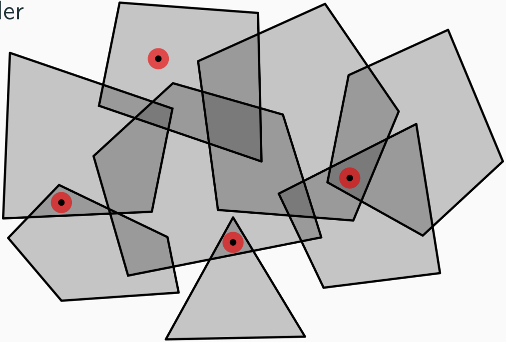
Efficiently stabbing convex polygons and variants of the Hadwiger-Debrunner (p,q)-theorem

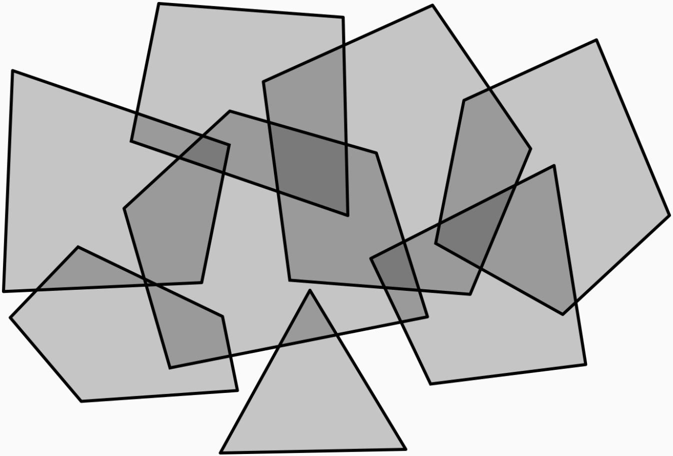
Justin Dallant, Patrick Schnider

March 17, 2020

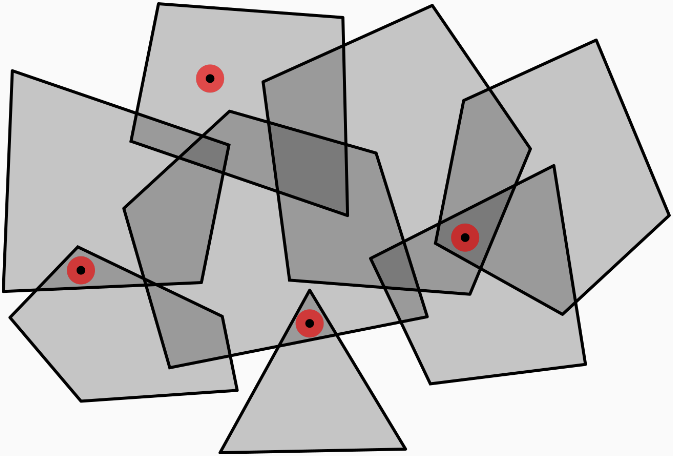
Würzburg



Stabbing convex polygons

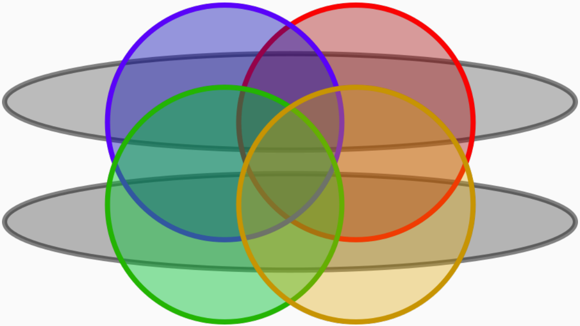


Stabbing convex polygons



(p, q)-property

For every choice of p sets, some q have a common intersection.



Family of 6 convex sets with the $(5, 4)$-property

Theorem (Hadwiger and Debrunner)
If $\mathcal{F}$ is a finite family of convex sets such that:

- $\mathcal{F}$ has the (p, q) property;
- p and q are "close enough";

Then $\mathcal{F}$ can be stabbed with $p - q + 1$ points.

