On Minimal-Perimeter Latice Animals

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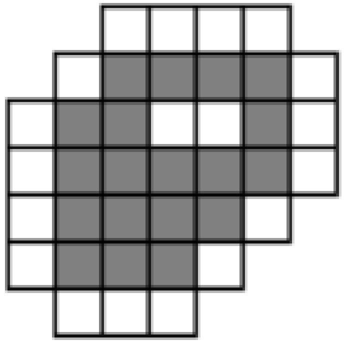
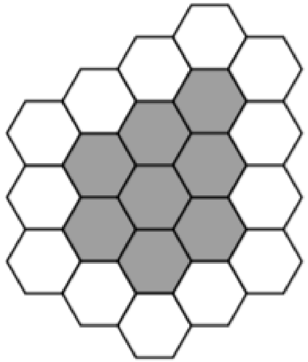
EuroCG 2020, Würzburg, Germany

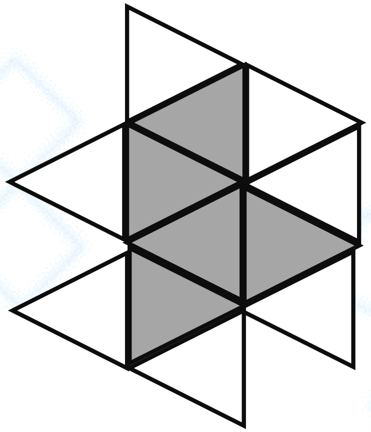
What is a Lattice Animal?



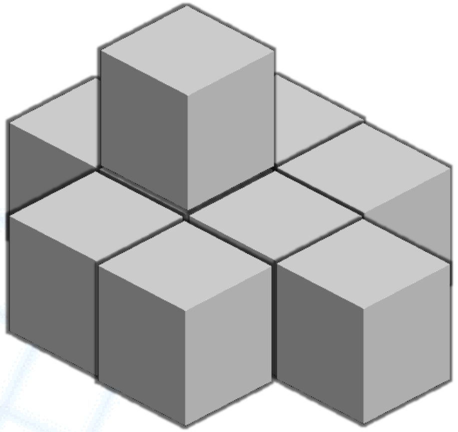
Polyominoes



Polyhexes



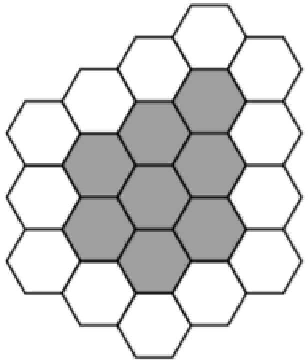
Polyiamonds

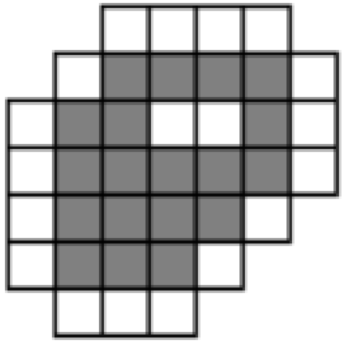


Polycubes

Definitions

Term	Definition	Notation
Lattice Animal	A set of connected cells on some lattice	Q





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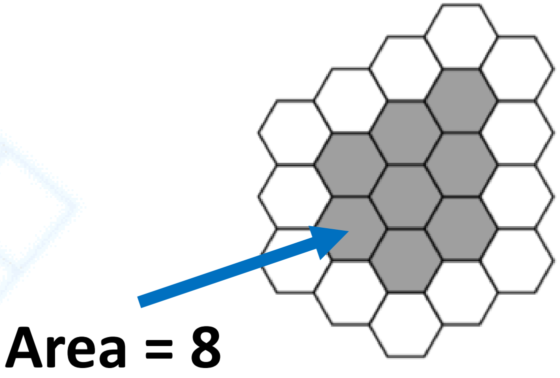
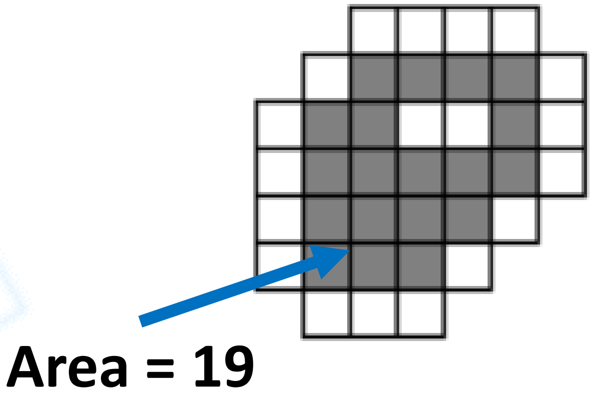
Lattice Animal	A set of connected cells on some lattice	Q
Area	The number of cells	$\|Q\|$





