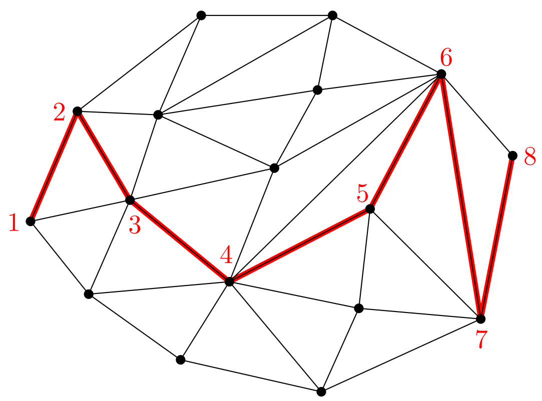
Counting and Enumeration in Geometry, March 20, 2020 — Exercises

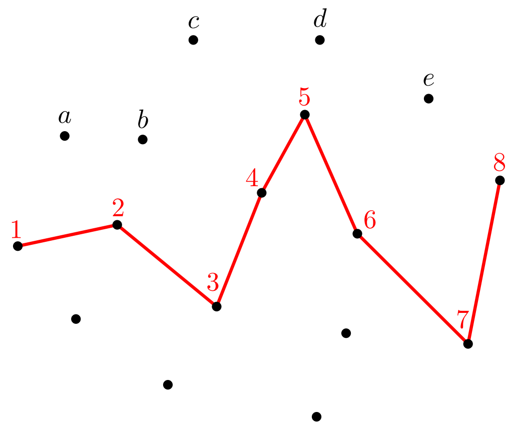
1. Sweeping a rope over a triangulation (Alvarez and Seidel, 2013)

   (a) What was the last new edge or pair of consecutive edges of the rope in the triangulation on the left? What is the next rope?

   (b) Which edge or pair of consecutive edges could have been the last added items, for some triangulation of the point set on the right for which the shown rope appears in the sweep?

   (c) For each resulting marked rope, determine the marked successor ropes.





2. Bipolar orientations

   A bipolar orientation is a planar DAG (directed acyclic graph) with a single source (vertex without incoming edges) and a single sink (vertex without outgoing edges), drawn in the plane such that the both the source and the sink lie on the outer face.

   (a) Prove that around every vertex v except the source and the sink, the outgoing edges form a connected subsequence in the cyclic order around v.

   (b) Prove that in the clockwise order around each face, the edges of the face cycle can be partitioned into a contiguous subsequence of forward (clockwise) edges and a contiguous subsequence of backward (counterclockwise) edges.

3. Vertical order among convex regions (Guibas and Yao, 1980)

   We have a finite set of disjoint convex polygons.

   (a) Among all polygons whose rightmost point p is visible from above (in the sense that the vertical upward ray from p is disjoint from all polygons), choose the polygon P whose rightmost point p is leftmost. Prove that P can be translated vertically upward to infinity without colliding with the other polygons.

   (b) We say that P is *below* Q if there are points $p \in P$ and $q \in Q$ on the same vertical line with p below q. Show that this relation has no cycles.

   (c) What is the situation in three dimensions?

4. Sweeping a rope over a line arrangement

   We have n non-vertical lines, no two of which are parallel. We want to sweep a rope across the arrangement of lines, starting from the boundary of the bottom face and ending at the boundary of the top face, flipping it over a single face at a time.

   Prove that this can be done with a rope that has always at most $O(n)$ edges.

   (Experiments indicate that the true maximum is $2n-2$, including the two unbounded rays that every rope contains.)

5. Matchings in a convex chain

   Let $S = (P_1, \ldots, P_n)$ be a sequence of points in convex position. For $i = 0, 1, \ldots, n$, let ℓ_i be a line that separates $P_1, \ldots, P_i$ from $P_{i+1}, \ldots, P_n$. For a matching M of S, we denote by M_i the part of M to the left of ℓ_i (on the side of the points $P_1, \ldots, P_i$). In such a "partial matching", edges of S that cross ℓ_i appear as dangling edges ("half-edges"): Only their left endpoint is determined. Let B_i^k denote the number of partial matchings M_i with k dangling edges.

   (a) Find a recursion that computes the numbers $B_{i+1}^0, B_{i+1}^1, \ldots$ from $B_i^0, B_i^1, \ldots$.

   (b) Show that the matchings of S are in one-to-one correspondence with the so-called *Motzkin paths*: A Motzkin path of length n is a path from $(0,0)$ to $(n,0)$ that uses steps of the form $(1,-1)$, $(1,0)$ and $(1,1)$ and never goes below the x-axis.

6. Counting perfect matchings of point sets (Wettstein 2014)

   We sweep a rope over a perfect matching of a points set by adding the leftmost matching edge e for which every matching edge that lies below e (in the sense of exercise 3b) is already on or below the rope.

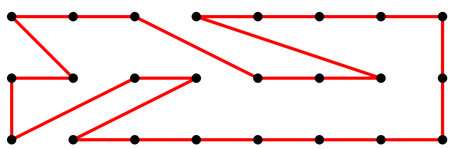
   We mark the left endpoint of e on the rope.

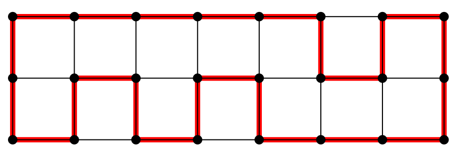
   Show that the number of perfect matchings of a point set can be determined in $O(n^3 2^n)$ time and $O(n 2^n)$ space.

The following exercises from an early draft were abandoned because they were deemed too difficult.

7. Hamilton cycles

   (a) Find a set of recursion formulas for counting the number H_n of noncrossing Hamilton cycles of the $3 \times n$ grid point set, like the one shown in the left figure for $n = 8$.





   (b) Determine or estimate the growth rate $\lim \sqrt[n]{H_n}$, possibly with the help of a computer.

   (c) If you are familiar with generating functions, you may try to derive a more precise asymptotic formula for H_n.

8. Hamilton paths

   If the previous exercise was too hard, do it for the $2 \times n$ grid, or for the $3 \times n$ grid *graph*, as shown in the right figure. If it was too easy, repeat it for Hamilton *paths*.