Edge Guarding Plane Graphs

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Abstract

Let \(G = (V,E)\) be a plane graph. We say that a face \(f\) of \(G\) is guarded by an edge \(vw \in E\) if at least one vertex from \(\{v,w\}\) is on the boundary of \(f\). For a planar graph class \(\mathcal{G}\) we ask for the minimal number of edges needed to guard all faces of any \(n\)-vertex graph in \(\mathcal{G}\). In this extended abstract we provide new bounds for two planar graph classes, namely the quadrangulations and the stacked triangulations.

1 Introduction

In 1975, Chvátal [4] laid the foundation for the widely studied field of art gallery problems by answering how many guards are needed to observe all interior points of any given \(n\)-sided polygon \(P\). Here a guard is a point \(p\) in \(P\) and it can observe any other point \(q\) in \(P\), if the line segment \(pq\) is fully contained in \(P\). He shows that \(\lfloor n/3 \rfloor\) guards are sometimes necessary and always sufficient. Fisk [7] revisited Chvátal’s Theorem in 1978 and gave a very short and elegant new proof by introducing diagonals into the polygon \(P\) to obtain a triangulated, outerplanar graph. Such graphs are 3-colorable and in each 3-coloring all faces are incident to vertices of all three colors, so the vertices of the smallest color class can be used as guard positions. Bose et al. [3] studied the problem to guard the faces of a plane graph instead of a polygon. A plane graph is a graph \(G = (V,E)\) with an embedding in \(\mathbb{R}^2\) with not necessarily straight edges and no crossings in the interior of any two edges. Here a face \(f\) is guarded by a vertex \(v\), if \(v\) is on the boundary of \(f\). They show that \(\lfloor n/2 \rfloor\) vertices (so called vertex guards) are sometimes necessary and always sufficient for \(n\)-vertex plane graphs.

We consider a variant of this problem introduced by O’Rourke [9]. He shows that only \(\lfloor n/4 \rfloor\) guards are necessary in Chvátal’s original setting if each guard is assigned to an edge of the polygon that he can patrol along instead of being fixed to a single point. Considering plane graphs again, an edge guard is an edge \(vw \in E\) and guards all faces having \(v\) and/or \(w\) on their boundary. For a given planar graph class \(\mathcal{G}\), we ask for the minimal number of edge guards needed to guard all faces of every plane \(n\)-vertex graph in \(\mathcal{G}\).

General (not necessarily triangulated) \(n\)-vertex plane graphs might need at least \(\lfloor n/3 \rfloor\) edge guards, even when requiring 2-connectedness [3]. The best known upper bounds have recently been presented by Biniaz et al. [1] and come in two different fashions: First, any \(n\)-vertex plane graph can be guarded by \(\lfloor 3n/8 \rfloor\) edge guards found in an iterative process. Second, a coloring approach yields an upper bound of \(\lfloor n/3 + \alpha/9 \rfloor\) edge guards where \(\alpha\) counts the number of quadrangular faces in \(G\). Looking at \(n\)-vertex triangulations, Bose et al. [3] provide a lower bound of \(\lfloor (4n - 8)/13 \rfloor\) edge guards. A corresponding upper bound of \(\lfloor n/3 \rfloor\) edge guards was published earlier in the same year by Everett and Rivera-Campo [6].

This note is based on the master’s thesis of the first author [8] and we present our results on quadrangulations and stacked triangulations. For both planar graph classes we give a lower and an upper bound for the number of edge guards. All graphs considered below are assumed to be plane, i.e. given with a fixed plane embedding.

This is an extended abstract of a presentation given at EuroCG’20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.
2 Main Results

2.1 Quadrangulations

Quadrangulations are the maximal plane bipartite graphs and every face is bounded by exactly four edges. All coloring approaches developed previously [1, 6] fail on graphs containing quadrangular faces. The previously best known upper bounds are the ones given by Biniaz et al. [1] for general plane graphs, \(3n/8\) respectively \([n/3 + \alpha/9]\), where \(\alpha\) is the number of quadrilateral faces. For \(n\)-vertex quadrangulations we have \(\alpha = n - 2\), so \([n/3 + (n-2)/9] = [(4n - 2)/9] > [3n/8]\) for \(n \geq 4\). In this section we provide a better upper and a not yet matching lower bound. Closing the gap remains an open problem.

▶ **Theorem 2.1.** For \(k \in \mathbb{N}\) there exists a quadrangulation \(Q_k\) with \(n = 4k + 2\) vertices needing \(k = (n-2)/4\) edge guards.

