Minimum Convex Partition of Degenerate Point Sets is NP-Hard*

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Abstract

Given a point set P and a natural integer k, are k closed convex polygons sufficient to partition the convex hull of P such that each polygon does not contain a point in P? What if the vertices of these polygons are constrained to be points of P? By allowing degenerate point sets, where three points may be on a line, we show that the first decision problem is NP-hard and the second NP-complete.

1 Introduction

The CG Challenge 2020 organised by Demaine, Fekete, Keldenich, Krupke and Mitchell [2], is about finding good solutions to the problem of *Minimum Convex Partition* (MCP). We give a definition equivalent to theirs, which fits better for the purpose of this paper.

▶ **Definition 1** (Minimum Convex Partition problem)**.** Given a set P of points in the plane and a natural number k, is it possible to find at most k closed convex polygons whose vertices are points of P, with the following properties:

- The union of the polygons is the convex hull of P,
- The interiors of the polygons are pairwise disjoint,
- No polygon contains a point of P in its interior.

The organisers of the CG Challenge 2020 mention that the complexity of this problem is unknown. Some partial results are known, under the additional assumption that no three points are on a line. For some more constrained point sets, Fevens, Meijer and Rappaport gave a polynomial time algorithm [3]. Keeping only the assumption that no three points are collinear, Knauer and Spillner have shown a $\frac{30}{11}$-approximation algorithm [6]. They also ask for the complexity of the Minimum Convex Partition problem. On a related note, Sakai and Urrutia have shown that for every set of n points, there exists a convex partition with at most $\frac{4}{3}n - 2$ polygons [8]. Although they do not mention it, it is straightforward to combine their result with the method of Knauer and Spillner to obtain a $\frac{8}{3}$-approximation algorithm. Concerning lower bounds, García-Lopez and Nicolás have given a construction for point sets for which any convex partition has at least $\frac{35}{32}n - \frac{3}{2}$ polygons [4]. An integer linear programming formulation of the problem, along with experimental results, has been recently introduced by Barboza, Souza and Rezende [1].

All those results, concerning algorithms and bounds, are shown for point sets in general position. However this is not assumed in the CG Challenge 2020. In this paper, we show that Minimum Convex Partition of degenerate point sets is NP-complete by a reduction from a modified version of planar 3-SAT. The complexity of the problem for point sets in general position is still open.

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This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

We also show the NP-hardness of a similar problem, which we call *Minimum Convex Tiling* problem (MCT). The problem is exactly as in Definition 1, but the constraint about the vertices of the polygons is removed (i.e. they need not be points of P). This can make a difference as shown in Figure 1. Equivalently, the MCT problem corresponds to the MCP problem when Steiner points are allowed. A *Steiner point* is a point that does not belong to the point set given as input, and which can be used as a vertex of some polygons. Our proofs are very similar for the two problems. Due to lack of space, some parts of the NP-hardness proof of MCT, and how to adapt it for MCP, are postponed in the Appendix.

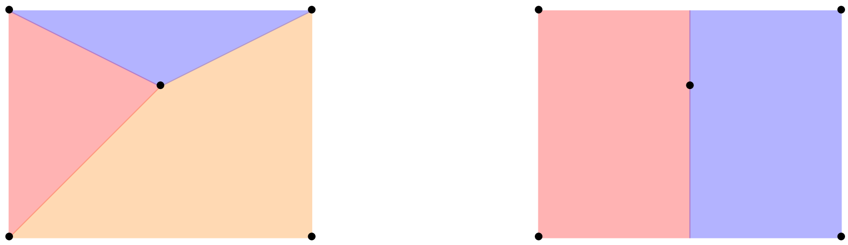

Figure 1 A minimum partition with three convex polygons, and a tiling with two.

