Graph Planarity Testing with Hierarchical Embedding Constraints

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EuroCG 2020

36th European Workshop on Computational Geometry
March 16-18, 2020 - Würzburg, Germany

Constraints in Graph Drawings

- In many contexts, data can be represented as networks of interconnected elements
- Information visualization is often based on graph representations
- Graph representations need to take into account layout rules

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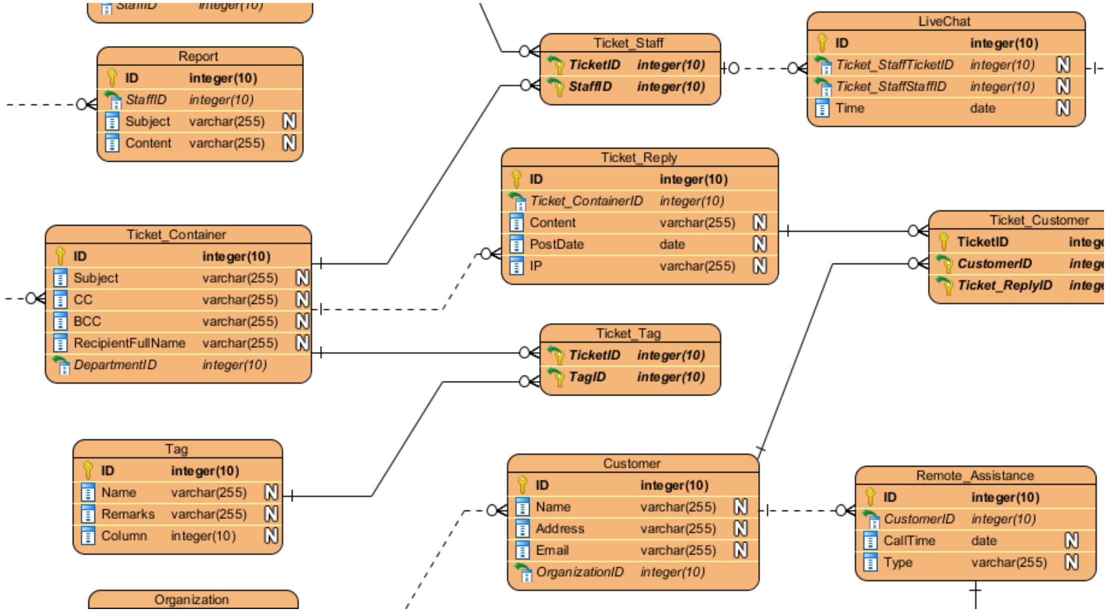

Image: https://www.visual-paradigm.com

Database diagrams

links between attributes should enter the tables only at the left or right side

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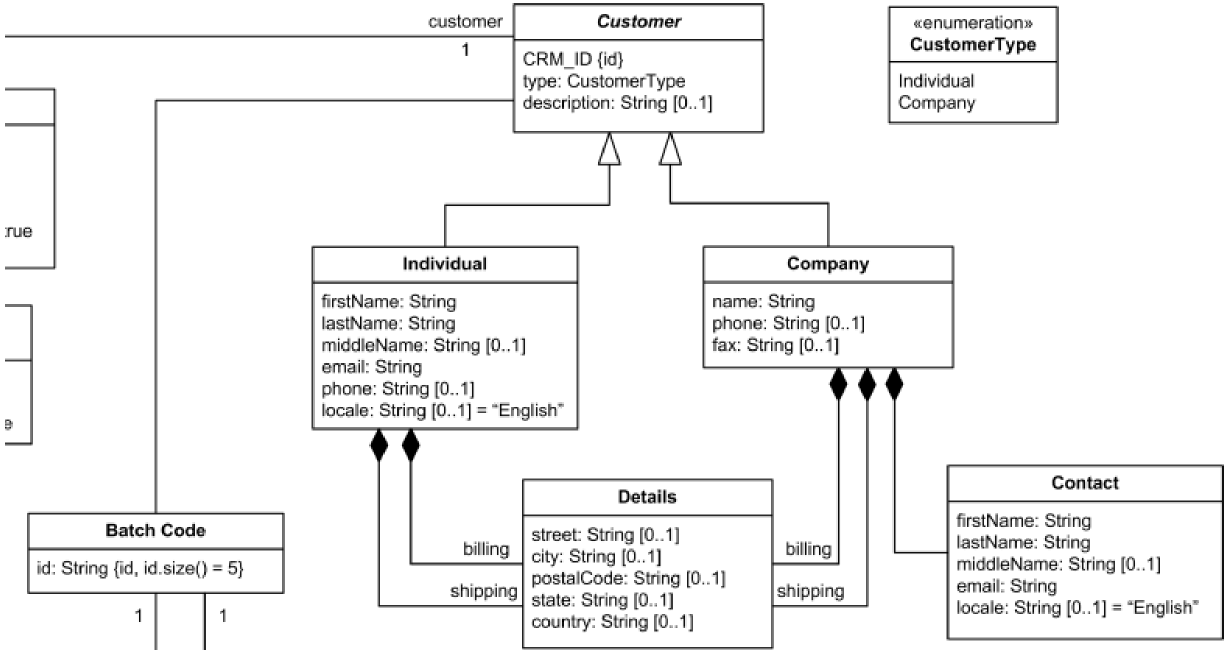

Image: https://www.uml-diagrams.org/

UML class diagrams

generalization edges should leave a class object at the top and enter a base class object at the bottom

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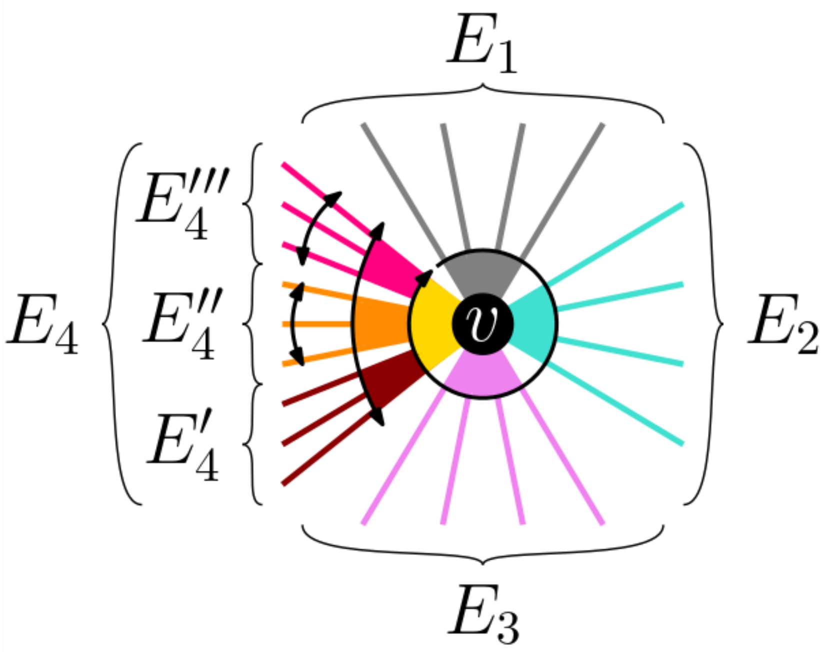

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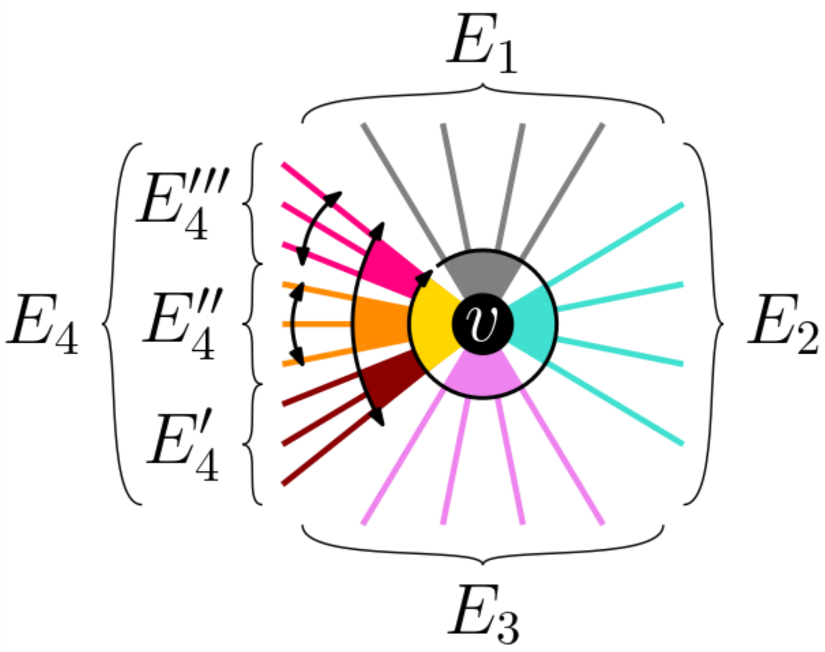