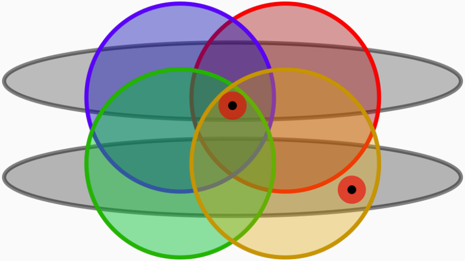
Family of 6 convex sets with the $(5, 4)$-property

Hadwiger-Debrunner (p, q)-Theorem

Theorem (Hadwiger and Debrunner)
If $\mathcal{F}$ is a finite family of convex sets such that:

- $\mathcal{F}$ has the (p, q) property;
- p and q are "close enough";

Then $\mathcal{F}$ can be stabbed with $p - q + 1$ points.



Family of 6 convex sets with the $(5, 4)$-property

Problem

How fast can we compute such $p - q + 1$ points for a family of n polygons in the plane?

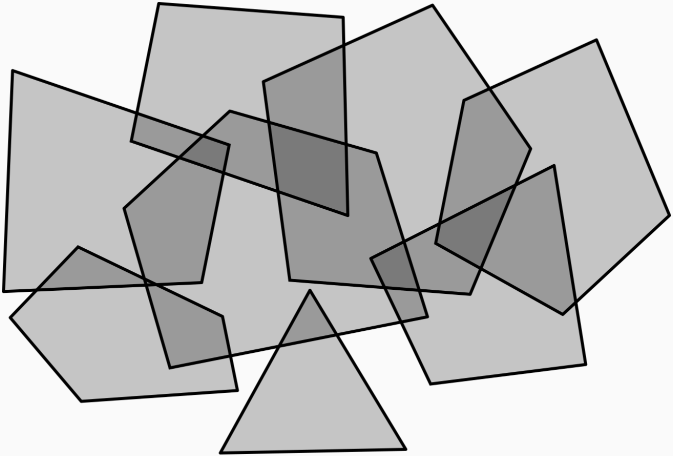
Problem

How fast can we compute such $p - q + 1$ points for a family of n polygons in the plane?

Restriction:
Polygons of constant size.

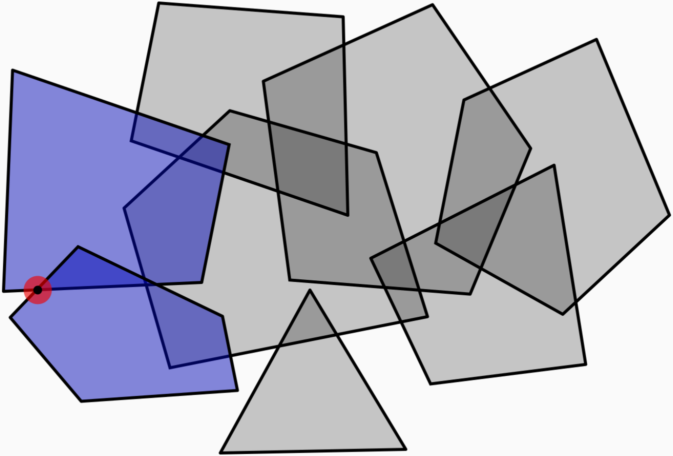
A first algorithm

A family of n convex polygons with the (p, q)-property.



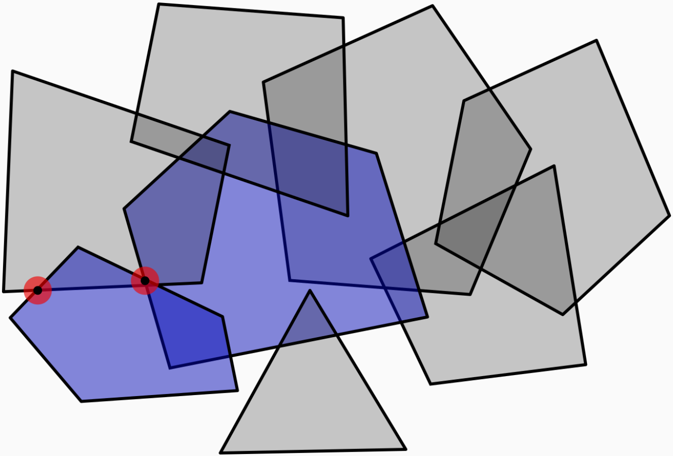
A first algorithm

Candidate points: leftmost point in pairwise intersections.



