36th European Workshop on Computational Geometry

Disjoint tree-compatible plane perfect matchings

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Compatibility of graphs

Setting: set $S$ of $2n$ points in the plane in general position
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disjoint compatible
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  ○ vertices: all plane perfect matchings on $S$
  ○ edge $(M_i, M_j) \iff M_i$ and $M_j$ are compatible
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- compatibility graph for matchings is connected
  convex point set: [C. Hernando, F. Hurtado and M. Noy; 2002.]
  general point set: [M.E. Houle, F. Hurtado, M. Noy and E. Rivera-Campo; 2005.]
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- diameter is $O(\log n)$ [ABDGHHKMRSSUW; 2009.]
  and $\Omega(\log n / \log \log n)$ [A.Razen; 2008.]
Disjoint compatibility of matchings
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- **disjoint** compatibility graph:
  - vertices: all plane perfect matchings on $S$
  - edge ($M_i, M_j$) $\iff M_i, M_j$ are disjoint compatible
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- disjoint compatibility graph for matchings on point sets of $2n \geq 6$ points in convex position is disconnected

[O. Aichholzer, A. Asinowski and T. Miltzow; 2015.]
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**Alternative way of defining compatibility?**
Disjoint tree-compatibility of matchings
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• consider ‘compatibility’ via
disjoint compatible plane spanning trees
Disjoint tree-compatibility of matchings

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not compatible!
Disjoint tree-compatibility of matchings

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matchings are \textbf{disjoint tree-compatible}
(for short: tree-compatible)
Disjoint tree-compatibility of matchings

- consider 'compatibility' via disjoint compatible plane spanning trees

- disjoint tree-compatibility graph $G_{2n}$:
  - vertices: all plane perfect matchings on $S$
  - edge $(M_i, M_j) \iff M_i, M_j$ disjoint tree-compatible
Disjoint tree-compatibility of matchings

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**ATTENTION:** different from (disjoint) compatibility!

- disjoint tree-compatible $\not\implies$ compatible
- disjoint compatible $\not\implies$ disjoint tree-compatible
Disjoint tree-compatibility of matchings

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**ATTENTION:** different from (disjoint) compatibility!

\[
\text{disjoint compatible} \not\Rightarrow \text{compatible} \\
\text{disjoint compatible} \not\Rightarrow \text{disjoint tree-compatible}
\]
$G_8$
Disjoint tree-compatible plane perfect matchings

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- more sophisticated arguments yield bound 5
Lower bound
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(i) $M$ tree-compatible to $E_1 \Rightarrow$ no green perimeter edge (analogously for $E_2$)
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(i) $M$ tree-compatible to $E_1 \Rightarrow$ no green perimeter edge (analogously for $E_2$)

(ii) $M$ tree-compatible to $O_1 \Rightarrow$ at most one green perimeter edge, which is $g$ (analogously for $O_2$ and $r$)
Lower bound

(iii) $M$ and $M'$ tree-compatible $\Rightarrow$ at least two perimeter edges in common
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Summary / Open problems
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- The disjoint tree-compatibility graph $G_{2n}$ is connected if and only if $2n \geq 10$.
- The diameter in that case is either 4 or 5.
- **Conjecture.** The diameter for all $2n \geq 18$ is 4.
  
  \[(\text{diam}(G_{2n}) = 5 \text{ for } n \in \{5, 6, 7, 8\} \text{ and diam}(G_{18}) = 4)\]
- Is $G_{2n}$ connected for general point sets (which $n$)?
- Compatibility via other graph classes?

Ongoing work: disjoint path-compatibility
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