Non-Planarity Measures and Small Cycles

EuroCG 2020 – PhD School

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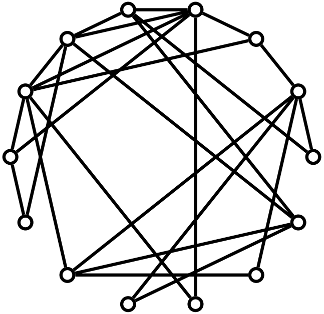
Planarity



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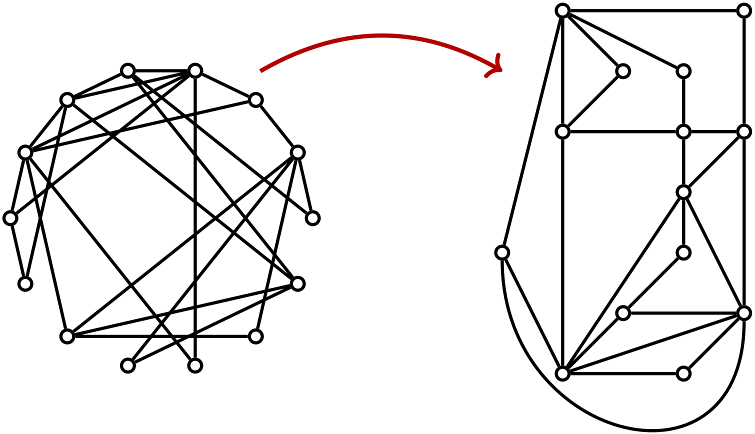

Planarity helps **a lot** algorithmically!

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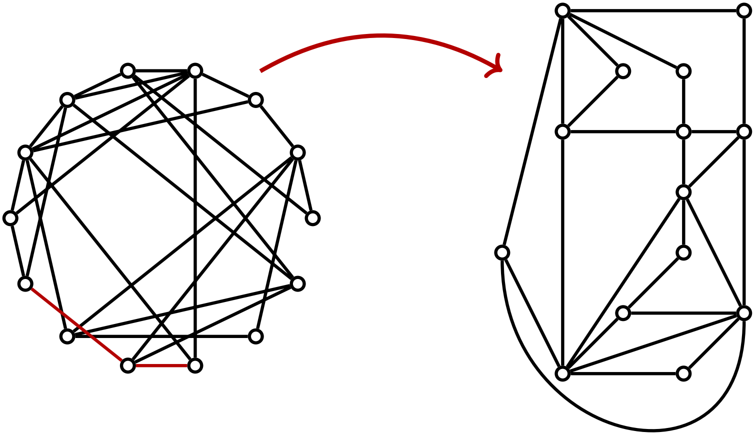

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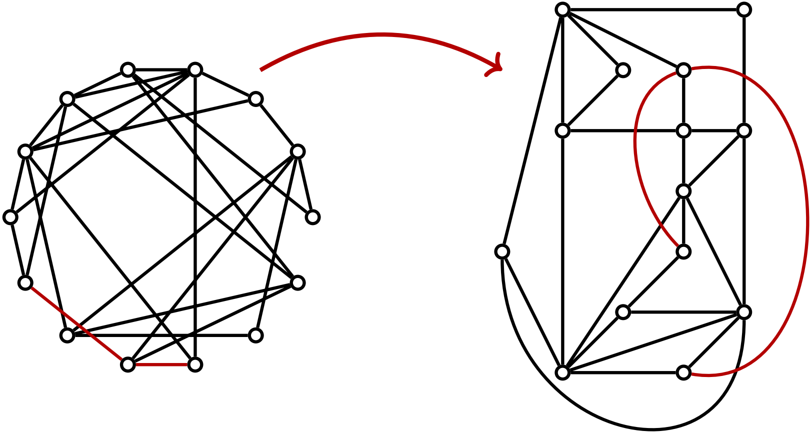
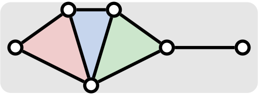


Euler

Theorem (Euler's formula)

Given a planarly drawn graph with n vertices, m edges, and f faces, then:

$$n - m + f = 2$$



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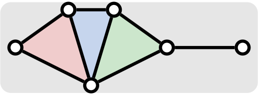
Consider the number of face-edge incidences:

≥ 3 edges per face

≤ 2 faces per edge



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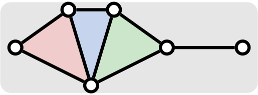

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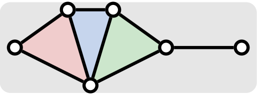

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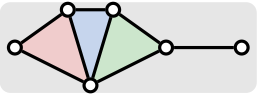