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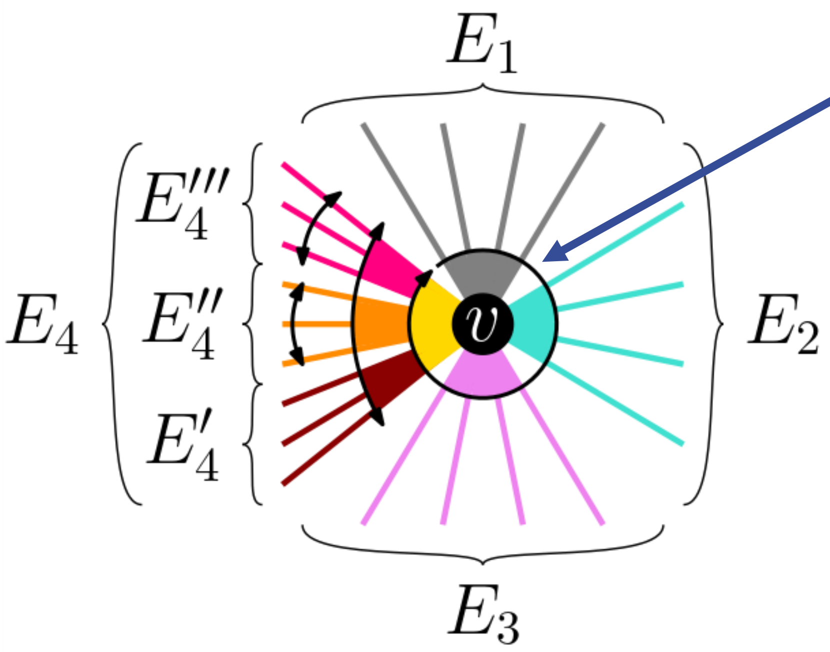

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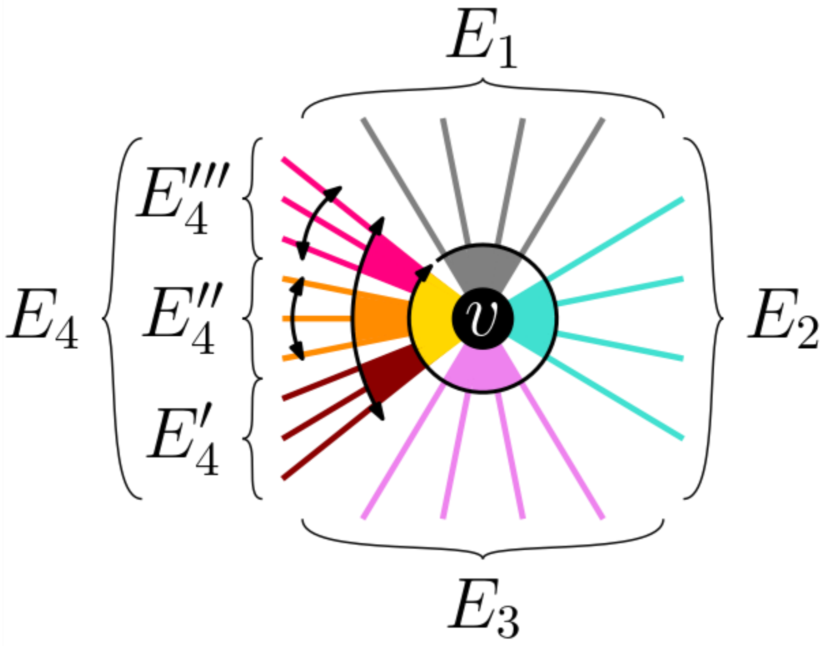

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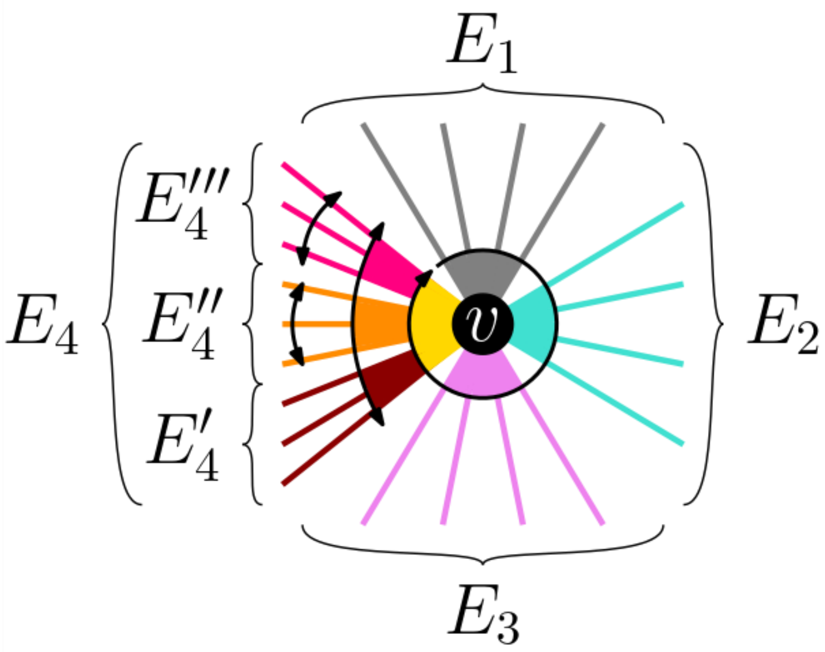

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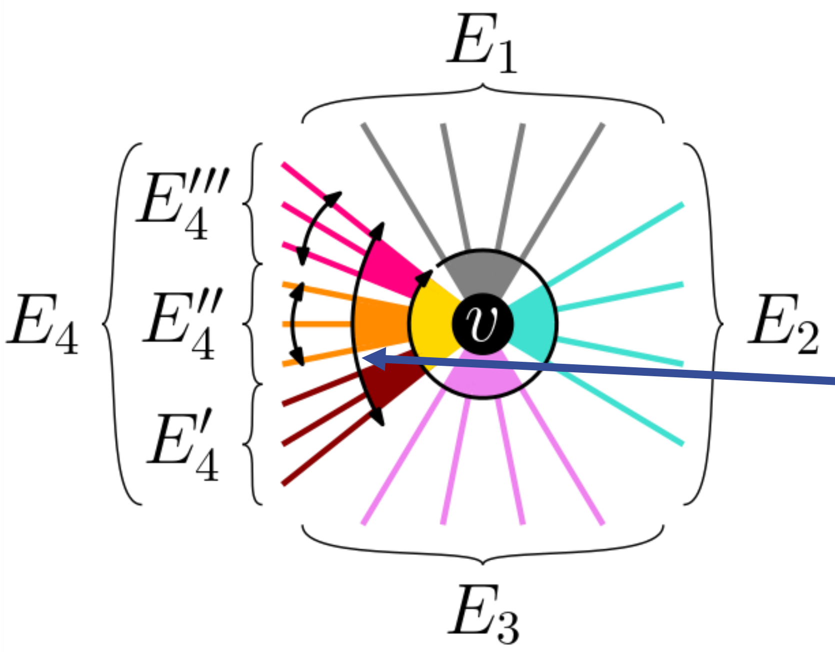

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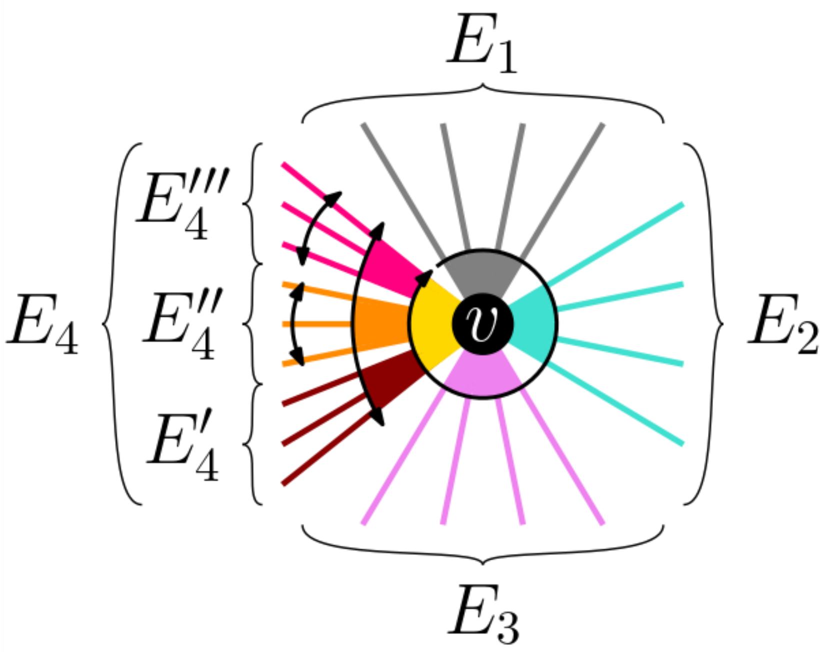

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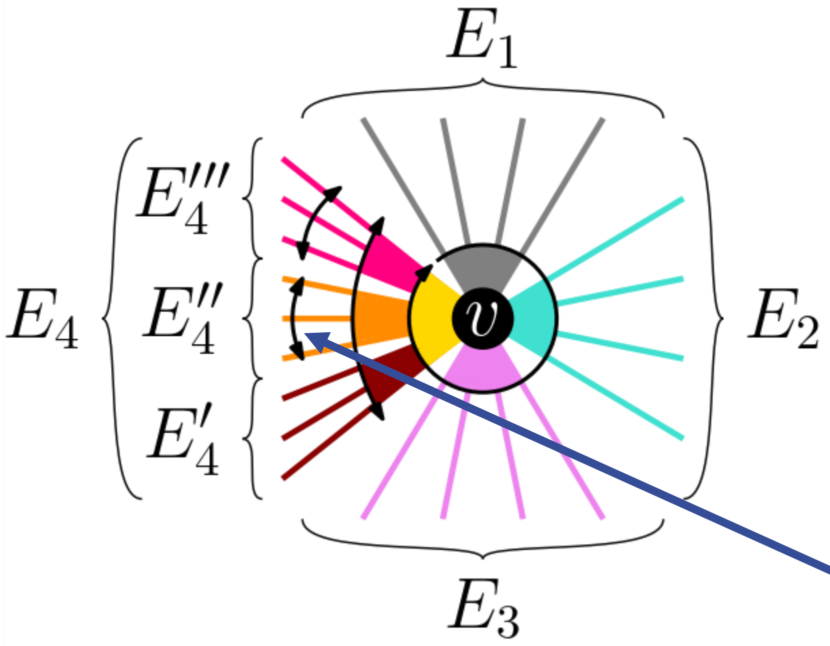

