Rotational symmetric flexible placements of graphs

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Abstract

We study the existence of an \( n \)-fold rotational symmetric placement of a symmetric graph in the plane allowing a continuous deformation that preserves the symmetry and the distances between adjacent vertices. We show that such a flexible placement exists if and only if the graph has a NAC-colouring satisfying an additional property on the symmetry; a NAC-colouring is a surjective edge colouring by two colours such that every cycle is either monochromatic, or there are at least two edges of each colour.

Rigid graphs are those which have only finitely many non-congruent placements in the plane with the same edge lengths as a generic placement. These graphs can, however, have non-generic special choices of a placement such that there are infinitely many non-congruent placements in the plane with the same edge lengths. We call such a placement flexible.

The study of flexible placements of generically rigid graphs has a long history. Dixon found two types of flexible placements of the bipartite graph \( K_{3,3} \) [3, 17, 14]. Walter and Husty [15] proved that these are indeed all (assuming that vertices do not overlap). Figure 1 shows some special symmetric cases of these two constructions applied to \( K_{4,4} \). Further examples of graphs with flexible placements are Burmester’s focal point mechanism [1], a 12-vertex graph studied by Kempe [11], and two constructions by Wunderlich [16, 18].

![Figure 1](image1.png)

**Figure 1** The vertices of \( K_{4,4} \) can be placed symmetrically on orthogonal lines to make the graph flexible with 2-fold rotational symmetry (left). A 2-fold rotationally symmetric flexible instance of \( K_{4,4} \) is obtained by placing the vertices of each part to a rectangle so that the two rectangles have the same intersection of diagonals and parallel/orthogonal edges (middle). Although there is a 4-fold rotationally symmetric choice of rectangles (right), the deformed placements preserving the edge lengths are only 2-fold symmetric. The colours indicate equality of edge lengths in a placement.

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This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.
In recent works [7, 8, 4] a deeper analysis of existence of flexible placements is done via graph colourings. There is a special type of edge colourings, called NAC-colourings (“No Almost Cycles”, see [7]), which classify the existence of a flexible placement in the plane and give a construction of the motion. Furthermore, determining the NAC-colourings of a given graph and the possible constructions that come from it can be done easily by using the SAGEMath package FlexRiLoG [5]. In [6] we used these methods for constructing flexible placements for symmetric graphs as in Figure 2. However, we did not take advantage of the symmetry for the construction, and instead had to construct the framework manually.

Symmetry plays an important role in art and design, and often appears in nature also. Due to this, there is a large body of work focused on symmetric frameworks and their properties in the context of rigidity theory; see [10, 12]. In particular, we shall be focusing on graphs and frameworks that display $n$-fold rotational symmetry, such as in Figure 2.

In this extended abstract, we formalise the NAC-colouring method for rotationally symmetric flexible placements, i.e. in such a way that the motion preserves the symmetry. By combining Theorem 3.1 and Theorem 3.2, we obtain the following result:

Let $G$ be a $C_n$-symmetric connected graph. Then $G$ has a $C_n$-symmetric NAC-colouring if and only if there exists a $C_n$-symmetric flexible framework $(G, p)$ in $\mathbb{R}^2$.

1 Preliminaries

We briefly recall some basic notions from rigidity theory and define NAC-colourings in this section. All graphs $G = (V(G), E(G))$ in the paper are connected and $|E(G)| \geq 1$.

**Definition 1.1.** A framework in $\mathbb{R}^2$ is a pair $(G, p)$ where $G$ is a (finite simple) graph and $p : V(G) \to \mathbb{R}^2$ is a placement of $G$, a possibly non-injective map such $p(u) \neq p(v)$ if $uv \in E(G)$. We define frameworks $(G, p)$ and $(G, q)$ to be equivalent if for all $uv \in E(G)$,

$$||p(u) - p(v)|| = ||q(u) - q(v)||.$$  \hspace{1cm} (1)

We define two placements $p, q$ of $G$ to be congruent if (1) holds for all $u, v \in V(G)$.

An equivalent definition for congruence is as follows; $p$ and $q$ are congruent if there exists an Euclidean isometry $M$ of $\mathbb{R}^2$ such that $Mq(v) = p(v)$ for all $v \in V(G)$.

