Simple Drawings of $K_{m,n}$ Contain Shooting Stars

Oswin Aichholzer$^1$, Alfredo García$^2$, Irene Parada$^1$, Birgit Vogtenhuber$^1$, and Alexandra Weinberger$^1$

1 Graz University of Technology, Austria  
[oaich|iparada|bvogt|weinberger]@ist.tugraz.at  
2 Departamento de Métodos Estadísticos and IUMA, Universidad de Zaragoza. olaverri@unizar.es

Abstract

Simple drawings are drawings of graphs in which all edges have at most one common point (either a common endpoint, or a proper crossing). It has been an open question whether every simple drawing of a complete bipartite graph $K_{m,n}$ contains a plane spanning tree as a subdrawing. We answer this question to the positive by showing that for every simple drawing of $K_{m,n}$ and for every vertex $v$ in that drawing, the drawing contains a shooting star rooted at $v$, that is, a plane spanning tree with all incident edges of $v$.

1 Introduction

A simple drawing is a drawing of a graph on the sphere $S^2$ or, equivalently, in the Euclidean plane where (1) the vertices are distinct points in the plane, (2) the edges are non-self-intersecting continuous curves connecting their incident points, (3) no edge passes through vertices other than its incident vertices, (4) and every pair of edges intersects at most once, either in a common endpoint, or in the relative interior of both edges, forming a proper crossing. Simple drawings are also called good drawings [3, 5] or (simple) topological graphs [7, 8]. In semi-simple drawings, the last requirement is softened such that edges without common endpoints are allowed to cross several times. Note that in any simple or semi-simple drawing, there are no tangencies between edges and incident edges do not cross. If a drawing does not contain any crossing at all, it is called plane.

The search for plane subdrawings of a given drawing has been a widely considered topic for simple drawings of the complete graph $K_n$ which still holds tantalizing open problems. For example, Rafla [10] conjectured that every simple drawing of $K_n$ contains a plane Hamiltonian cycle, a statement which is by now known to be true for $n \leq 9$ [1] and several classes of simple drawings (e.g., 2-page book drawings, monotone drawings, cylindrical drawings), but still remains open in general. A related question concerns the least number of pairwise disjoint edges in any simple drawing of $K_n$. The currently best lower bound of $\Omega(n^{1/2-\varepsilon})$, for any $\varepsilon > 0$, for this number has been obtained by Ruiz-Vargas [11], improving over several previous bounds [8, 9, 12], while the trivial upper bound of $n/2$ would be implied by a positive answer to Rafla’s conjecture. A structural result of Fulek and Ruiz-Vargas [6] implies that every simple drawing of $K_n$ contains a plane sub-drawing with at least $2n - 3$ edges.

* O.A., I.P., and A.W. partially supported by the Austrian Science Fund (FWF) grant W1230. A.G. supported by MINECO project MTM2015-63791-R and Gobierno de Aragón under Grant E41-17 (FEDER). I.P. and B.V. partially supported by the Austrian Science Fund within the collaborative DACH project Arrangements and Drawings as FWF project I 3340-N35. This work has been initiated at the 6th Austrian-Japanese-Mexican-Spanish Workshop on Discrete Geometry which took place in June 2019 near Strobl, Austria. We thank all the participants for the great atmosphere and fruitful discussions.

36th European Workshop on Computational Geometry, Würzburg, Germany, March 16–18, 2020. This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.
In this work, we consider the search for plane spanning trees in drawings of complete bipartite graphs. For complete graphs, plane spanning trees trivially exist in both simple and semi-simple drawings, as incident edges don’t cross, and as every vertex is incident to all other vertices. Hence the star of any vertex, which consists of all edges incident to that vertex and all vertices, is a plane spanning tree.

The task of finding plane spanning trees in drawings of complete bipartite graphs $K_{m,n}$ turns out to be more involved. In fact, not every semi-simple drawing of a complete bipartite graph contains a plane spanning tree; see Figure 1.

![Figure 1](image-url) A semi-simple drawing of $K_{2,3}$ that does not contain a plane spanning tree.