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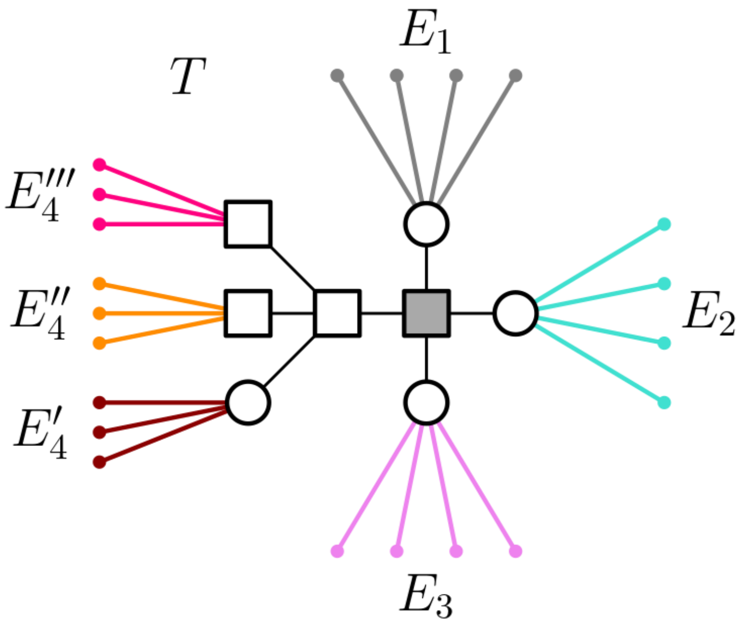
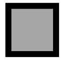
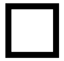
- The edges of and E_4'' have only two possible orders that are the reverse of one another

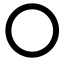
FPQ-trees

- Represent a family of permutations on a set of elements
  - Each element is a leaf



 F-nodes: The order of children is fixed

 Q-nodes: The order of children may be reversed

 P-nodes: The order of children may be arbitrarily permuted

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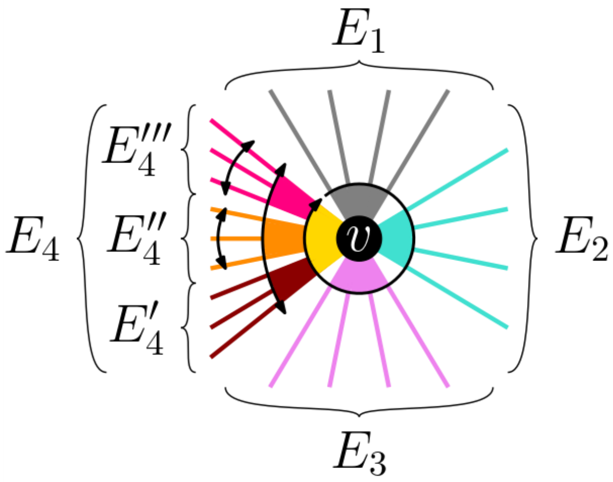

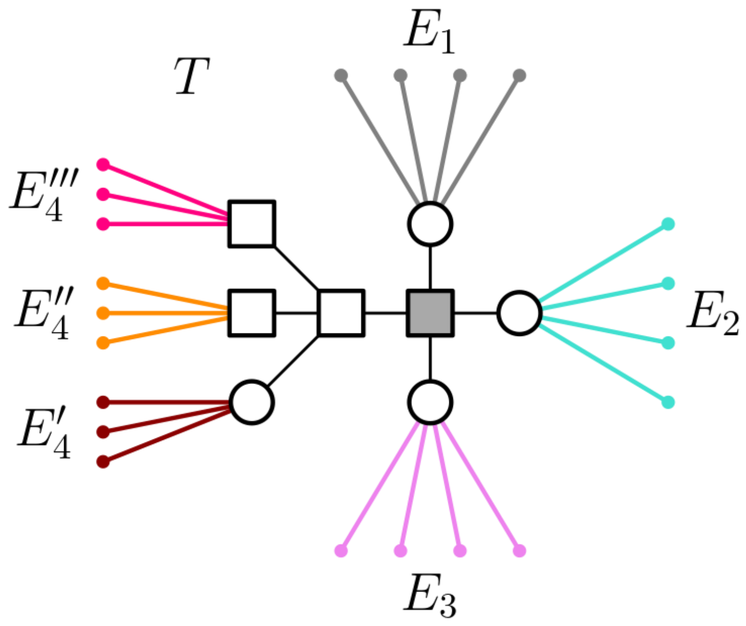

- **F-nodes**: The order of children is fixed
- **Q-nodes**: The order of children may be reversed
- **P-nodes**: The order of children may be arbitrarily permuted

- Embeddings constraints are modeled by means of FPQ-trees
  - Represent the cyclic orders of the edges incident to a vertex
  - Each edge is a leaf in T

Graph Planarity Testing

- Edge crossings negatively affect the readability of graph representations

  Cognitive experiments:
  - Purchase - 1997
  - Purchase, Carrington, Allder - 2002
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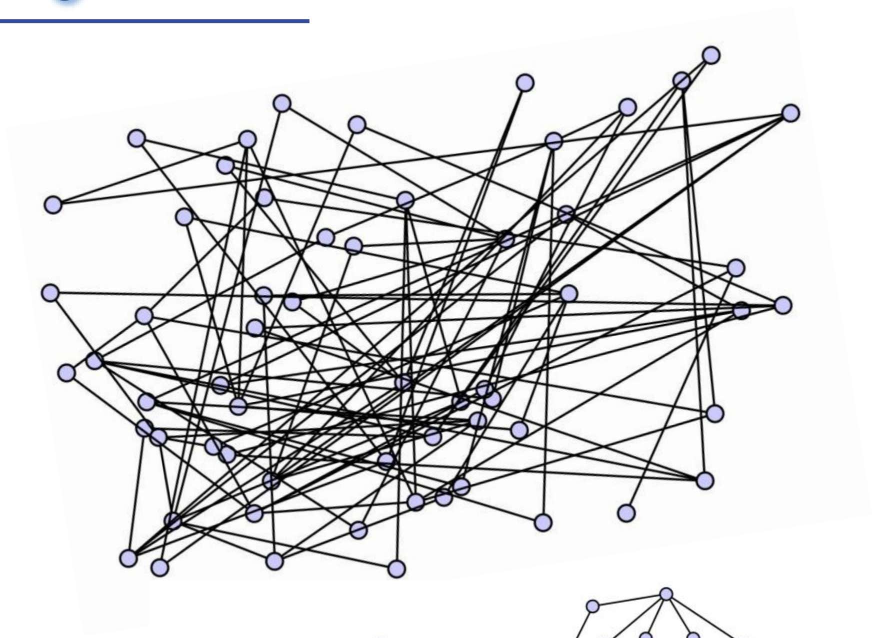

