All Simple Drawings of $K_{m,n}$ Contain Shooting Stars

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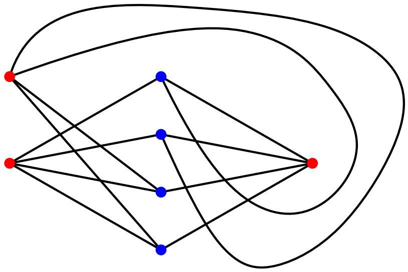
Simple Drawings

Simple drawings (also called *good drawings* or *simple topological graphs*) are drawings of graphs on the sphere or in the Euclidean plane where:

- Vertices drawn as distinct points; edges drawn as arcs.



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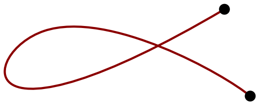

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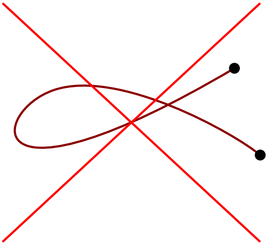

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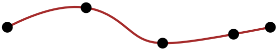

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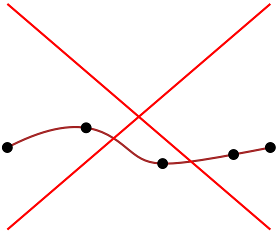

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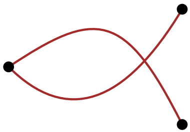

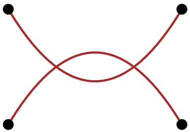

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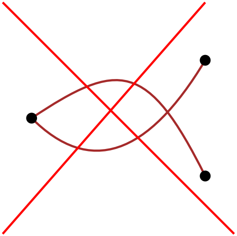

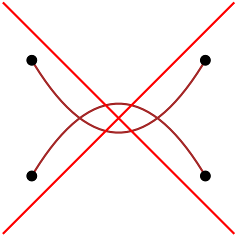

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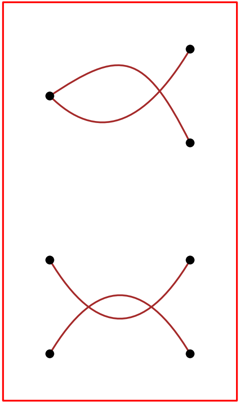

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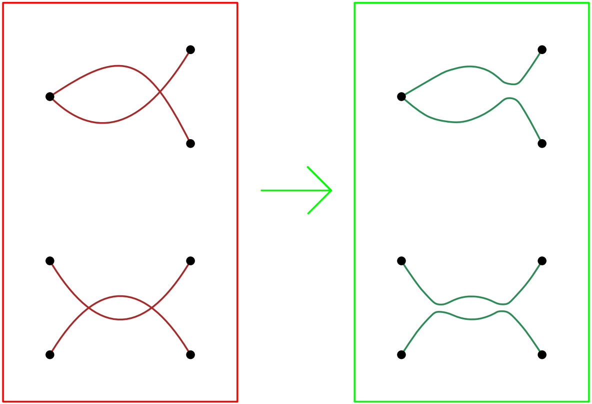
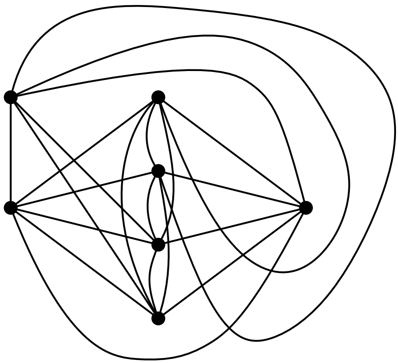


Question

Do all simple drawings of $K_{m,n}$ contain plane spanning trees?