Definitions

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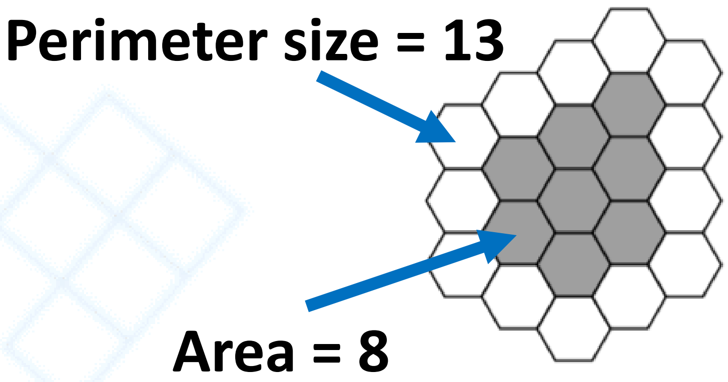
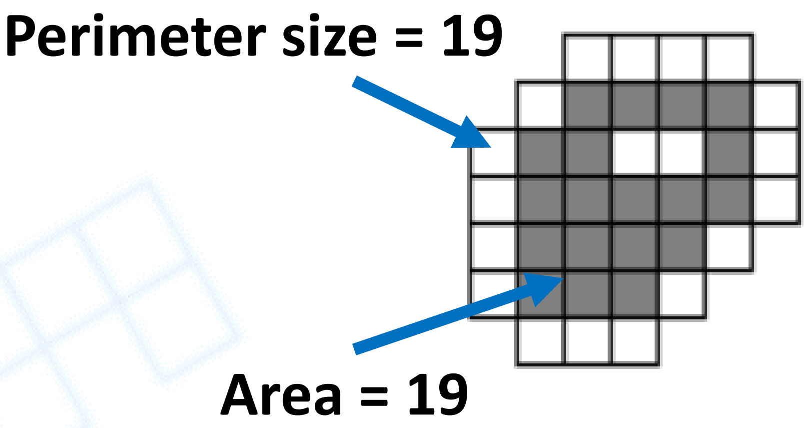
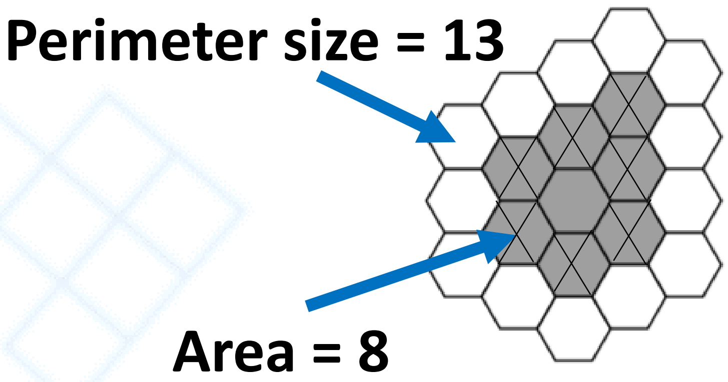
Lattice Animal	A set of connected cells on some lattice	Q
Area	The number of cells	$\|Q\|$
Perimeter	Empty adjacent cells	$\mathcal{P}(Q)$

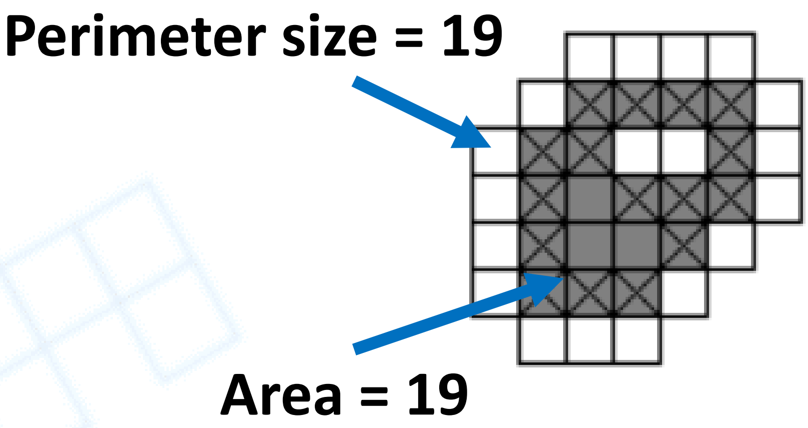




Definitions

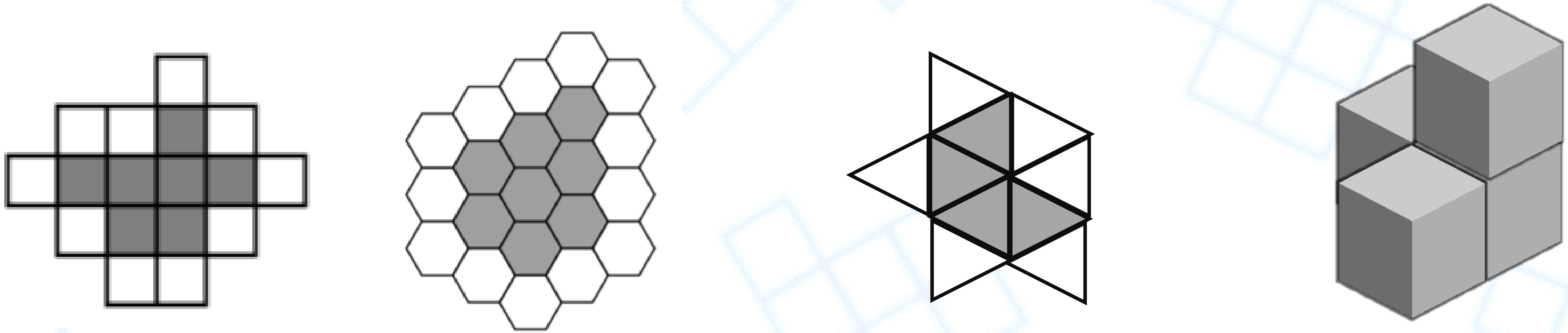
Term	Definition	Notation
Lattice Animal	A set of connected cells on some lattice	Q
Area	The number of cells	$\|Q\|$
Perimeter	Empty adjacent cells	$\mathcal{P}(Q)$
Border	Lattice animal cells with empty adjacent cells	$\mathcal{B}(Q)$