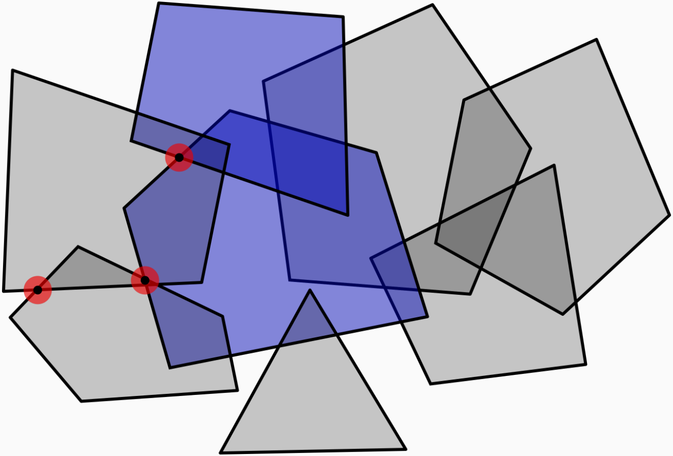
A first algorithm

Candidate points: leftmost point in pairwise intersections.



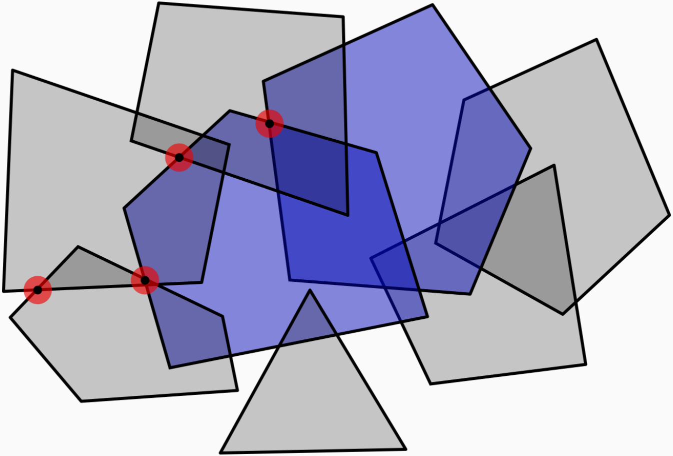
A first algorithm

Candidate points: leftmost point in pairwise intersections.



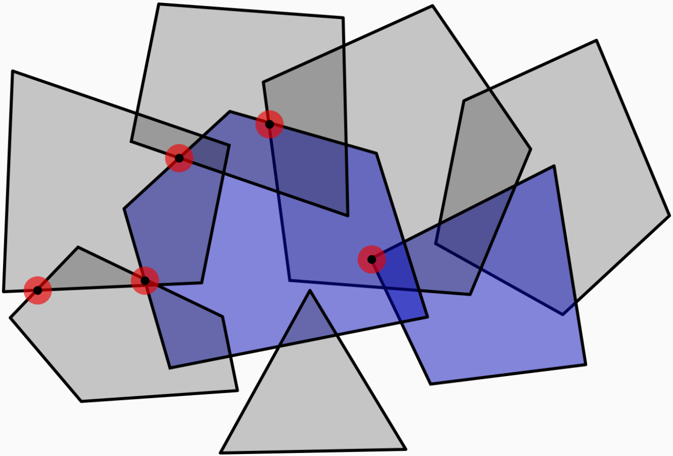
A first algorithm

Candidate points: leftmost point in pairwise intersections.