Our proof builds upon gadgets introduced by Lingas [7]. He used them to prove NP-hardness of two decision problems: *Minimum Rectangular Partition for rectangles with point holes* and *Minimum Convex Partition for polygons with polygon holes*. In the second problem, Steiner points are allowed. However, as noted by Keil [5], one can easily adapt Lingas' proof not to use Steiner points. That is what we do in a second part to prove NP-hardness of the MCP problem. The two proofs of Lingas are similar, and consist in a reduction from the following variation of planar 3-SAT. The instances are a CNF formula F with set of variables X and set of clauses C, and a planar bipartite graph $G = (X \cup C, E)$, such that there is an edge between a variable $x \in X$ and a clause $c \in C$ if and only if x or $\bar{x}$ is a literal of c. Moreover, each clause contains either two literals or three, and if it contains three the clause must contain at least one positive and one negative literal. Lingas refers to this decision problem as the *Modified Planar* 3*-SAT* (MPLSAT). Lingas states that this problem is NP-complete, and we provide a proof of it in the Appendix. The main result of this paper is as follows:

▶ **Theorem 2.** *MPLSAT can be reduced in polynomial time to MCP, and to MCT.*

As it is easy to see that MCP is in NP, Theorem 2 implies that MCP is NP-complete. The question whether MCT is in NP is still open.

2 Construction of the point set

We do the reduction by constructing a point set in three steps. First we construct a non-simple polygon, in a very similar way as in Lingas' proof, with some more constraints. Secondly, we add some line segments to build a grid around the polygon, and finally we discretise all line segments into sets of evenly spaced collinear points. The idea of the first part is to mimic Lingas' proof. The second part makes the correctness proof easier, and the last part transforms the construction into our setting. The aim of the grid is to force the sets in a minimum convex tiling to be rectangular.

We use the gadgets introduced by Lingas, namely cranked wires and junctions [7]. A wire is shown in Figure 2. It consists of a loop delimited by two polygons, one inside the other.

In Lingas' construction, the two polygons are simple, and a wire is therefore a polygon with one hole. Moreover in his proof the dimensions of the cranks do not matter. In our case, the polygon inside is not simple, and each line segment has unit length. Each wire is bent several times with an angle of 90°, as shown in Figure 2, in order to close the loop.

