $\xi \in \tilde{S}_x, x \in G$. Then g reduces A to $L(\mathcal{A}, c)$, completing the proof.

Theorem 3. *The relation "$L(\mathcal{A}, c) \in \mathbf{\Sigma}^0_1(F)$" is decidable.*

Proof. By Corollary 1, we can first decide whether $L(\mathcal{A}, c) \in B(\mathbf{\Sigma}^0_1)_k$. If not, then $L(\mathcal{A}, c) \notin \mathbf{\Sigma}^0_1(F)$. If yes, then, by Theorem 1, $\equiv_0$ respects the labeling c. Moreover, from the proof of that theorem and definition of the Boolean hierarchy of partitions it follows easily that $L(\mathcal{A}, c) \in \mathbf{\Sigma}^0_1(\widetilde{C}_{\mathcal{A}}, \leq_0, \tilde{c})$. By the proof of Theorem 3.1 in [Se04] we can effectively find a k-forest $G \leq \widetilde{C}_{\mathcal{A}}$ with $\mathbf{\Sigma}^0_1(G) = \mathbf{\Sigma}^0_1(\widetilde{C}_{\mathcal{A}})$. By Lemma 1, $L(\mathcal{A}, c)$ is actually CA-complete in $\mathbf{\Sigma}^0_1(G)$. By Theorem 2, $L(\mathcal{A}, c) \in \mathbf{\Sigma}^0_1(F)$ iff $G \leq F$. Since the relation $G \leq F$ is decidable, this completes the proof.

5 Reducibilities on Regular Partitions

In this section we consider the reducibilities from Section 2 on the ω-regular partitions. We start with a characterization of some structures of CA-degrees.

Theorem 4. *For any $k \geq 2$, the quotient-structures of $(B(\mathbf{\Sigma}^0_1)_k; \leq_{CA})$, $(B(\mathbf{\Sigma}^0_1)_k \cap \mathcal{R}_k; \leq_{CA})$, $(B(\mathcal{L}_0)_k; \leq_{CA}))$ and $(\mathcal{F}_k; \leq)$ are isomorphic.*

Proofsketch. By the proof of Theorem 2, the map $F \mapsto B_F$ induces an isomorphism from (the quotient-structure of) $(\mathcal{F}_k; \leq)$ into $(B(\mathcal{L}_0)_k; \leq_{CA})$. From the proof of Theorem 2.2.4 in [H96] it follows that each $C \in B(\mathbf{\Sigma}^0_1)_k$ is CA-equivalent to B_F for some $F \in \mathcal{F}_k$. Obviously, $B(\mathcal{L}_0)_k \subseteq B(\mathbf{\Sigma}^0_1)_k \cap \mathcal{R}_k \subseteq B(\mathbf{\Sigma}^0_1)_k$, completing the proof.

The next result generalizes the corresponding fact from [Wag79] and shows that the effective "automatic" reducibilities from Section 2 coincide with their non-effective counterparts on some classes of ω-regular partitions. Since the sets $B(\mathcal{L}_0)_k$ and $B(\mathbf{\Sigma}^0_1)_k$ are closed under operations $\oplus, p_0, \ldots, p_{k-1}$ introduced in Section 2, we have the following assertion which follow from results mentioned at the end of Section 2.

Lemma 2. *(i) For any $k \geq 2$, the sets $B(\mathcal{L}_0)_k$, $B(\mathbf{\Sigma}^0_1)_k \cap \mathcal{R}_k$ and $B(\mathbf{\Sigma}^0_1)_k$ are dc-semilattices under $\leq_{CA}, \oplus, p_0, \ldots, p_{k-1}$.*

Theorem 5. *(i) For any $k \geq 2$, the relations $\leq_{CS}$ and $\leq_{DS}$ coincide on $\mathcal{R}_k$.*

(ii) For any $k \geq 2$, the relations $\leq_{CA}$ and $\leq_{DA}$ coincide on $B(\mathbf{\Sigma}^0_1)_k \cap \mathcal{R}_k$.

Proof. (i) One implication is obvious. The other follows easily from the Büchi-Landweber theorem. Indeed, let $A, B \in \mathcal{R}_k$ and $A \leq_{CS} B$, i.e. $A = B \circ f$ for some synchronous continuous function f on X^ω. Then $\forall \xi(\langle \xi, f(\xi)\rangle \in C)$ where $C = \{\langle \xi, \eta\rangle \mid \forall i < k(\xi \in A^{-1}(i) \leftrightarrow \eta \in B^{-1}(i))\}$. Since C is regular, by Büchi-Landweber theorem there is a synchronous DS-function g on X^ω such that $\forall \xi(\langle \xi, g(\xi)\rangle \in C)$. Therefore, $A \leq_{DS} B$ via g.

(ii) Proof of this assertion is more complicated than the corresponding proof in [Se98] because for $k > 2$ there is no analog of the well-known "semilinear principle" stating that for all Borel sets A, B at least one of relations $A \leq_{CS} B$, $\overline{B} \leq_{CS} A$ holds true. But it is possible to find another argument. Namely, let $A \leq_{CA} B$ for some $A, B \in B(\mathbf{\Sigma}^0_1)_k \cap \mathcal{R}_k$. If $p_i(B) \equiv_{CA} B$ for some $i < k$, then, by Theorem 4 and Lemma 2, $B \equiv_{CA} B_T$ for some k-tree T (because, by [Se04], k-trees correspond exactly to join-irreducible elements of $(\mathcal{F}_k, \sqcup)$). By inspecting the proof of Theorem 2.2.4 in [H96] it is not hard to show that $A \leq_{CS} B$, hence $A \leq_{DS} B$ by (i) and therefore $A \leq_{DA} B$.