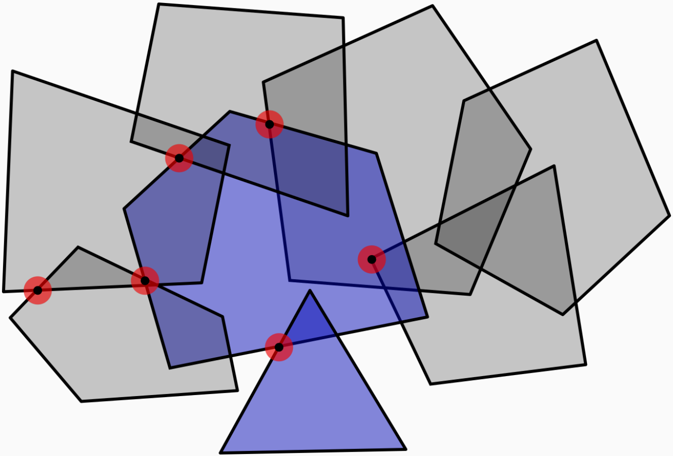
A first algorithm

Candidate points: leftmost point in pairwise intersections.



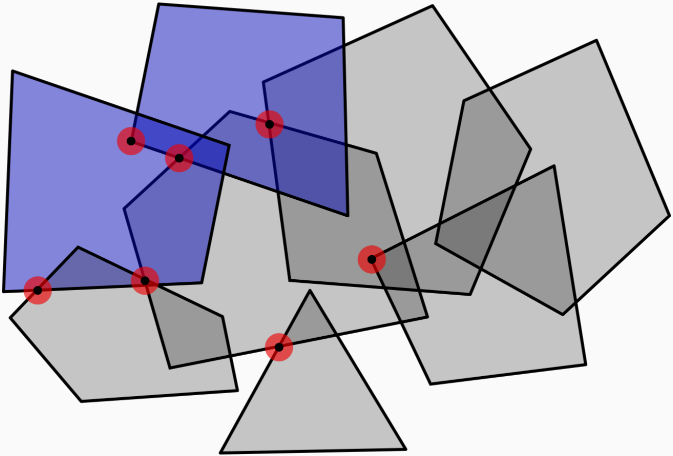
A first algorithm

Candidate points: leftmost point in pairwise intersections.



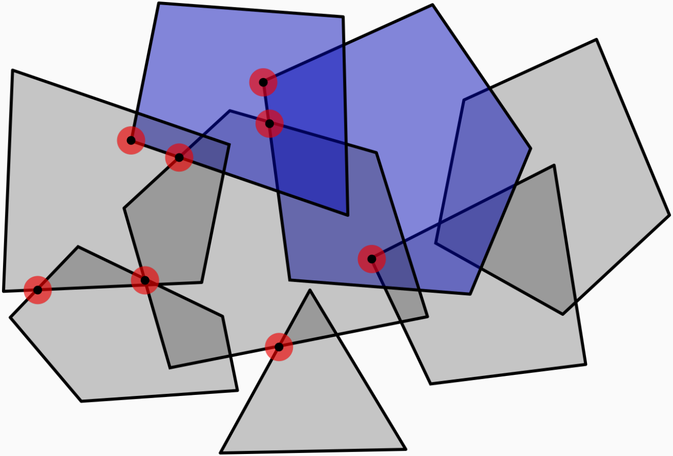
A first algorithm

Candidate points: leftmost point in pairwise intersections.



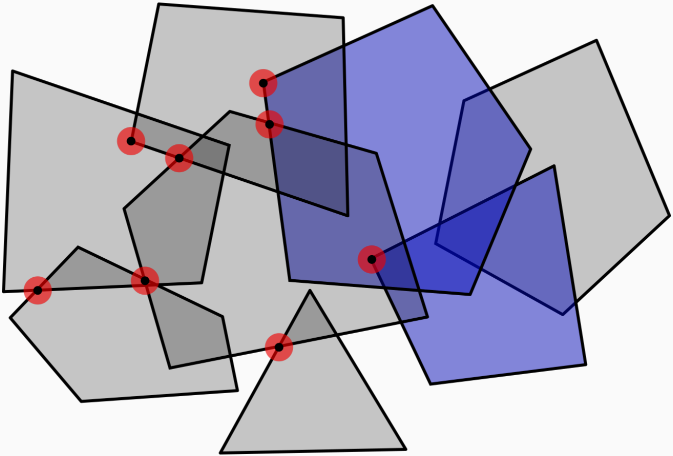
A first algorithm

Candidate points: leftmost point in pairwise intersections.



