Holes and Islands
in Random Point Sets

Martin Balko, Manfred Scheucher, Pavel Valtr
a finite point set $S$ in the plane is in general position if $\nexists$ collinear points in $S$
$k$-Gons

a finite point set $S$ in the plane is in general position if $\not\exists$ collinear points in $S$

throughout this presentation, every set is in general position
$k$-Gons

a finite point set $S$ in the plane is in general position if $\not\exists$ collinear points in $S$

a $k$-gon (in $S$) is the vertex set of a convex $k$-gon

5-gon 6-gon
$k$-Gons

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**Theorem** (Erdős and Szekeres 1935).

$\forall k \in \mathbb{N}, \exists$ a smallest integer $ES(k)$ such that every set of $ES(k)$ points contains a $k$-gon.
A $k$-hole (in $S$) is the vertex set of a convex $k$-gon containing no other points of $S$.

- A green pentagon is a 5-hole.
- A red hexagon is not a 6-hole.
$k$-Holes

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Erdős, 1970’s: For $k$ fixed, does every sufficiently large point set contain $k$-holes?
$k$-Holes

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Erdős, 1970’s: For $k$ fixed, does every sufficiently large point set contain $k$-holes?

- 3 points $\Rightarrow \exists$ 3-hole
- 5 points $\Rightarrow \exists$ 4-hole
- 10 points $\Rightarrow \exists$ 5-hole [Harborth ’78]
- $\exists$ arbitrarily large point sets with no 7-hole [Horton ’83]
- Sufficiently large point sets $\Rightarrow \exists$ 6-hole [Gerken ’08 and Nicolás ’07, independently]
Counting $k$-Holes

$h_k(n) := \text{minimum \ # of } k\text{-holes among all sets of } n \text{ points}$

- $h_3(n)$ and $h_4(n)$ both in $\Theta(n^2)$
  - [Bárány and Füredi ’87, Bárány and Valtr ’04]

- $h_5(n)$ in $\Omega(n \log^{4/5} n)$ and $O(n^2)$
  - [Aichholzer, Balko, Hackl, Kynčl, Parada, S., Valtr, and Vogtenhuber ’17]

- $h_6(n)$ in $\Omega(n)$ and $O(n^2)$
  - [Gerken ’08, Nicolás ’07]

- $h_k(n) = 0$ for $k \geq 7$  
  - [Horton ’83]
Holes in Higher Dimensions

- $\exists$ $d$-dimensional Horton sets not containing $k$-holes for sufficiently large $k = k(d)$ [Valtr ’92]

- minimum number of empty simplices $(d + 1)$-holes) in $n$-point set in $\mathbb{R}^d$ is in $\Theta(n^d)$ [Bárány and Füredi ’92]
Random Point Sets

- Random point sets give the upper bound $O(n^d)$
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- $EH_{d,k}^K(n) :=$ expected number of $k$-holes in sets of $n$ points chosen independently and uniformly at random from convex shape $K \subset \mathbb{R}^d$
Random Point Sets

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- $EH_{d,k}^K(n) :=$ expected number of $k$-holes in sets of $n$ points chosen independently and uniformly at random from convex shape $K \subset \mathbb{R}^d$

- Bárány and Füredi (1987) showed

\[
EH_{d,d+1}^K(n) \leq (2d)^{2d^2} \cdot \binom{n}{d} \leq O(n^d)
\]
Our Results I

- extend bound to larger holes, and even to islands
- \( I \subseteq S \) is an island (in \( S \)) if \( S \cap \text{conv}(I) = I \)
- “hole = gon + island”
Our Results I

• extend bound to larger holes, and even to islands

**Theorem 1.** Let $d \geq 2$ and $k \geq d + 1$ be integers, and let $K$ be a convex body in $\mathbb{R}^d$. If $S$ is a set of $n$ points chosen uniformly and independently at random from $K$, then the expected number of $k$-islands in $S$ is at most

$$2^{d-1} \cdot \left(2d^{2d-1} \binom{k}{\lfloor d/2 \rfloor} \right)^{k-d-1} \cdot (k-d) \cdot \frac{n(n-1)\cdots(n-k+2)}{(n-k+1)^{k-d-1}} \cdot O(n^d)$$
Our Results I

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**Theorem 1.** Let \( d \geq 2 \) and \( k \geq d + 1 \) be integers, and let \( K \) be a convex body in \( \mathbb{R}^d \). If \( S \) is a set of \( n \) points chosen uniformly and independently at random from \( K \), then the expected number of \( k \)-islands in \( S \) is at most

\[
2^{d-1} \cdot \left( \frac{k}{2d^{d-1}} \right)^{k-d-1} \cdot (k-d) \cdot \frac{n(n-1) \cdots (n-k+2)}{(n-k+1)^{k-d-1}}
\]

- In particular:
  \( \exists \) sets of \( n \) points in \( \mathbb{R}^d \) with \( O(n^d) \) \( k \)-islands
Our Results II

• the bound from Theorem 1 is asymptotically optimal, but the leading constant can be improved for $k$-holes.

