A \((1 + \varepsilon)\)-approximation for the minimum enclosing ball problem in \(\mathbb{R}^d\) *

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**Abstract**

Given a set of points \(P\) in \(\mathbb{R}^d\) for an arbitrary \(d\), the 1-center problem or minimum enclosing ball problem (MEB) asks to find a ball \(B^*\) of minimum radius \(r^*\) which covers all of \(P\). Kumar et. al. [5] and Bádoiu and Clarkson [1] simultaneously developed core-set based \((1 + \varepsilon)\)-approximation algorithms. While Kumar et. al. achieve a slightly better theoretical runtime of \(O(nd/\varepsilon + \varepsilon^6 \log 1/\varepsilon)\), Bádoiu and Clarkson have a stricter bound of \([2/\varepsilon]\) on the size of their core-set, which strongly affects run-time constants.

We give a gradient-descent based algorithm running in time \(O(nd/\varepsilon)\) based on a geometric observation that was used first for a 2-center streaming algorithm by Kim and Ahn [4]. Our approach can be extended to the \(k\)-center problem to obtain a \((1 + \varepsilon)\)-approximation in time \(O(ndk 2^{1/\varepsilon})\).

1 Introduction

Given a set of points \(P \subset \mathbb{R}^d\), the *minimum enclosing ball problem* (MEB), also known as the 1-center problem, asks to find a ball \(B^*\) of minimum radius \(r^*\) containing all of \(P\) and is an important subproblem in clustering. While it can be solved in worst-case linear time for fixed \(d\) [6], the dependence on \(d\) is exponential and hence not practical for high dimensional real-world applications. However, Bádoiu et al. [3] presented a \((1 + \varepsilon)\)-approximation algorithm for arbitrary \(d\) running in time \(O(nd/\varepsilon^2 + \varepsilon^4 \log 1/\varepsilon)\) using core-sets of size at most \(1/\varepsilon^2\) independent of \(d\). An \(\varepsilon\)-core-set is a subset \(S \subset P\), such that a ball of radius \((1 + \varepsilon)r^*\) around the center of a minimum enclosing ball of \(S\) covers \(P\). Their algorithm can be extended to approximate the \(k\)-center problem, but the running time is then exponential in \(k\) and the size of the core-set; so having a tight bound on the size of the core-set is paramount. In fact, no polynomial time approximation scheme for the \(k\)-center problem in high dimensions can exist if \(P \not\in \text{NP}\), see [7].

Kumar et al. [5] improved these results to finding \(\varepsilon\)-core-sets of size \(O(1/\varepsilon)\) in time \(O(nd/\varepsilon^2 + \varepsilon^4 \log 1/\varepsilon)\). Independently Bádoiu and Clarkson [1] achieved an algorithm with a similar running time of \(O(nd/\varepsilon + \varepsilon^3)\) while having a stricter bound of \([2/\varepsilon]\) on the size of their core-set, which significantly affects run-time especially when extending to the \(k\)-center problem. Bádoiu and Clarkson [1] also gave a simple gradient-descent algorithm obtaining a \((1 + \varepsilon)\)-approximation in time \(O(nd/\varepsilon^2)\) and later showed that a tight bound of \([1/\varepsilon]\) on the size of \(\varepsilon\)-core-sets exists, see [2]. The gradient-descent algorithm has the advantage of not computing minimum enclosing balls for several subsets of \(P\) of size \(O(1/\varepsilon)\) which improves the constants involved in the calculation of each step and simplifies implementation. Bádoiu and Clarkson [2] also performed runtime experiments on both the gradient-descent and the different core-set-based algorithms which results showed that the gradient-descent algorithm is competitive in reality, as it converges significantly faster than its theoretical bound suggests.

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36th European Workshop on Computational Geometry, Würzburg, Germany, March 16–18, 2020. This is an extended abstract of a presentation given at EuroCG’20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.
We combine the analysis of these core-set-based algorithms with the ideas of the gradient-descent algorithm and extend structural observations made by Kim and Ahn [4] for the Euclidean 2-center problem in a streaming model to obtain a new efficient gradient-descent algorithm that converges to a \((1 + \varepsilon)-\)approximation in time \(O(nd/\varepsilon^2)\). It can be applied to the \(k\)-center problem in a similar fashion as the core-set based algorithms. Hence, it gives an alternative \((1 + \varepsilon)-\)approximation in time \(O(ndk2^{d/\varepsilon})\) for that problem without the use of core-sets and, possibly, a faster algorithm on real-world data.