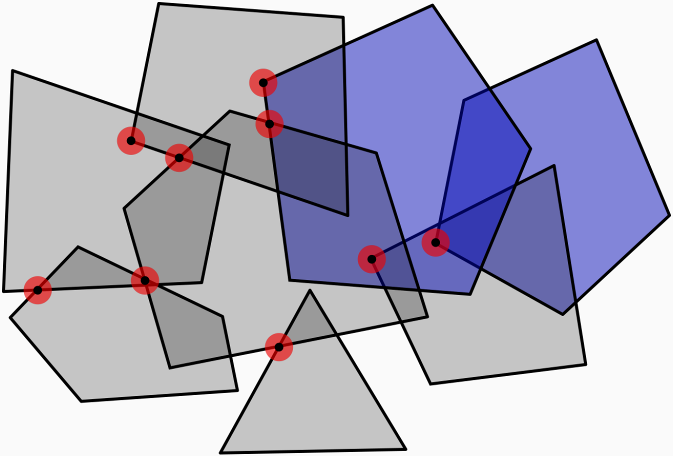
A first algorithm

Candidate points: leftmost point in pairwise intersections.



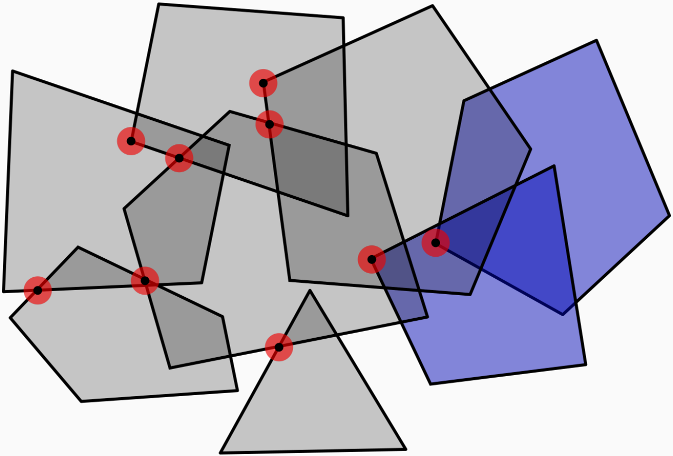
A first algorithm

Candidate points: leftmost point in pairwise intersections.



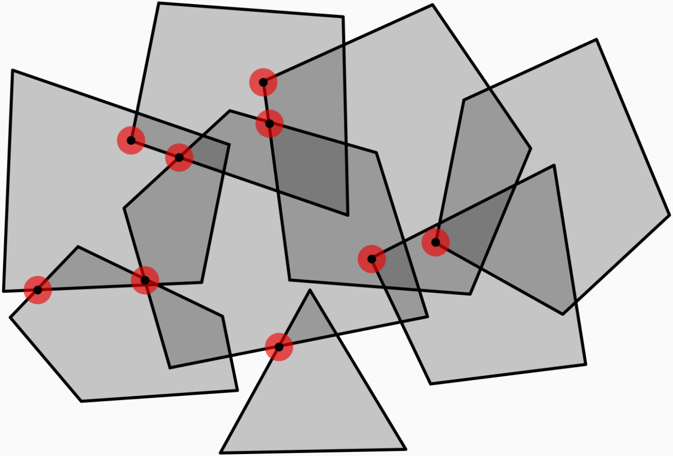
A first algorithm

Candidate points: leftmost point in pairwise intersections.