Now let $p_i(B) \not\equiv_{CA} B$. By Theorem 4, $B \equiv_{CA} B_0 \oplus \cdots \oplus B_n$ for some join-irreducible partitions $B_0, \ldots, B_n$ strictly below B. If A is join-irreducible then $A \leq_{CA} B_l$ for some $l \leq n$, and the assertion follows by induction on $rk(B)$. Otherwise, consider representation $A \equiv_{CA} A_0 \oplus \cdots \oplus A_m$ for some join-irreducible $A_0, \ldots, A_m$. Then $A_j \leq_{CA} B$ for all $j \leq m$. By the case just considered, $A_j \leq_{DA} B$ for all $j \leq m$. Therefore, $A \leq_{DA} B$, completing the proof of the theorem.

As an easy corollary of the last theorem, we obtain the following generalization of Theorem 6.9 in [Se98] (and Theorem 4.7 in [Ba92]) for some levels of the hierarchy of partitions.

Theorem 6. *(i) For all $k \geq 2$ and $F \in \mathcal{F}_k$, $F(\mathcal{L}_0) = F(\mathbf{\Sigma}^0_1) \cap \mathcal{R}_k$.*
(ii) For any $k \geq 2$, $B(\mathcal{L}_0)_k = B(\mathbf{\Sigma}^0_1)_k \cap \mathcal{R}_k$.

Proof. (i) The inclusion from left to right is obvious. Conversely, let $A \in F(\mathbf{\Sigma}^0_1)$ and $A \in \mathcal{R}_k$. Then $A \leq_{CA} B_F$ and, by the proof of Theorem 2, $B_F \in F(\mathcal{L}_0)$. By Theorem 5, $A \leq_{DA} B_F$. The class $F(\mathcal{L}_0)$ is clearly closed downwards under $\leq_{DA}$. Hence, $A \in F(\mathcal{L}_0)$. Since $\bigcup\{\mathcal{L}_0(F) \mid F \in \mathcal{F}_k\} = B(\mathcal{L}_0)_k$ and $\bigcup\{\mathbf{\Sigma}^0_1(F) \mid F \in \mathcal{F}_k\} = B(\mathbf{\Sigma}^0_1)_k$, the assertion (ii) follows from (i). This completes the proof.

So far, our results for ω-regular partitions generalized the corresponding results for ω-regular languages. Now we present a result that has completely different formulations for ω-regular languages and for ω-regular k-partitions for $k > 2$. Recall that *first-order theory* $FO(A)$ of a structure A of signature σ is the set of first-order sentences of signature σ which are true in A.

Consider the question whether the first-order theory $FO(\mathcal{R}_k; \leq_{CA})$ of the quotient-structure of $(\mathcal{R}_k; \leq_{CA})$ is decidable. The question may be also asked for reasonable substructures of this structure. For $k = 2$ the theory $FO(\mathcal{R}_k; \leq_{CA})$ (as well as the theory $FO(B(\mathcal{L}_0)_k; \leq_{CA})$) is decidable because, by a main fact in [Wag79], the quotient-structures of these preorderings are almost well-ordered, and the first-order theory of any ordinal is known to be decidable. It turns out that for $k > 2$ the situation is quite different.

Theorem 7. *(i) For any $k \geq 3$, the theory $FO(B(\mathcal{L}_0)_k; \leq_{CA})$ is undecidable and, moreover, is computably isomorphic to the first-order arithmetic $FO(\omega; +, \cdot)$.*

(ii) For any $k \geq 3$, $FO(\mathcal{R}_k; \leq_{CA})$ is undecidable and, moreover, $FO(\omega; +, \cdot)$ is m-reducible to $FO(\mathcal{R}_k; \leq_{CA})$.

Proof. (i) follows from Theorem 4 and the corresponding fact about the theory $FO(\mathcal{F}_k; \leq_{CA})$ established in [KS06].

(ii) By (i) and Theorem 4, it suffices to m-reduce $FO(B(\mathbf{\Sigma}^0_1)_k \cap \mathcal{R}_k; \leq_{CA})$ to $FO(\mathcal{R}_k; \leq_{CA})$. For this it suffices to show that the set $B(\mathbf{\Sigma}^0_1)_k \cap \mathcal{R}_k$ is first-order definable in $(\mathcal{R}_k; \leq_{CA})$. Let $\lambda(u)$ be the formula $\exists y(y < u) \wedge \forall y < u \exists z(y < z < u)$ stating that u is not minimal and has no immediate predecessor. Let $\mu(u)$ be the formula $\lambda(u) \wedge \forall v < u \neg\lambda(u)$ stating that u is minimal among the non-minimal elements having no immediate predecessor. Finally, let $p = k(k-1)$, $u_1, \ldots, u_p$ be different variables, and $\phi(x)$ be the formula

$$\exists u_1 \cdots \exists u_p ((\bigwedge_{i \neq j} u_i \neq u_j) \wedge (\bigwedge_i (\mu(u_i) \wedge u_i \not\leq x))).$$

By Theorem 1, $\phi(x)$ defines $B(\mathbf{\Sigma}^0_1)_k \cap \mathcal{R}_k$ in $(\mathcal{R}_k; \leq_{CA})$. This completes the proof.

6 Conclusion

We have shown that several facts about the Wagner hierarchy of regular ω-languages may be extended to the case of ω-regular partitions. We expect that actually the full theory around the Wagner hierarchy may be extended similarly, though this of course needs some additional technical work.

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