Complete graphs (e.g. K_7)



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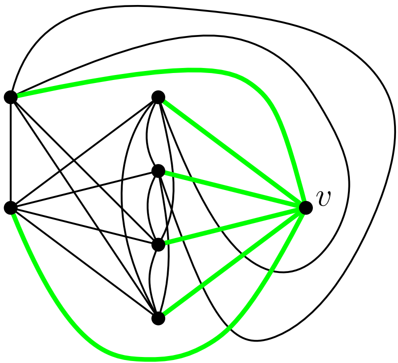
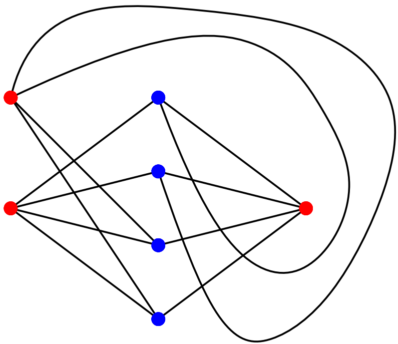


Definition

Star of v ... all edges incident to v.

Complete Bipartite Graphs (e.g. $K_{3,4}$)



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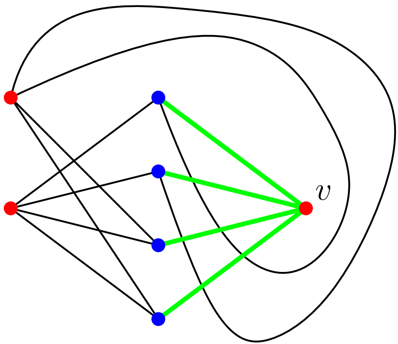

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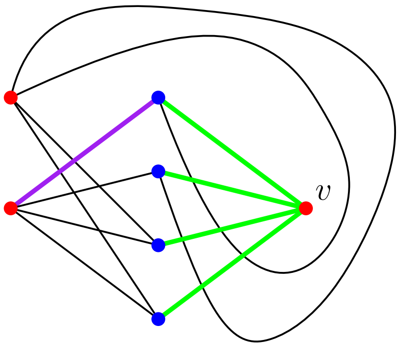

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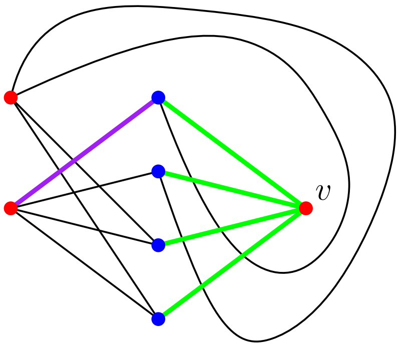

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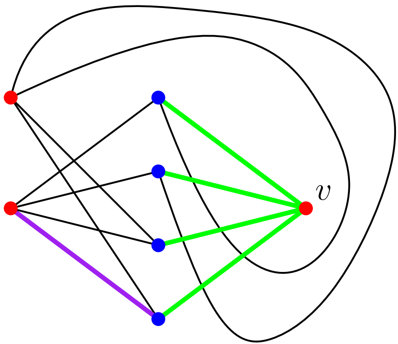

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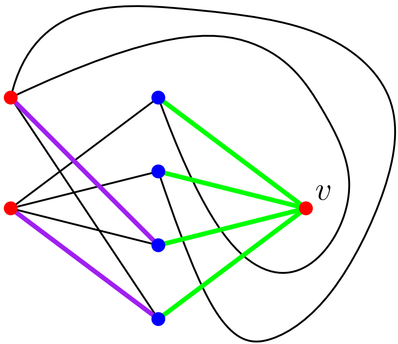

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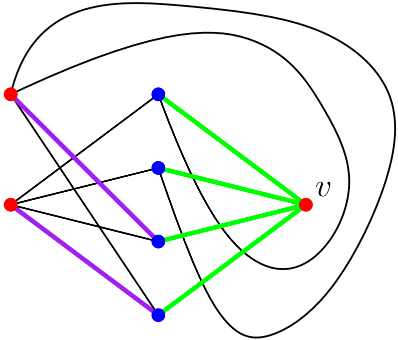

Definition

A *shooting star rooted at* v is a plane spanning tree that contains the star of vertex v, i.e., all edges incident to v.

