Between Two Shapes, Using the Hausdorff Distance

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Directed Hausdorff distance $A \rightarrow B$. 
Directed Hausdorff distance $A \rightarrow B$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig.png}
\caption{Directed Hausdorff distance between sets $A$ and $B$.}
\end{figure}
Directed Hausdorff distance $A \rightarrow B$. 
Directed Hausdorff distance $B \rightarrow A$. 

![Diagram](image-url)
Directed Hausdorff distance $B \rightarrow A$. 

![Diagram showing Directed Hausdorff distance](image-url)
Directed Hausdorff distance $B \rightarrow A$. 
Undirected Hausdorff distance $A \leftrightarrow B$. 
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$d_H = 1$
Directed Hausdorff distance $B \rightarrow A$. 
Directed Hausdorff distance $A \rightarrow B$. 
Find $S$ with minimal Hausdorff distance to $A$ and $B$. 

A \hspace{2cm} S \hspace{2cm} B
Find $S$ with minimal Hausdorff distance to $A$ and $B$.

Result: distance $1/2$ is always possible.
$d_H = 1$
\[ A \oplus D_{1/2} \]
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\[ B \oplus D_{1/2} \]

\[ S_{1/2} \]
$A \oplus D_{3/4}$

$B \oplus D_{1/4}$

$S_{3/4}$
Claim:
- $d_H(A, S) = \alpha$
- $d_H(B, S) = 1 - \alpha$
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To show:
- $S \subseteq A \oplus D_\alpha$
- $S \subseteq B \oplus D_{1-\alpha}$
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- $S \subseteq A \oplus D_{\alpha}$
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- $A \subseteq S \oplus D_{\alpha}$
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Lemma: \( S_\alpha = (A \oplus D_\alpha) \cap (B \oplus D_{1-\alpha}) \) has \( d_H(A, S_\alpha) = \alpha \) and \( d_H(B, S_\alpha) = 1 - \alpha \).
Lemma: $S_\alpha = (A \oplus D_\alpha) \cap (B \oplus D_{1-\alpha})$ has $d_H(A, S_\alpha) = \alpha$ and $d_H(B, S_\alpha) = 1 - \alpha$. 

- $S_\alpha \subseteq A \oplus D_\alpha$
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- $S_\alpha \subseteq A \oplus D_\alpha$ ✓
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- $S_\alpha \subseteq A \oplus D_\alpha$  ✓
- $A \subseteq S_\alpha \oplus D_\alpha$
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- $S_\alpha \subseteq A \oplus D_\alpha$  ✓
- $A \subseteq S_\alpha \oplus D_\alpha$  ✓
convex
convex
$O(n + m)$ complexity
non-convex

convex
non-convex connected \( O(n + m) \) complexity convex
non-convex

possibly disconnected

\( O(nm) \) complexity
maximal $S$
minimal $S$
$A \oplus D_{1/2}$

$B \oplus D_{1/2}$

$S_{1/2}$
\[ A \oplus D_{3/4} \]

\[ A \oplus D_{1/4} \]

\[ S_{3/4} \]

\[ B \oplus D_{1/4} \]
\[ B \oplus D_{7/8} \]

\[ B \oplus D_{1/8} \]
Conclusion:
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- We can compute a shape with $d_H(A, S) = \alpha$ and $d_H(B, S) = 1 - \alpha$
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- The morph obtained by varying $\alpha$ is 1-Lipschitz continuous
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Future work:

- Extend to more than two input sets
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  - Minimum required $\alpha$ may be 1
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Future work:

- Extend to more than two input sets
  - Minimum required $\alpha$ may be 1
- Not yet clear how to do morphing between three shapes
Input sets \( \{A_1, \ldots, A_k\} \)
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Let $S_\alpha = \bigcap_i A_i \oplus D_\alpha$
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