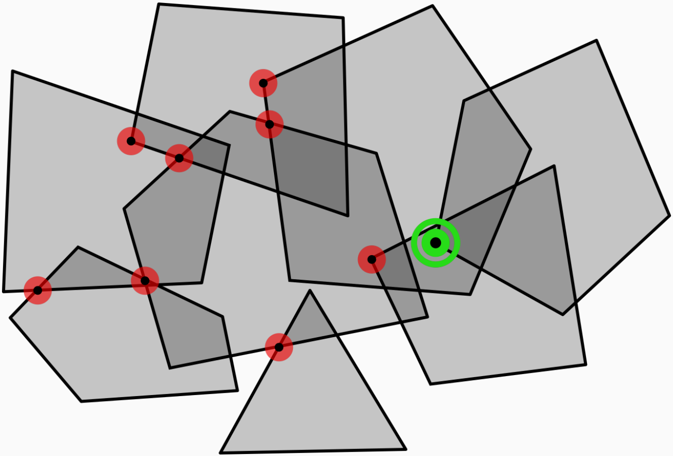
A first algorithm

Candidate points: leftmost point in pairwise intersections.



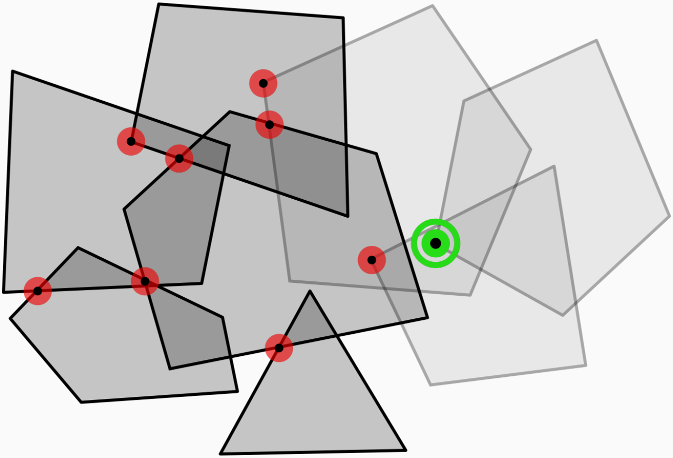
A first algorithm

First stabbing point: rightmost candidate point.



A first algorithm

Delete stabbed polygons and restart for next stabbing point.



A first algorithm

If p and q are constants:

Runtime for this algorithm

$\simeq$

Time to compute the first stabbing point

A first algorithm

If p and q are constants:

$$\text{Runtime for this algorithm} \simeq \text{Time to compute the first stabbing point}$$

Naively: $\mathcal{O}(n^2)$ time.

An optimization technique by T.Chan

Call s_1 = first stabbing point.

An optimization technique by T.Chan

Call $s_1 =$ first stabbing point.

Using a technique by Chan (1999):

$$\text{Decide if } s_1 \text{ lies to the right of a vertical line in } T(n) \text{ time}$$
$$\Longrightarrow$$
$$\text{Compute the first stabbing point in } \mathcal{O}(T(n)) \text{ expected time}$$

An optimization technique by T.Chan

Call $s_1 =$ first stabbing point.

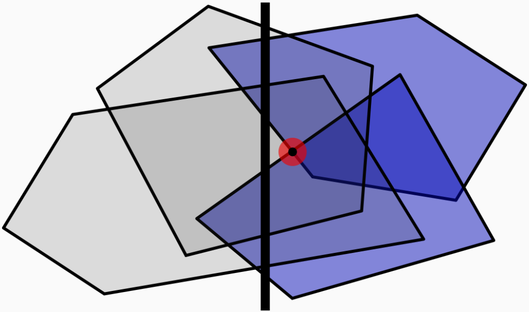
Using a technique by Chan (1999):

$$\text{Decide if } s_1 \text{ lies to the right of a vertical line in } T(n) \text{ time}$$
$$\Longrightarrow$$
$$\text{Compute the first stabbing point in } \mathcal{O}(T(n)) \text{ expected time}$$

Can we decide this in subquadratic time?

The decision problem

Rephrasing the question
Are there any two intersecting polygons whose intersection lies entirely to the right of some vertical line ℓ?



