On the edge-length ratio of 2-trees[*]

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──── **Abstract** ────

We study planar straight-line drawings of graphs that minimize the ratio between the length of
the longest and the shortest edge. We answer a question of Lazard et al. [Theor. Comput. Sci.
770 (2019), 88–94] and, for any given constant r, we provide a 2-tree which does not admit a
planar straight-line drawing with a ratio bounded by r. When the ratio is restricted to adjacent
edges only, we prove that any 2-tree admits a planar straight-line drawing whose edge-length
ratio is at most $4 + \varepsilon$ for any arbitrarily small $\varepsilon > 0$.

1   Introduction

Straight-line drawings of planar graphs are thoroughly studied both for their theoretical
interest and their applications in a variety of disciplines (see, e.g., [7, 13]). Different quality
measures for planar straight-line drawings have been considered in the literature, including
area, angular resolution, slope number, average edge length, and total edge length (see,
e.g., [9, 10, 12]).

This paper studies the problem of computing planar straight-line drawings of graphs
where the length ratio of the longest to the shortest edge is as small as possible. We recall
that the problem of deciding whether a graph admits a planar straight-line drawing with
specified edge lengths is NP-complete even when restricted to 3-connected planar graphs [8]
and the completeness persists in the case when all given lengths are equal [5]. In addition,
deciding whether a degree-4 tree has a planar drawing such that all edges have the same
length and the vertices are at integer grid points is NP-complete [1].

In the attempt of relaxing the edge length conditions which make the problem hard,
Hoffmann et al. [10] propose to minimize the ratio between the longest and the shortest
edges among all straight-line drawings of a graph. While the problem remains hard for
general graphs (through approximation of unit disk graphs [6]), Lazard et al. prove [11]
that any outerplanar graph admits a planar straight-line drawing such that the length ratio
of the longest to the shortest edges is strictly less than 2. This result is tight in the sense
that for any $\varepsilon > 0$ there are outerplanar graphs that cannot be drawn with an edge-length
ratio smaller than $2 - \varepsilon$. Lazard et al. also ask whether their construction could be extended
to the class of series-parallel graphs.

We answer this question in the negative sense, by showing that a subclass of series-
parallel graphs, called 2-trees, does not allow any planar straight-line drawing of bounded
edge-length ratio. (The class of 2-trees is defined constructively: an edge is a 2-tree, and

modifying a 2-tree by adding a new vertex connected to two neighboring vertices is also a 2-tree.) In fact, a corollary of our main result is the existence of an $\Omega(\log n)$ lower bound for the edge-length ratio of planar straight-line drawings of 2-trees. Motivated by this negative result, we consider a local measure of edge-length ratio and prove that when the ratio is restricted only to the adjacent edges, any series-parallel graph admits a planar straight-line drawing with local edge-length ratio at most $4+\varepsilon$, for any arbitrarily small $\varepsilon > 0$. The proof of this upper bound is constructive, and it gives rise to a linear-time algorithm assuming a real RAM model of computation. (The omitted proofs can be found in [2].)

It is worth noticing that Borrazzo and Frati recently showed that any 2-tree on n vertices can be drawn with edge-length ratio $O(n^{0.695})$ [3]. This, together with our $\Omega(\log n)$ result, defines a non-trivial gap between upper and lower bound on the edge-length ratio of planar straight-line drawings of partial 2-trees.

2   Preliminaries

We consider finite nonempty planar graphs and their planar straight-line drawings. Once a straight-line drawing of G is given, with a slight abuse of notation we use the same symbol for a vertex U and the point U representing the vertex U in the drawing, as well as for an edge UV and the corresponding segment UV of the drawing. For points U and V, let $|UV|$ denote the Euclidean distance between U and V. For three mutually adjacent vertices U, V and W of a graph G, the symbol $\triangle UVW$ denotes the triangle of the corresponding drawing of G. For a polygon Q, we denote its perimeter by $P(Q)$ and its area by $A(Q)$.

The *edge-length ratio* of a planar straight-line drawing of a graph G is the ratio between the length of the longest and the shortest edge of the drawing.

▶ **Definition 2.1.** The *edge-length ratio* $\rho(G)$ of a planar graph G is the infimum edge-length ratio taken over all planar straight-line drawings of G.

A vertex is called simplicial when its neighborhood forms a clique. A complete graph on $k + 1$ vertices is a k-tree; a graph constructed from a k-tree by adding a simplicial vertex to a clique of size k is also a k-tree. A partial k-tree is a subgraph of a k-tree.