**Definition 1.2.** Let $(G, p)$ be a framework. A flex (in $\mathbb{R}^2$) of $(G, p)$ is a continuous path $t \mapsto p_t$, $t \in [0, 1]$, in the space of placements of $G$ such that $p_0 = p$ and each $(G, p_t)$ is equivalent to $(G, p)$. If $p_t$ is congruent to $p$ for all $t \in [0, 1]$ then $p_t$ is trivial. We define $(G, p)$ to be flexible if there is a non-trivial flex of $(G, p)$ in $\mathbb{R}^2$, and rigid otherwise.

Figure 2 A symmetric graph which has a 3-fold rotationally symmetric flexible placement.
It was shown in [13] that a framework \((G, p)\) with a generic placement of vertices (see [9]) is rigid if and only if \(G\) contains a Laman graph as a spanning subgraph. This does not inform us whether a graph will have a flexible placement; for example, any generic placement of \(K_{4,4}\) is rigid, however as shown by Figure 1 we can construct flexible placements for it. To determine whether a graph has flexible placements we introduce the following.

\[\text{Definition 1.3.}\] An edge colouring \(\delta : E(G) \to \{\text{red}, \text{blue}\}\) of a graph \(G\) is a NAC-colouring if \(\delta(E(G)) = \{\text{red}, \text{blue}\}\) and for each cycle in \(G\), either all edges have the same colour, or there are at least two red and two blue edges. NAC-colourings \(\delta, \bar{\delta}\) of \(G\) are conjugated if \(\bar{\delta}(e) \neq \delta(e)\) for all \(e \in E(G)\).

\[\text{Remark.}\] The colourings considered within this paper are not required to have incident edges coloured differently, contrary to the common graph-theoretical terminology.

Having these definitions, we can recall the result [7, Theorem 3.1].

\[\text{Theorem 1.4.}\] A graph has a flexible placement in \(\mathbb{R}^2\) if and only if it has a NAC-colouring.

## 2 Rotational symmetry

The following definitions specify the rotational symmetric setting where Definition 2.1 gives a combinatorial description of symmetry of a graph and Definition 2.2 describes geometric symmetry of a framework. Note, that in figures we describe vertices in graphs by filled disks and in frameworks with circles.

\[\text{Definition 2.1.}\] Let \(G\) be a graph and \(n \geq 2\). Let the \(n\)-fold rotation group, \(\mathcal{C}_n := \langle \omega : \omega^n = 1 \rangle\) act on \(G\), i.e., there exists an injective group homomorphism \(\theta : \mathcal{C}_n \to \text{Aut}(G)\), where \(\text{Aut}(G)\) is the automorphism group of \(G\). We define \(\gamma v := \theta(\gamma)(v)\) for \(\gamma \in \mathcal{C}_n\); similarly, for any edge \(e = uv \in E(G)\), we define \(\gamma e := \gamma u \gamma v\). We shall define \(v \in V(G)\) to be an invariant vertex if \(\gamma v = v\) for all \(\gamma \in \mathcal{C}_n\), and partially invariant if \(\gamma v = v\) for some \(\gamma \in \mathcal{C}_n, \gamma \neq 1\). The graph \(G\) is called \(\mathcal{C}_n\)-symmetric if:

(a) a vertex is invariant if and only if it is partially invariant, and
(b) the set of invariant vertices of \(G\) forms an independent set.

\[\text{Definition 2.2.}\] Let \((G, p)\) be a framework in \(\mathbb{R}^2\), \(G\) be \(\mathcal{C}_n\)-symmetric and \(\tau : \mathcal{C}_n \to O(2, \mathbb{R})\) be a symmetry map, i.e., an injective group homomorphism, given by

\[
\tau(\omega) = \begin{bmatrix}
\cos(2\pi/n) & \sin(2\pi/n) \\
-\sin(2\pi/n) & \cos(2\pi/n)
\end{bmatrix}.
\]

If \(p(\gamma v) = \tau(\gamma)p(v)\) for each \(v \in V(G)\) and \(\gamma \in \mathcal{C}_n\), then \((G, p)\) is a \(\mathcal{C}_n\)-symmetric framework; likewise, we define \(p\) to be a \(\mathcal{C}_n\)-symmetric placement of \(G\).

We note that if \((G, p)\) is \(\mathcal{C}_n\)-symmetric then it is \(\mathcal{C}_m\)-symmetric for all \(m|n\).

\[\text{Definition 2.3.}\] Let \((G, p)\) be a \(\mathcal{C}_n\)-symmetric framework in \(\mathbb{R}^2\). If there is a non-trivial flex \(p_0\) of \((G, p)\) such that each \((G, p_0)\) is \(\mathcal{C}_n\)-symmetric, then \((G, p)\) is \(\mathcal{C}_n\)-symmetric flexible (or \(n\)-fold rotation symmetric flexible), and \(\mathcal{C}_n\)-symmetric rigid otherwise.