**Proof.** Define \(Q_k = (V, E)\) with \(V := \{s, t\} \cup \bigcup_{i=1}^{k} \{a_i, b_i, c_i, d_i\}\) and \(E := \bigcup_{i=1}^{k} \{sa_i, sc_i, ta_i, tc_i, a_i b_i, a_i d_i, c_i b_i, c_i d_i\}\) as the union of \(k\) vertex disjoint 4-cycles and two extra vertices connecting them. Figure 1a shows this and a planar embedding. Now for any two distinct \(i, j \in \{1, \ldots, k\}\) the two quadrilateral faces \((a_i, b_i, c_i, d_i)\) and \((a_j, b_j, c_j, d_j)\) are only connected via paths through \(s\) or \(t\). Therefore, no edge can guard two or more of them and we need at least \(k\) edge guards for \(Q_k\). On the other hand it is easy to see that \(\{sa_1, \ldots, sa_k\}\) is an edge guard set of size \(k\), so \(Q_k\) needs exactly \(k\) edge guards. ▶

The following Lemma is from Bose et al. [2] and we cite it using the terminology of Biniaz et al. [1]. A **guard coloring** of a plane graph \(G\) is a non-proper 2-coloring of its vertex set, such that each face \(f\) of \(G\) has at least one boundary vertex of each color and at least one monochromatic edge (i.e. an edge where both endpoints receive the same color). They prove that a guard coloring exists for all graphs without any quadrangular faces.

▶ **Lemma 2.2 ([2, Lemma 3.1]).** If there is a guard coloring for an \(n\)-vertex plane graph \(G\), then \(G\) can be guarded by \([n/3]\) edge guards.

▶ **Theorem 2.3.** Every quadrangulation can be guarded by \([n/3]\) edge guards.

**Proof.** Let \(G\) be a quadrangulation. We show that there is a guard coloring for \(G\), which is sufficient by Lemma 2.2. Consider the dual graph \(G^* = (V^*, E^*)\) of \(G\) with its inherited plane embedding, so each vertex \(f^* \in V^*\) is placed inside the face \(f\) of \(G\) corresponding to it. Since every face of \(G\) is of degree four, its dual graph \(G^*\) is 4-regular. Using Petersen’s
2-Factor Theorem \cite{10}\footnote{Diestel \cite[Corollary 2.1.5]{5} gives a very short and elegant proof of this theorem in his book. He only considers simple graphs there, but all steps in the proof (including the given proof of Hall’s Theorem \cite[Theorem 2.1.2]{5,11}) also work for multigraphs like $G^*$ that have at most two edges between any pair of vertices.} we get that $G^*$ contains a 2-factor $H$ (a spanning 2-regular subgraph). Any vertex of $H$ is of degree 2, so $H$ is a set of vertex-disjoint cycles that can be nested inside each other. Now define a 2-coloring $\text{col} : V \rightarrow \{0, 1\}$ for the vertices of $G$: For each $v \in V$ let $c_v$ be the number of cycles $C$ of $H$ such that $v$ belongs to the region of the embedding surrounded by $C$. The color of $v$ is determined by the parity of $c_v$ as $\text{col}(v) := c_v \mod 2$.

We claim that this yields a guard coloring of $G$: Any edge $e = ab \in E$ has a corresponding dual edge $e^*$. If $e^* \in E(H)$, then $e$ crosses exactly one cycle edge, so $|c_a - c_b| = 1$ and therefore $\text{col}(a) \neq \text{col}(b)$. Otherwise $e \notin E(H)$, so its two endpoints are in the same cycles, thus $\text{col}(a) = \text{col}(b)$ and $e$ is monochromatic. Because $H$ is a 2-factor, each face has exactly two monochromatic edges. ☐

Figure 1b shows an example quadrangulation with a 2-factor in its dual graph. From here it is easy to color the vertices in green and orange to obtain a guard coloring.

In order to bridge the gap between the lower ($\lceil (n - 2)/4 \rceil$) and the upper bound ($\lceil n/3 \rceil$), we also consider the subclass of 2-degenerate quadrangulations in the master’s thesis \cite[Theorem 5.9]{8}:

\begin{itemize}
  \item \textbf{Theorem 2.4.} Every $n$-vertex 2-degenerate quadrangulation can be guarded by $\lceil n/4 \rceil$ edge guards.
\end{itemize}

Note that this bound is best possible, as the quadrangulations constructed in Theorem 2.1 are 2-degenerate.