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Bipartite graphs (girth ≥ 4): ≥ 4 edges per face $\implies m \leq 2n - 4$

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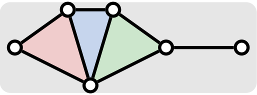

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Girth g: $\geq g$ edges per face $\implies m \leq \frac{g}{g-2}(n-2)$

high girth $\rightarrow$ few edges

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Theorem (Kuratowski, 1930)

*Graph G is **non-planar***

$$\iff$$

*G contains a K_5- or $K_{3,3}$-**subdivision** as a subgraph.*

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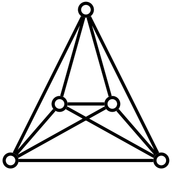
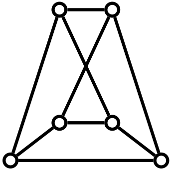


K_5



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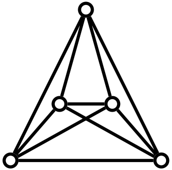

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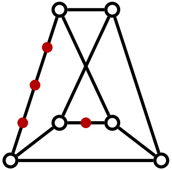

$K_{3,3}$-subdivision

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What does close mean?

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Can we gain an algorithmic advantage is graph is close to planar?
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- **Crossing Number:** min. number of crossings in the plane $\mathrm{cr}(G)$
- **Skewness:** min. number of edge deletions to obtain planar graph $\mathrm{sk}(G)$
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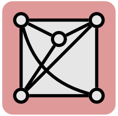
Relating Measures of Non-Planarity

Lemma

$\mathrm{cr}(G) \geq \mathrm{sk}(G) \geq \gamma(G)$



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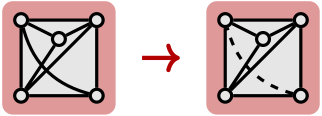

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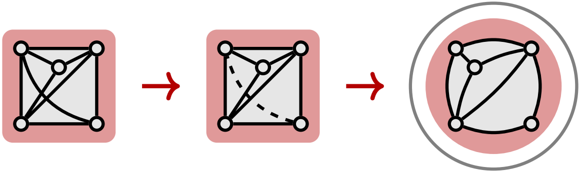

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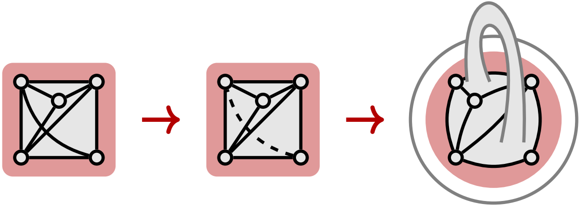

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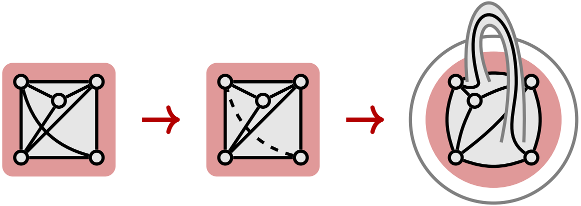

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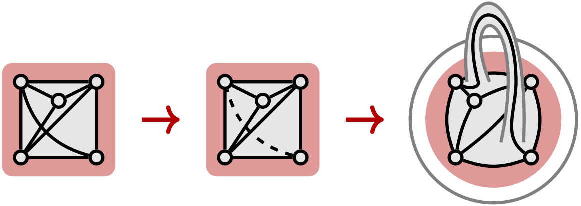

... and the gaps can be **large**!

Exploiting Low Non-Planarity

$$\{G : \gamma(G) \leq k\} \supset \{G : \mathrm{sk}(G) \leq k\} \supset \{G : \mathrm{cr}(G) \leq k\}$$

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- finer-grained parameterization may be more practical
- more tractable in practice, reasonable exact algorithms & strong heuristics

How to **exactly** compute
$\mathrm{cr}(G)$, $\mathrm{sk}(G)$, and $\gamma(G)$?

Integer Linear Programming

Formulate the problem as an **ILP**

$$
\begin{aligned}
\max \quad & c^\top x && (1)\\
s.t. \quad & Ax \le b && (2)\\
& x \ge 0 && (3)\\
& x \in \mathbb{Z}^n && (4)
\end{aligned}
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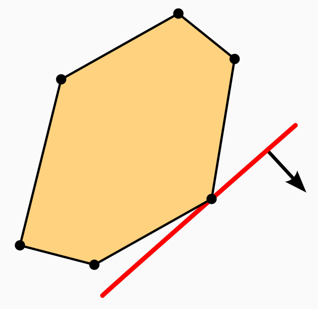