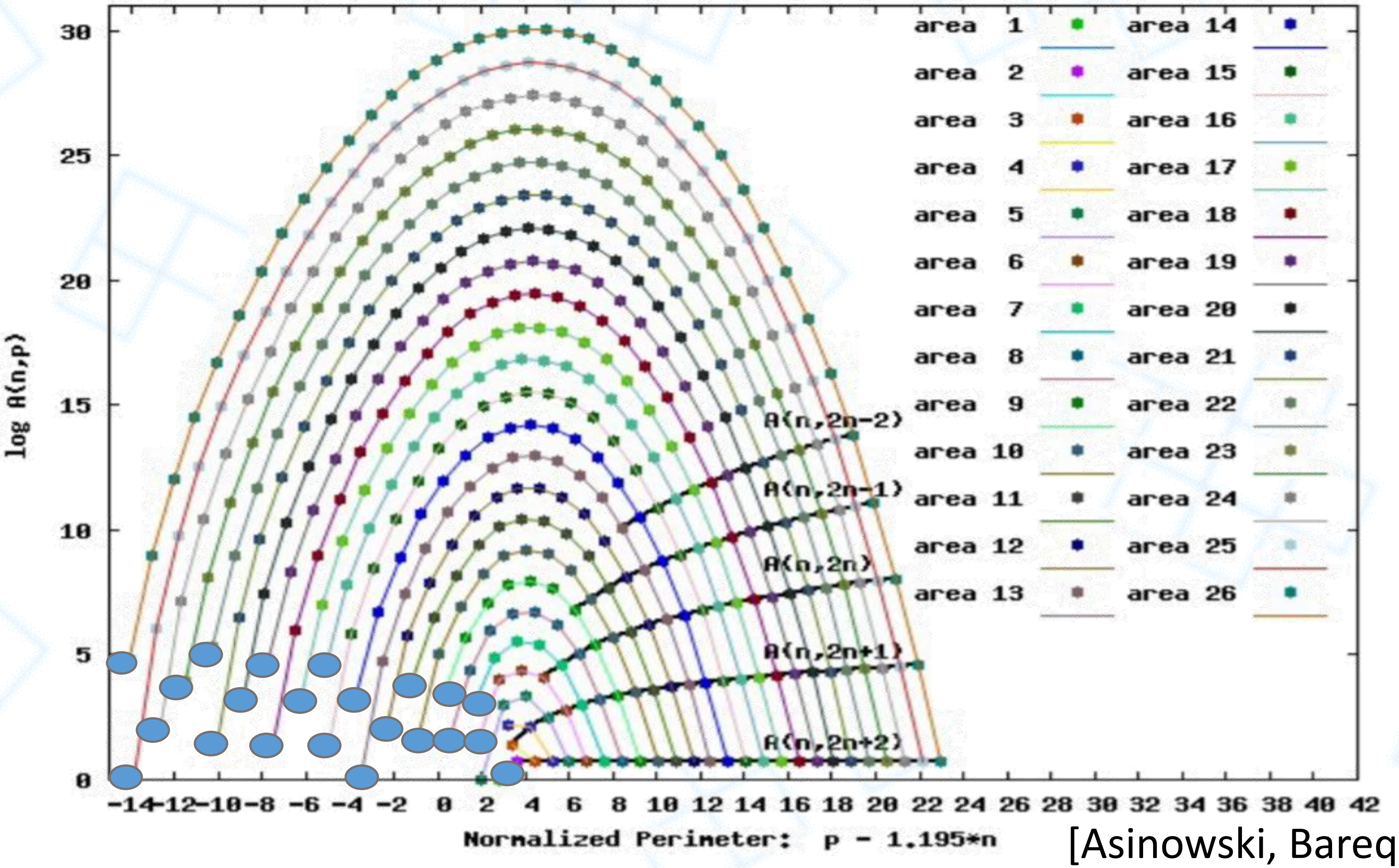


Minimal-Perimeter Lattice Animals

- **Definition:** a minimal-perimeter lattice animal (MPA) is a lattice animal which have the minimum possible perimeter from within all lattice animals of the same size.
- **Examples:**

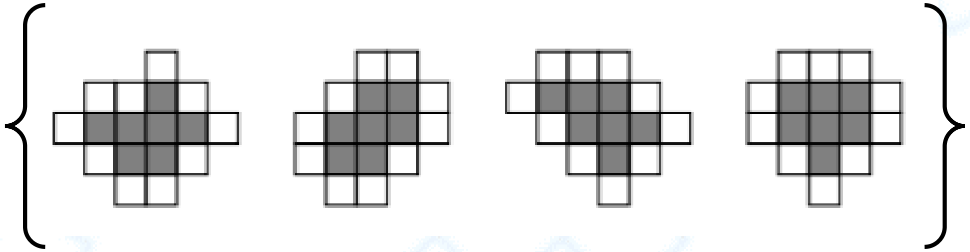


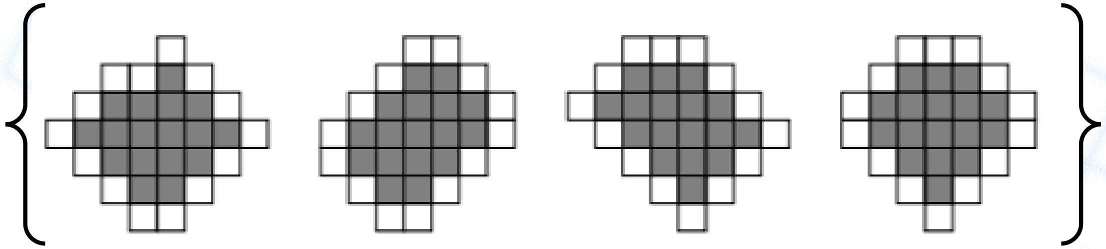
Motivation

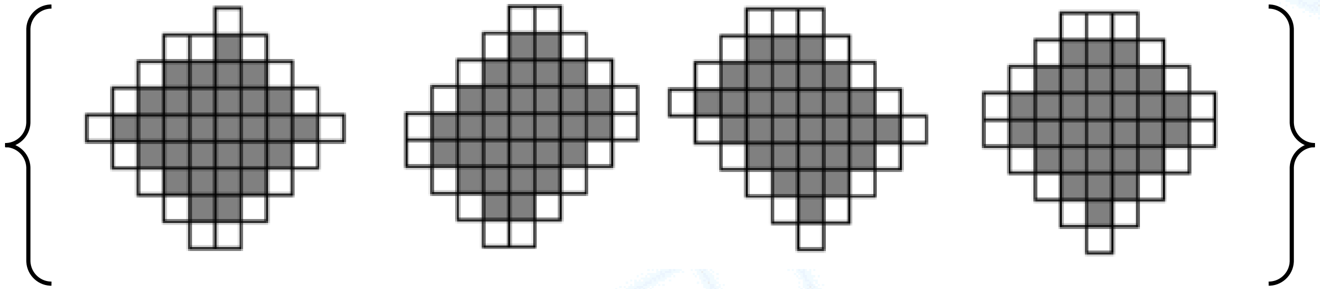


[Asinowski, Barequet and Zheng. 2017]

