Balanced Independent Sets on Colored Interval Graphs

Sujoy Bhore, Jan-Henrik Haunert, Fabian Klute, Guangping Li, Martin Nöllenburg
Boundary labeling

- labels are at the boundary of the focus region
- a leader connects a label with its corresponding POI
- task: select a large conflict-free labeling

Algorithms for labeling focus regions.
Boundary labeling

- labels represent objects of multiple categories
- task: select a good mixture of different object types
Model

input:

- a set of $n$ colored axis-parallel unit squares touching a disk $D$
- rectangle: icon
Model

input:

- a set of $n$ colored axis-parallel unit squares touching a disk $D$
- rectangle: icon

interval representation of its intersection model
Model

input:

- a set $I$ of $n$ intervals on the real line
- each interval is colored by a coloring $c: I \rightarrow \{1, \ldots, k\}$
Model

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- a set $I$ of $n$ intervals on the real line
- each interval is colored by a coloring $c: I \to \{1, \ldots, k\}$

**goal:** $f$-Balanced Independent Set ($f$-BIS)
- an independent set $M \subseteq I$
- $M$ contains exactly $f$ elements from each of $k$ color classes
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1-BIS
1-BIS Problem: NP hardness
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reduction from 3-bounded 3SAT

Each variable \(x_i\) appears in \(\leq 3\) clauses

Each clause \(C_j\) has 2 or 3 literals

\[
(x_1 \lor x_2 \lor x_4) \land (x_1 \lor x_3 \lor x_4) \land (x_3 \lor x_4) \land (\overline{x_1} \lor \overline{x_2} \lor x_3)
\]
1-BIS Problem: NP hardness

reduction from 3-bounded 3SAT

- each variable $x_i$ appears in $\leq 3$ clauses
- each clause $C_j$ has 2 or 3 literals

- gadgets:

\[
\begin{align*}
C_1 & : (x_1 \lor x_2 \lor x_4) \land (\overline{x_1} \lor x_3 \lor x_4) \\
C_2 & : (x_1 \lor x_3 \lor x_4) \land (x_3 \lor x_4) \\
C_3 & : (\overline{x_1} \lor \overline{x_2} \lor x_3) \\
C_4 &
\end{align*}
\]
1-BIS Problem: NP hardness

reduction from 3-bounded 3SAT

- each variable $x_i$ appears in $\leq 3$ clauses
- each clause $C_j$ has 2 or 3 literals

- gadgets:
  - clause: color

\[ (x_1 \lor x_2 \lor x_4) \land (x_1 \lor x_3 \lor x_4) \land (x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_3) \]
1-BIS Problem: NP hardness

reduction from 3-bounded 3SAT

- each variable $x_i$ appears in $\leq 3$ clauses
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- gadgets:
  - clause: color
  - variable: one (colored) interval for each occurrence
  - intersection: each pair of opposite literals

\[
\begin{align*}
\text{variable: } & x_1 \\
\text{clause: } & C_1 \\
\text{variable: } & x_2 \\
\text{clause: } & C_2 \\
\text{variable: } & x_3 \\
\text{clause: } & C_3 \\
\text{variable: } & x_4 \\
\text{clause: } & C_4
\end{align*}
\]

\[
\begin{align*}
& (x_1 \lor \overline{x_2} \lor x_4) \land (x_1 \lor x_3 \lor \overline{x_4}) \land (x_3 \lor x_4) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3})
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$C_1$ $C_2$ $C_3$ $C_4$
1-BIS Problem: NP hardness

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\end{align*}
\]
1-BIS Problem: NP hardness

- Correctness

\[(x_1 \lor \overline{x_2} \lor x_4) \land (\overline{x_1} \lor x_3 \lor \overline{x_4}) \land (x_3 \lor x_4) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3})\]

- 1-BIS \implies:
1-BIS Problem: NP hardness

- **Correctness**

\[
\begin{align*}
(x_1 \lor \overline{x_2} \lor x_4) & \land (\overline{x_1} \lor x_3 \lor \overline{x_4}) & \land (x_3 \lor x_4) & \land (\overline{x_1} \lor \overline{x_2} \lor x_3)
\end{align*}
\]

- **1-BIS**:\,

\[
\begin{align*}
C_1 & \\
C_2 & \\
C_3 & \\
C_4 & \\
\end{align*}
\]
1-BIS Problem: NP hardness

- **Correctness**

\[(x_1 \lor x_2 \lor x_4) \land (\overline{x_1} \lor x_3 \lor \overline{x_4}) \land (x_3 \lor x_4) \land (\overline{x_1} \lor \overline{x_2} \lor x_3)\]

- **1-BIS ⇒**: evaluate the chosen literals as true
1-BIS Problem: NP hardness

- **Correctness**

- **1-BIS** \(\Rightarrow\): evaluate the chosen literals as true

- **assignment:**
1-BIS Problem: NP hardness

Correctness

\[
\begin{align*}
C_1 & : (x_1 \lor \overline{x_2} \lor x_4) \land (\overline{x_1} \lor x_3 \lor \overline{x_4}) \land (x_3 \lor x_4) \land (\overline{x_1} \lor \overline{x_2} \lor x_3) \\
C_2 & \\
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C_4 &
\end{align*}
\]

1-BIS ⇒: evaluate the chosen literals as true

\[\{x_1: T, x_2: F, x_3: T, x_4: F\}\]
1-BIS Problem: NP hardness

**Correctness**

\[
\begin{align*}
(x_1 \lor \overline{x_2} \lor x_4) \land (\overline{x_1} \lor x_3 \lor \overline{x_4}) \land (x_3 \lor x_4) \land (\overline{x_1} \lor \overline{x_2} \lor x_3)
\end{align*}
\]