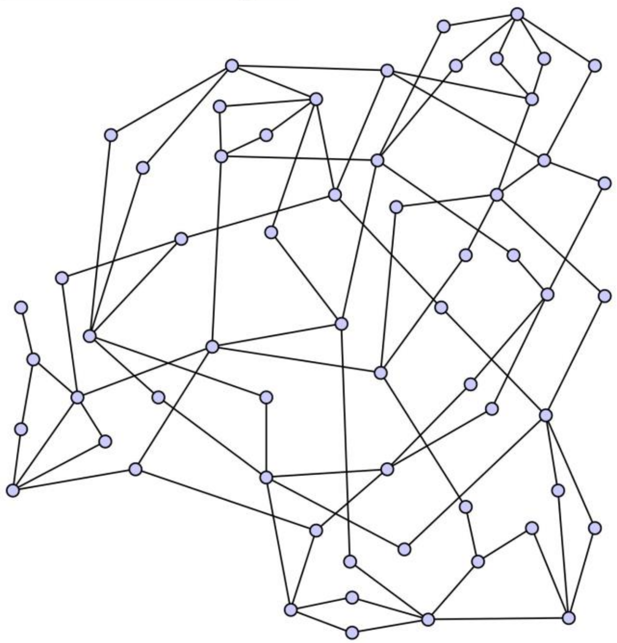

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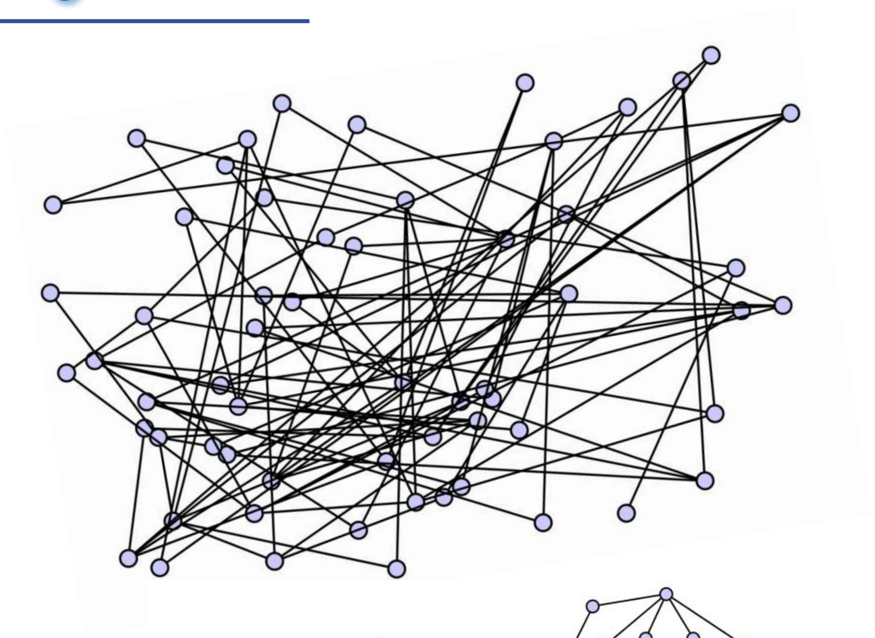
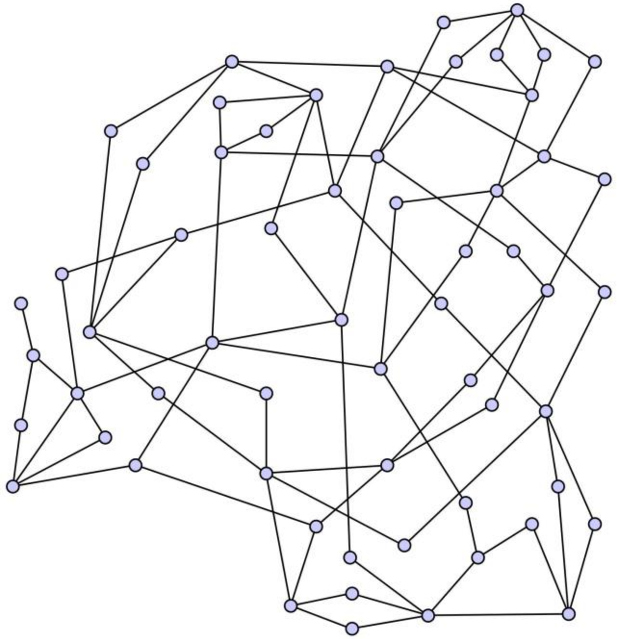
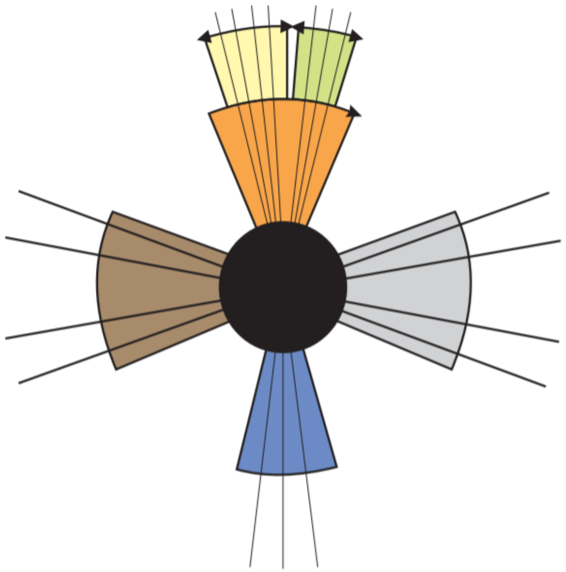


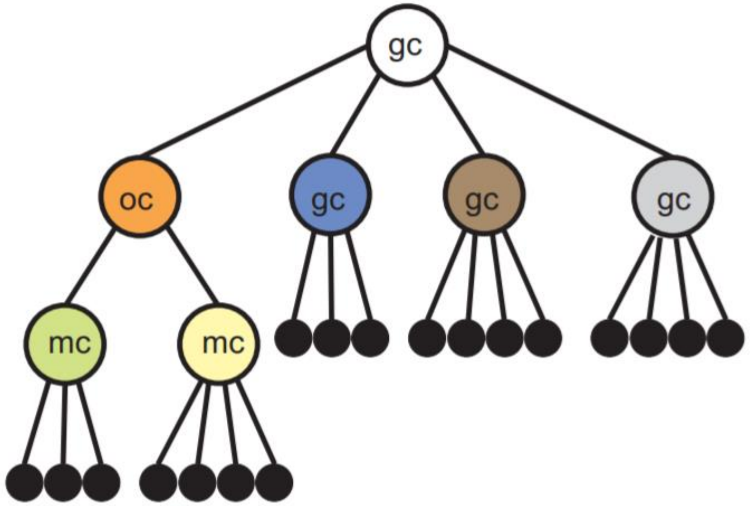


- The graph planarity testing problem is at the heart of graph algorithms and of their applications
  - Remark. Minimizing the total number of crossings in a graph drawing is NP-hard [Garey, Johnson - 1983]

Graph Planarity Testing + Embedding Constraints

- Introduced by [Gutwenger, Klein, Mutzel - 2008]
- They model each hierarchical embedding constraint as a constraint tree





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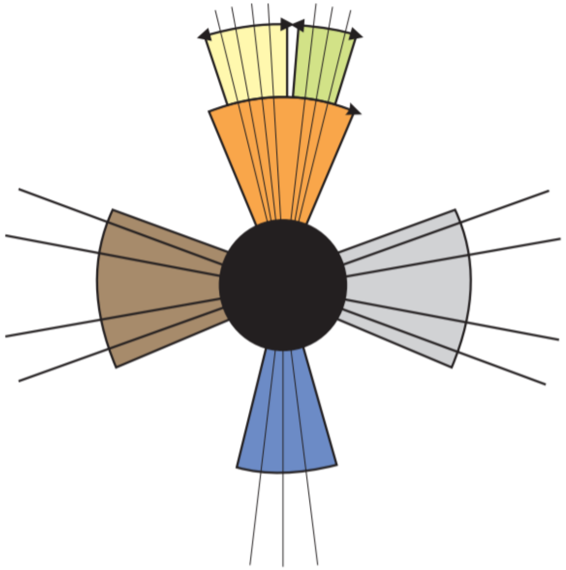

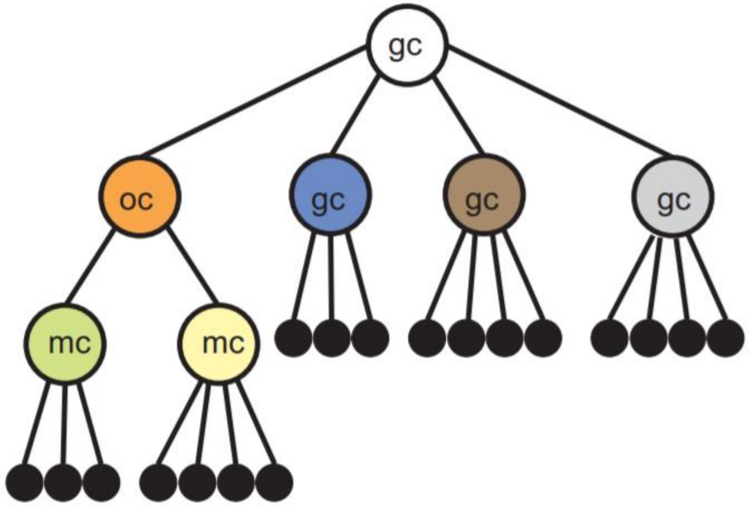

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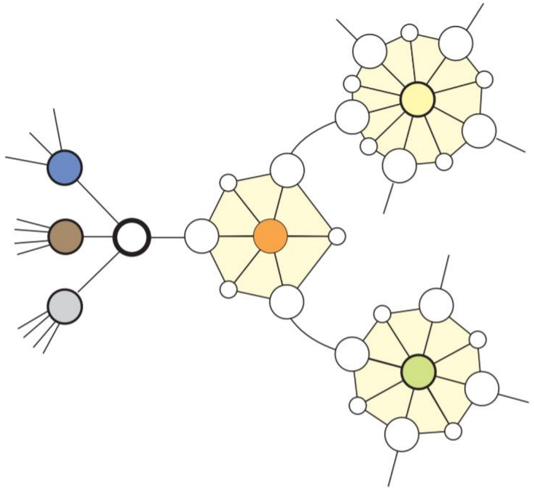

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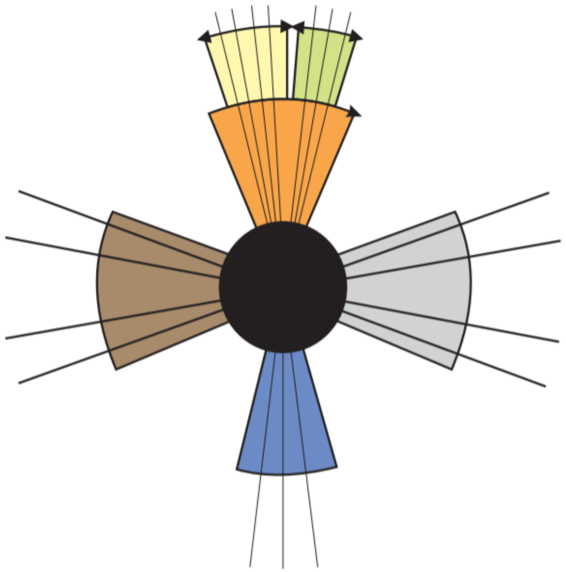