3   Edge-length ratio of 2-trees

We recall that 2-trees are planar graphs. The main result of this section is the following.

▶ **Theorem 3.1.** *For any $r \geq 1$, there exists a 2-tree G whose edge-length ratio $\rho(G) \geq r$.*

To prove Theorem 3.1, for a given r we argue that a sufficiently large 2-tree, drawn with the longest edge having length r, contains a triangle with area at most $\frac{1}{2}$ (Lemma 3.2). Then, inside this triangle of small area we build a sequence of triangles with perimeters decreasing by $\frac{1}{2}$ in each step (Lemmas 3.8 and 3.9), which results in a triangle with an edge of length less than 1.

We consider a special subclass $\mathcal{G} = \{G_0, G_1, \dots\}$ of 2-trees with labeled vertices and edges constructed as follows: G_0 is the complete graph K_3 whose vertices and edges are given the label 0. The graph G_{i+1} is obtained by adding five simplicial vertices to each edge of label i of G_i. Each newly created vertex and edge gets label $i + 1$. See Fig. 1 for an example where the black vertices and edges have label 0, the blue ones have label 1, and the red ones have label 2.

A *separating triangle of level i* in a straight-line drawing of a 2-tree G is an unordered triple $\{U, V, W\}$ of its mutually adjacent vertices such that the vertex W of label i was added
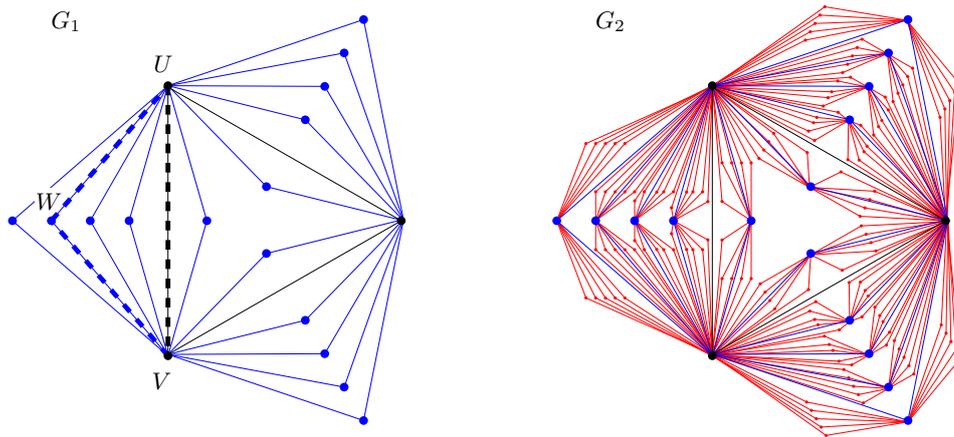
Figure 1 The 2-trees G_1 and G_2. Black color corresponds to label 0, blue to 1, and red to 2. Separating triangle Δ_1 is emphasized by a dashed line in G_1.

as a simplicial vertex to the edge UV in the recursive construction of G and the triangle $\triangle UVW$ splits the plane into two regions, each containing at least two other vertices with label i which are simplicial to the edge UV. In particular, the triangle $\triangle UVW$ contains two vertices of G with label i in its interior. For example, in Fig. 1 (a) vertices $\{U, V, W\}$ form a separating triangle.

▶ **Lemma 3.2.** *For any $k > i \geq 1$, any planar straight-line drawing of the graph G_k and an edge e of G_k labeled by i there exists a separating triangle of level $i + 1$ containing the endpoints of e.*

We proceed to show that any drawing of G_k contains a triangle of sufficiently small area. To this aim, we construct a sequence of nested triangles such that each element's area is half of the previous element's area. We denote as Δ_i a separating triangle in an embedding of G_k such that all its edges have labels at most i, with $i \leq k - 1$.

▶ **Lemma 3.3.** *For any $k \geq 1$, any planar straight-line drawing of G_k contains a sequence of triangles $\Delta_1, \Delta_2, \ldots, \Delta_k$, where for any $i \in \{1, \ldots, k\}$ the triangle Δ_i is a separating triangle of level i, and for each $i > 1$, in addition, Δ_i is in the interior of Δ_{i-1} and $A(\Delta_i) \leq \frac{1}{2} A(\Delta_{i-1})$.*

▶ **Corollary 3.4.** *For any given $r > 1$, there is a k such that every planar straight-line drawing of G_k with edge lengths at most r contains a separating triangle of area at most $\frac{1}{4}$.*

We call *thin* any triangle with edges of length at least 1 and area at most $\frac{1}{4}$. Any thin triangle has height at most $\frac{1}{2}$ and hence one obtuse angle of size at least $\frac{2\pi}{3}$ and two acute angles, each of size at most $\frac{\pi}{6}$.