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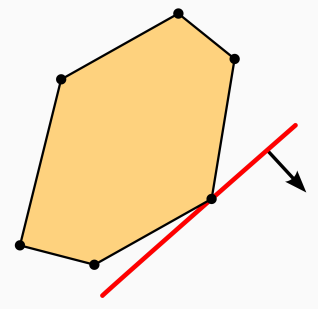
- fast solvers using branch-and-cut



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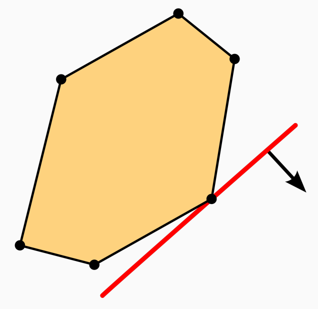

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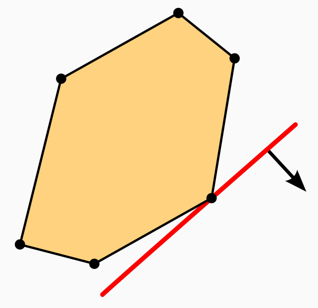

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Roadmap (of this talk)

Skewness:	Original ILP $\rightarrow$ Improvement by considering Short Cycles
Genus:	Original ILP $\rightarrow$ Improvement by considering Short Cycles

Skewness $\mathbf{sk}(G)$

= Maximum Planar Subgraph (MPS)

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Kuratowski-based ILP [Mutzel, 1994]

$$\min \sum_{e \in E} s_e$$

$$\text{s.t.} \sum_{e \in K} s_e \geq 1 \qquad \forall \text{ Kuratowski subdivisions } K \subseteq E$$

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exponentially many

Variable is **1** iff edge e is **deleted**

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Solve via branch-and-cut with heuristic separation.

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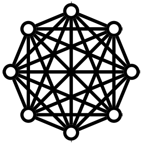

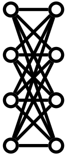

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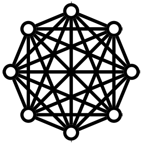

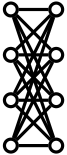

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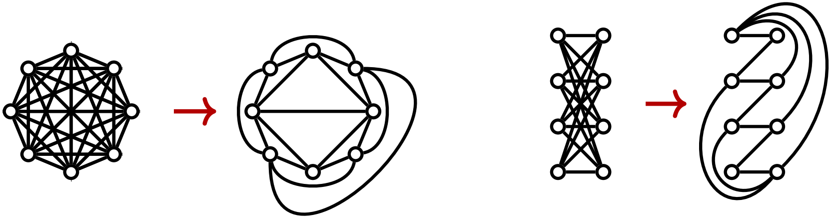

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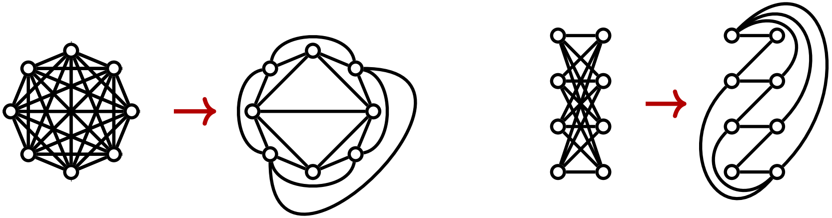

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- On complete (complete bipartite) graphs, **optimality** follows directly from **Euler's formula**
  $\Rightarrow$no Kuratowski constraints needed

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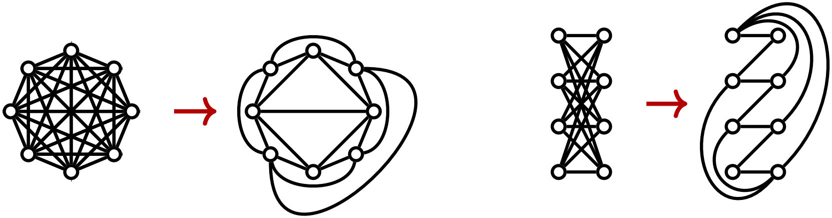

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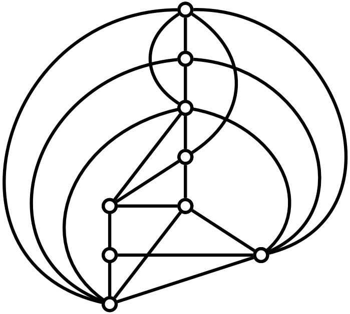
  $\Rightarrow$no Kuratowski constraints needed
- Real-world graphs typically **neither** have large girth **nor** is their MPS tri- or quadrangulated...