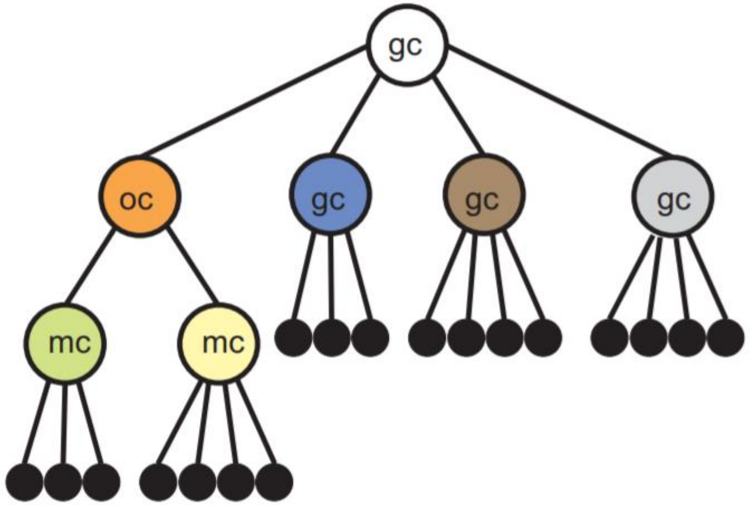

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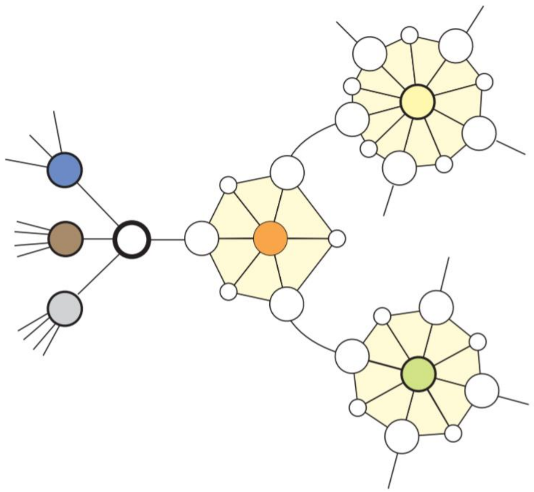
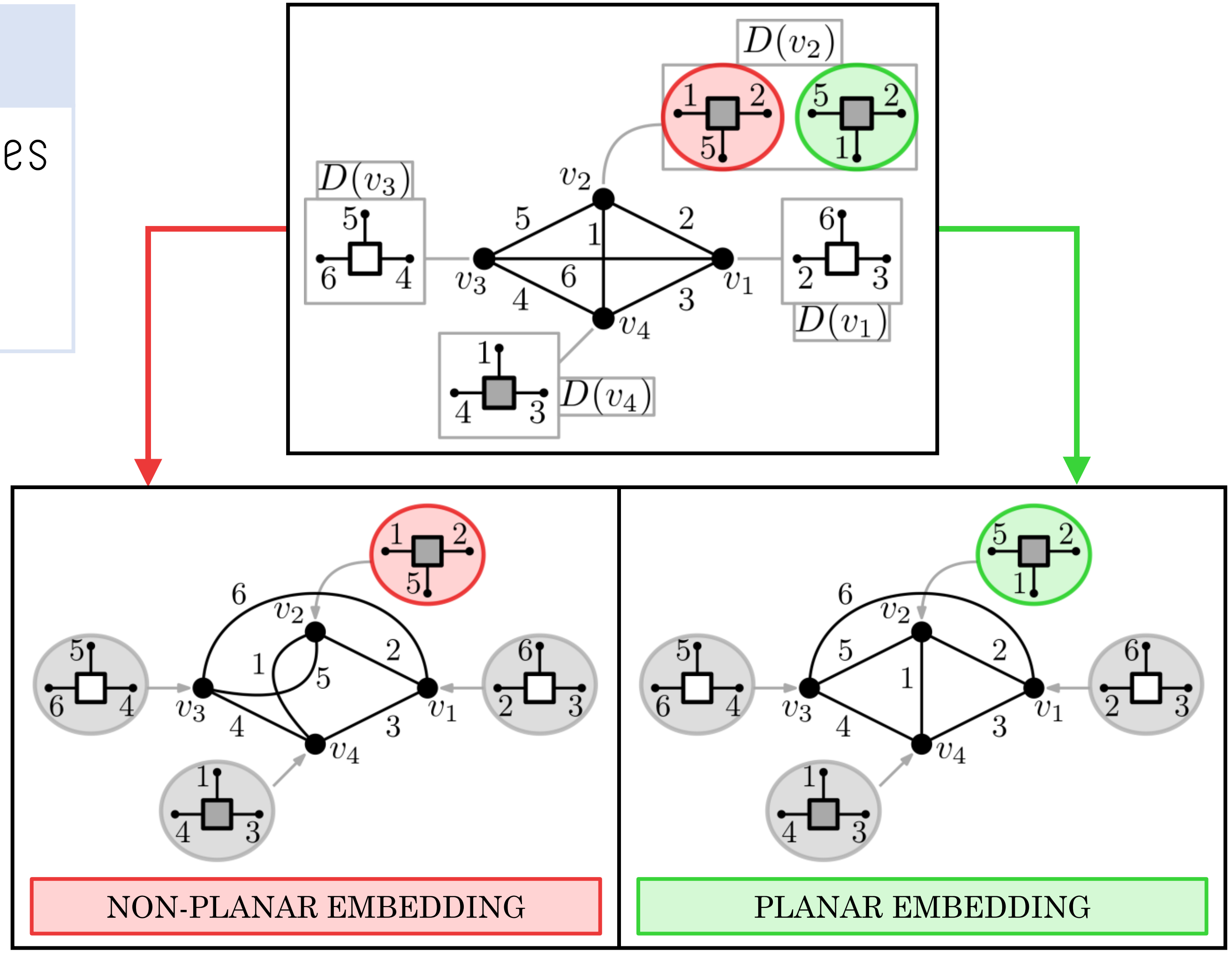


- Constrained planarity testing is linear-time solvable
- Constraint trees ≡ FPQ-trees

FPQ-Choosable Planarity Testing

FPQ-Choosable Graph

A (multi-)graph G and a mapping D that associates each vertex v of G with a set $D(v)$ of FPQ-trees whose leaves represent the edges incident to v.



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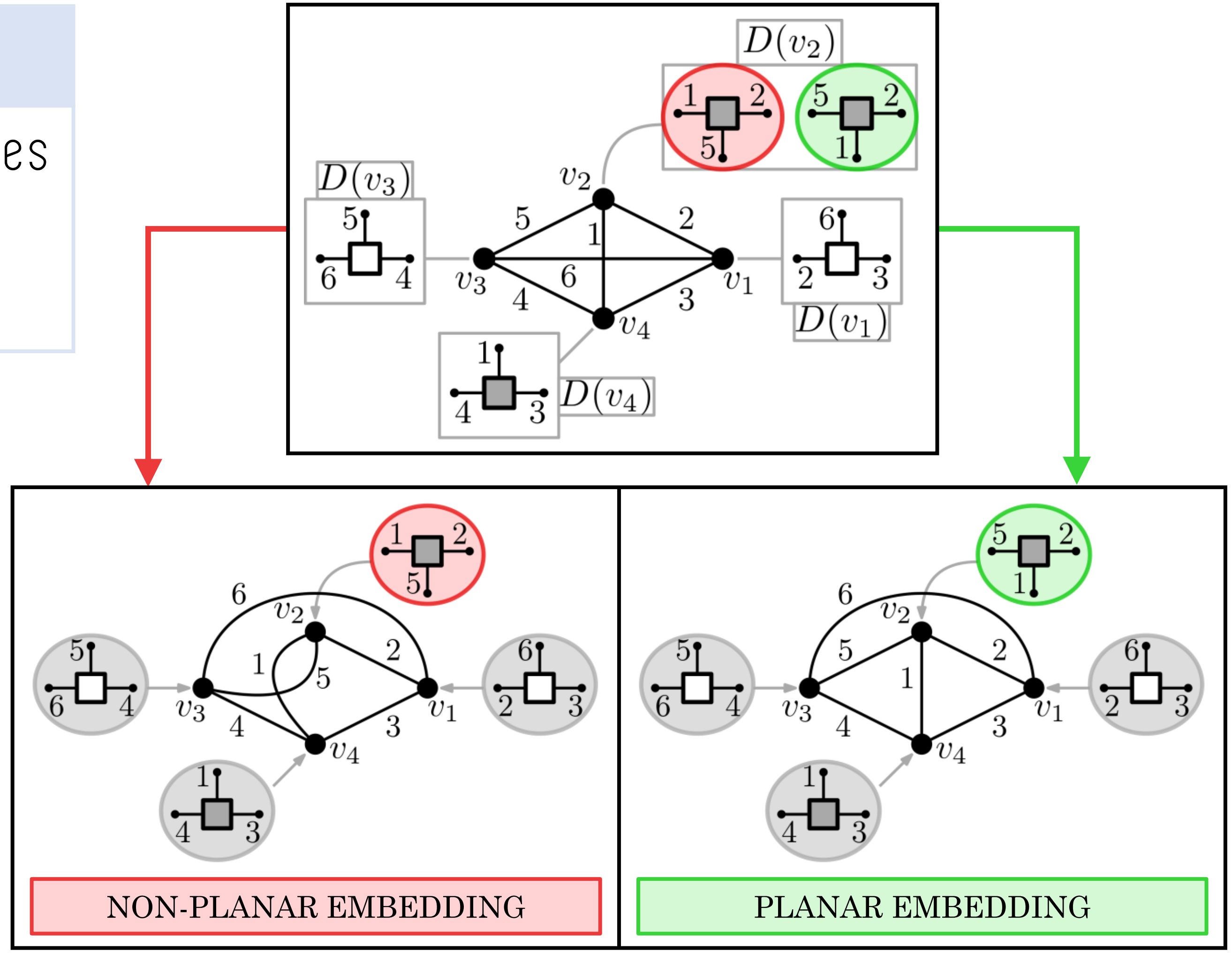
QUESTION: Does G admit a planar embedding such that, for each vertex v, the cyclic order of the edges incident to v is encoded by an FPQ-tree in $D(v)$?