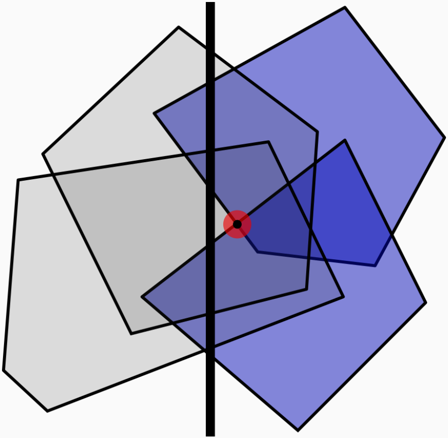
W.l.o.g.
All polygons intersect ℓ.

The decision problem

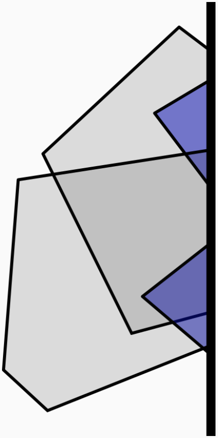
Agarwal et al. (2002):
Counting the number of pairwise intersections between n convex polygons of constant size can be done in $\mathcal{O}(n^{4/3} \log^{2+\epsilon}(n))$ time.

We can use this to test if two polygons intersect exclusively to the right of ℓ.

The decision problem



6 pairwise intersections



5 pairwise intersections

Putting everything back together

Theorem
We can compute $p - q + 1$ points stabbing $\mathcal{F}$ in $\mathcal{O}(n^{4/3} \log^{2+\epsilon}(n))$ expected time.

For polyhedra in $\mathbb{R}^3$, a similar method yields an algorithm running in $\mathcal{O}(n^{13/5+\epsilon})$ expected time.

Adapting the Hadwiger-Debrunner Theorem to other settings

Ordered-Helly System
A set system with sufficient conditions to carry out a proof similar to the one for the Hadwiger-Debrunner Theorem.

Adapting the Hadwiger-Debrunner Theorem to other settings

Ordered-Helly System
A set system with sufficient conditions to carry out a proof similar to the one for the Hadwiger-Debrunner Theorem.

With this structure, an analogue to the Hadwiger-Debrunner (p, q)-theorem can be proven:

Adapting the Hadwiger-Debrunner Theorem to other settings

Ordered-Helly System
A set system with sufficient conditions to carry out a proof similar to the one for the Hadwiger-Debrunner Theorem.

With this structure, an analogue to the Hadwiger-Debrunner (p, q)-theorem can be proven:

Theorem
If $\mathcal{F}$ is a family of sets of an Ordered-Helly System such that:

- $\mathcal{F}$ has the (p, q) property;
- p and q are "close enough";

Then $\mathcal{F}$ can be stabbed with $p - q + 1$ points.

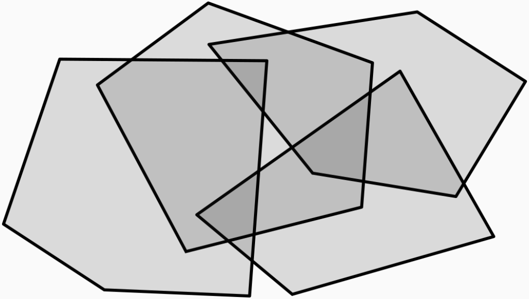
Crucial condition: Existence of a Helly number

S **has Helly number** h**:**
If $\mathcal{F} \subset S$ such that not all sets in $\mathcal{F}$ share a common point, then some h sets in $\mathcal{F}$ do not share a common point.

Crucial condition: Existence of a Helly number

S **has Helly number** h**:**
If $\mathcal{F} \subset S$ such that not all sets in $\mathcal{F}$ share a common point, then some h sets in $\mathcal{F}$ do not share a common point.

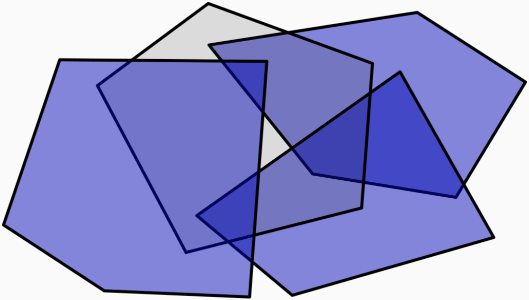
Helly's theorem : for convex sets in the plane, $h = 3$.