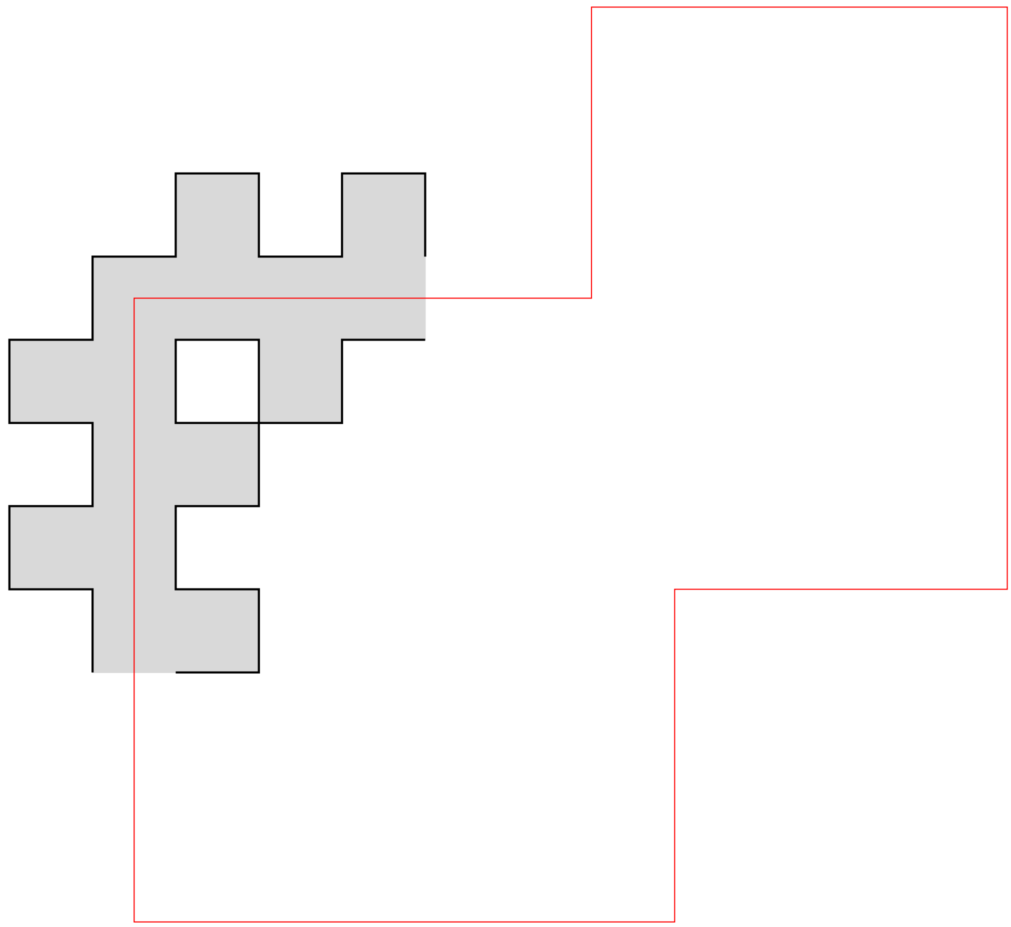

Figure 2 A cranked wire, edges are in black and its interior is in grey. The wire follows the whole red loop, but for sake of simplicity, only a section of the wire at a bend is drawn.

The wires are used to encode the values of the variables, with one wire for each variable. We are interested in two possible tilings of a wire, called vertical and horizontal, which are shown in Figure 3.

As in Lingas' proof, we interpret the vertical tiling as setting the variable to *true*, and the horizontal as *false*. Lingas proved the following:

▶ **Lemma 3** (Lingas [7]). *A minimum tiling with convex sets of a wire uses either vertical or horizontal rectangles but not both. Any other tiling requires at least one more convex set.*

The second tool is called a junction, and it serves to model a clause. Figure 4 depicts a junction corresponding to a clause of three literals. A junction has three arms, represented as dashed black line segments. A junction for a clause of two literals is obtained by blocking one of the arms of the junction. The blue line segments have length $1 + \varepsilon$, for a fixed ε arbitrarily small. Therefore, the red line segments are not aligned with the long black line segment to the left of the junction. A junction can be in four different orientations, which

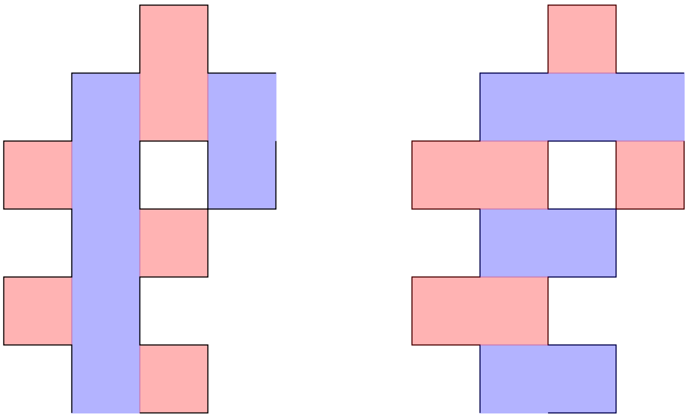

Figure 3 A section of a wire and its optimal tilings: vertical (left) and horizontal (right).

can be obtained successively by making rotations of 90°. Let us consider the orientation of the junction in Figure 4. One wire is connected from above, one from below, and one from the left. A wire can only be connected to a junction at one of its bends (see Figure 2). We then remove the line segment corresponding to the arm of the junction, as illustrated in Figure 4.