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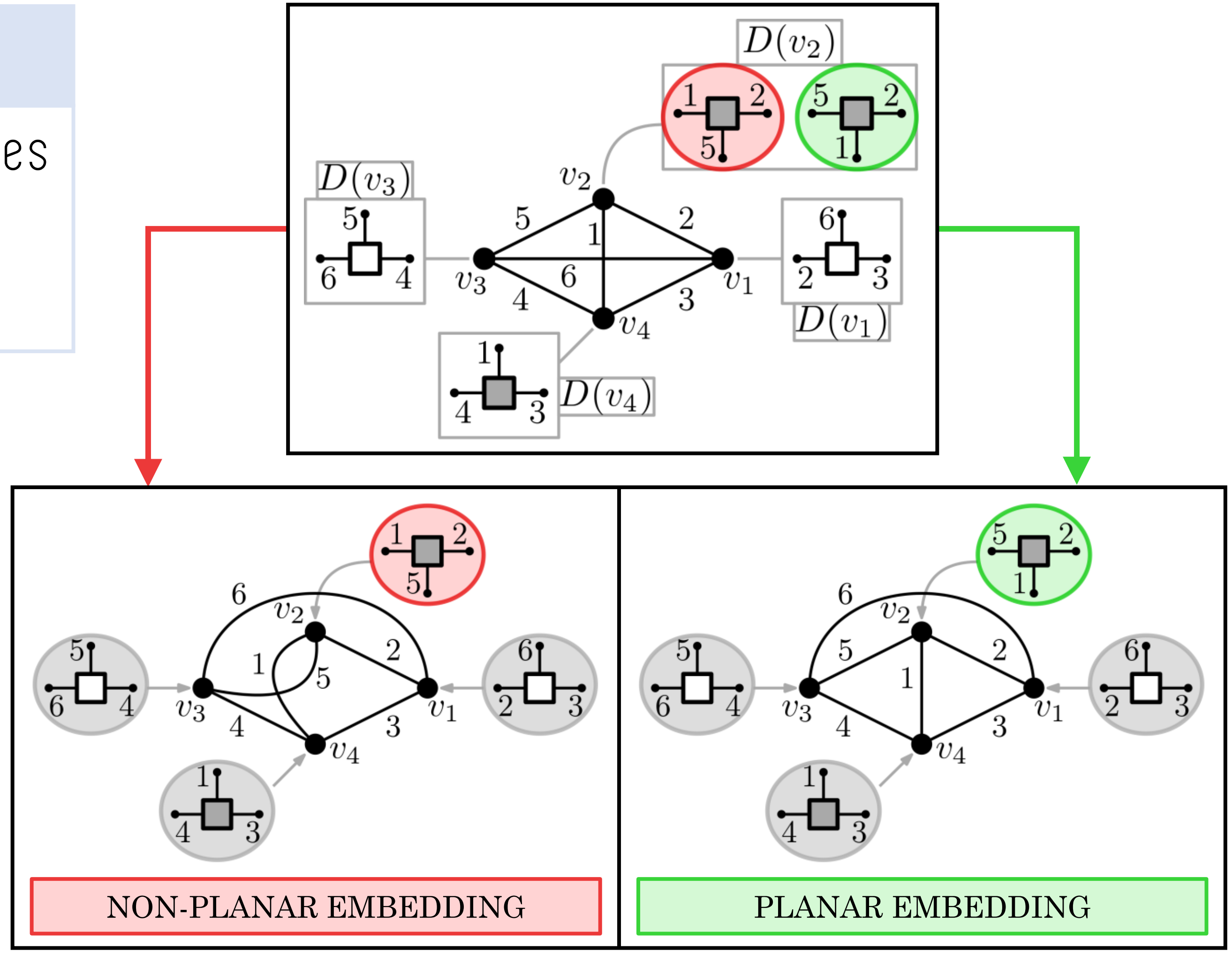

Remark. If $|D(v)| = 1$ for each v, then the problem can be solved in linear time [Gutwenger et al. - 2008]

Our Results

Parameters	Complexity
D_{max}	NP-complete - (Theorem 1)
t	W[1]-hard - (Theorem 2)
D_{max}, t	FPT - (Theorem 3)

D_{max}: Maximum number of FPQ-trees per vertex

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Theorem 3. FPQ-Choosable Planarity Testing is FPT for biconnected graphs, where the parameters are t and D_{max} $\longrightarrow$ $O(D_{max}{}^{\frac{9}{4}t} \cdot n^2 + n^3)$-time algorithm

Theorem 1

FPQ-Choosable Planarity Testing with a bounded number of FPQ-trees per vertex (>1) is NP-complete. It remains NP-complete even whe the FPQ-trees have only P-nodes.

- Reduction from the 3-edge-coloring problem for triconnected cubic non-planar graphs

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Theorem 2

FPQ-Choosable Planarity Testing parameterized by treewidth is W[1]-hard. It remains W[1]-hard even when the FPQ-trees have only P-nodes.

- Parameterized reduction from the list coloring problem

Theorem 3

FPQ-Choosable Planarity Testing is FPT for biconnected graphs, where the parameters are t and $D_{max} \longrightarrow O({D_{max}}^{\frac{9}{4}t} \cdot n^2 + n^3)$-time algorithm

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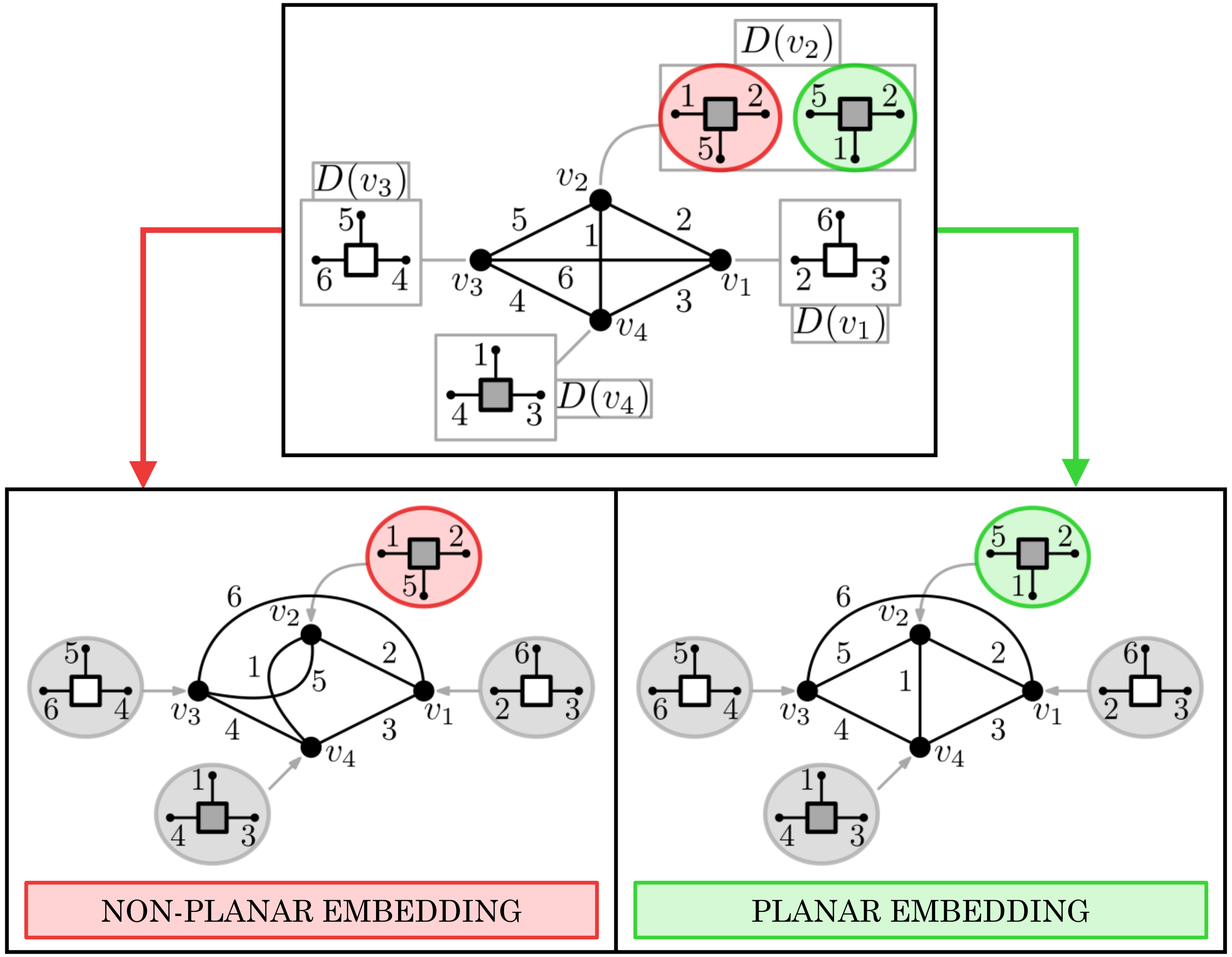
3. At each step of the visit, equip the current node μ with the set $\Psi(\mu)$ of admissible tuples
4. Do we reach the root?
   - YES $\Rightarrow$ (G, D) is FPQ-choosable planar
   - NO: We find a node such that $\Psi(\mu) = \emptyset \Rightarrow (G, D)$ is not FPQ-choosable planar

