Geometric bistellar moves relate triangulations of Euclidean, hyperbolic and spherical manifolds

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Abstract

A geometric triangulation of a Riemannian manifold is a triangulation where the interior of each simplex is totally geodesic. Bistellar moves are local changes to the triangulation which are higher dimensional versions of the flip operation of triangulations in a plane. We show that geometric triangulations of a compact hyperbolic, spherical or Euclidean n-dimensional manifold are connected by geometric bistellar moves (possibly adding or removing vertices), after taking sufficiently many derived subdivisions. For dimension 2 and 3, we show that geometric triangulations of such manifolds are directly related by geometric bistellar moves.

1 Introduction and Notations

If we do not allow adding or removing vertices, it is known that the flip graph of Euclidean triangulations of 2-dimensional polytopes is connected. This forms the basis for the Lawson edge flip algorithm to obtain a Delaunay triangulation. For 5-dimensional polytopes, Santos [7] has given examples of triangulated polytopes with disconnected flip graphs. The problem of connectedness of such a flip graph for 3-dimensional polytopes is still open.

In this article we show that if we allow bistellar moves which add or remove vertices then the flip graph is connected in dimension 3 not just for polytopes but also for geometric triangulations of any compact Euclidean, spherical or hyperbolic manifold. This can be used to show that any quantity calculated from a geometric triangulation which is invariant under geometric Pachner moves is in fact an invariant of the manifold. Furthermore in dimension greater than 3, any two such triangulations are related by bistellar moves after taking suitably many derived subdivisions.

In [4] we give an alternate approach using shellings to relate geometric triangulations via topological bistellar moves (so intermediate triangulations may not be geometric). This gives a tighter bound on the number of such flips required to relate the geometric triangulations, than the one which can be calculated from the algorithm in this article.

A *geometric triangulation* of a Riemannian manifold is a finite triangulation where the interior of each simplex is a totally geodesic disk. Every constant curvature manifold has a geometric triangulation. A *common subcomplex* of simplicial triangulations K_1 and K_2 of M is a simplicial complex structure L on a subspace of M such that $K_1|_{|L|} = K_2|_{|L|} = L$. The main result in this article is the following:

▶ **Theorem 1.1.** *Let K_1 and K_2 be geometric simplicial triangulations of a compact constant curvature manifold M with a (possibly empty) common subcomplex L with $|L| \supset \partial M$. When M is spherical we assume that the diameter of the star of each simplex is less than π. Then*

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for some $s \in \mathbb{N}$, the s-th derived subdivisions $\beta^s K_1$ and $\beta^s K_2$ are related by geometric bistellar moves which keep $\beta^s L$ fixed.

In dimension 2 and 3, every internal stellar move can be realised by geometric bistellar moves (see for example Lemma 2.11 of [3]), so we get the following immediate corollary:

▶ **Corollary 1.2.** *Let K_1 and K_2 be geometric simplicial triangulations of a closed constant curvature 2 or 3-manifold M. When M is spherical we assume that the diameter of the star of each simplex is less than π. Then K_1 is related to K_2 by geometric bistellar moves.*

An *abstract simplicial complex* consists of a finite set K^0 (the vertices) and a family K of subsets of K^0 (the simplexes) such that if $B \subset A \in K$ then $B \in K$. A *simplicial isomorphism* between simplicial complexes is a bijection between their vertices which induces a bijection between their simplexes. A *realisation* of a simplicial complex K is a subspace $|K|$ of some $\mathbb{R}^N$, where K^0 is represented by a finite subset of $\mathbb{R}^N$ and vertices of each simplex are in general position and represented by the linear simplex which is their convex hull. Every simplicial complex has a realisation in $\mathbb{R}^N$ where N is the size of K_0, by representing K_0 as a basis of $\mathbb{R}^N$. Any two realisations of a simplicial complex are simplicially isomorphic. For A a simplex of K, we denote by ∂A the *boundary complex* of A. When the context is clear, we shall use the same symbol A to denote the simplex and the simplicial complex $A \cup \partial A$. We call K a *simplicial triangulation* of a manifold M if there exists a homeomorphism from a realisation $|K|$ of K to M. The simplexes of this triangulation are the images of simplexes of $|K|$ under this homeomorphism.

▶ **Definition 1.3.** For A and B simplexes of a simplicial complex K, we denote their join $A \star B$ as the simplex $A \cup B$. As the join of totally geodesic disks in a constant curvature manifold gives a totally geodesic disk, operations involving joins are well-defined in the class of geometric triangulations of a constant curvature manifold.