Minimal-Perimeter Lattice Animals

$M_7 =$


$M_{17} =$


$M_{31} =$


Inflation of Polyominoes

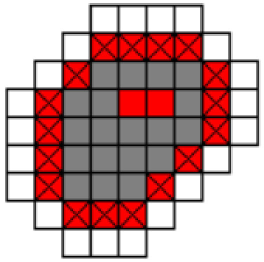
Inflation of a polyomino Q, $I(Q)$, is

$$I(Q) = Q \cup \mathcal{P}(Q)$$

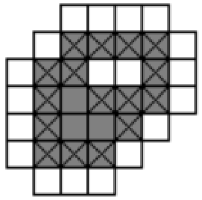
The deflated polyomino $\mathrm{D}(Q)$ is

$$D(Q) = Q \backslash \mathcal{B}(Q)$$

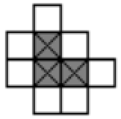
$I(Q)$



Q

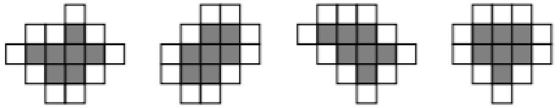


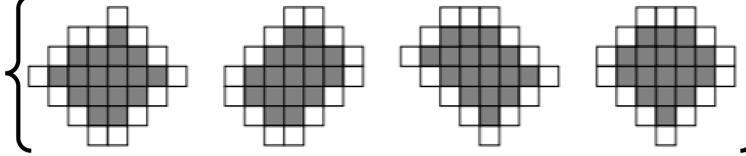
$D(Q)$

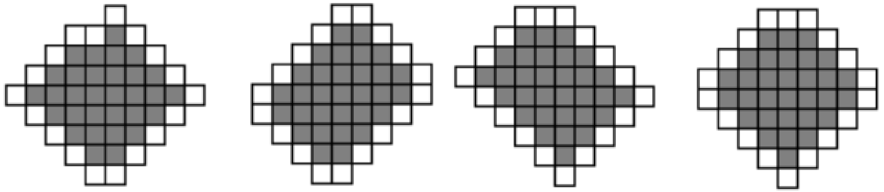


Inflation of Polyominoes

- Theorem: For $n \geq 3$ and any $k \in N$, $|M_n| = |M_{n+k\epsilon(n)+2k(k-1)}|$

$M_7 = \{$  $\}$

$M_{17} = \{$  $\}$

$M_{31} = \{$  $\}$

$|M_{2477537}| = 4$

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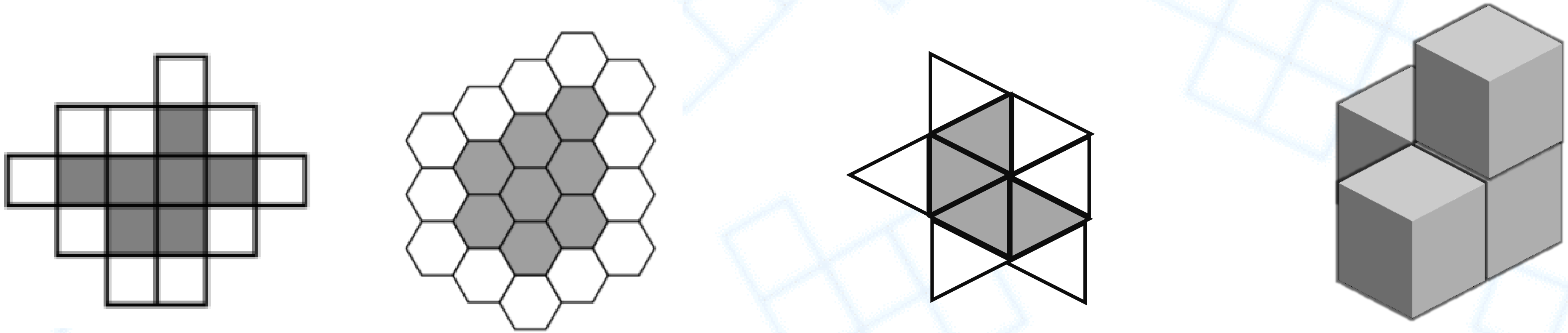
- **Examples:**



Genralization to Lattice Animals

- Does inflation induce a bijection in other lattices?
- The following set of conditions are sufficient:

1) The minimal perimeter size is monotonically increasing (w.r.t the area)
2) $|\mathcal{P}(Q)| = |\mathcal{B}(Q)| + c$ for some c ← **Heaviest requirements**
3) Deflation of a MPA creates a valid lattice animal