Question

Do all simple drawings of $K_{m,n}$ contain plane spanning trees?

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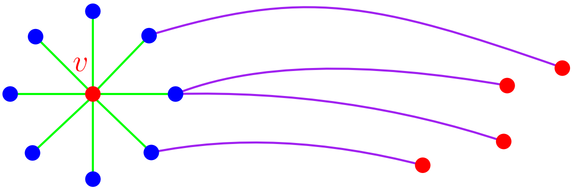
Theorem

Let D be a simple drawing of $K_{m,n}$ and v be a vertex of D. Then D contains a shooting star rooted at v.



Sketch of Proof

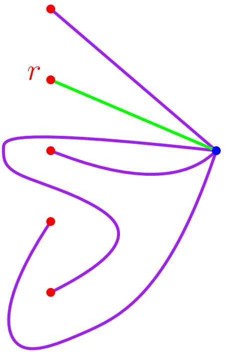
Let D be a simple drawing of $K_{m,n}$, with sides of the bipartiton R and B;

Show: There is a shooting star rooted at $r \in R$;

Induction on $|B| = n$;

Sketch of Proof

Induction Base $n = 1$:



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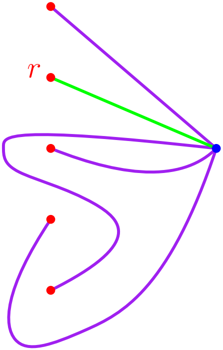
<u>Induction Base</u> $n = 1$:



<u>Assumption:</u>
Every simple drawing of $K_{m,n'}$ with $n' < n$ contains a shooting stars and the root can be arbitrarily chosen (we will choose r).

Sketch of Proof

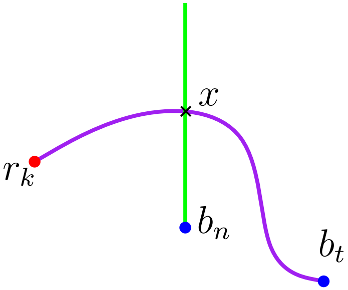
Induction Step $(n-1) \to n$

Stereographic projection from r;

Sketch of Proof

Induction Step $(n-1) \to n$

Stereographic projection from r;
First crossing;



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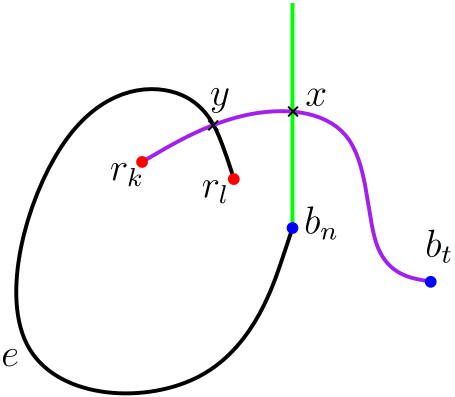

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Induction Step $(n-1) \rightarrow n$

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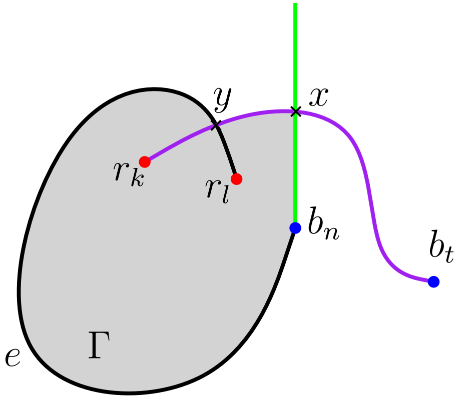