The link of a simplex A in a simplicial complex K is the simplicial complex defined by $lk(A, K) = \{B \in K : A \star B \in K\}$. The (closed) star of A in K is the simplicial complex defined by $st(A, K) = A \star lk(A, K)$.

▶ **Definition 1.4.** Suppose that A is an r-simplex in a simplicial complex K of dimension n then a stellar subdivision on A gives the geometric triangulation $(A, a)K$ by replacing $st(A, K)$ with $a \star \partial A \star lk(A, K)$ for $a \in int(A)$. The inverse of this operation $(A, a)^{-1}K$ is called a stellar weld and they both are together called stellar moves. When $lk(A, K) = \partial B$ for some $(n - r)$-dimensional geometric simplex $B \notin K$, then the bistellar move $\kappa(A, B)$ consists of changing K by replacing $A \star \partial B$ with $\partial A \star B$. It is the composition of a stellar subdivision and a stellar weld, namely $(B, a)^{-1}(A, a)$. Another way of defining this is to take a disk subcomplex D of K which is simplicially isomorphic to a disk D' in $\partial\Delta^{n+1}$ and the flip consists of replacing it with the disk $\partial\Delta^{n+1} \setminus int(D')$.

The derived subdivision βK of K is obtained from K by performing a stellar subdivision on all r-simplexes, and ranging r inductively from n down to 1.

All stellar and bistellar moves we shall consider are geometric in nature. See Fig 1 for a combinatorial bistellar move which is not geometric.

As the supports in $\mathbb{R}^N$ of two triangulations of a manifold may be different so when the manifold is not a polytope we can not take a linear cobordism between them. A subtle point here is that even if we obtain a common geometric refinement of two geometric triangulations, the refinement may not be a simplicial subdivision of the corresponding simplicial complexes. To see a topological subdivision which is not a simplicial subdivision, observe that there

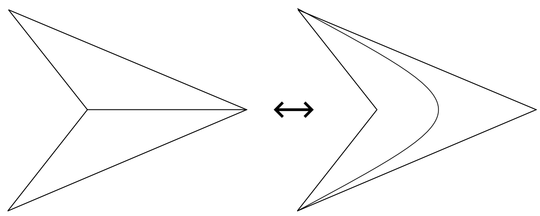

Figure 1 A 2-2 combinatorial bistellar move which is not geometric.

exists a simplicial triangulation K of a 3-simplex Δ which contains in its 1-skeleton a trefoil with just 3 edges [5]. If K were a simplicial subdivision of Δ there would exist a linear embedding of Δ in some $\mathbb{R}^N$ which takes simplexes of K to linear simplexes in $\mathbb{R}^N$. As the stick number of a trefoil is 6, there can exist no such embedding. While there may not exist such a global embedding of a geometric triangulation K as a simplicial complex in $\mathbb{R}^N$ which takes geometric subdivisions to linear (simplicial) subdivisions, for constant curvature manifolds there does exist such a local embedding on stars of simplexes of K. This is the property we exploit in this note.

2 Star-convex flat polyhedra

▶ **Definition 2.1.** We define a flat polyhedron P in $\mathbb{R}^n$ to be the realisation of a simplicial complex in $\mathbb{R}^n$ which is homeomorphic to an n-dimensional closed ball. We call a flat polyhedron P in $\mathbb{R}^n$ strictly star-convex with respect to a point x in its interior if for any $y \in P$, the interior of the segment $[x, y]$ lies in the interior of P.

We call a triangulation K of P regular if there is a function $h : |K| \to \mathbb{R}$ that is linear on each simplex of K and strictly convex across codimension one simplexes of K, i.e., if points x and y are in neighboring top-dimensional simplexes of K then the segment connecting $h(x)$ and $h(y)$ is above the graph of h (except at the end points).

In their proof of the weak Oda conjecture, Morelli [6] and Wlodarczyk [8] proved that any two triangulations of a convex polyhedron are related by a sequence of stellar moves. As interior stellar moves can be given by bistellar moves in dimension 3, Izmestiev and Schlenker [3] have improved on this result to show the following:

▶ **Theorem 2.2** (Lemma 2.11 of [3]). *Any two triangulations of a convex polyhedron P in $\mathbb{R}^3$ can be connected by a sequence of geometric bistellar moves, boundary geometric stellar moves and continuous displacements of the interior vertices.*

With their techniques however, even when the two triangulations agree on the boundary, we still need boundary stellar moves to relate them. Our aim in this section is to show that their techniques can be tweaked to give a boundary relative version for triangulations of strictly star-convex flat polyhedra in any dimension. The main theorem of this section is the following:

▶ **Theorem 2.3.** *Let $P \subset \mathbb{R}^n$ be a strictly star-convex flat polyhedron. Let K_1 and K_2 be triangulations of P that agree on the boundary. Then for some $s \in \mathbb{N}$, their s-th derived subdivisions $\beta^s K_1$ and $\beta^s K_2$ are related by geometric bistellar moves.*

We use the following simple observation in the proof:

▶ **Lemma 2.4** (Lemma 4, Ch 1 of [9])**.** *Let K and L denote two simplicial complexes with $|K| \subset |L|$. Then there exists $r \in \mathbb{N}$ and a subdivision K' of K such that K' is a subcomplex of $\beta^r L$.*

▶ **Lemma 2.5.** *Let K denote a triangulated flat polyhedron. Then for some $s \in \mathbb{N}$, its s-th derived subdivision $\beta^s K$ is regular.*

Proof. Let Δ be an n-simplex with $|\Delta| \supset |K|$. By Lemma 2.4, there exists an $r \in \mathbb{N}$ and subdivision K' of K which is a subcomplex of $\beta^r\Delta$. As Δ is trivially a regular triangulation, so its stellar subdivision $\beta^r\Delta$ is also regular (see for instance [2]). Restricting its regular function to the subcomplex K' we get K' to be regular, as codimension one simplexes of K' are also codimension one simplexes of $\beta^r\Delta$. As $|K| = |K'|$ so applying Lemma 2.4 a second time, we get $s \in \mathbb{N}$ such that $\beta^s K$ is a subdivision of K'. Finally as $\beta^s K$ is the subdivision of a regular subdivision K' of K so by Claim 3 in proof of Theorem 1 of [1], $\beta^s K$ is a regular triangulation. ◀

Proof of 2.3. The techniques in this proof are essentially those of Morelli and Wlodarczyk as detailed in Section 2 of [3].

Choose $a \in \mathbb{R}^{n+1}$ outside K_1 such that the orthogonal projection map $pr : \mathbb{R}^{n+1} \to \mathbb{R}^n$ takes the support of $C(K_1) = a \star K_1 \subset \mathbb{R}^{n+1}$ onto P and takes a to the interior of an n-simplex of K_1. By Lemma 2.5, there exists $s \in \mathbb{N}$ so that $K = \beta^s C(K_1)$ is a regular simplicial cobordism between $\beta^s K_1$ and $\beta^s C(\partial K_1)$. Choose new vertices of the derived subdivision K such that for any simplex $A \in K$ of dimension less than $n+1$, $pr(A)$ is a simplex of the same dimension as A.

Let $h : |K| \to \mathbb{R}$ be a regular function for K. If a simplex σ' has some point above a simplex σ then $\frac{\partial h}{\partial x_{n+1}}$ on σ' is greater than $\frac{\partial h}{\partial x_{n+1}}$ on σ. So inductively removing simplexes in non-increasing order of the vertical derivative of h we ensure that the projection of the upper boundary onto P is always one-to-one. That is, we get a sequence of triangulations $\Sigma_0 = K, \Sigma_1, \ldots, \Sigma_N = K_1$ such that $\Sigma_{i+1} = \Sigma_i \setminus \sigma_i$ and the orthogonal projection $pr : \partial^+\Sigma_i \to P$ from the upper boundary of Σ_i onto P is one-to-one for every i. Removing an $n+1$-simplex σ_i from K corresponds to a bistellar move on $\partial^+\Sigma_i$. As the projection map is linear so it also corresponds to a bistellar move taking $pr(\partial^+\Sigma_i)$ to $pr(\partial^+\Sigma_{i+1})$. Therefore $pr(\partial^+\Sigma_0) = \beta^s C(\partial K_1)$ is bistellar equivalent to $pr(\partial^+\Sigma_N) = \beta^s K_1$. Consequently, $\beta^s K_1$ is bistellar equivalent to $\beta^s K_2$ via $\beta^s C(\partial K_1) = \beta^s C(\partial K_2)$. ◀

3 Geometric manifolds

▶ **Definition 3.1.** Let K be a geometric triangulation of a Riemannian manifold M and let L be a subcomplex of K. We call K locally geodesically-flat relative to L if for each simplex A of $K \setminus L$, $st(A, K)$ is simplicially isomorphic to a star-convex flat polyhedron in $\mathbb{R}^n$ by a map which takes geodesics to straight lines.

▶ **Definition 3.2.** Let L be a subcomplex of K containing ∂K and let αK be a subdivision of K which agrees with K on L. Let $\beta_r^\alpha K$ be the subdivision of K such that, if A is a simplex in L or $dim(A) \leq r$, then $\beta_r^\alpha A = \alpha A$. If A is not in L and $dim(A) > r$ then $\beta_r^\alpha A = a \star \beta_r^\alpha \partial\alpha A$, i.e. it is subdivided as the cone on the already defined subdivision of its boundary. Observe that $\beta_n^\alpha K$ is αK while $\beta_0^\alpha K = \beta_L K$ is a derived subdivision of K relative to L.