### 2 An Algorithm for the Euclidean 1-center problem

In this section we present an algorithm for the Euclidean 1-center problem for high dimensions and show the following theorem:

> **Theorem 2.1.** Given a set \(P \in \mathbb{R}^d\), one can compute a \((1 + \varepsilon)\)-approximation of the minimum enclosing ball in time \(O(nd/\varepsilon^2)\) with the gradient-descent-algorithm \textsc{GradientMEB}.

Let \(P \subset \mathbb{R}^d\) for any \(d \geq 2\) be a set of \(n\) points. Let \(B(c, r)\) denote a ball of radius \(r\) centered at \(c\) and let \(r(B)\) and \(c(B)\), denote the radius and center of a ball \(B\), respectively. We denote by \(pq\) the straight line segment between two points \(p\) and \(q\) and by \(|pq|\) the length of \(pq\). Finally, we denote the boundary of a closed set \(A\) by \(\partial A\).

Let \(B^* = B(c^*, r^*)\) be the optimal solution of the 1-center problem for a set \(P\). The core idea of the algorithm is as follows. We will start with an arbitrary point \(p_1\) from \(P\) as a starting center \(m_1\). For any center \(m_i\) constructed, the radius \(r_i\) necessary to cover all of \(P\) with a ball \(B(m_i, r_i)\) is defined by the farthest point in \(P\) from \(m_i\). Therefore, in every subsequent step we pick that farthest point as \(p_{i+1}\) and construct a new center \(m_{i+1}\) on the line segment between \(p_{i+1}\) and \(m_i\) to reduce \(r_i\). We will use a central structural property proven in Lemma 2.2 to show how to construct \(m_i = m(p_{i+1}, m_i)\) in such a way that we can give a bound on its distance \(|m_i c^*|\) to the optimal center \(c^*\) decreasing with every step \(i\). The exact definition of \(m(p_{i+1}, m_i)\) will hence be given after that Lemma.

**Algorithm 1 GradientMEB**

**Input:** Set of points \(P \subset \mathbb{R}^d\).

**Output:** A center \(c\) such that \(B(c, (1 + \varepsilon)r^*)\) covers \(P\)

\[
p_1 \leftarrow \text{arbitrary point from } P
\]

\[
m_1 \leftarrow p_1
\]

\[
\text{bestRadius} \leftarrow \infty
\]

for \(i = 1\) to \(\lceil \frac{d}{2\varepsilon} \rceil\) do

\[
p_{i+1} \leftarrow \text{farthest point from } m_i \text{ in } P
\]

if \(|m_ip_{i+1}| < \text{bestRadius}\) then

\[
\text{bestCenter} \leftarrow m_{i-1}
\]

\[
\text{bestRadius} \leftarrow |m_{i-1}p_i|
\]

\[
m_{i+1} \leftarrow m(p_{i+1}, m_i)
\]

return bestCenter

We will start with the proof of the central structural property and the construction of \(m(p_{i+1}, m_i)\) and then show that after at most \(k = \lceil \frac{d}{2\varepsilon} \rceil\) such steps, \(|m_k c^*| < \varepsilon r^*\) and hence \(B(m_k, (1 + \varepsilon)r^*)\) covers all of \(P\).

Both together will proof the correctness of \textsc{GradientMEB}. As the algorithm runs for \(\lceil \frac{d}{2\varepsilon} \rceil\) rounds, finding \(p_i\) each round takes \(O(nd)\) and the computation of \(m_i\) takes \(O(d)\), this will also proof Theorem 2.1.
Lemma 2.2. Given two d-dimensional balls \( B \) and \( B' \) with radii \( r \) and \( r' \) around the same center point \( c \) with \( r > r' \) with \( d \geq 2 \). Let \( p \in \partial B \) and \( p' \in \partial B' \) with \( |pp'| = l \geq r \). Let \( B'' \) be the d-dimensional ball centered around \( c \) that is tangential to \( pp' \). We denote that tangential point with \( m \) and the distances \( |p'm| \) with \( l_1 \) and \( |pm| \) with \( l_2 \), so \( l = l_1 + l_2 \). Consider any line segment \( p_1p_2 \) with \( |p_1p_2| > l \), \( p_1 \in B' \) and \( p_2 \in B \). Then any point \( m^* \) on \( p_1p_2 \) with \( |p_1m^*| \geq l_1 \) and \( |p_2m^*| \geq l_2 \) lies inside \( B'' \).