\(C_1\) \hspace{1cm} \(C_2\) \hspace{1cm} \(C_3\) \hspace{1cm} \(C_4\)

1-BIS \Rightarrow: evaluate the chosen literals as true

\(\Leftarrow\) assignment: choose a positive evaluated literal in each \(C_i\)

\(\{x_1: T, x_2: F, x_3: T, x_4: F\}\)
1-BIS Problem: NP hardness

- **Correctness**

\[ (x_1 \lor \overline{x_2} \lor x_4) \land (x_1 \lor x_3 \lor \overline{x_4}) \land (x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_3) \]

- **1-BIS** ⇒: evaluate the chosen literals as true

- \( \left\{ x_1: T, x_2: F, x_3: T, x_4: F \right\} \)
$f$-BIS: An \textit{FPT} Algorithm by $(f, k)$

- sorted set of intervals $\mathcal{I} = \{I_1, \ldots, I_n\}$ sorted by right-endpoints
$f$-BIS: An FPT Algorithm by $(f, k)$

- sorted set of intervals $\mathcal{I} = \{I_1, \ldots, I_n\}$

- $\text{pred}(I_j)$: rightmost interval completely left to $I_j$ (if it exists)
**$f$-BIS: An FPT Algorithm by $(f, k)$**

- **sorted set of intervals** $\mathcal{I} = \{I_1, \ldots, I_n\}$

- **pred($I_j$)**: rightmost interval completely left to $I_j$ (if it exists)

- **cardinality vector** $C_{\mathcal{I}'}$: $k$-dimensional vector $(c_1, \ldots, c_i, \ldots, c_k)$
  - cardinality of intervals of color $i$ in $\mathcal{I}'$

- $C_{\mathcal{I}'}$ is valid: $\mathcal{I}'$ is independent and $c_i \leq f$
**f-BIS: An FPT Algorithm by \((f, k)\)**

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- **\(U_j\):** union of valid cardinality vectors of \(\{I_1, \ldots, I_j\}\)

  - \(U_0 = \{(0, \ldots, 0)\}\)

  - \(U_j = U_{j-1} \cup \{u \oplus \hat{e}_c(I_j) \mid u \in U_{\text{pred}(I_j)}\}\)
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  \((0, \ldots, 1, \ldots, 0)\)
**f-BIS: An FPT Algorithm by \((f, k)\)**

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- \(O(n \log n)\)
**$f$-BIS: An FPT Algorithm by $(f, k)$**

- sorted set of intervals $\mathcal{I} = \{I_1, \ldots, I_n\}$

$$O(n \log n)$$

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  $$O(|U_n| \times \alpha(|U_n|))$$

- $\text{pred}(I_j)$: rightmost interval completely left to $I_j$ (if it exists)
\textit{f-BIS: An FPT Algorithm by (f, k)}

- sorted set of intervals $\mathcal{I} = \{I_1, \ldots, I_n\}$

- $\text{pred}(I_j)$: rightmost interval completely left to $I_j$ (if it exists)

- \textit{cardinality vector} $C_{\mathcal{I}'}$: $k$-dimensional vector $(c_1, \ldots, c_i, \ldots, c_k)$
  
  \begin{itemize}
    \item $C_{\mathcal{I}'}$ is valid: $\mathcal{I}'$ is independent and $c_i \leq f$
    \item $U_j$: union of valid cardinality vectors of $\{I_1, \ldots, I_j\}$ \quad $|U_j| = O(f^k)$
    \item $U_0 = \{(0, \ldots, 0)\}$
    \item $U_j = U_{j-1} \cup \{u \oplus \hat{e}_c(I_j) \mid u \in U_{\text{pred}(I_j)}\}$ \quad $O(|U_n| \times \alpha(|U_n|))$
  \end{itemize}
**f-BIS: An FPT Algorithm by** \((f, k)\)

- sorted set of intervals \(\mathcal{I} = \{I_1, \ldots, I_n\}\)

\慎重\(pred(I_j)\): rightmost interval completely left to \(I_j\) (if it exists)

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  \[|U_j| = O(f^k)\]

- \(U_0 = \{(0, \ldots, 0)\}\)

- \(U_j = U_{j-1} \cup \{u \oplus \hat{e}_{c(I_j)} \mid u \in U_{pred(I_j)}\}\)

  \[O(|U_n| \times \alpha(|U_n|))\]

- runtime: \(O(n \log n + nf^k \alpha(f^k))\)

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Conclusion

- our results
  - $f$-Balanced Independent Set:
    - NP-hardness
    - FPT by $(f, k)$
Conclusion

- our results

  - $f$-Balanced Independent Set:
    - $NP$-hardness
    - $FPT$ by $(f, k)$
  
  - $FPT$ by the Vertex Cover Number

- relevant problems:
  - 2-approximation for 1-Max-Colored Independent Sets
  - $NP$-hardness of $f$-Balanced Dominating Set
Conclusion

- our results
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- relevant problems:
  - 2-approximation for $1$-Max-Colored Independent Sets
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- open problems:
  - balanced set on intersection graphs (e.g. boxicity graphs)
Conclusion

- our results
  - \( f \)-Balanced Independent Set:
    - \( \mathsf{NP} \)-hardness
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- open problems:
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guangping@ac.tuwien.ac.at