If the tiling of the wire connected from above is vertical, then one of the rectangles can be prolonged into the junction. The same holds for the wire connected from below. On the contrary, a rectangle can be prolonged from the wire connected from the left only if the tiling is horizontal. If a rectangle can be prolonged, we say that the wire *sends true*, otherwise it *sends false*. If a clause contains two positive literals x, y and one negative $\bar{z}$, the corresponding junction is as in Figure 4, or the 180° rotation of it. The wire corresponding to z is connected from the left or right, and the wires corresponding to x and y are connected from above and below, or vice versa. Therefore, the wire corresponding to x (respectively y) sends *true* if and only if x (respectively y) is set to *true*. On the contrary, the wire corresponding to z sends *true* if and only if z is set to *false*. If the clause has two negative literals, then the junction is horizontal, and the junction behaves likewise.

Lingas proved that when minimising the number of convex polygons in a tiling, for each junction at least one adjacent wire sends *true*. Before stating Lingas' lemma exactly, we need to explain the first step of the construction of the point set.

2.1 Construction of the polygon with holes

Let us consider one instance (F, G) of MPLSAT. Lingas states that the planarity of G implies that the junctions and the wires can be embedded as explained above, and so that they do not overlap [7]. Thus we obtain a polygon with holes, that we denote by Π. He adds without proof that the dimensions of Π are polynomially related to $|V|$, where V denotes the vertex set of G. We show in the appendix how to embed the polygon with holes into a grid Λ, such that each edge consists of line segments of Λ. Moreover Λ is of size $\Theta(|V|^2)$. We can now state Lingas' lemma:

▶ **Lemma 4** (Lingas [7]). *In a minimum tiling with convex sets of Π, a junction contains wholly at least three convex sets. The junction contains wholly exactly three if and only if at least one of the wires connected to the junction sends* true*.*

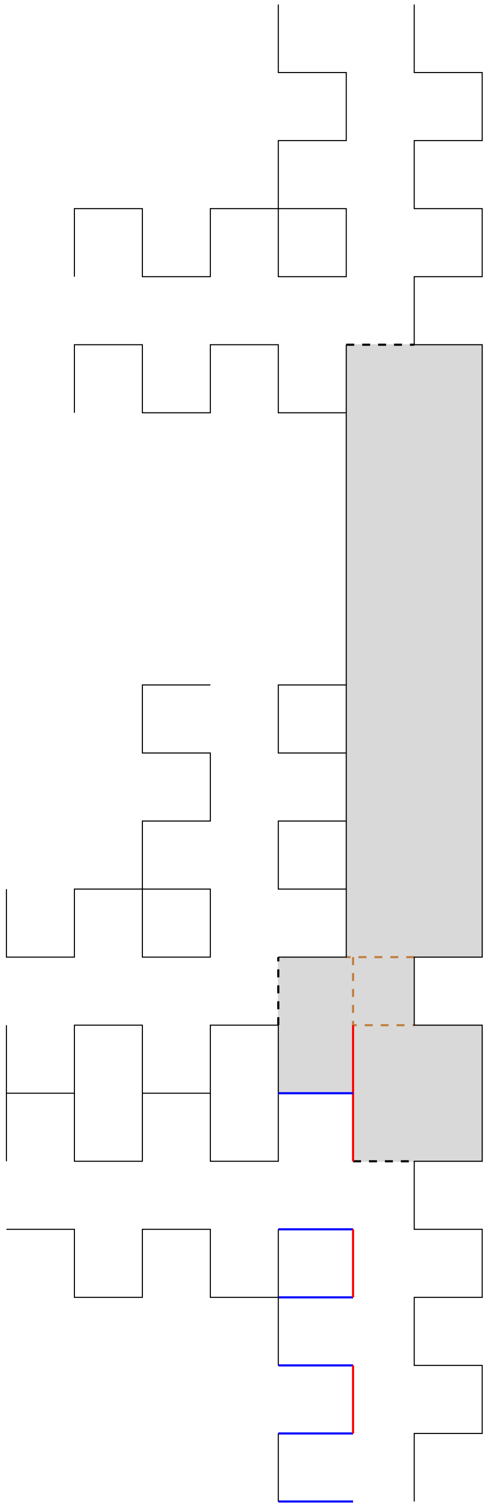

Figure 4 A junction for the MCT problem.