• for empty simplices in $\mathbb{R}^d$, we have a better bound

\[ EH_{d,d+1}^K(n) \leq 2^{d-1} \cdot d! \cdot \binom{n}{d} \]

• for 4-holes in $\mathbb{R}^2$, we have $EH_{2,4}^K(n) \leq 12n^2 + o(n^2)$
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- very recently, Reitzner and Temesvari proved an asymptotically tight bound for $EH_{d,d+1}^K(n)$ if $d = 2$ or if $d \geq 3$ and $K$ is an ellipsoid.
Our Results III

- Theorem 1 is the first nontrivial bound for $k$-islands in $\mathbb{R}^d$ for $d > 2$

- In the plane, the $O(n^2)$ bound is achieved by Horton sets [Fabila-Monroy and Huemer '12]

- However, $d$-dimensional Horton sets with $d > 2$ do not give the $O(n^d)$ bound on $k$-islands
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**Theorem 3.** Let $d \geq 2$ and let $k$ be fixed positive integers. Then every $d$-dimensional Horton set $H$ with $n$ points contains at least $\Omega(n^{\min\{2^{d-1},k\}})$ $k$-islands. If $k \leq 3 \cdot 2^{d-1}$, then $H$ even contains at least $\Omega(n^{\min\{2^{d-1},k\}})$ $k$-holes.
• we cannot have \( O(n^d) \) for \( k \)-islands if \( k \) is not fixed

**Theorem 3.** Let \( d \geq 2 \) and let \( K \) be a convex body in \( \mathbb{R}^d \). Then, for every set \( S \) of \( n \) points chosen uniformly and independently at random from \( K \), the expected number of islands in \( S \) is \( 2^\Theta(n^{(d-1)/(d+1)}) \).
Idea of the proof of Theorem 1

Rest of this presentation:

idea how to prove the bound $O(n^2)$ on the expected number of $k$-islands in a set $S$ of $n$ points chosen uniformly and independently at random from convex body $K \subset \mathbb{R}^2$ with area $\lambda(K) = 1$
We prove an $O(1/n^{k-2})$ bound on the probability that a $k$-tuple $I = (p_1, \ldots, p_k)$ determines $k$-island with 2 additional properties:

- (P1) $p_1, p_2, p_3$ form largest triangle $\triangle$ in $I$
- (P2) $p_4, \ldots, p_{3+a}$ inside $\triangle$; rest outside & incr. dist. to $\triangle$
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• First, $\triangle$ contains precisely $p_4, \ldots, p_{3+a}$ with prob. $O(1/n^{a+1})$

$\iff p_1, \ldots, p_{3+a}$ form an island in $S$ satisfying (P1) and (P2)
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\[
\int_{h=0}^{2/\ell} \left( \frac{h\ell}{2} \right)^a \left( 1 - \frac{h\ell}{2} \right)^{n-3-a} dh
\]

a points inside  
$n - 3 - a$ outside
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\[
\int_0^1 x^a (1-x)^{n-3-a} dx = \frac{a! \cdot (n-3-a)!}{(a + n - 3 - a + 1)!} \approx a! \cdot n^{(n-3-a)-(n-2)}
\]

(Beta-function)
• We prove an $O(1/n^{k-2})$ bound on the probability that a $k$-tuple $I = (p_1, \ldots, p_k)$ determines $k$-island with 2 additional properties:
  
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• First, $\triangle$ contains precisely $p_4, \ldots, p_{3+a}$ with prob. $O(1/n^{a+1})$
  
• Next, conditioned on the fact that $p_1, \ldots, p_{i-1}$ determines island satisfying (P1) and (P2), $p_1, \ldots, p_i$ determines island sat. (P1) and (P2) with prob. $O(1/n)$
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$\Rightarrow$ $I$ determines $k$-island with (P1) and (P2) prob. at most

$$O \left( \frac{1}{n^{a+1}} \cdot \frac{1}{n^{k-(3+a)}} \right) = O(1/n^{k-2})$$
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Finally, since there are $n \cdot (n-1) \cdots (n-k+1)$ possibilities to select $I$, we obtain the desired bound $O(n^k \cdot n^{2-k}) = O(n^2)$ on the expected number of $k$-islands in $S$
THANK YOU
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