Key Observations



“real-world” graph

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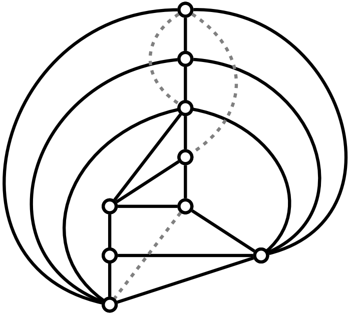

MPS of “real-world” graph

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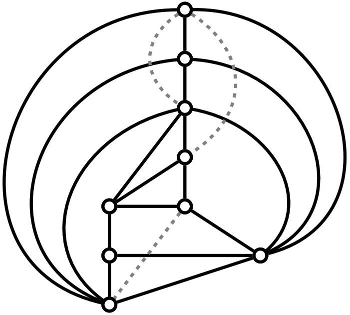

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- few **large faces** in MPS (or few **short cycles** in input)

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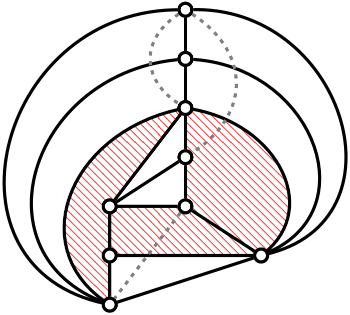

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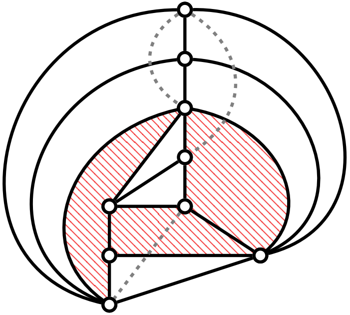

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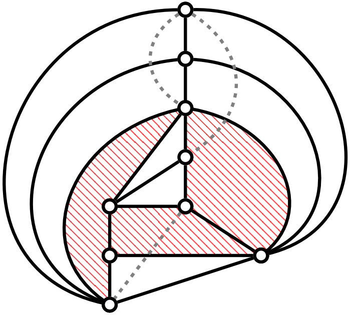

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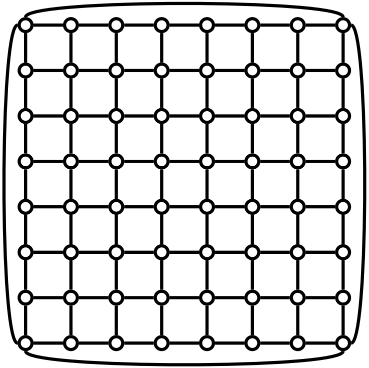

grid-like graph

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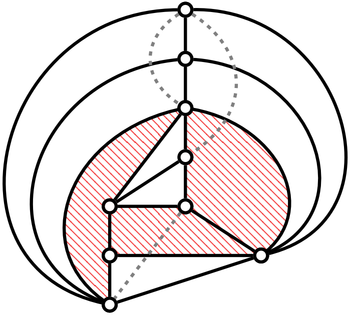

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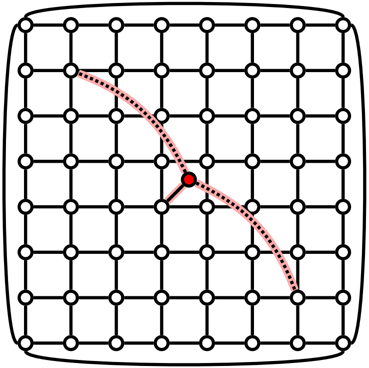

triconnected artificial graph
with no biconnected MPS

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Idea for Cycle Model

Assume: MPS is biconnected.

$\mathcal{C}_G :=$ set of cycles in input G.

new variable is **1** iff α is a face in the solution

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Problems:

- We cannot enumerate **all** cycles in practice
- MPS is not biconnected

Obstacle: Non-Biconnectedness

Lemma

(MPS will not be outerplanar.) To each a face-walk β we can associate a sub-cycle $\alpha \subseteq \beta$ such that each cycle is used at most once.