For simple drawings of $K_{m,n}$, the existence of plane spanning trees has so far only been proven for specific types of drawings. It is not too hard to see that monotone drawings always contain plane spanning trees. Aichholzer et al. showed in [2] that simple drawings of $K_{2,n}$ and $K_{3,n}$, as well as so-called outer drawings of $K_{m,n}$ [4], always contain plane spanning trees of a special type, which they called shooting stars. A shooting star rooted at $v$ is a plane spanning tree with root $v$ that has height 2 and contains the star of vertex $v$.

In the present work, we show that every simple drawing of $K_{m,n}$ contains shooting stars rooted at an arbitrary vertex of $K_{m,n}$.

**Theorem 1.1.** Let $D$ be a simple drawing of the complete bipartite graph $K_{m,n}$ and let $r$ be an arbitrary vertex of $K_{m,n}$. Then $D$ contains a shooting star rooted at $r$.

2 Proof of Theorem 1.1

**Proof.** We can assume that $D$ is drawn on a point set $S = R \cup B$, $R = \{r_1, \ldots, r_m\}$, $B = \{b_1, \ldots, b_n\}$, in which the points in the two vertex partitions $R$ and $B$ are colored red and blue, respectively. Without loss of generality let $r = r_1$.

![Figure 2](image-url) A simple drawing of $K_{3,3}$ (left) and its stereographic projection from $r_1$ (right).

To simplify the figures, we consider the drawing $D$ on the sphere and apply a stereographic projection from $r$ onto a plane. In that way, the edges in the star of $r$ are represented as (not necessarily straight-line) infinite rays; see Figure 2. We will depict them in blue. In the
following, we consider all edges oriented from their red to their blue endpoint. In order to specify how two edges cross each other, we introduce some notation. Consider two crossing edges \( e_1 = r_kb_k \) and \( e_2 = r_jb_j \) and let \( \times \) be their crossing point. Consider the arcs \( \times r_i \) and \( \times b_k \) on \( e_1 \) and \( \times r_j \) and \( \times b_l \) on \( e_2 \). We say that \( e_2 \) crosses \( e_1 \) in **clockwise direction** if the clockwise cyclic order of these arcs around the crossing \( \times \) is \( \times r_i, \times r_j, \times b_k, \) and \( \times b_l \). Otherwise, we say that \( e_2 \) crosses \( e_1 \) in **counterclockwise direction**; see Figure 3.

![Figure 3](image)

**Figure 3** Edge \( e_2 \) crosses edge \( e_1 \) in clockwise direction (left) or counterclockwise direction (right).

We prove Theorem 1.1 by induction on \( n \). For \( n = 1 \) and any \( m \geq 1 \), the whole drawing \( D \) is a shooting star rooted at any vertex, and in particular at \( r \).

Assume that the existence of shooting stars rooted at any vertex has been proven for any simple drawing of \( K_{m,n'} \) with \( n' < n \). Let \( M \) be a subset of the edges of \( D \) connecting each vertex \( r \neq r \) to some blue vertex in \( B \) (with exactly one edge for every red vertex), such that (i) \( M \cup \{ \bigcup_{j=2}^n rb_j \} \) does not contain any crossing and (ii) the number of crossings of \( M \) with edge \( rb_1 \) is the minimum possible. Observe that the set \( M \) is well defined, since, by the induction hypothesis, the subdrawing of \( D \) obtained by deleting the blue vertex \( b_1 \) and its incident edges contains a shooting star rooted at \( r \). Thus there exists a set \( M_1 \) of edges from \( D \) connecting each \( r_i \neq r \) to some blue vertex in \( B \setminus \{ b_1 \} \) such that \( M_1 \cup \{ \bigcup_{j=2}^n rb_j \} \) does not contain any crossing. As arbitrarily many of the edges in \( M_1 \) might cross \( rb_1 \), \( M_1 \) might not fulfill condition (ii). We will show that \( M \) does not contain any crossing with \( rb_1 \). Since \( M \cup \{ \bigcup_{j=2}^n rb_j \} \) does not contain crossings by construction and as \( rb_1 \) does not cross any edges of \( \bigcup_{j=2}^n rb_j \) in a simple drawing, it follows that the edges in \( M \cup \{ \bigcup_{j=1}^n rb_j \} \) form the desired shooting star.