Crucial condition: Existence of a Helly number

S **has Helly number** h**:**
If $\mathcal{F} \subset S$ such that not all sets in $\mathcal{F}$ share a common point, then some h sets in $\mathcal{F}$ do not share a common point.

Helly's theorem : for convex sets in the plane, $h = 3$.



Examples of Ordered-Helly Systems

- Convex sets in $\mathbb{R}^d$;

Examples of Ordered-Helly Systems

- Convex sets in $\mathbb{R}^d$;
- Convex sets in $\mathbb{R}^d \times \mathbb{Z}^k$;

Examples of Ordered-Helly Systems

- Convex sets in $\mathbb{R}^d$;
- Convex sets in $\mathbb{R}^d \times \mathbb{Z}^k$;
- Subtrees of a given tree;

Examples of Ordered-Helly Systems

- Convex sets in $\mathbb{R}^d$;
- Convex sets in $\mathbb{R}^d \times \mathbb{Z}^k$;
- Subtrees of a given tree;
- Ideals of a given Poset;

Examples of Ordered-Helly Systems

- Convex sets in $\mathbb{R}^d$;
- Convex sets in $\mathbb{R}^d \times \mathbb{Z}^k$;
- Subtrees of a given tree;
- Ideals of a given Poset;
- Abstract convex geometries.

Open questions

- How to these methods translate when dropping the condition that p and q are "close enough"?

Open questions

- How to these methods translate when dropping the condition that p and q are "close enough"?
- Can we get lower runtimes (especially in dimension ≥ 3)?

Open questions

- How to these methods translate when dropping the condition that p and q are "close enough"?
- Can we get lower runtimes (especially in dimension ≥ 3)?
- Can we get similar runtimes for polygons of non-constant size?

Open questions

- How to these methods translate when dropping the condition that p and q are "close enough"?
- Can we get lower runtimes (especially in dimension ≥ 3)?
- Can we get similar runtimes for polygons of non-constant size?

Open questions

- How to these methods translate when dropping the condition that p and q are "close enough"?
- Can we get lower runtimes (especially in dimension ≥ 3)?
- Can we get similar runtimes for polygons of non-constant size?

Thank you for your attention.

Hadwiger-Debrunner (p, q)-Theorem

Definition of (p, q)-property
$\mathcal{F}$ has the (p, q)-property if $|\mathcal{F}| \geq p$ and for every choice of p sets in $\mathcal{F}$ there exist q among them which have a common intersection.

Theorem (Hadwiger and Debrunner)
Let $p \geq q \geq d + 1$ and $(d - 1)p < d(q - 1)$, and let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$.

If $\mathcal{F}$ has the (p, q)-property, then there exist $p - q + 1$ points in $\mathbb{R}^d$ stabbing $\mathcal{F}$.

Ordered-Helly System

A *base set* $\mathcal{B}$ with a total order $\preceq$, a family $\mathcal{C} \subset \mathcal{P}(\mathcal{B})$ of "*convex sets*" and a family $\mathcal{D} \subset \mathcal{C}$ of "*compact sets*", such that:

1. $\mathcal{D}$ is closed under intersections;
2. For all non-empty $S \in \mathcal{D}$, there exists a minimum $x \in S$ with respect to $\preceq$;
3. For all $t \in \mathcal{B}$, we have $\{x \in \mathcal{B} \mid x \preceq t \text{ and } x \neq t\} \in \mathcal{C}$;
4. There exists a Helly number h on $\mathcal{C}$.

Theorem
Let $p \geq q \geq h$ and $(h-2)p < (h-1)(q-1)$, and let $\mathcal{F}$ be a finite subfamily of $\mathcal{C}$.
If $\mathcal{F}$ has the (p, q)-property, then there exists a set of $p - q + 1$ elements in $\mathcal{B}$ stabbing $\mathcal{F}$.