On the Number of Delaunay Triangles occurring in all Contiguous Subsequences

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joined work with S. Funke
Motivation

- Subcomplexes of the Delaunay triangulation useful for representing the shape of objects from discrete samples
  - $\alpha$-shapes, $\beta$-skeleton, the crust
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- Restrict temporal samples to shorter time intervals
  - $\alpha$-shapes used to visualize the regions of storm events

[Bonerath et al. '19]
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- Precompute all Delaunay triangles occurring in all contiguous subsequences & index them w.r.t. time, possibly some other parameter ($\alpha$ value, ...) for faster retrieval

[Bonerath et al. ’19]
Some Delaunay Triangulations

- \( P = \{p_1, p_2, \ldots, p_n\} \), \( P_{i,j} := \{p_i, p_{i+1}, \ldots, p_j\} \)
- Example: Incremental construction of \( DT(P) \) via the sequence \( DT(P_{1,3}), DT(P_{1,4}), \ldots DT(P_{1,n}) \)
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\]

![Delaunay Triangulation Diagram](image)
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- $T_{i,j}$: triangles of $DT(P_{i,j})$
- $T := \bigcup_{i<j} T_{i,j}$
- $|T| = ?$
Some Delaunay Triangulations

- \( P = \{p_1, p_2, \ldots, p_n\} \), \( P_{i,j} := \{p_i, p_{i+1}, \ldots, p_j\} \)
- Another example with \(|T| \in \Theta(n^2)|

\[ \begin{array}{c}
  p_1 p_2 p_3 \\
  p_{n/2} \\
  p_n \\
  p_{n/2} + 1 \\
  p_n + 2
\end{array} \]
What is the expected number of Delaunay triangles in contiguous subsequences for arbitrary point sets $P$ ordered uniformly at random?
Counting Delaunay Edges and Triangles

• Let $E_T := \{e \mid \exists t \in T : e \text{ edge of } t\}$
• Assume non-degeneracy of $P$
  – No 4 co-circular points, no 3 co-linear points
• Proof:
  1. Bound the expected number of Delaunay edges
  2. Show linear dependence between the number of Delaunay triangles and Delaunay edges
Lemma 1: Any $e = \{p_i, p_j\} \in E_T$ appears in $DT(P_{i,j})$

There exists some triangle $t \in T$ which uses $e$, so for suitable $a \leq i, b \geq j$, $e$ appears in $DT(P_{a,b})$: 

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There exists some triangle \( t \in T \) which uses \( e \), so for suitable \( a \leq i, b \geq j \), \( e \) appears in \( DT(P_{a,b}) \):

\[ \Rightarrow e \in DT(P_{i,j}) \]
Lemma 2: For $j > i + 1$: $\Pr[e \in DT(P_{i,j})] < \frac{6}{j-i}$

- $DT(P_{i,j})$ is a planar graph with $j - i + 1$ nodes
  - Euler’s formula: $\leq 3(j - i + 1) - 6$ edges
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- All points in $P_{i,j}$ are equally likely to be $p_i/p_j$
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- All points in \( P_{i,j} \) are equally likely to be \( p_i/p_j \)
- So choosing \( p_i \) and \( p_j \) out of \( P_{i,j} \) is the same as choosing one edge (amongst all \( \binom{j-i+1}{2} \) possible edges) in a graph with \( j - i + 1 \) nodes and \( \leq 3(j - i + 1) - 6 \) edges
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  $\Rightarrow Pr[e \in DT(P_{i,j})] \leq \frac{3(j-i+1)-6}{\binom{j-i+1}{2}} < \frac{6}{j-i}$
Lemma 3: The expected size of $E_T$ is $\Theta(n \log n)$

Lower bound:

- Within $P_{1,1}, \ldots, P_{1,n}$, $p_1$'s nearest neighbor changes $\Theta(\log n)$ times in expectation
  - Applies to all $p_i$
- Nearest neighbor graph is a subgraph of the Delaunay triangulation

$$\Rightarrow E[|E_T|] \in \Omega(n \log n)$$
Lemma 3: The expected size of $E_T$ is $\Theta(n \log n)$