Admissible Tuple



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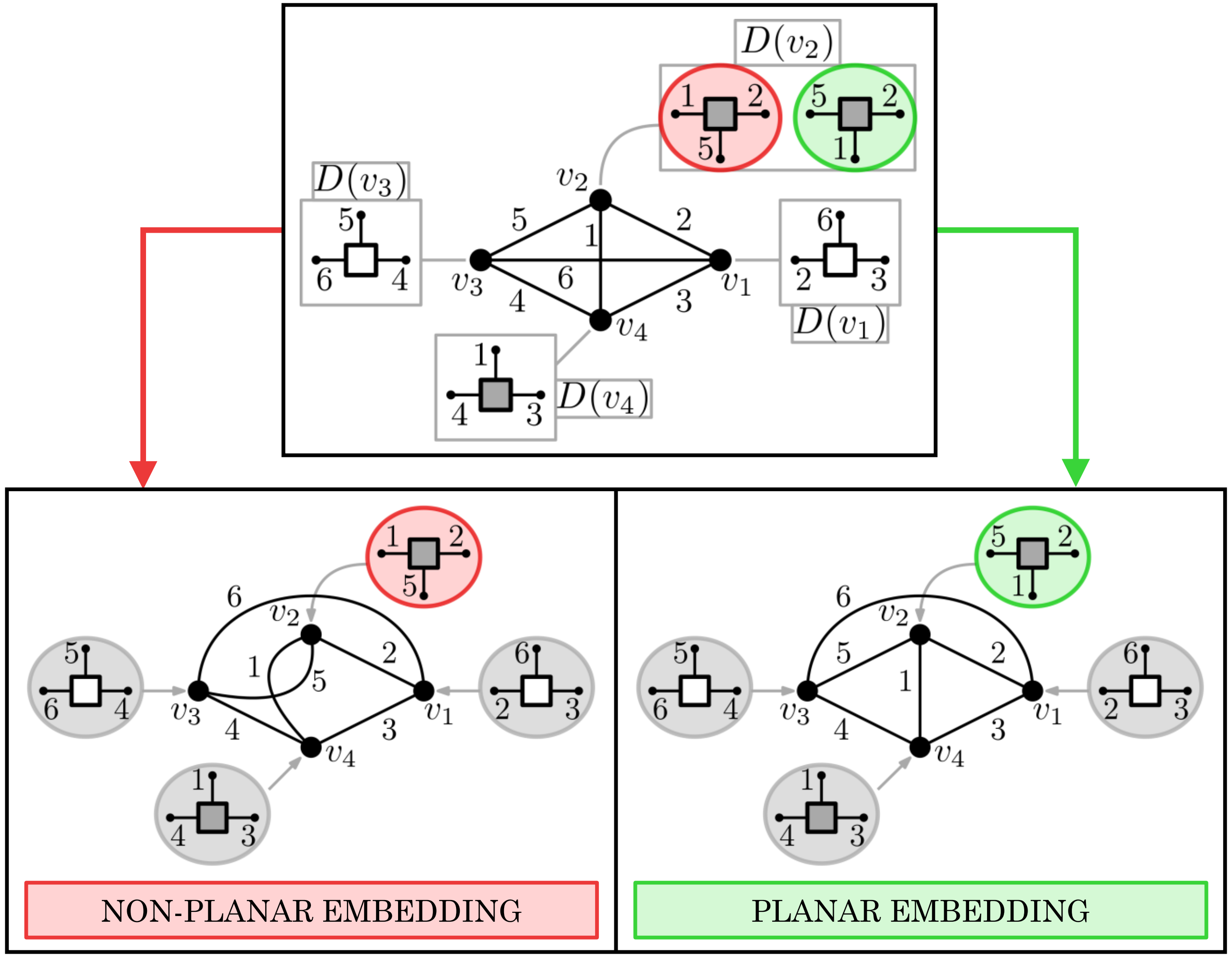

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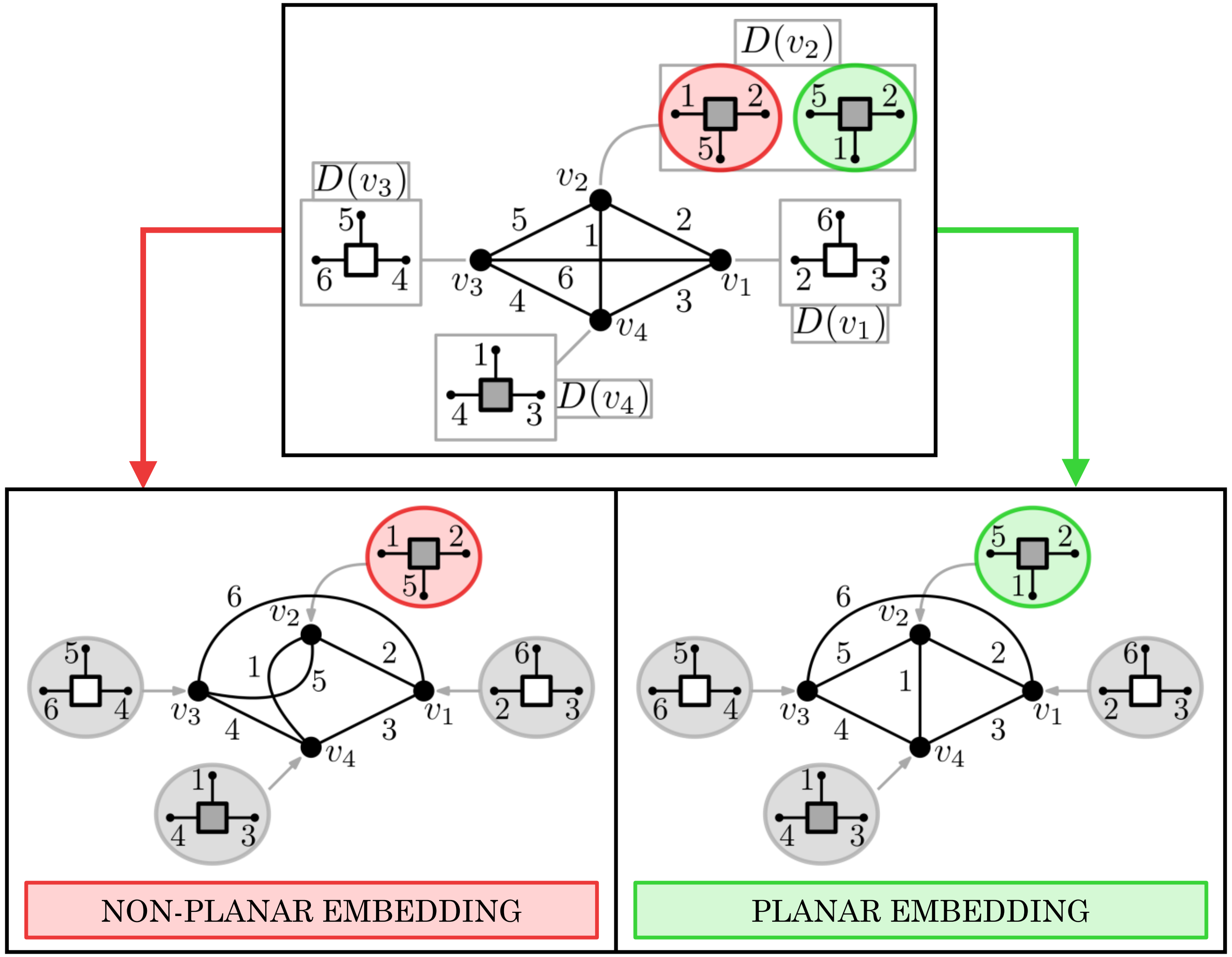
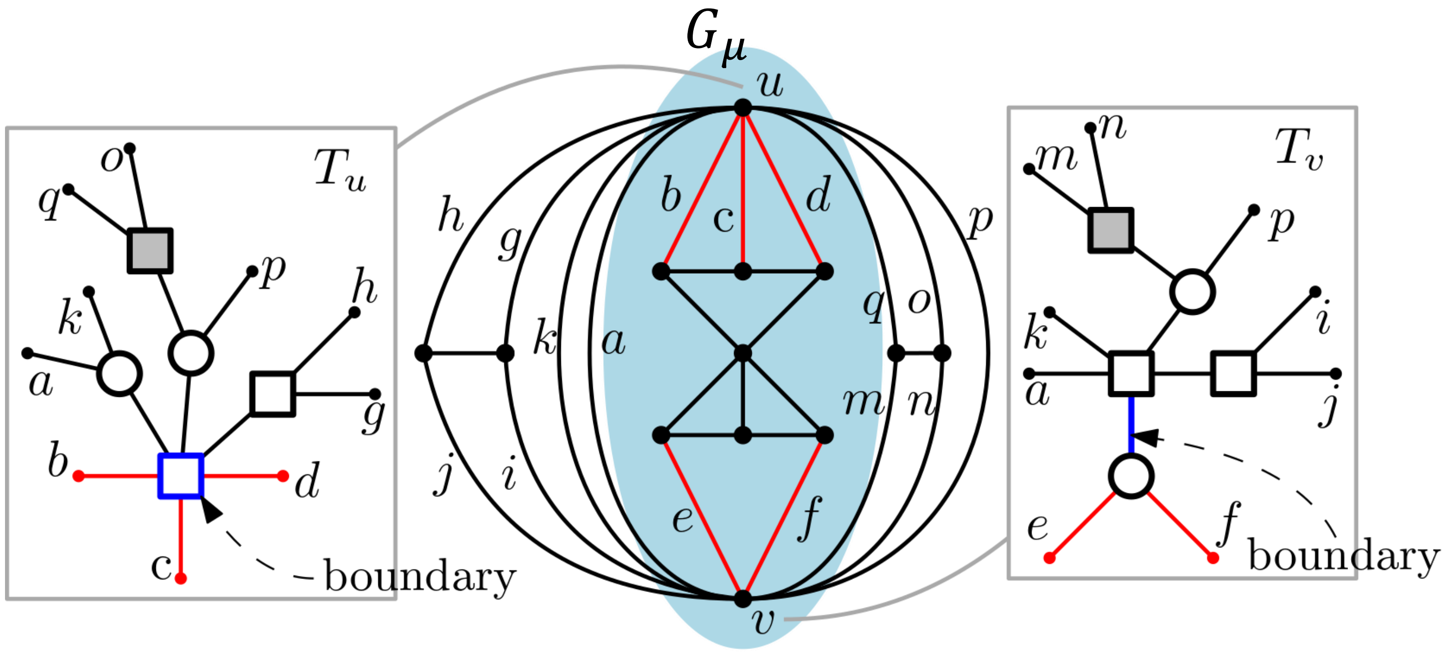