Proof structure

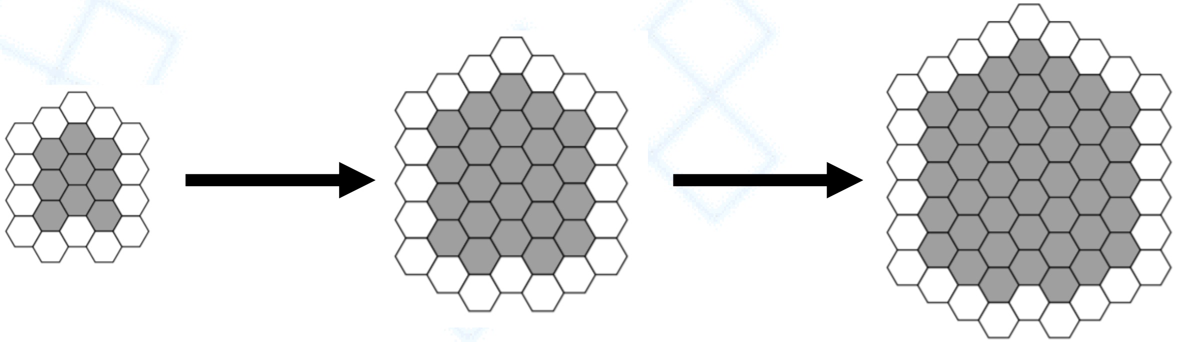
- The idea is to show a bijection between sets of MPAs.
- First direction: Inflation of an MPA creates a new (unique) MPA.
- Second direction: If one MPA of area n is created by an inflation, then any MPA of area n can be deflated to a smaller MPA.

Proof: First direction

- **Theorem:** For a minimal-perimeter animal $Q, I(Q)$ is a minimal-perimeter animal as well.
- <u>Proof idea</u>
  - Assume $I(Q)$ is not minimal-perimeter animal.
  - $\exists Q'$ s.t. $|Q'| = I(Q)$ and $|\mathcal{P}(Q')| < |\mathcal{P}(I(Q))|$
  - After some calculations (using condition #2) – $|D(Q')| > |Q|$, and $|\mathcal{P}(Q')| < |\mathcal{P}(Q)|$
  - Contradicts condition #1.
  - $\Rightarrow Q$ is not a minimal-perimeter animal.

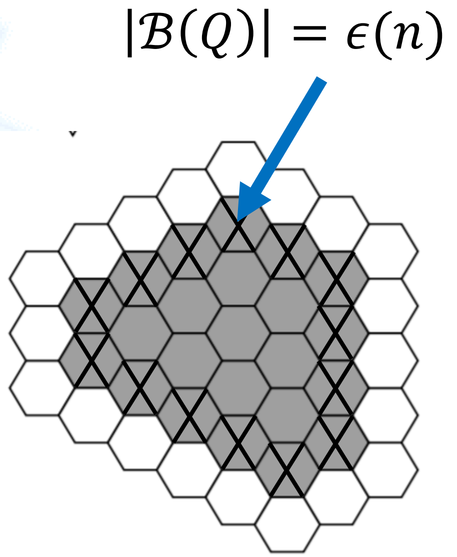
First direction: Corollary

- Inflation of MPAs creates an infinite chain of new MPAs.



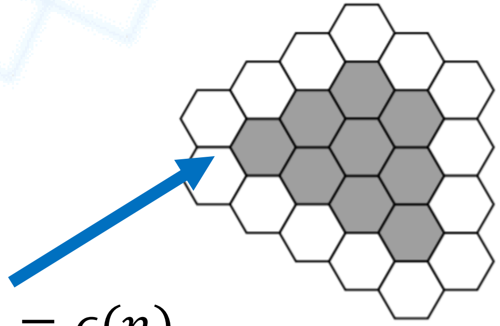
Second direction

- Lemma: If $Q \in M_{n+\epsilon(n)}$ then $D(Q) \in M_n$
- Proof:
- Let Q $\in M_{n+\epsilon(n)}$
- $|\mathcal{B}(Q)| = \epsilon(n)$, thus $|D(Q)| = n$.



Second direction

- Lemma: If $Q \in M_{n+\epsilon(n)}$ then $D(Q) \in M_n$
- Proof:
- Let $Q \in M_{n+\epsilon(n)}$
- $|\mathcal{B}(Q)| = \epsilon(n)$, thus $|D(Q)| = n$.
- $|\mathcal{P}(D(Q))| \geq \epsilon(n)$ and $\mathcal{P}(D(Q)) \subseteq \mathcal{B}(Q)$
- $\mathcal{P}(\mathrm{D(Q)}) = \mathcal{B}(Q) \Rightarrow I(D(Q)) = Q.$
- $\Rightarrow |M_n| \geq |M_{n+\epsilon(n)}|$



$|\mathcal{P}(Q)| = \epsilon(n)$

Genralization to Lattice Animals

- Does inflation induce a this bijection in other lattices?
- The following set of conditions are sufficient:

1) The minimal perimeter size is monotonically increasing
2) $|\mathcal{P}(Q)| = |\mathcal{B}(Q)| + c$ for some c ← **Heaviest requirements**
3) Deflation of a MPA creates a valid lattice animal

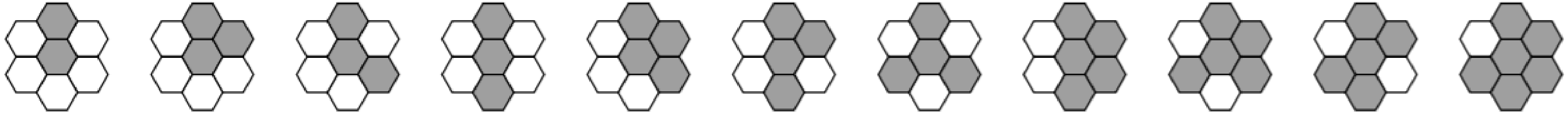
Proof for polyhexes

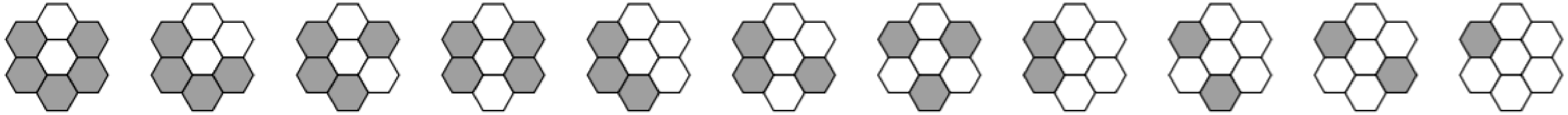
1) The minimal perimeter size is monotonically increasing
   Known [Vainsencher and Bruckstein, 2008]
2) $|\mathcal{P}(Q)| = |\mathcal{B}(Q)| + c$ for some c
3) Deflation of a minimal-perimeter polyhex creates a valid polyhex