Upper bound: Use linearity of expectation to sum over all potential edges of $E_T$
- Edges $\{p_i, p_{i+1}\}$ always exist
- Other edges $\{p_i, p_j\}$ exist with probability $< \frac{6}{j-i}$

\[
E[|E_T|] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr[\{p_i, p_j\} \in E_T]
\]

\[
\leq \sum_{i=1}^{n-1} \left[ 1 + \sum_{j=i+2}^{n} \frac{6}{j-i} \right] = (n - 1) + 6 \sum_{i=1}^{n-1} \sum_{j=2}^{n-i} \frac{1}{j}
\]

\[
\leq (n - 1) + 6 \sum_{i=1}^{n-1} H_n = O(n \log n)
\]
Delaunay edges used by many Delaunay triangles

\[ DT(P_{1,3}) \]
Delaunay edges used by many Delaunay triangles

\[ DT(P_{1,4}) \]
Delaunay edges used by many Delaunay triangles

\[ DT(P_{1,5}) \]
Delaunay edges used by many Delaunay triangles

$\text{DT}(P_{1,5}) \rightarrow \text{Edge } \{p_1, p_2\} \text{ used by many Delaunay triangles}$
Lemma 4: $|T| \in \Theta(|E_T|)$ for arbitrary orderings of $P$

- For each triangle in $T$, at most 3 edges exist in $E_T$
  \[ \Rightarrow |E_T| \leq 3|T| \]
- For upper bound on edges, charge triangles to edges:
  - Delaunay triangle $p_a p_b p_c$ ($a < b < c$) exists in $DT(P_{a,c})$
  - In $DT(P_{a,c})$, at most one other triangle uses edge $\{p_a, p_c\}$
  \[ \Rightarrow \text{Charging } T\text{'s triangles } p_a p_b p_c \text{ to } \{p_a, p_c\} \text{ ensures at most two triangles are charged to each edge in } E_T \]
  \[ \Rightarrow |T| \leq 2|E_T| \]
Putting it all together

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What is the expected number of Delaunay triangles in contiguous subsequences for arbitrary point sets $P$ ordered uniformly at random?

$$E[|E_T|] = \Theta(n \log n) \text{ and } |T| \in \Theta(|E_T|)$$

$$\Rightarrow E[|T|] = \Theta(n \log n)$$
## Experimental results & data

$n$ points sampled from the unit square, averaged over 20 runs

| $n$ | $|\bigcup_{j \leq n} T_{1,j}|$ | $|T|$ | $T$ computation time |
|-----|-------------------------------|------|----------------------|
| $2^{15}$ | 196,168                       | 2,860,956 | 6,309 ms              |
| $2^{16}$ | 392,592                       | 6,267,247 | 14,229 ms             |
| $2^{17}$ | 785,879                       | 13,622,094 | 32,817 ms            |
| $2^{18}$ | 1,572,292                     | 29,425,885 | 70,545 ms            |
| $2^{19}$ | 3,144,770                     | 63,210,634 | 155,370 ms          |
| $2^{20}$ | 6,290,562                     | 135,134,028 | 347,186 ms         |
| $2^{21}$ | 12,581,989                    | 287,719,166 | 771,705 ms        |
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More data

| $n$ | $|T|$      | $T$ time  | $|\bigcup_{j \leq n} T_{1,j}|$ | $T_{1,n}$ time |
|-----|-----------|-----------|-------------------------------|----------------|
| $2^{15}$ | 2,860,956 | 6,309 ms  | 196,168                       | 260 ms         |
| $2^{16}$ | 6,267,247 | 14,229 ms | 392,592                       | 745 ms         |
| $2^{17}$ | 13,622,094 | 32,817 ms | 785,879                       | 1,779 ms       |
| $2^{18}$ | 29,425,885 | 70,545 ms | 1,572,292                     | 4,068 ms       |
| $2^{19}$ | 63,210,634 | 155,370 ms| 3,144,770                     | 9,008 ms       |
| $2^{20}$ | 135,134,028 | 347,186 ms| 6,290,562                     | 20,374 ms      |
| $2^{21}$ | 287,719,166 | 771,705 ms| 12,581,989                    | 44,082 ms      |