Smallest Universal Covers for Families of Triangles

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Introduction

Def. A *universal cover* for a given family of objects is a *convex* set that contains a *congruent copy* of each object in the family. That is, translations, rotations, and reflections are allowed.

Aim. Given a family of objects, find a smallest universal cover, i.e., a universal cover of smallest *area*.

In general, finding a smallest universal cover is hard:
- Sets of unit diameter (a.k.a. Lebesgue’s Universal Cover Problem)
- Unit curves
- Unit convex curves
- Sets of unit perimeter
Introduction

Thm. The smallest universal cover for the family of all triangles of unit diameter is a triangle and it is unique. \[K83\]

Thm. The same is true for the family of all triangles of unit perimeter. \[FW00\]

 Conj. For any family $\mathcal{T}$ of triangles of bounded diameter, there is a triangle $Z$ that is a smallest universal cover for $\mathcal{T}$. 

![Diagram of triangle cover](image)
Results

Conj. For any family $\mathcal{T}$ of triangles of bounded diameter, there is a triangle $Z$ that is a smallest universal cover for $\mathcal{T}$.

Thm. For any two triangles, there is a triangle that is a smallest universal cover.

Thm. For triangles of unit circumradius, the unique smallest universal cover is a triangle.

Thm. There exist three triangles whose smallest universal cover is not determined by any two of them.
Thm. Let $S$ and $T$ be triangles. Then there is a triangle $Z$ that is a smallest universal cover for the family $\{S, T\}$.
Two Triangles

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Pf. $S' = $ the smallest triangle similar to $S$ s.t. $T$ fits into $S'$. If $S' = S$ done; otherwise:

Lemma. If a convex set $X$ maximally fits into a convex set $Y$, then there are at least four incidences between vertices of $X$ and edges of $Y$. [AAS98]
Two Triangles

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\[ \text{Diagram:} \]
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Lemma. Let $\mathcal{T}$ be a family of triangles, and let $Z$ be a universal cover for $\mathcal{T}$. Let $S \in \mathcal{T}$, and let $S'$ be the smallest universal cover for $\mathcal{T}$ that is similar to $S$. If $\frac{|S'|}{|S|} = \left(\frac{|Z|}{|S|}\right)^2$, then $Z$ is a smallest universal cover for $\mathcal{T}$. 
Two Triangles

Thm. Let $S$ and $T$ be triangles. Then there is a triangle $Z$ that is a smallest universal cover for the family $\{S, T\}$.

Pf. $S' = \text{the smallest triangle similar to } S \text{ s.t. } T \text{ fits into } S'$.

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Lemma. Let $\mathcal{T}$ be a family of triangles, and let $Z$ be a universal cover for $\mathcal{T}$. Let $S \in \mathcal{T}$, and let $S'$ be the smallest universal cover for $\mathcal{T}$ that is similar to $S$. If $\frac{|S'|}{|S|} = \left(\frac{|Z|}{|S|}\right)^2$, then $Z$ is a smallest universal cover for $\mathcal{T}$.

![Diagram of triangles and their relationship](image.png)
Triangles of Unit Circumradius

\( T_0 \) = the equilateral triangle (i.e., \( T_1(60^\circ) = T_0 \)).

\( T_1(\theta) \) = the isosceles triangle of base angle \( \theta \).

\( T(\theta) \) = the smallest universal cover for \( T_0 \) and \( T_1(\theta) \).
**Triangles of Unit Circumradius**

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\( T(\theta) \) = the smallest universal cover for \( T_0 \) and \( T_1(\theta) \).

For some \( 75^\circ < \theta_m < 80^\circ \), if \( 60^\circ \leq \theta \leq \theta_m \) or \( \theta \geq 80^\circ \):
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When \( \theta = 80^\circ \):
Triangles of Unit Circumradius

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$T_1(\theta) = \text{the isosceles triangle of base angle } \theta.$

$T(\theta) = \text{the smallest universal cover for } T_0 \text{ and } T_1(\theta).$

$T^* = T(80^\circ)$ is the largest one.

Thm. $T^*$ is the smallest universal cover for the family $\mathcal{T}$ of triangles of unit circumradius.
Triangles of Unit Circumradius

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\( T^* = T(80^\circ) \) is the largest one.

Thm. \( T^* \) is the smallest universal cover for the family \( \mathcal{T} \) of triangles of unit circumradius.

Sketch. 1) \( T^* \) covers every triangle of unit circumradius.
Triangles of Unit Circumradius

$T_0 =$ the equilateral triangle (i.e., $T_1(60^\circ) = T_0$).

$T_1(\theta) =$ the isosceles triangle of base angle $\theta$.

$T(\theta) =$ the smallest universal cover for $T_0$ and $T_1(\theta)$.

$T^* = T(80^\circ)$ is the largest one.

Thm. $T^*$ is the smallest universal cover for the family $\mathcal{T}$ of triangles of unit circumradius.

Sketch. 1) $T^*$ covers every triangle of unit circumradius.

2) $T^*$ is a smallest universal cover for $\mathcal{T}$.

$\therefore T^*$ is the smallest universal cover for $T_0$ and $T_1(80^\circ)$. 
Triangles of Unit Circumradius

\[ T_0 = \text{the equilateral triangle (i.e., } T_1(60^\circ) = T_0). \]

\[ T_1(\theta) = \text{the isosceles triangle of base angle } \theta. \]

\[ T(\theta) = \text{the smallest universal cover for } T_0 \text{ and } T_1(\theta). \]

\[ T^* = T(80^\circ) \] is the largest one.

Thm. \( T^* \) is the smallest universal cover for the family \( \mathcal{T} \) of triangles of unit circumradius.

Sketch. 1) \( T^* \) covers every triangle of unit circumradius.

2) \( T^* \) is a smallest universal cover for \( \mathcal{T} \).
\[ \therefore \] \( T^* \) is the smallest universal cover for \( T_0 \) and \( T_1(80^\circ) \).

3) \( T^* \) is the unique smallest universal cover for \( \mathcal{T} \).
\[ \therefore \] Any smallest universal cover for \( \{T_1(\theta)\} \) should be congruent to \( T^* \).
Thm. There exist three triangles whose universal cover is not determined by any two of them.

Conj. $Z$ is the smallest universal cover for these three triangles.