- faces in the solution $\overset{1:1}{\longleftrightarrow}$ cycles
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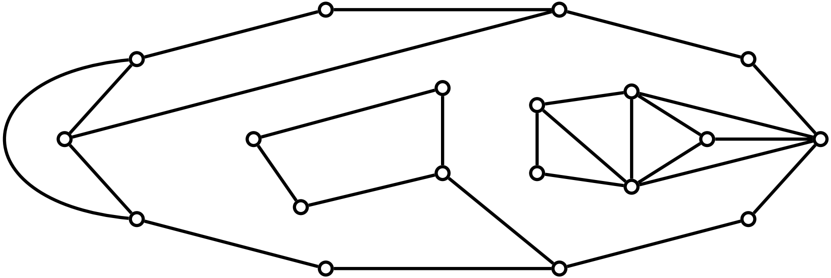

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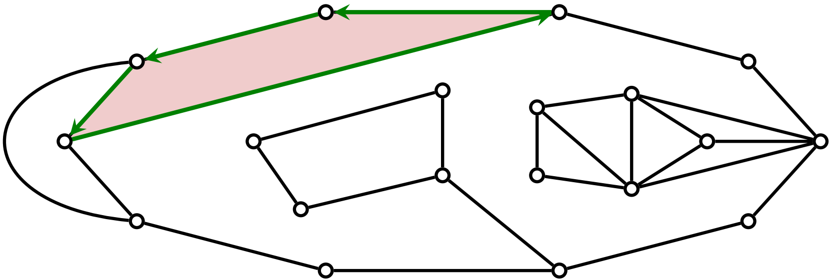

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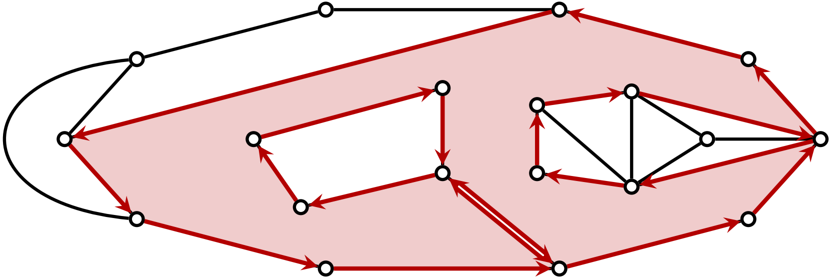

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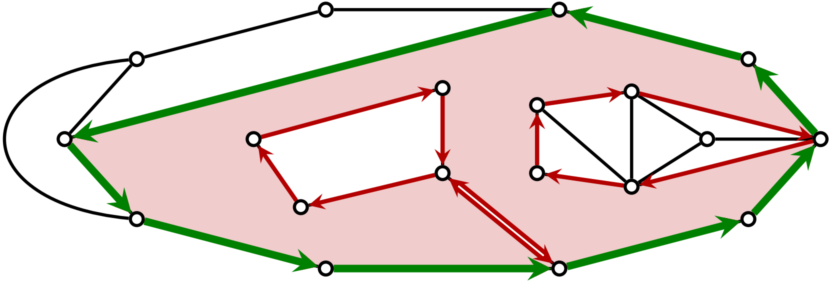

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Pick $D \in \mathbb{N}$ such that number of variables is “reasonable”.

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$$(D+1)(|V(G)| - 2) + \sum_{d=3,4,\ldots,D} (D + 1 - d) \sum_{\alpha \in \mathcal{C}_G^{=d}} c_\alpha \geq (D-1) \sum_{e \in E} (1 - s_e)$$

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The ILP formulations become strictly stronger when increasing D (but the number of variables increases drastically, as well).

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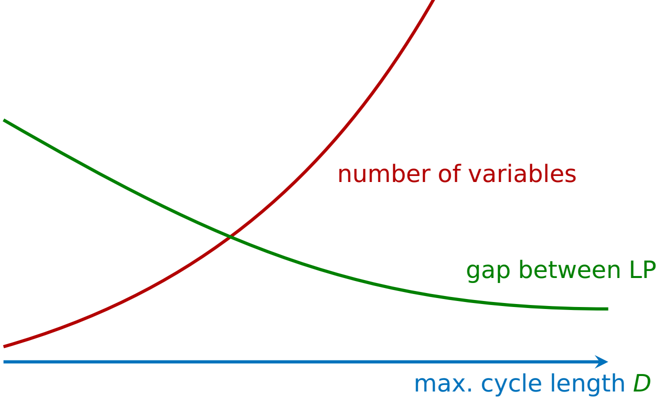