Proof. For \( d = 2 \) we first show that \( |p_1p_2| \) intersects \( B'' \) at all. It is clear that we can rotate and reflex \( pp' \) without changing \( B'' \) as long as its length \( l \) stays the same. Hence we can assume without loss of generality, that \( p' \), \( c \) and \( p_2 \) are collinear with \( p_1 \in cp' \). Then \( p_2 \) must lie in \( B \setminus B(p', l) \) which means \( p_1p_2 \) intersects \( B'' \) as illustrated in Figure 1.

![Figure 1](image1.png)

Now consider a point \( m^* \) on \( p_1p_2 \) with \( |p_1m^*| \geq l_1 \) and \( |p_2m^*| \geq l_2 \). Assume \( m^* \) is not contained in \( B'' \). Then either \( p_1m^* \cap B'' = \emptyset \) and \( p_2m^* \cap B'' \neq \emptyset \) or the other way around as \( p_1p_2 \) intersects \( B'' \) somewhere.

Let’s first assume, \( p_1m^* \cap B'' = \emptyset \) and \( p_2m^* \cap B'' \neq \emptyset \). In this case \( p_1 \in B' \setminus B'' \). Let \( p_t \) be a point on the boundary of \( B'' \) such that \( p_1p_t \) is tangential to \( B'' \) and \( m^* \) lies within the triangle \( p_1p_t \). Clearly \( |p_1p_t| > |p_1m^*| \). But by construction of \( B'' \), \( |p'm| \geq |p_1p_t| \), which contradicts \( |p'm| = l_1 \leq |p_1m^*| \). This is illustrated in Figure 2 (assuming without loss of generality that \( p' \) is collinear with \( p_1p_2 \), as this does not not affect the construction of \( B'' \)).

![Figure 2](image2.png)

The other case can be shown equivalently by switching \( p_1 \) and \( p_2 \) and replacing \( p' \) and \( B' \) and \( l_1 \) with \( p \) and \( B \) and \( l_2 \), respectively.

For \( d > 2 \) we can show the lemma by choosing the 2-dimensional plane passing through \( c \) and the line segment \( p_1p_2 \) and then follow the same arguments as for \( d = 2 \).
Note, that the point $m'$ on $p_1p_2$ with $\frac{|m'p_1|}{|p_1p_2|} = \frac{1}{l} = \frac{|m'p'|}{|pp'|}$ fulfills $|m'p_1| \geq l_1$ and $|m'p_2| \geq l_2$. The following corollary follows from that observation, Lemma 2.2 and the Pythagorean theorem and defines a way to calculate that point without actually knowing $r^*$.

\textbf{Corollary 2.3.} Let $B$ and $B'$ be two balls in $\mathbb{R}^d$ with $c(B) = c(B')$ and radii $r(B) = r^*$ and $r(B') = r' = \delta r^*$ for some $0 < \delta \leq 1$. Then the line segment $pp'$ between any two points $p \in \partial B$ and $p' \in \partial B'$ with distance $|pp'| = l = (1 + \varepsilon)r^*$ is tangential to $B'' = B(c,r_m)$ with

$$r_m \leq r^* \sqrt{1 - \left(\frac{(1 + (1 + \varepsilon))^2 - \delta^2}{2(1 + \varepsilon)}\right)^2}$$

at a point $m^*$ with $l_1 := |m^*p'|$.

Let $p_1 \in B'$ and $p_2 \in B$ with $|p_1p_2| \geq l = (1 + \varepsilon)r^*$.

Then

$$m(p_1,p_2) := p_1 + (p_2 - p_1)\frac{l_1}{l} = p_2 + (p_1 - p_2)\frac{\delta^2 + (1 + \varepsilon)^2 - 1}{2(1 + \varepsilon)^2}$$

lies in the ball $B(c,r_m)$ and can be calculated independent of $r^*$, only knowing $p_1$, $p_2$, $\delta$ and $\varepsilon$.