Easy to see…

Proof for polyhexes

- How to prove that $\mathcal{P}(Q) = \mathcal{B}(Q) + c$?
- Classify each cell or perimeter cell to one of the following patterns:

Regular cells: 

Perimeter cells: 

- Show that: $\mathcal{P}(\mathrm{Q}) = \mathcal{B}(Q) + 3 \cdot \#$  $+ 2 \cdot \#$  $+ \#$ 

 $3 \cdot \#$  $2 \cdot \#$  $- \#$

Proof for polyhexes

- How to prove that $\mathcal{P}(Q) = \mathcal{B}(Q) + c$?
- Show that: $\mathcal{P}(\mathrm{Q}) = \mathcal{B}(Q) + 3 \cdot \#$  $+ 2 \cdot \#$  $+ \#$ 

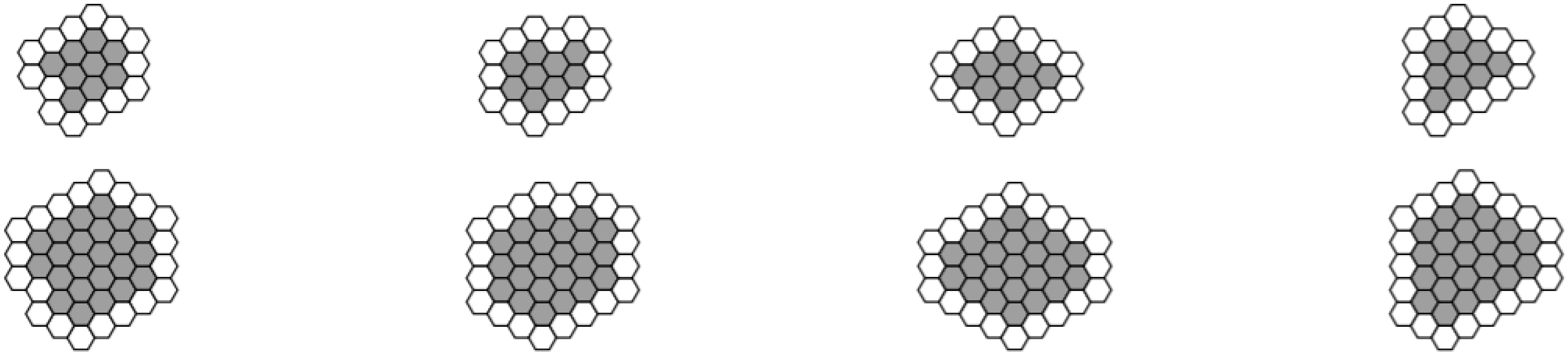
   $3 \cdot \#$  $2 \cdot \#$  $- \#$

- Use some calculations to get:

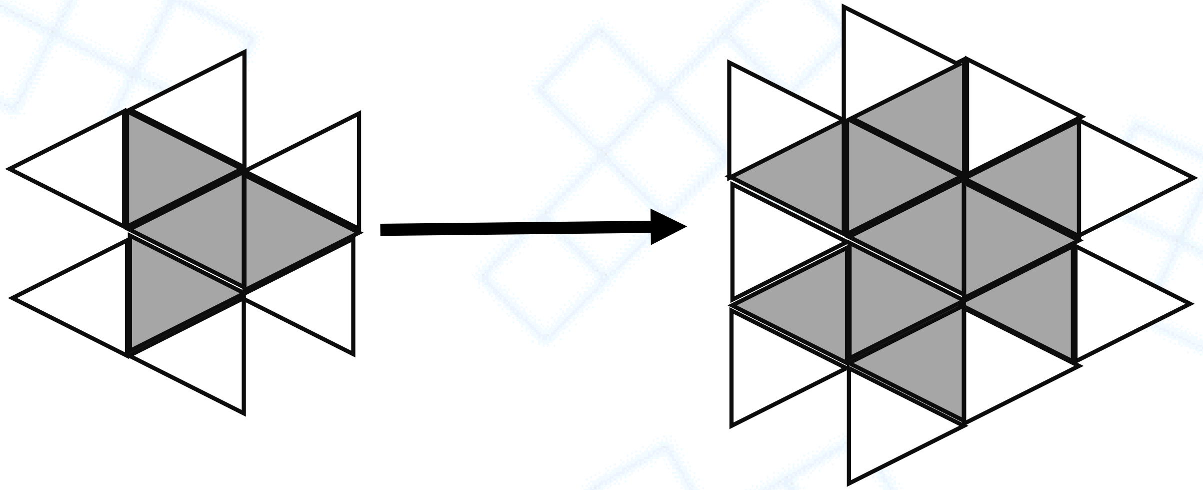
$$\mathcal{P}(Q) = \mathcal{B}(Q) + 6$$

- (For polyominoes it is $\mathcal{P}(Q) = \mathcal{B}(Q) + 4$)