▶ **Lemma 3.5.** *Let $\triangle UVW$ be a thin triangle, where the longest edge is UV and let $Z \in \triangle UVW$ be such that $|ZW| \geq 1$. Then one of the angles $\angle UWZ$ or $\angle VWZ$ is obtuse.*

Now we focus our attention on the perimeters of the considered triangles.

▶ **Observation 3.6.** *Let a triangle $\triangle UVW$ be placed in the interior of a polygon Q. Then the perimeter of the triangle is bounded by the perimeter of the polygon, i.e., $P(\triangle UVW) \leq P(Q)$.*
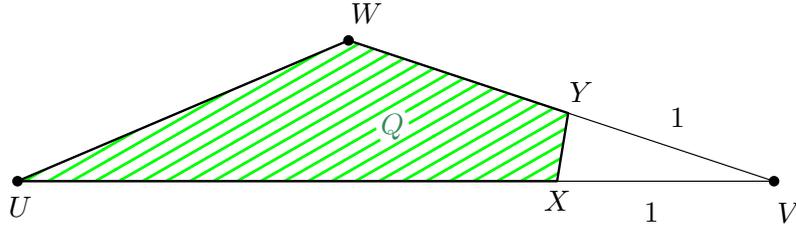
■ **Figure 2** Cutting-off the triangle $\triangle XVY$.

▶ **Lemma 3.7.** *Let $\triangle UVW$ be a thin triangle, where the longest edge is UV. Then the polygon Q, created by cutting off an isosceles triangle $\triangle XVY$ with both edges XV and VY of length 1, has perimeter $P(Q) \leq P(\triangle UVW) - 1$.*

See Fig. 2 for an example of cutting off. We now show that a separating triangle with a small area is guaranteed to contain a separating triangle of a significantly smaller perimeter.

▶ **Lemma 3.8.** *Let G_k have a planar straight-line drawing with edge length at least 1 and let $\triangle UVW$ be a thin separating triangle of level $i \leq k - 1$. Assume that the edge UV is of level $i - 1$ and that it is incident to the obtuse angle of $\triangle UVW$. Then $\triangle UVW$ contains a separating thin triangle Q of level $i + 1$ whose perimeter satisfies $P(Q) \leq P(\triangle UVW) - 1$.*

▶ **Lemma 3.9.** *Let G_k have a planar straight-line drawing with edge length at least 1 and let $\triangle UVW$ be a thin separating triangle of level $i \leq k - 2$. Assume that the edge UV is of level $i - 1$ and that it is* not *incident to the obtuse angle of $\triangle UVW$. Then $\triangle UVW$ contains a separating thin triangle Q of level at most $i + 2$ whose perimeter satisfies $P(Q) \leq P(\triangle UVW) - 1$.*

Now we combine all lemmas together to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. For given r we choose $k = \log_2 \left(\frac{\sqrt{3}}{4} r^2 \right) + 3$ and consider the graph G_{k+4r}. Assume for a contradiction that G_{k+4r} allows a drawing of edge-length ratio at most r. Without loss of generality assume that the longest edge of such drawing has length r and hence the shortest has length at least 1. In the drawing of the graph G_{k+4r} consider a sequence of separating triangles $\Delta_1, \ldots, \Delta_{k+4r}$ where $\Delta_1, \ldots, \Delta_k$ are chosen as shown in Lemma 3.3.

By Corollary 3.4, the triangle Δ_k is thin. Observe that the side-length and area constraints imply that it could be drawn inside a rectangle $r \times \frac{1}{8}$, hence it has perimeter at most $2r + \frac{1}{4}$ by Observation 3.6.

For any $i \in \{0, 1, \ldots, 2r - 1\}$ either the edge of level $k + 2i - 1$ in Δ_{k+2i} is incident to the obtuse angle of Δ_{k+2i} or not:

- In the first case we apply Lemma 3.8 to get $P(\Delta_{k+2i+1}) \leq P(\Delta_{k+2i}) - 1$. As Δ_{k+2i+2} is inside Δ_{k+2i+1}, we get $P(\Delta_{k+2i+2}) \leq P(\Delta_{k+2i}) - 1$.
- Otherwise we apply Lemma 3.9 to derive $P(\Delta_{k+2i+2}) \leq P(\Delta_{k+2i}) - 1$ directly.