2.2 Discretisation of the line segments

To construct the point set, we first construct a collection of line segments. We then discretise this collection by replacing each line segment by a set of collinear points.

Let us consider our polygon with holes Π that lies in the grid Λ. The grid consists of points with integer coordinates, and line segments between points that are at distance 1. We consider the collection of line segments consisting of Π union each line segment of Λ whose interior is not contained in the interior of Π. Notice that therefore we have line segments outside Π, but also inside its holes. Moreover, the collection of line segments that we obtain, denoted by Φ, is a subgraph of the grid graph Λ.

Now we define K as twice the number of unit squares in Λ plus 1. Finally, we replace each line segment in Φ by K points evenly spaced. We denote this point set by P.

3 Proof of correctness

We have constructed P in order to have the following property:

▶ **Lemma 5.** *In a minimum convex tiling Σ of P, for each convex set $S \in \Sigma$, the interior of S does not intersect Φ.*

Let K' denote the number of unit squares in Φ, plus the minimum number of rectangles in a partition of the wires, plus three times the number of clauses. Using Lemmas 3 and 4 shown by Lingas coupled with Lemma 5, we immediately obtain the following theorem:

▶ **Theorem 6.** *The formula F is satisfiable if and only if there exists a convex tiling of P with K' polygons.*

Since P and K' can be computed in polynomial time, Theorem 6 implies Theorem 2 for the MCT problem. Due to lack of space, we postpone most of the proof of Lemma 5 to the Appendix. Nonetheless, we state and prove here the lemma giving the key idea of the proof. We use a packing argument, and claim that in a convex tiling Σ of P, if a convex set $S \in \Sigma$ has large area, then most of its area is contained in a unique cell of Φ. In the Appendix we show that in a minimum convex tiling, all convex sets have large area, and that each of them fills the cell that contains it. For a set S, let $\mathrm{A}(S)$ be the area of S.

▶ **Lemma 7.** *Let L and L' be two squares in Λ, and S be a convex polygon whose interior does not contain any point in P. If $\mathrm{A}(S \cap L) > 1/K$, and the boundary of S crosses a line segment of Φ between L and L', then $\mathrm{A}(S \cap L') \leq 1/K$.*

Proof. The proof is illustrated in Figure 5. By assumption, S goes between two points p and q at distance $1/K$. Let us consider the two line segments s and s' of the boundary of S that intersect the line ℓ spawned by p and q. Assume for contradiction that the lines spawned by s and s' do not intersect, or intersect on the side of ℓ where L lies. This implies that $S \cap L$ is contained in a parallelogram that has area $1/K$, as illustrated in Figure 6. Indeed such a parallelogram has base $1/K$ and height 1, therefore $\mathrm{A}(S \cap L) \leq 1/K$. This shows that the lines spawned by s and s' intersect on the side of ℓ where L' lies. Using the same arguments as above, this implies $\mathrm{A}(S \cap L') \leq 1/K$. ◀

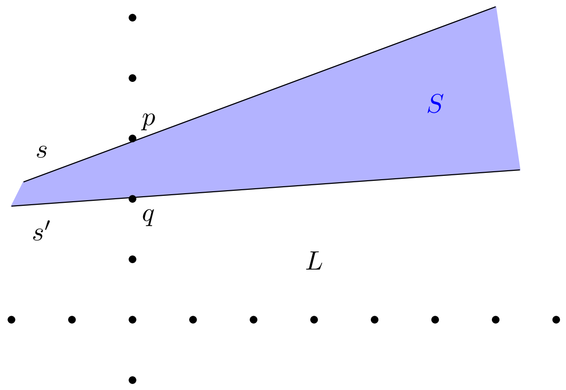

Figure 5 If $A(S \cap L) > 1/K$, the two lines spawned by s and s' intersect on the left side.