Assume for a contradiction that \( rb_1 \) crosses at least one edge in \( M \). When traversing \( rb_1 \) from \( b_1 \) to \( r \), let \( x \) be the first crossing point of \( rb_1 \) with an edge \( r_kb_k \) in \( M \). Without loss of generality, when orienting \( rb_1 \) from \( r \) to \( b_1 \), \( r_kb_k \) from \( r_k \) to \( b_k \), \( r_kb_k \) crosses \( rb_1 \) in counterclockwise direction (otherwise we can mirror the drawing).

Suppose first that the arc \( r_kx \) (on \( r_kb_k \) and oriented from \( r_k \) to \( x \)) is crossed in counterclockwise direction by an edge incident to \( b_1 \) (and oriented from the red endpoint to \( b_1 \)). Let \( e = r_kb_k \) be such an edge whose crossing with \( r_kb_k \) at a point \( y \) is the closest to \( x \). Otherwise, let \( e \) be the edge \( r_kb_k \) and \( y \) be the point \( r_k \). In the remaining figures, we represent in blue the edges of the star of \( r \), in red the edges in \( M \), and in black the edge \( e \).

We distinguish two cases depending on whether \( e \) crosses an edge of the star of \( r \). The idea in both cases is to define a region \( \Gamma \) and, inside it, redefine the connections between red and blue points to reach a contradiction.

**Case 1**: \( e \) does not cross any edge of the star of \( r \). Let \( \Gamma \) be the closed region of the plane bounded by the arcs \( yb_1 \) (on \( e \)), \( b_1x \) (on \( rb_1 \)), and \( xy \) (on \( r_kb_k \)); see Figure 4. Observe that all the blue points \( b_j \) lie outside the region \( \Gamma \) and that for all the red points \( r_i \) inside region \( \Gamma \), the edge \( r_ib_1 \) must be in \( \Gamma \). Let \( M' \) denote the set of edges \( r_ib_1 \) with \( r_i \in \Gamma \) and note that \( r_kb_k \in M' \). Consider the set \( M'' \) of red edges obtained from \( M' \) by replacing, for each red point \( r_i \in \Gamma \), the (unique) edge incident to \( r_i \) in \( M' \) by the edge \( r_ib_1 \) in \( M_\Gamma \), and keeping the
other edges in \( M \) unchanged. In particular, the edge \( r_k b_1 \) has been replaced by the edge \( r_k b_1 \). The edges in \( M_\Gamma \) neither cross each other nor cross any of the blue edges \( r b_j \). Moreover, we now show that the non-replaced edges in \( M \) must lie completely outside \( \Gamma \). These edges can neither cross \( r_k b_1 \) (by definition of \( M \)) nor the arc \( b_1 x \) (on \( r b_1 \)). Thus, if they are incident to \( b_1 \) they cannot cross the boundary of \( \Gamma \), and otherwise their endpoints lie outside \( \Gamma \) and they can only cross one arc of the boundary. Therefore, \( M' \) satisfies that \( M' \cup \{ \bigcup_{j=2}^{m} r b_j \} \) does not contain any crossing, and has fewer crossings with \( r b_1 \) than \( M \). This contradicts the definition of \( M \) as the one with the minimum number of crossings with \( r b_1 \).

**Case 2:** \( e \) crosses the star of \( r \). When traversing \( e \) from \( r_k \) or \( r_l \) (depending on the definition of \( e \)) to \( b_1 \), let \( I = \{ \alpha, \beta, \ldots, \rho \} \) be the indices of the edges of the star of \( r \) in the order as they are crossed by \( e \) and let \( y_{\alpha}, \ldots, y_{\rho} \) be the corresponding crossing points on \( e \). Note that, when orienting \( e \) from \( r_k \) or \( r_l \) to \( b_1 \), the edges \( r b_{\xi}, \xi \in I \), oriented from \( r \) to \( b_{\xi} \), cross \( e \) in counterclockwise direction, since they can neither cross \( r_k b_1 \) (by definition of \( M \)) nor \( r b_1 \).