One can extend this definition to a sequence $m_1,m_2,\ldots,m_k$ based on a sequence of points $p_1,\ldots,p_k$ with $p_i \in P$ with $|p_im_{j-1}| \geq (1 + \varepsilon)r^*$ for all $i \geq j > 1$.

Let

$$\delta_i := \begin{cases} 1, & \text{if } i = 1, \\ \sqrt{1 - \left(\frac{(1 + (1 + \varepsilon))^2 - \delta^2_i}{2(1 + \varepsilon)}\right)^2}, & \text{otherwise.} \end{cases}$$

and

$$m_i := \begin{cases} p_1, & \text{if } i = 1, \\ (m_{i-1},m_{i-1}) = m_{i-1} + (p_i - m_{i-1})\frac{\delta_i^2 + (1 + \varepsilon)^2 - 1}{2(1 + \varepsilon)^2}, & \text{otherwise.} \end{cases}$$

As all points in $P$ lie in the ball $B(c^*,r^*)$, it follows by induction from Corollary 2.3 that $m_i$ lies in the ball $B(c^*,\delta_i r^*)$.

\textsc{GradientMEB} starts with an arbitrary point $p_1$ from $P$ as $m_1$ and always uses the farthest point from $m_1$ in $P$ as $p_i$. That way, at each round we either have $|m_ip_{i+1}| > (1 + \varepsilon)r^*$ or $m_i$ is already a $(1 + \varepsilon)$-approximation. As we do not know, which of both holds at any round, we just return the best $m_i$ out of all rounds.

It remains to prove, that any sequence of points with $|p_im_{j-1}| \geq (1 + \varepsilon)r^*$ for all $i \geq j > 1$ contains at most $k \leq \lceil \frac{2}{\varepsilon} \rceil$ points before $m_i \in B(c^*,\varepsilon r^*)$.

If at step $i$, $|m_{i-1}p_i| = (1 + \varepsilon)r^*$, $p_i \in \partial B^*$ and $m_{i-1} \in \partial B(c^*,\delta_{i-1}r^*)$, then $m_i \in \partial B(c^*,\delta_i r^*)$ by Lemma 2.2. In that case, $m_i \in B(c^*,\varepsilon r^*)$ if and only if $\delta_i < \varepsilon$. As this is the worst-case, we can assume we were given our sequence of points $p_i \in P$ by an adversary, always fulfilling $|m_{i-1}p_i| = (1 + \varepsilon)r^*$ and $p_i \in \partial B^*$, which gives $m_{i-1} \in \partial B(c^*,\delta_{i-1})$ by induction.

We use a similar proof as [1] for their core-set based algorithm. For this we consider the line segments $a_i := m_{i-1}m_i$ with $|a_i| = \alpha_i(1 + \varepsilon)r^*$ and $b_i := m_ip_i$ with $|b_i| = \beta_i(1 + \varepsilon)r^*$ that together form $m_{i-1}p_i$ as illustrated in Figure 3.

As $m_i$ converges towards $c^*$ with increasing $i$, $\beta_i$ increases and $\alpha_i$ decreases. However, $\beta_i$ can be at most $\frac{1}{\delta_i(1 + \varepsilon)}$ by construction as $b_i$ forms a right-angled triangle with $c^*p_i$ as the hypotenuse, so $\beta_i(1 + \varepsilon)r^* = |b_i| < |c^*p_i| = r^*$. We will now show a lower bound on $\beta_i$ and prove that it exceeds $\frac{1}{\delta_i(1 + \varepsilon)}$ for $i \geq \frac{2}{\varepsilon} - 1$. In that case, there does not exist a point $p_i$ with $|m_{i-1}p_i| = (1 + \varepsilon)r^*$ and $p_i \in \partial B^*$ that our adversary could have given us. This
can only happen if the intersection of \( \partial B^* \) and \( \partial B(m_{i-1}, (1 + \varepsilon)r^*) \) is empty, and therefore, \( \partial B(m_{i-1}, (1 + \varepsilon)r^*) \) covers \( B^* \).

\[ |b_i| = (1 + \varepsilon) r^* - |a_i| \Rightarrow \beta_i = 1 - \alpha_i. \quad (5) \]