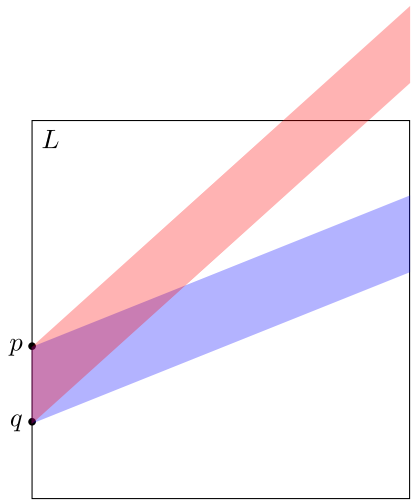

Figure 6 The area of the parallelograms is $1/K$.

4 Open problems

It is still open whether MCT is in NP. It may be that the coordinates of a vertex of a polygon require exponentially many bits to be written. We also do not know the complexity of MCP and MCT when it is assumed that no three points are collinear. A key property used for our proof can be summarised as follow: When a rectangle with large area is delimited by a lot of points, it is optimal to take this rectangle in the convex tiling or partition. But this cannot be achieved when no three points are on a line, as illustrated in Figure 7. In any convex partition the red edges are forced. Even in a convex tiling, one can observe that they are needed for the tiling to be minimum. This implies that three convex sets are necessary. But adding the convex set consisting of the points in convex position would add one non-necessary convex set. Adding more points to delimit the convex set cannot change the fact that taking this convex set in the tiling would not be optimal. The construction can easily be adapted for the MCP problem. Therefore it is not clear how one could force some convex sets to be in the partition or tiling.

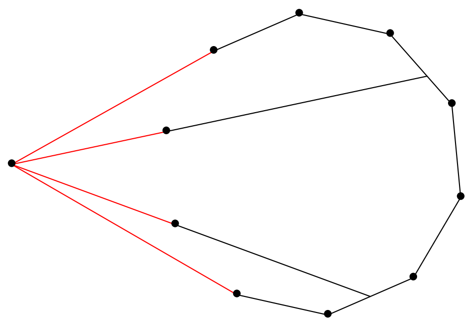

Figure 7 An optimal tiling splits the area delimited by points in convex position.

References

1 Allan S. Barboza, Cid C. de Souza, and Pedro J. de Rezende. Minimum convex partition of point sets. In *International Conference on Algorithms and Complexity*, pages 25–37. Springer, 2019. doi:/10.1007/978-3-030-17402-6_3.

2 Erik Demaine, Sándor Fekete, Phillip Keldenich, Dominik Krupke, and Joseph S. B. Mitchell. CG:SHOP 2020. https://cgshop.ibr.cs.tu-bs.de/competition/cg-shop-2020. Accessed: 12/02/2020.

3 Thomas Fevens, Henk Meijer, and David Rappaport. Minimum convex partition of a constrained point set. *Discrete Applied Mathematics*, 109(1-2):95–107, 2001. doi:10.1016/S0166-218X(00)00237-7.

4 Jesús García-López and Carlos M Nicolás. Planar point sets with large minimum convex decompositions. *Graphs and Combinatorics*, 29(5):1347–1353, 2013. doi:10.1007/s00373-012-1181-z.

5 J Mark Keil. Decomposing a polygon into simpler components. *SIAM Journal on Computing*, 14(4):799–817, 1985. doi:10.1137/0214056.

6 Christian Knauer and Andreas Spillner. Approximation algorithms for the minimum convex partition problem. In *Scandinavian Workshop on Algorithm Theory*, pages 232–241. Springer, 2006. doi:10.1007/11785293_23.

7 Andrzej Lingas. The power of non-rectilinear holes. In *International Colloquium on Automata, Languages, and Programming*, pages 369–383. Springer, 1982. doi:10.1007/BFb0012784.

8 Toshinori Sakai and Jorge Urrutia. Convex decompositions of point sets in the plane. *arXiv preprint arXiv:1909.06105*, 2019.