- **Assignment** A is a function that assigns to each vertex v an FPQ-tree $T_v \in D(v)$
- A is **compatible** with G if there exists a planar embedding $\mathcal{E}$ such that, for each v, $\mathcal{E}$ induces a cyclic order of its incident edges that is described by T_v
- A is **consistent** with $\mathcal{E}$

Admissible Tuple

For each internal node μ of $\mathcal{T}$ with poles u and v:

- G_μ is the pertinent graph
- The boundary of T_u is the element that separates the edges that belong to G_μ and the edges that are external to G_μ
- The boundary can be either a Q-node (or F-node) or an edge



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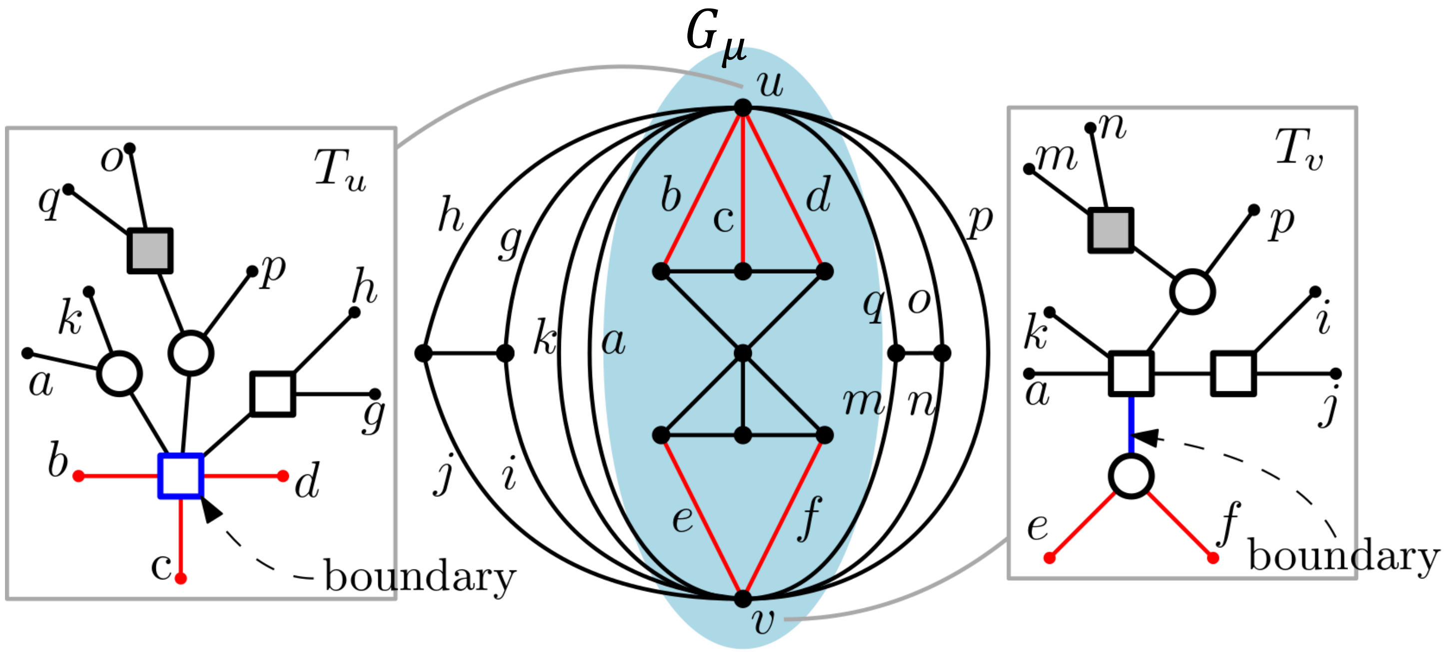

- If the boundary of T_u is a Q- (or F-) node, it imposes an orientation o_u that defines the permutation of its children
- We establish a default orientation and we call it the clockwise orientation

Admissible Tuple

Tuple of a node μ: $\langle T_u, T_v, o_u, o_v \rangle \in D(u) \times D(v) \times \{0,1\} \times \{0,1\}$

clockwise   counter-clockwise

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A tuple is admissible for μ if there exists an assignment A_μ that is consistent with a planar embedding $\mathcal{E}_\mu$ of G_μ

- $\Psi(\mu)$ is the set of admissible tuples for μ
- $\Psi(\mu)$ is computed from the set of admissible tuples of the children of μ
  - Depending on whether μ is an S-, P-, Q-, or R-node

Theorem 3

FPQ-Choosable Planarity Testing is FPT for biconnected graphs, where the parameters are t and $D_{max} \longrightarrow O({D_{max}}^{\frac{9}{4}t} \cdot n^2 + n^3)$-time algorithm

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FPQ-Choosable Planarity Testing is FPT for biconnected graphs, where the parameters are t and D_{max} $\longrightarrow$ $O({D_{max}}^{\frac{9}{4}t}\, n^2 + n^3)$-time algorithm

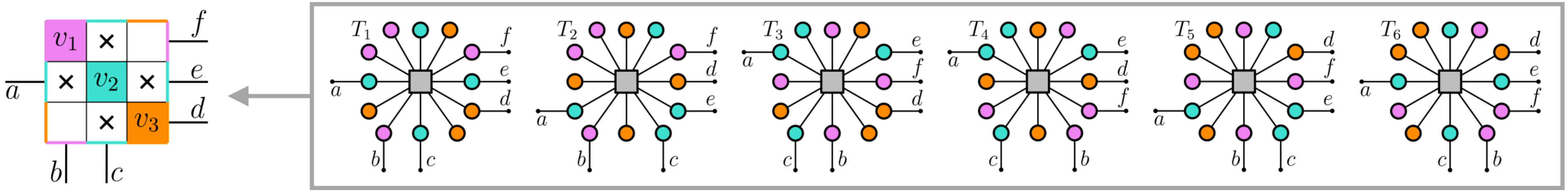
For R-nodes, in order to compute the set of admissible tuples:

- We execute the sphere-cut decomposition of the skeleton of μ
  - It has branchwidth at most b (the branchwidth of G)
- For a graph G with treewidth t and branchwidth $b > 1$, it holds

$$b - 1 \leq t \leq \left\lfloor \frac{3}{2} b \right\rfloor - 1$$ [Robertson, Seymour - 1991]

Remarks

Let G be a clustered n-vertex graph whose clusters have size at most k. Let t be the treewidth of G. If the (multi-)graph obtained by collapsing each cluster of G into a vertex is biconnected, there exists an $O(k!^{\frac{9}{4}t} \cdot n^2 + n^3)$-time algorithm to test whether G is NodeTrix planar with fixed sides.



Each FPQ-tree allows a possible permutation described by the matrix

Open Problems

- Theorem 1 is based on a reduction that associates $\mathbf{6}$ FPQ-trees to each vertex.

  What is the time complexity if $\mathbf{2} \leq D_{max} \leq \mathbf{5}$?

- Is it possible to extend Theorem 3 to simply connected graphs ?

- Improve the time complexity of Theorem 3.

- Apply our approach to other hybrid representation models.

Thank you for your attention

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