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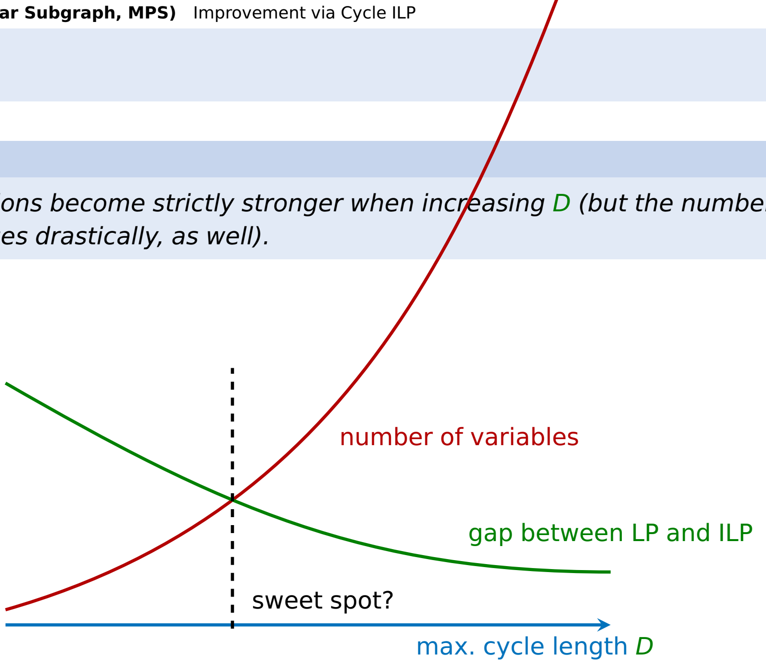

Supplemental Constraint Zoo

Once you have cycle variables,
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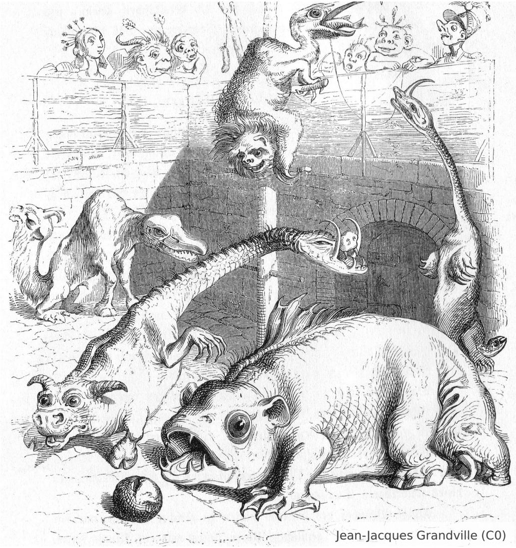
Goal: Tightly link edge- and cycle-variables



Jean-Jacques Grandville (C0)

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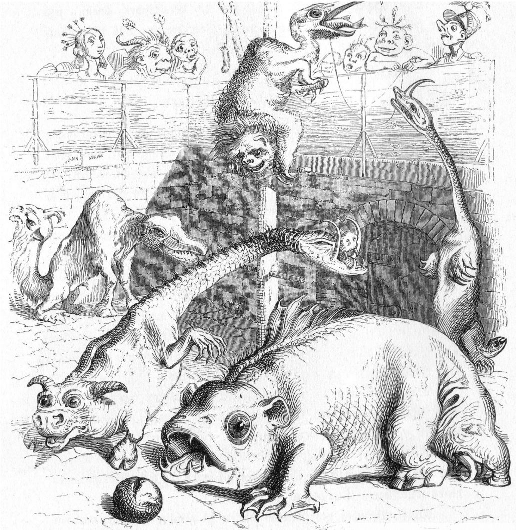
For example:

- pseudo-tree extension
- cycle-edge cons.
- Kuratowski-cycle cons.
- k-cycles-path cons.
- cycle-clique cons.
- cycle-two-paths cons.



Jean-Jacques Grandville (C0)

Experimental Setting

Framework

- C++, GCC 6.3
- OGDF 2018.03 [www.ogdf.net]
- SCIP 6.0 [scip.zib.de]
- CPLEX 12.8

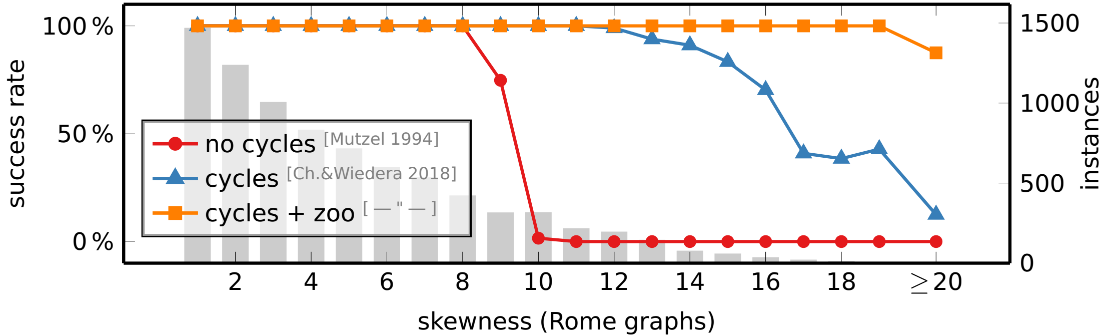
Computations

- Xeon Gold 6134
- limit: 20 minutes, 8 GB

Instances

- Rome [Di Battista et al., 1995]
- Expander [Steger & Wormald, 1997]
- North [North, 1995]
- SteinLib [Koch et al., 2000]