Therefore, $P(\Delta_{k+4r}) \leq P(\Delta_k) - 2r \leq 2r + \frac{1}{4} - 2r = \frac{1}{4}$, a contradiction to the assumption that all triangles of G_{k+4r} have all sides of length at least one.   ◀

Note that the graph G_{k+4r} has $O^*((5^4)^r)$ vertices. The dependency between the edge-length ratio and the number of vertices could be rephrased as follows:

▶ **Corollary 3.10.** *The edge-length ratio over the class of n-vertex 2-trees is $\Omega(\log n)$.*

We recall that Borrazzo and Frati prove that every partial 2-tree with n vertices admits a planar straight-line drawing whose edge-length ratio is in $O(n^{0.695})$ (Corollary 1 of [4]).

4 Local edge-length ratio of 2-trees

The aesthetic criterion studied in the previous section took into account any pair of edges. By our construction of nested triangles, it might happen that two edges attaining the maximum length ratio might be far in the graph distance (in the Euclidean distance they are close as the triangles are nested). This observation leads us to the question, whether the two edges might be forced to appear close or whether 2-trees allow drawings where the length ratio of any two adjacent edges could be bounded by a constant. For this purpose we define the local variant of the edge-length ratio as follows:

The *local edge-length ratio* of a planar straight-line drawing of a graph G is the maximum ratio between the lengths of two adjacent edges (sharing a common vertex) of the drawing.

▶ **Definition 4.1.** The *local edge-length ratio* $\rho_l(G)$ of a planar graph G is the infimum local edge-length ratio taken over all planar straight-line drawings of G.

$$\rho_l(G) = \inf_{\text{drawing of } G} \quad \max_{UV, VW \in E_G} \frac{|UV|}{|VW|}$$

Observe that the local edge-length ratio $\rho_l(G)$ is by definition bounded by the global edge-length ratio $\rho(G)$. In particular, every outerplanar graph G allows a drawing witnessing $\rho_l(G) \leq 2$ [11]. We extend this positive result to a class of all 2-trees with a slightly increased bound on the ratio.

▶ **Theorem 4.2.** *The local edge-length ratio of any n-vertex 2-tree G is $\rho_l(G) \leq 4$. Also, for any arbitrarily small positive constant ε, a planar straight-line drawing of G with local edge-length ratio at most $4 + \varepsilon$ can be computed in $O(n)$ time assuming the real RAM model of computation.*
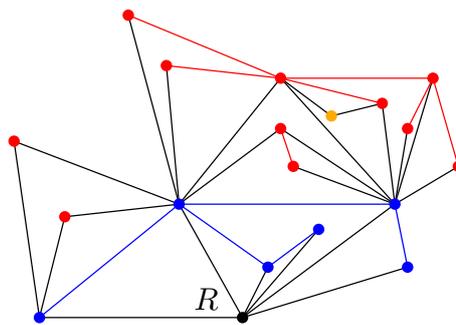
Figure 3 Decomposition rooted in vertex R of a 2-tree. Each connected component of colored vertices forms a part of the decomposition.

The proof of Theorem 4.2 is based on a construction that for a given 2-tree G and any $\varepsilon > 0$ provides a straight-line drawing of local edge-length ratio $4 + \varepsilon$. The general idea of the construction is as follows: We cover G by subgraphs, called parts, so that these parts could be arranged into a rooted tree (see Fig. 3). For any vertex u the parts containing u form a subtree of height one, where the root of this subtree has one edge in common with each of its children and such edge is always incident to u. For each part, we reserve

a suitable area where this part can be drawn and then describe how to draw it there with the local edge-length ratio at most $2 + \frac{\varepsilon}{2}$. For any two adjacent edges, either they belong to the same part or to two parts that have a parent-child or sibling relationship in the tree. By this reasoning we can prove that the local edge-length ratio is at most 4.

5   Open Problems

1. Corollary 3.10 of this paper gives a logarithmic lower bound while Corollary 1 of [4] gives a sub-linear upper bound on the edge-length ratio of planar straight-line drawings of partial 2-trees. We find it interesting to close the gap between upper and lower bound.
2. Theorem 4.2 gives an upper bound of 4 on the local edge-length ratio of partial 2-trees. It would be interesting to establish whether such an upper bound is tight.
3. The construction in Theorem 4.2 creates drawings where the majority of angles are very close to 0 or π radians. Hence, it would make sense to study the interplay between (local or global) edge-length ratio and angular resolution in planar straight-line drawings.

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