In addition, by the construction of \( m_i \) as illustrated in Figure 3 and the Pythagorean theorem, it holds

\[ \beta_i^2 (1 + \varepsilon)^2 = 1^2 - \delta_i^2 \]
\[ = 1 - (\delta_{i-1}^2 - \alpha_i^2 (1 + \varepsilon)^2) \]
\[ = 1 - ((1 - \beta_{i-1}^2)(1 + \varepsilon)^2) - \alpha_i^2 (1 + \varepsilon)^2) \]
\[ \Rightarrow \beta_i^2 = \alpha_i^2 + \beta_{i-1}^2. \quad (6) \]

Combining these two equations we get

\[ 1 - \alpha_i = \sqrt{\beta_{i-1}^2 + \alpha_i^2} \]
\[ 1 - 2\alpha_i + \alpha_i^2 = \beta_{i-1}^2 + \alpha_i^2 \]
\[ \Rightarrow \alpha_i = \frac{1 - \beta_{i-1}^2}{2}. \quad (7) \]

Applying Equation 5 again we obtain the recurrence

\[ \beta_i = \frac{1 + \beta_{i-1}^2}{2}. \quad (8) \]

If we substitute \( \gamma_i = \frac{1}{1 - \beta_i} \Leftrightarrow \beta_i = \frac{\gamma_i - 1}{\gamma_i} \) in Equation 8, we get

\[ \gamma_i = \frac{\gamma_{i-1} - 1}{1 - 1/(2\gamma_{i-1})} = \gamma_{i-1}(1 + \frac{1}{2\gamma_{i-1}^2} + \frac{1}{4\gamma_{i-1}^4} + \cdots) \geq \gamma_{i-1} + \frac{1}{2}. \quad (9) \]

As we have \( \beta_1 = 1/2 \) and hence \( \gamma_1 = 2 \), we know \( \gamma_i \geq (3+i)/2 \) and hence \( \beta_i \geq 1 - \frac{2}{3+i} \). To obtain \( \beta_i \geq 1/(1+\varepsilon) \) it suffices to have \( i \geq \lfloor 2/\varepsilon \rfloor - 1 \).

This also concludes the proof of Theorem 2.1.
2.1 Extension to the 2-center problem

We employ a strategy quite similar to the approach in [1]. We aim to construct two series of centers \( m_{1,j} \) and \( m_{2,k} \) based on two series of points from the two optimal balls \( B_1^* \) and \( B_2^* \).

We start with an arbitrary point \( p_1 \) and set \( m_{1,1} = p_1 \) as we can assume \( p_1 \in B_1^* \) without loss of generality. In every further step, we pick a point \( p_i \) farthest from the two current centers \( m_{1,j} \) and \( m_{2,k} \). As long as we have no center for \( B_2^* \), we pick the point furthest from \( m_{1,j} \). We then employ a guessing oracle that tells us whether \( p_i \) belongs to \( B_1^* \) or \( B_2^* \). Depending on its answer, we add the point to the sequence for the respective ball then calculate a new center \( m_{1,j+1} \) or \( m_{2,k+1} \) as in our 1-center algorithm.

After at most \( 2\lceil \frac{2}{\varepsilon} \rceil \) picks, we obtain a \((1 + \varepsilon)\)-approximation. As we do not have a guessing oracle, we just exhaust all possible guesses and return the best solution encountered, which results in a running time of \( O(nd2^{\varepsilon}) \).

2.2 Extension to the \( k \)-center problem for \( k > 2 \)

The \( k \)-center algorithm is a straightforward extension of the 2-center algorithm. As we need to guess at most \( k\lceil \frac{2}{\varepsilon} \rceil \) points to obtain \((1 + \varepsilon)\)-approximation and have to exhaust \( k \) possibilities each, our algorithm runs in time \( O(ndk2^{\varepsilon}) \).

3 Conclusion

We provided a new efficient gradient-descent \((1 + \varepsilon)\)-approximation algorithm for MEB in arbitrary dimensions running in time \( O(nd/\varepsilon) \), which is strictly better than previous core-set based approaches with running times \( O(nd/\varepsilon + \frac{1}{\varepsilon} \log \frac{1}{\varepsilon}) \) as long as \( nd \in o(\frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon}) \). Like the core-set based algorithms it can be extended to the \( k \)-center problem with a running time of \( O(ndk2^{\varepsilon}) \), which makes the gradient-descent based algorithm theoretically equivalent to the core-set based approach with possibly better run-time constants by combining similar analysis with new geometric observations.

References