We define the following for edge colourings of \(\mathcal{C}_n\)-symmetric graphs.

\[\text{Definition 2.4.}\] Let \(G\) be a \(\mathcal{C}_n\)-symmetric graph with colouring \(\delta\). A red, resp. blue, component is a connected component of \(G^\delta_{\text{red}} := (V(G), \{e \in E(G) : \delta(e) = \text{red}\})\), resp. \(G^\delta_{\text{blue}}\). A blue or red component \(H \subset G\) is partially invariant if there exists \(\gamma \in \mathcal{C}_n \setminus \{1\}\) such that \(\gamma H = H\), and invariant if \(\gamma H = H\) for all \(\gamma \in \mathcal{C}_n\) (see Figure 3 for an example).
We focus on the following class of NAC-colourings suitable for dealing with symmetries.

Definition 2.5. Let $G$ be a $C_n$-symmetric graph with NAC-colouring $\delta$. We define $\delta$ to be a $C_n$-symmetric NAC-colouring if $\delta(\gamma e) = \delta(e)$ for all $e \in E(G)$ and $\gamma \in C_n$ and no two distinct blue, resp. red, partially invariant components are connected by an edge (see Figure 4 for examples).

Example 2.6. The cartesian product of $K_4$ and $K_2$ has a single NAC-colouring (up to conjugation) $\delta$, see Figure 5. The graph is $C_2$-symmetric under the symmetry $\theta$, where $\theta(\omega)$ is the permutation $(1,6)(2,5)(3,8)(4,7)$; further, the NAC-colouring $\delta$ is a $C_2$-symmetric NAC colouring with respect to $\theta$. The graph is also $C_4$-symmetric under the symmetry $\theta'$, where $\theta'(\omega)$ is the permutation $(1,6,3,8)(5,2,7,4)$; however, the NAC-colouring $\delta$ is not a $C_4$-symmetric NAC colouring with respect to $\theta'$, since the blue partially invariant components are connected by edges.
3 Necessary and sufficient conditions for rotationally symmetric flexibility

In this section we show the construction of $C_n$-symmetric motions from $C_n$-symmetric NAC-colourings. The inverse direction is also true.

Theorem 3.1. If $(G, p)$ is a $C_n$-symmetric flexible framework in $\mathbb{R}^2$, $G$ being a connected graph, then $G$ has a $C_n$-symmetric NAC-colouring.

The proof of Theorem 3.1 follows in a similar vein to the proof of [7, Theorem 3.1] by using methods from valuation theory. There are more complexities involved however, since we also require that the colouring obtained respects the symmetry of the graph and partially invariant components are not connected by edges.

Theorem 3.2. Let $G$ be a $C_n$-symmetric connected graph. If $G$ has a $C_n$-symmetric NAC-colouring $\delta$, then there exists a $C_n$-symmetric flexible framework $(G, p)$ in $\mathbb{R}^2$.

Proof. The proof is based on the “zigzag” grid construction from [7] with a specific choice of the grid. Let $R^j_0, R^j_1, \ldots, R^j_{n_j}$ be the red components of $G^j_{\text{red}}$ that are not partially invariant. We can assume that $R^j_i = \omega^i R^j_0$ for $0 \leq i < n_j$ and $1 \leq j \leq m$. Similarly, let $B^j_0, B^j_1, \ldots, B^j_k$ be the blue components of $G^j_{\text{blue}}$ that are not partially invariant and $B^j_i = \omega^i B^j_0$ for $0 \leq i < n_j$ and $1 \leq j \leq k$.

Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_k$ be points in $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that $a_j \neq \tau(\omega)^i a_{j'}$ and $b_j \neq \tau(\omega)^i b_{j'}$ for $j \neq j'$ and $1 \leq i < n_j$. We define functions $\overline{\pi}, \overline{b}: V(G) \to \mathbb{R}^2$ by

$$
\overline{\pi}(v) = \begin{cases} 
\tau(\omega)^i a_j & \text{if } v \in R^j_i, \\
(0, 0) & \text{otherwise},
\end{cases} \quad \text{and} \quad \overline{b}(v) = \begin{cases} 
\tau(\omega)^i b_j & \text{if } v \in B^j_i, \\
(0, 0) & \text{otherwise}.
\end{cases}
$$

We note that a vertex is mapped to the origin by $\overline{\pi}$ (respectively, $\overline{b}$) if and only if it lies in a red (respectively, blue) partially invariant component. We now obtain for each $t \in [0, 2\pi]$ a placement $p_t$ of $G$, where

$$
p_t(v) := \begin{pmatrix} \cos t & -\sin t \\
\sin t & \cos t \end{pmatrix} \pi(v) + \overline{b}(v).
$$

Let $uv \in E(G)$. If $\delta(uv)$ is red (resp. blue) then $\pi(u) = \pi(v)$ (resp. $\overline{b}(u) = \overline{b}(v)$). Hence, the edge length $\|p_t(u) - p_t(v)\|$ is independent of $t$.