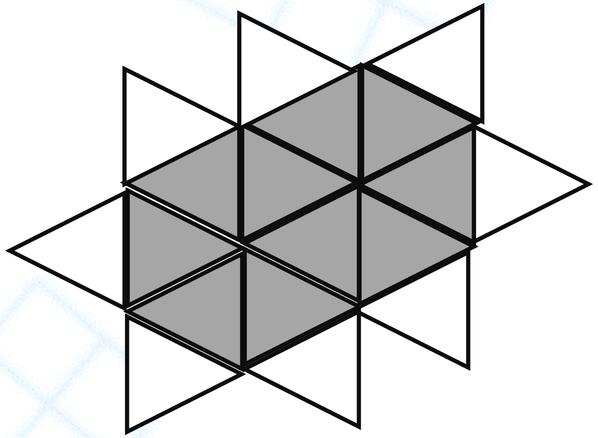
Proof for polyhexes



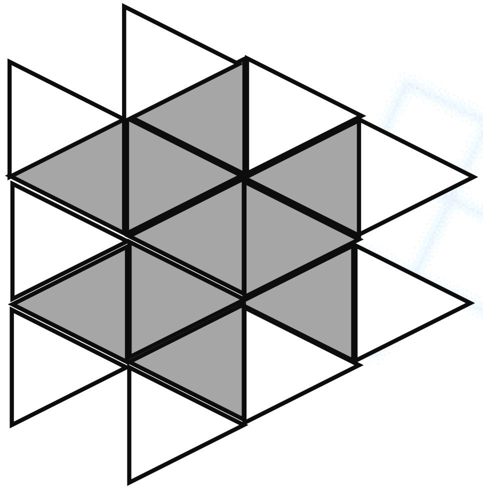
Polyiamonds



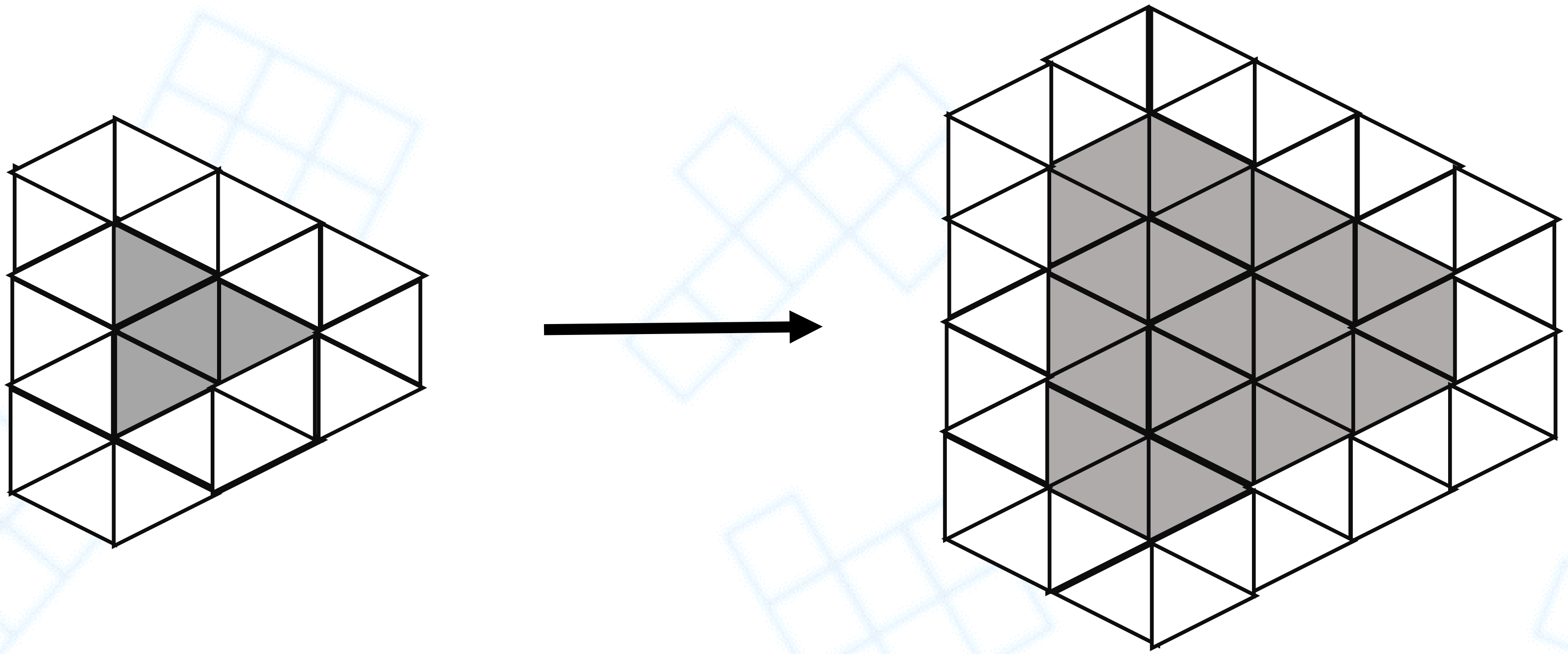
Polyiamonds



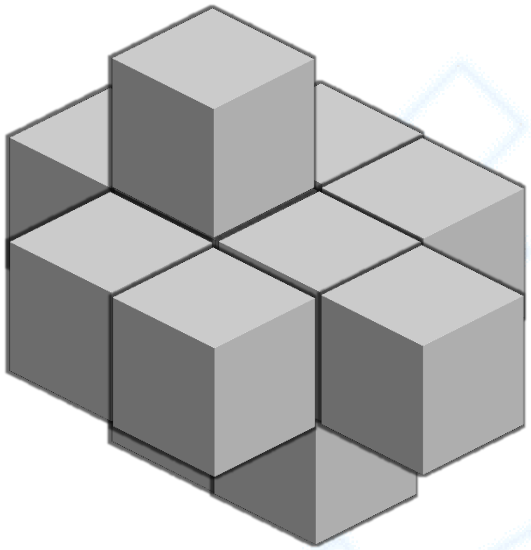
vs.