The three arcs \( y_{\alpha} (on \ r b_{\alpha}), \ y_{\beta} b_1 (on \ e), \) and \( b_1 r \) divide the plane into two (closed) regions, \( \Pi_{\text{left}} \), containing vertex \( r_k \), and \( \Pi_{\text{right}} \), containing vertex \( b_1 \). For each \( \xi \in I \), let \( M_{\xi} \) be the set of red edges of \( M \) incident to some red point in \( \Pi_{\text{right}} \) and to \( b_{\xi} \). Note that all the edges in \( M_{\xi} \) (if any) must cross the edge \( e \). When traversing \( e \) from \( r_k \) or \( r_l \) to \( b_1 \), we denote by \( x_{\xi}, z_{\xi} \) the first and the last crossing points of \( e \) with the edges of \( M_{\xi} \cup r b_{\xi} \), respectively; see Figure 5 for an illustration. We remark that both \( x_{\xi} \) and \( z_{\xi} \) might coincide with \( y_{\xi} \) and, in particular, if \( M_{\xi} = \emptyset \) then \( x_{\xi} = y_{\xi} = z_{\xi} \).

We now define some regions in the drawing \( D \). Suppose first that there are edges in \( M \) (oriented from the red to the blue point) that cross \( r b_1 \) (oriented from \( r \) to \( b_1 \)) in clockwise direction. Let \( r_{\xi} b_{\eta} \) be the edge in \( M \) whose clockwise crossing with \( r b_1 \) at a point \( x' \) is the...
There exist simple drawings of $K_m,n$ in which every plane subdrawing has at most as many edges as a shooting star. For example, consider a straight line drawing of $K_{m,n}$ where all vertices are in convex position such that all red points are next to each other in the convex hull; see Figure 6 (left). The convex hull is a $(m+n)$-gon which shares only two edges with the drawing of $K_{m,n}$; see Figure 6 (right). All other edges of the drawing of $K_{m,n}$ are diagonals of the polygon. As there can be at most $(m+n)-3$ pairwise non-crossing diagonals in a convex $(m+n)$-gon, any plane subdrawing of this drawing of $K_{m,n}$ contains at most $m+n-1$ edges.

**Figure 6** Left: A simple drawing of $K_{m,n}$ where no plane subdrawing can have more edges than a shooting star. Right: A convex $(n+m)$-gon (in green lines) around the simple drawing of $K_{m,n}$.

Furthermore, both requirements from Theorem 1.1 — simplicity of the drawing and having
All Simple Drawings of $K_{m,n}$ Contain Shooting Stars

a complete bipartite graph – are in fact necessary: As mentioned in the introduction, not all semi-simple drawings of $K_{m,n}$ contain a plane spanning tree. Further, if in the example in Figure 6, we delete one of the two edges of $K_{m,n}$ on the boundary of the convex hull, then any plane subdrawing has at most $m + n - 2$ edges. Hence the resulting drawing cannot contain any plane spanning tree.

4 Remarks on an Algorithm

The proof of Theorem 1.1 contains an algorithm with which we can find shooting stars in given simple drawings. We start with constructing the shooting star for a subdrawing that is a $K_{m,1}$ and then inductively add more vertices. Every time we are adding a new vertex, the shooting star of the step before is a set fulfilling all requirements of $M_1 \cup \left( \bigcup_{j=2}^{n} rB_j \right)$ in the proof. By replacing edges as described in the proof, we obtain a new set with the same properties and fewer crossings. We continue replacing edges until we obtain a set of edges ($M$ in the proof) that form a shooting star for the extended vertex set. We remark that the runtime of this algorithm might be exponential, as finding the edges of $M_G$ might require solving the problem for the subgraph induced by $\Gamma$. However, we believe that there exists a polynomial-time algorithm for this task.

Open Problem 1. Given a simple drawing of the complete bipartite graph, is there a polynomial-time algorithm to find a plane spanning tree contained in the drawing?

References