Experimental Evaluation



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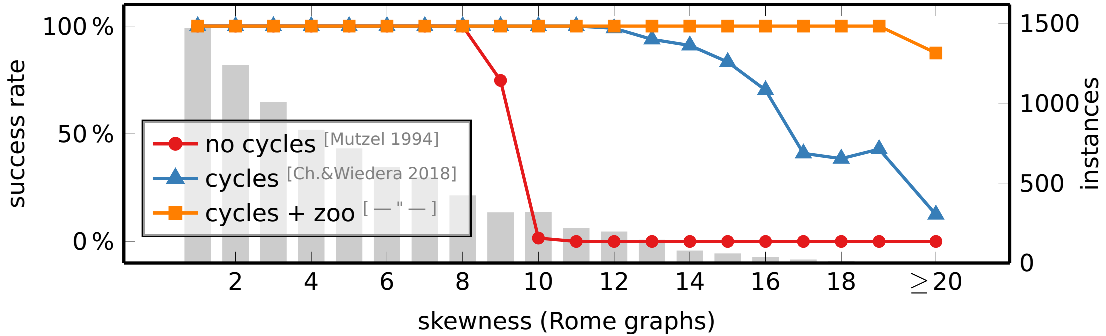

Without vs. with cycles (+zoo)	Rome	North	Expander	SteinLib
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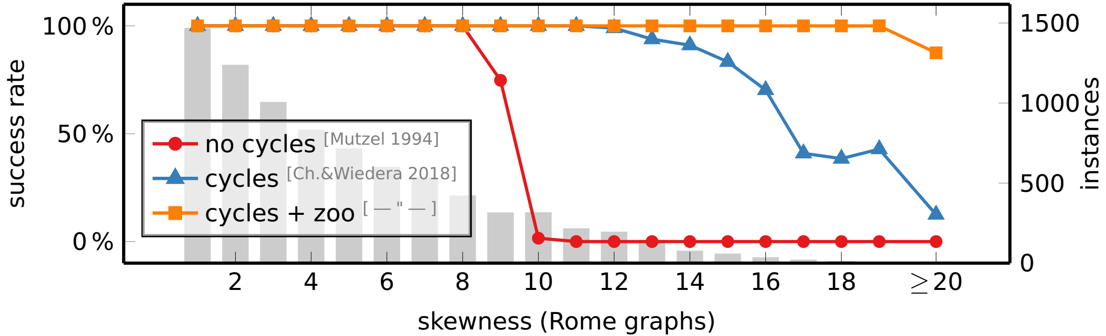

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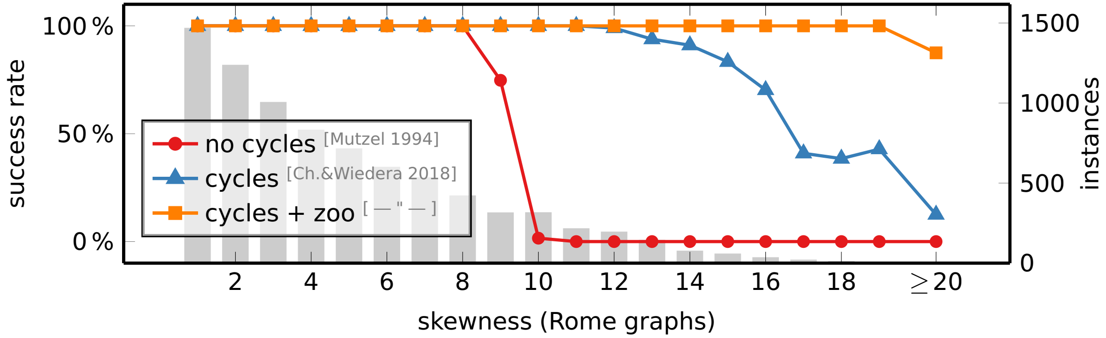

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D (raise until ≥1000 vars)	11	8	6	7

Genus $\gamma(G)$

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computing it is quite hard practice...

- $\exists$ linear time FPT-algorithm for bounded genus g [Mohar 1999], but **doubly exponential** in g and no known implementation even for toroidal case
- no (reasonable) heuristics

Faces in Embeddings of Higher Genus

Theorem (Euler's formula)

... is sensitive to the genus of the surface on with the drawing is: $n - m + f = 2 - 2\gamma$.