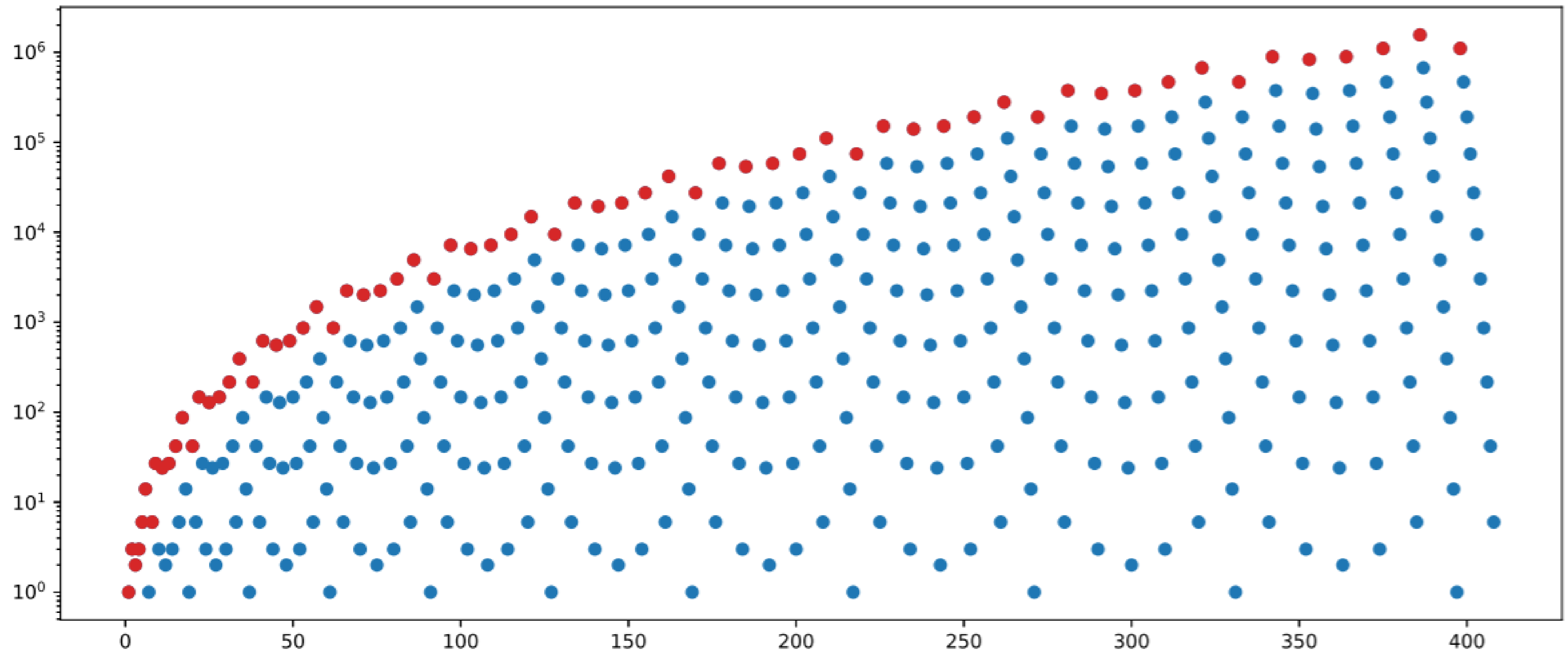
Polyiamonds



Polycubes

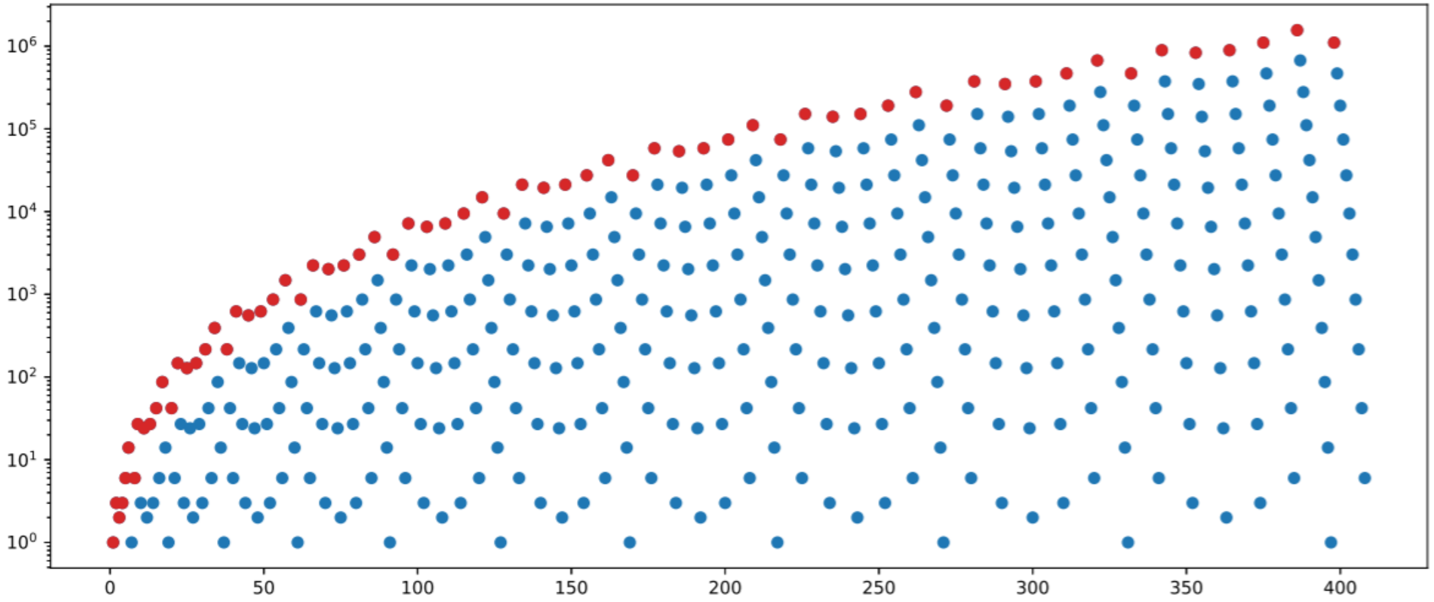


Counting Minimal-Perimeter Polyhexes



Questions

- Is there a bijection between sets of minimal-perimeter polycubes?
- Are all the conditions are necessary?



Thank you!