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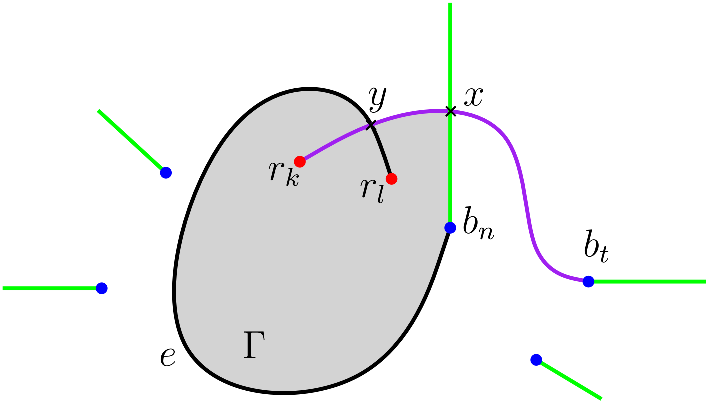

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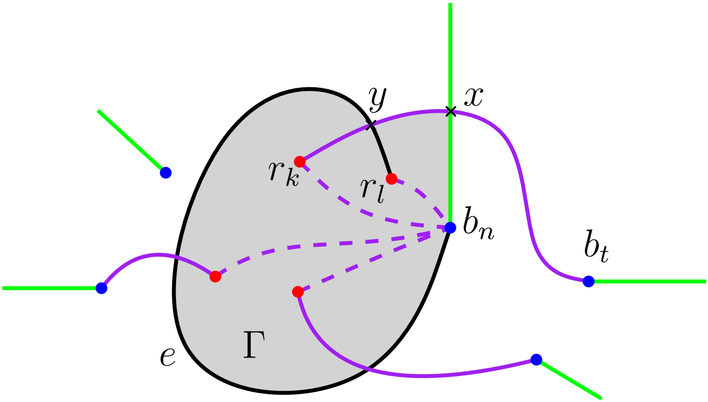

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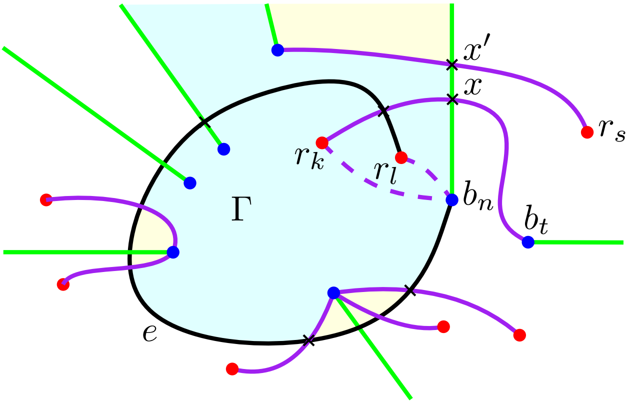

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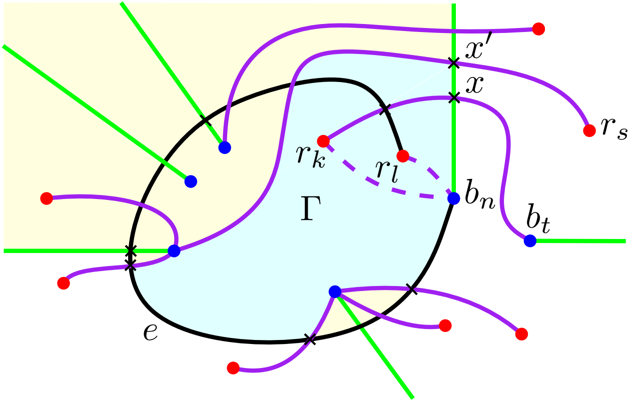

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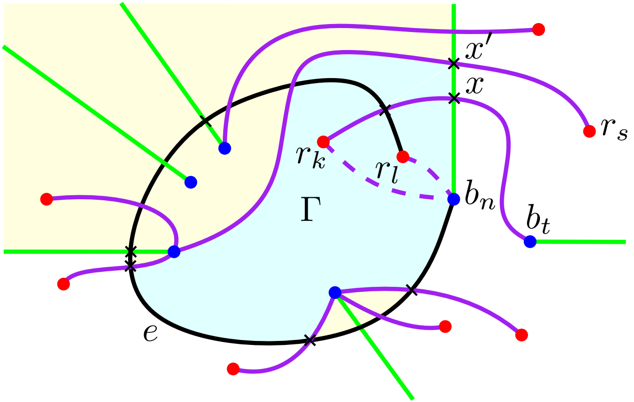

□

Tightness

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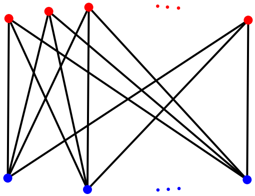

In some simple drawings of $K_{m,n}$ any plane structure has at most as many edges as a shooting star.