Next, we have to show that no two adjacent vertices are mapped to the same point by the placement $p_0$. Assume that $(\pi(u), \overline{b}(u)) = (\pi(v), \overline{b}(v))$ for some vertices $u, v$. Suppose this is due to the fact that $u$ and $v$ belong to the same red and same blue (possibly partially invariant) component. Hence, $uv \notin E(G)$, otherwise $\delta$ is not a NAC-colouring ($uv$ would yield a cycle with a single edge in one color). On the other hand, if $u$ and $v$ are in two different red (resp. blue) components, then $\pi(u) = \pi(v) = (0, 0)$ (resp. $\overline{b}(u) = \overline{b}(v) = (0, 0)$).

By our construction of $\pi$ (resp. $\overline{b}$), it follows that $u, v$ both lie in partially invariant red (resp. blue) components. Since these components are partially invariant, $uv \notin E(G)$ by the assumption that $\delta$ is $C_n$-symmetric.

As edge lengths are preserved and no two vertices connected by an edge are mapped to the same point, $(G, p) := (G, p_0)$ is a framework with a flex $p_t$. Further, the flex is not trivial by surjectivity of $\delta$, thus $(G, p)$ is a flexible framework.

Finally, we show that $p_t$ is $C_n$-symmetric. If $v \in R^j_i \cap B^k_i$, then

$$
\omega v \in \tau(\omega) R^j_i \cap \tau(\omega) B^k_i = R^{(i+1) \mod n}_j \cap B^{(k+1) \mod n}_i.
$$
Hence, \( \overline{\pi}(\omega v) = \tau(\omega)\overline{\pi}(v) \) and \( \overline{b}(\omega v) = \tau(\omega)\overline{b}(v) \). The same equalities hold also if \( v \) belongs to a partially invariant component, since then \( \omega v \) is also in a partially invariant component. Using commutativity of rotation matrices, we conclude the proof by (2):

\[
p_t(\omega v) = \tau(\omega) \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \overline{\pi}(v) + \tau(\omega)\overline{b}(v) = \tau(\omega)p_t(v).
\]

\( \blacktriangleright \)

\textbf{Example 3.3.} By using the construction described in Theorem 3.2 we can construct the \( \mathcal{C}_n \)-symmetric flexible frameworks given in Figures 6, 7 and 8.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{A flexible \( \mathcal{C}_2 \)-symmetric placement for a given \( \mathcal{C}_2 \)-symmetric NAC-colouring.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{A flexible \( \mathcal{C}_3 \)-symmetric placement for a given \( \mathcal{C}_4 \)-symmetric NAC-colouring.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{A flexible \( \mathcal{C}_4 \)-symmetric placement for a given \( \mathcal{C}_4 \)-symmetric NAC-colouring.}
\end{figure}

\textbf{Example 3.4.} We consider the graph in Figure 9 with the given \( \mathcal{C}_3 \)-symmetric NAC-colouring. Then the red 3-cycle forms a red partially invariant component. Therefore its position is fixed during the motion.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{A flexible \( \mathcal{C}_3 \)-symmetric placement for a given \( \mathcal{C}_3 \)-symmetric NAC-colouring.}
\end{figure}

The constructed framework given by the proof of Theorem 3.2 may not be proper flexible, i.e., have no overlapping vertices, see Figure 10. As outlined in [8], there are necessary and sufficient conditions for determining when a graph will or will not have a proper flexible placements, although they have as of yet not been adapted fully to the symmetric case.
Figure 10 A flexible $C_2$-symmetric placement for a given $C_2$-symmetric NAC-colouring for which two vertices overlap.

Remark. While we have only dealt with frameworks with rotational symmetry, there are other types of symmetry for the plane, namely reflectional and translational symmetry. Although flexible placements that preserve translational symmetry have very recently been investigated [2], not much is known for flexible placements that preserve reflectional symmetry or preserve both reflectional and rotational symmetry.

References

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