Excercise 1. Skewness: Cycle vs. Paths

We are given a non-planar biconnected graph $G = (V, E)$ without multiple edges or self-loops as an input. Recall that in the Skewness ILP we have a binary variable $s_e$ for each $e \in E$ that is 1 if and only if $e$ is deleted to obtain a planar graph. Thus, edges in the computed maximum planar subgraph have $s_e = 0$. Furthermore, there is a constant $D \in \mathbb{N}$ such that we have binary variables $c_\alpha$ for each cycle $\alpha$ with $|\alpha| \leq D$. Variable $c_\alpha$ is 1 if and only if $\alpha$ represents a face in the computed maximum planar subgraph.

Consider a cycle $\alpha$ in $G$ with $|\alpha| \leq D$. Let $a_1, b_1, a_2, b_2$ denote four of its distinct (but not necessarily neighboring) vertices in this cyclic order around $\alpha$. Assume that there are two vertex-disjoint paths $A = a_1 \rightarrow a_2$ and $B = b_1 \rightarrow b_2$, both of which are internally-vertex-disjoint from $\alpha$.

What can you say about feasible solutions w.r.t. the variables involved in this subgraph? Try to write your insight as a linear constraint.

▶ Sketch of solution.

If the cycle is realized as a face, then both paths $A, B$ need to be outside. But then they would cross. Thus, either the cycle is not chosen as a face, or at least one edge in one of the paths needs to be removed. As a linear constraint we may write:

$$c_\alpha \leq \sum_{e \in A \cup B} s_e.$$

We call these the cycle-two-paths constraints; they are part of the mentioned zoo.

Excercise 2. Skewness: Paths vs. Cycles

Consider a vertex $v \in V$ with incident edges $F \subset E$, $|F| \geq 3$. Let $e_1, e_2 \in F$. Assume that $G$ contains two cycles $\alpha, \beta$ of length at most $D$, both of which traverse $v$ exactly once. Cycle $\alpha$ enters $v$ via $e_1$ and leaves via $e_2$. Inversely, $\beta$ enters via $e_2$ and leaves via $e_1$.

(a) Assume that the maximum planar subgraph would be biconnected. What could you say about feasible solutions w.r.t. the variables involved in the subgraph $F \cup \alpha \cup \beta$?

(b) How does the fact that the MPS may be non-biconnected ruin this argument?

(c) Consider a path internally-vertex-disjoint from $\alpha \cup \beta$ that connects $v$ to another vertex on $\alpha \cup \beta$. How and why can you now use the insight of (a) (and (b))? Try to write it as a linear constraint.

▶ Sketch of solution.

ad a) If both $\alpha$ and $\beta$ constitute faces, the fact that they both use $e_1$ and $e_2$ in different orientations tells us that $v$ has to have degree 2 in the MPS; this means that under these circumstances every single edge of $F' := F \setminus \{e_1, e_2\}$ needs to be deleted. We may write this as multiple constraints:

$$c_\alpha + c_\beta - 1 \leq s_e \quad \forall e \in F'.$$
ad b) In a non-biconnected graph, we may have faces that are not (simple) cycles, but closed walks—i.e., some vertices/edges may be repeated. Since we only want to consider cycles in the ILP, $\alpha$ may be the “representative” of a closed walk $\beta$ with $\alpha \subset \beta$. In $\beta$, the edges $F'$ may be traversed but we simply “forget” this when only looking at $\alpha$.

ad c) Let $P$ be any path as described in the question. Now, the argument from (a) holds in the following sense: if all of $P$ would be in the MPS, as well as (all of) $\alpha$ and $\beta$, then no edge of $P$ can be traversed twice by a single face; this would constitute a contradiction. Thus, either $\alpha$, $\beta$ or $P$ needs to have one edge removed in the solution. We may write this as

$$c_\alpha + c_\beta - 1 \leq \sum_{e \in P} s_e.$$ 

We call these the path-two-cycles constraints; they are part of the mentioned zoo.

Excercise 3. Skewness: Faces and Kuratowski

Recall the Skewness ILP requires $\sum_{e \in K} s_e \geq 1$ for every Kuratowski subdivision $K$ in $G$.

(a) Assume there is a cycle $\alpha \in K$ with $|\alpha| \leq D$. Can you rewrite the above Kuratowski constraint to use $c_\alpha$?

(b) Generalize the constraint to not consider a single cycle, but an (arbitrary) set of cycles $C$.

Sketch of solution.

ad a) Either some edge of $K$ outside of $\alpha$ needs to be removed, or $\alpha$ needs to be broken. We may thus write

$$\sum_{e \in K \setminus \alpha} s_e \geq c_\alpha.$$ 

ad b) If all cycles are chosen, we need to remove at least one remaining edge in $K$. But a single non-chosen cycle may invalidate that:

$$\sum_{e \in K \setminus \bigcup_{\alpha \in C} \alpha} s_e \geq \sum_{\alpha \in C} c_\alpha - (|C| - 1).$$

We call these Kuratowski-cycle constraints; they are part of the mentioned zoo.

Excercise 4. Genus: Closed Walks with Simplicial Elements

In embeddings, in particular those on a higher genus surface, a face may have simplicial edges or vertices, i.e., they appear multiple times along the facial walk around a face (face tracing).

(a) What is the minimum length of a closed walk that contains a simplicial edge? What for a simplicial vertex? Try to argue as concisely as possible.

(b) Answer the above question in dependency on the graph’s girth.

Sketch of solution.

a) 8 and 6 edges, respectively:

b) Let $g$ be the girth of $G$. The cycles at the left/right need to have length at least $g$. Thus $2g + 2$ and $2g$, respectively.