Requirements

Theorem

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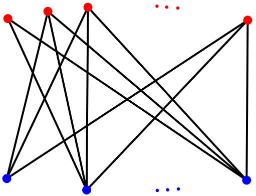

Some simple drawings of $K_{m,n}$ minus one edge do not contain plane spanning trees.

Requirements

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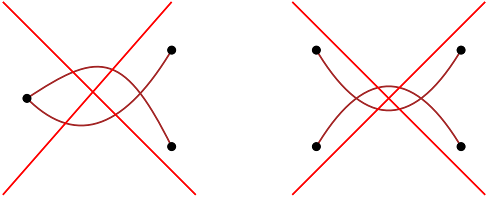
- Any pair of edges intersects at most once.



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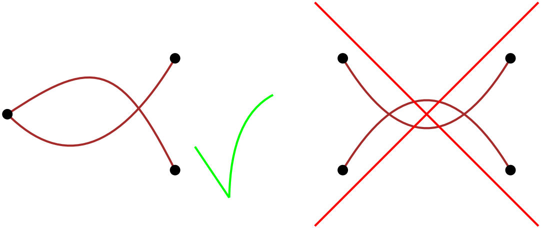

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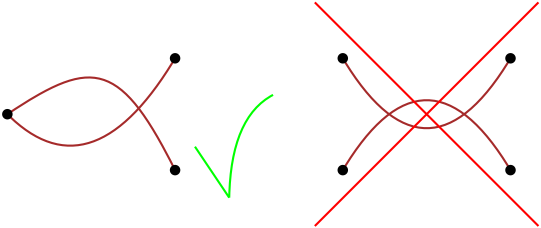

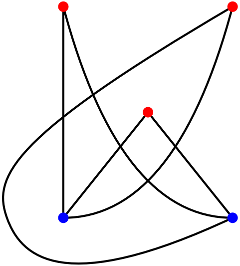

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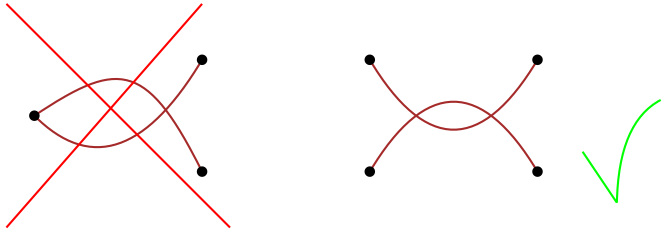

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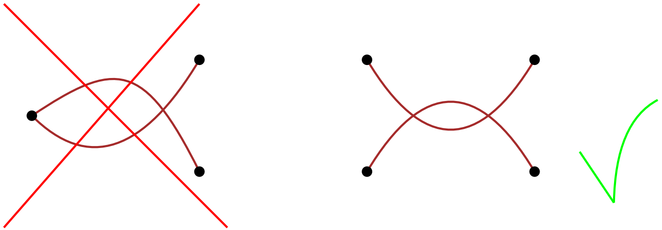

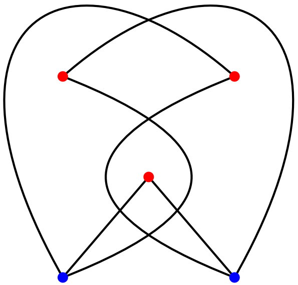

Summary

- All simple drawings of $K_{m,n}$ contain plain spanning trees, even shooting stars rooted at arbitrary v.
- Requirements are necessary.
  Maximum plane subdrawings can be trees.