▶ **Lemma 3.3.** *Let K be a locally geodesically-flat simplicial complex relative to a subcomplex L which contains ∂K. Let αK be a geometric subdivision of K which agrees with K on L.*

Then there exists $s \in \mathbb{N}$ for which $\beta^s \alpha K$ is related to $\beta^s K$ by bistellar moves which keep $\beta^s L$ fixed.

Proof. For A a positive dimensional r-simplex in $K \setminus L$, $st(A, \beta_r^\alpha K)$ is a strictly star-convex subset of $st(A, K)$. As K is locally geodesically-flat relative to L, there exists a geodesic embedding taking $st(A, \beta_r^\alpha K)$ to a strictly star-convex flat polyhedron of $\mathbb{R}^n$. By Theorem 2.3, $\beta^s st(A, \beta_r^\alpha K)$ is bistellar equivalent to $\beta^s C(\partial st(A, \beta_r^\alpha K))$. As A is not in L so no interior simplex of $st(A, \beta_r^\alpha K)$ is in L and consequently these bistellar moves keep $\beta^s L$ fixed. Taking all simplexes A in $K \setminus L$ of dimension $r = n$, we get a sequence of bistellar moves taking $\beta^s \beta_r^\alpha K$ to $\beta^s \beta_{r-1}^\alpha K$. Ranging r from n down to 1, we inductively obtain a sequence of bistellar moves taking $\beta^s \alpha K = \beta^s \beta_n^\alpha K$ to $\beta^s \beta_L K = \beta^s \beta_0^\alpha K$, which keeps $\beta^s L$ fixed.

And finally, arguing as above with the trivial subdivision $\alpha K = K$, we get $\beta^s \beta_L K$ from $\beta^s K$ by bistellar moves which keep $\beta^s L$ fixed. ◀

The following simple observation allows us to treat the star of a simplex in a geometric triangulation as the linear triangulation of a star-convex polytope in $\mathbb{R}^n$ and bistellar moves in the manifold as bistellar moves of the polytope.

▶ **Lemma 3.4.** *Let K be a geometric simplicial triangulation of a spherical, hyperbolic or Euclidean n-manifold M and let L be a subcomplex of K containing ∂K. When M is spherical we require the star of each positive dimensional simplex of $K \setminus L$ to have diameter less than π. Then K is locally geodesically-flat relative to L.*

Proof. Let K be a geometric triangulation of M and let B be the interior of the star of a simplex in $K \setminus L$. As K is simplicial, B is an open n-ball.

When M is hyperbolic, let $\phi : B \to \mathbb{H}^n$ be the lift of B to the hyperbolic space in the Klein model. As geodesics in the Klein model are Euclidean straight lines (as sets) so ϕ is the required embedding.

When M is spherical, let D be the southern hemisphere of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, let T be the hyperplane $x_{n+1} = -1$ and let $p : D \to T$ be the radial projection map (gnomonic projection) which takes spherical geodesics to Euclidean straight lines. As B is small enough, lift B to D and compose with the projection p to obtain the required embedding ϕ from B to $T \simeq \mathbb{E}^n$.

When B is Euclidean let ϕ be the lift of B to $\mathbb{R}^n$, which is an isometry. ◀

It is known (Theorem 4(c) of [1]) that for simplicial complexes of dimension at least 5 the number of derived subdivisions required to make the link of a vertex combinatorially isomorphic to a convex polyhedron is not (Turing machine) computable. So in particular, the stars of simplexes of a geometric triangulation may not even be combinatorially isomorphic to convex polyhedra, which is why we need to work with star-convex polyhedra instead.

Given a Riemannian manifold M, a *geometric polytopal complex* C of M is a finite collection of geometric convex polytopes whose union is all of M and such that for every $P \in C$, C contains all faces of P and intersection of two polytopes is a face of each of them.

Proof of 1.1. By Lemma 3.4, K_1 and K_2 are locally geodesically flat simplicial complexes. Let C be the geometric polytopal complex obtained by intersecting the simplexes of K_1 and K_2. Then $K = \beta_L C$, the derived subdivision of C relative to L is a common geometric subdivision of K_1 and K_2. By Lemma 3.3 then, there exists $s \in \mathbb{N}$ so that $\beta^s K_1$ and $\beta^s K_2$ are bistellar equivalent via $\beta^s K$ by bistellar moves which leave $\beta^s L$ fixed. ◀

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