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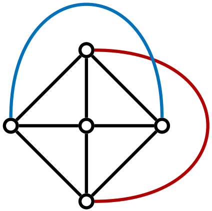
Face-Tracing-based ILP [Beyer et al. 2016]

- Minimize γ by finding an embedding ($=$ rotation system) that **maximizes** f.
- Count faces via **face tracing**.

Crossing-free Non-planar Graphs



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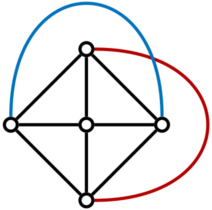

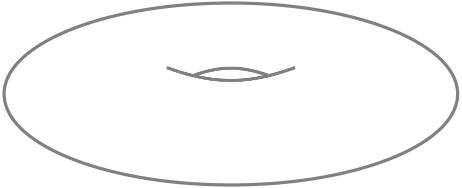

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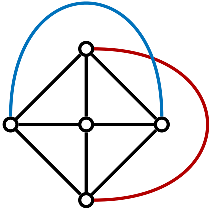
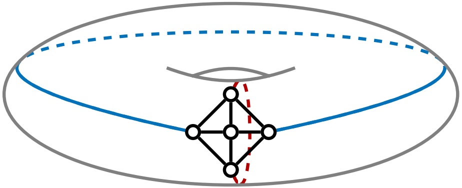
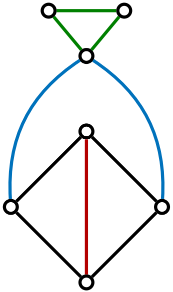




Embeddings and Face Tracing



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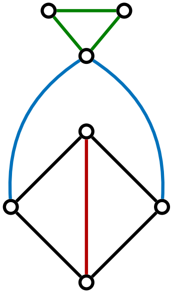

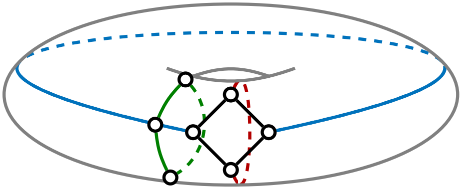

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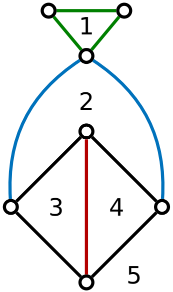

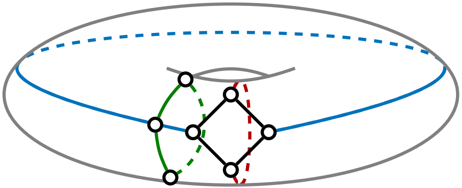

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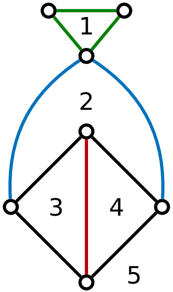

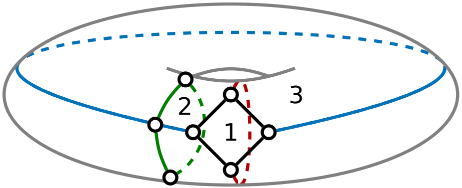

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all **walks** of length $\leq D$: $\mathcal{F}_D$

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Use small face α? $c_\alpha \in \{0,1\}$ $\forall \alpha \in \mathcal{F}_D$

also needs amendment, no details

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Now that we have cycle variables, we can again give additional add-ons to bind them tigher with the face-tracing (but fewer options than for skewness; no details now).

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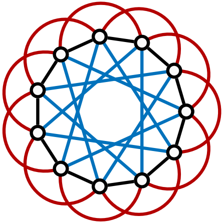
  Expander 5% $\to$ 24%
- speed-up on commonly solved:
  Rome 200x
  North 100x
  Expander 80x

Evaluation: Genera from Literature

The cycle-based ILP can confirm results from literature (all with non-trivial dual bounds):

- **Circulants with genus** ≤ 2 [Conder&Grande 2015]
  hardest case: $C_{11}(1, 2, 4)$
  3 pages analysis, 85h $\Longrightarrow$ 180h [Beyer et al. 2016] $\Longrightarrow$ **10s**



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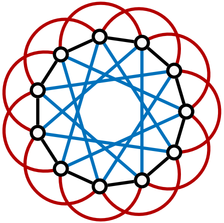

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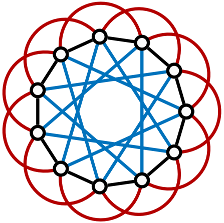
- $\mathbb{Z}_9 \times \mathbb{Z}_3$ **has genus** 4 [Brin et al. 1989]
  full paper $\Longrightarrow$ 5 minutes



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Thank you and good health!

In particular also to the organizers Sascha, Steven and Philipp!