### 2.2 Stacked Triangulations

The stacked triangulations (also known as Apollonian networks or planar 3-trees) are a subclass of the triangulations that can recursively be formed by the following rules:

(i) A triangle is a stacked triangulation and (ii) if $G$ is a stacked triangulation and $f$ an inner face, then the graph obtained by placing a new vertex into $f$ and connecting it with all three boundary vertices is again a stacked triangulation. We shall prove that the stacked triangulations are a non-trivial subclass of the triangulations that need strictly less than $\lceil n/3 \rceil$ edge guards (which is the best known upper bound for general triangulations).

\begin{itemize}
  \item \textbf{Theorem 2.5.} For even $k \in \mathbb{N}$ there is a stacked triangulation $G$ with $n = (7k + 4)/2$ vertices needing at least $k = (2n - 4)/7$ edge guards.
\end{itemize}

\textbf{Proof.} Let $S$ be a stacked triangulation with $k$ faces and therefore $(k + 4)/2$ vertices (by Euler’s formula). Subdivide each face $f$ of $S$ with three new vertices $a_f, b_f, c_f$ such that the resulting graph is a stacked triangulation and these three vertices form a new triangular face $t_f$, i.e. $f$ and $t_f$ do not share any boundary vertices. This subdivision is shown in Figure 2a for a single face $f$. Then $G$ has $n = (k + 4)/2 + 3k = (7k + 4)/2$ vertices. For any two distinct faces $f, g$ of $S$ the shortest path between any two boundary vertices of the new faces $t_f$ and $t_g$ has length at least 2, so no edge can guard both of them. Therefore $G$ needs at least $k$ edge guards. ☐

\begin{itemize}
  \item \textbf{Theorem 2.6.} Every $n$-vertex stacked triangulation can be guarded by $\lfloor 2n/7 \rfloor$ edge guards.
\end{itemize}
A proof of Theorem 2.6 is given in the master’s thesis [8, Theorem 4.14] but it is too long for this extended abstract. We restrict ourselves to briefly describing the main idea: We do induction on \( n \), the number of vertices. Given any \( n \)-vertex stacked triangulation, we find a triangle \( \Delta := \{x, y, z\} \subseteq V(G) \) containing at least \( k^- \geq 4 \) vertices inside of it but among all possible candidates one where \( k^- \) is minimal. Let \( V^- \subseteq V \) be the vertices in the interior of \( \Delta \). We remove \( V^- \) from \( G \), so \( \Delta \) becomes a face and we subdivide it with \( V^+ \) new vertices \( V^+ \). Call the resulting graph \( G^+ \). Applying the induction hypothesis on \( G^+ \) provides us with an edge guard set \( \Gamma' \) of size at most \( \lceil 2|V(G^+)|/7 \rceil \). We show that \( \Gamma' \) can be augmented to an edge guard set \( \Gamma \) for \( G \) with size \( |\Gamma| = |\Gamma'| + \ell \), such that \( \ell/(k^- - k^+) \leq 2/7 \), so that \( \Gamma \) has size at most \( |2n/7| \).

For example consider a stacked triangulation \( G \) with a separating triangle \( \Delta = \{x, y, z\} \) as shown in Figure 2b with \( k^- = 6 \) vertices \( V^- \) inside (the figure only shows the separating triangle and its interior vertices). Assume for now that \( V^+ = \emptyset \), so \( \Delta \) is a face in \( G^+ \). An edge guard set \( \Gamma' \) of \( G^+ \) guards \( \Delta \), for example we could have \( x \in V(\Gamma') \) and \( y, z \notin V(\Gamma') \). But then – after reinserting the vertices of \( V^- \) – no single edge can guard all the remaining faces.

So in this situation it is impossible to extend \( \Gamma' \) by a single edge to and edge guard set \( \Gamma \) for \( G \). The following lemma tells us how to choose \( V^+ \) instead, such that such a situation cannot arise.

**Lemma 2.7.** Let \( \{x, y, z\} \) be a face of a stacked triangulation \( G \). By stacking two new vertices into \( \{x, y, z\} \) we can obtain a stacked triangulation \( H \) such that for each edge guard set \( \Gamma \) of \( H \) there is an edge guard set \( \Gamma' \) with \( x, y \in V(\Gamma') \) and \( |\Gamma'| \leq |\Gamma| \).

**Proof.** Add vertex \( a \) with edges \( xa, ya, za \) and then vertex \( b \) with edges \( ab, xb, yb \) to obtain \( H \) (see Figure 2c). Now let \( \Gamma \) be any edge guard set for \( H \) not yet fulfilling the requirements, so \( |\{x, y\} \cap V(\Gamma)| \leq 1 \). If \( b \notin V(\Gamma) \) as part of an edge \( vb \), we can set \( \Gamma' := (\Gamma \setminus \{vb\}) \cup \{xy\} \). This is possible, because for any possible neighbor \( v \) of \( b \), edge \( xy \) guards a superset of the faces that \( vb \) guards. If otherwise \( b \notin V(\Gamma) \), we assume without loss of generality that \( x \in V(\Gamma) \) and \( y \notin V(\Gamma) \). Note that \( |\{x, y\} \cap V(\Gamma)| \geq 1 \), because face \( \{x, y, b\} \) must be guarded. Face \( \{a, b, y\} \) can then only be guarded by edge \( va \) where \( v \in \{x, z\} \). Since \( N(a) \subseteq N(y) \) we can set \( \Gamma' := (\Gamma \setminus \{va\}) \cup \{vy\} \). In both cases \( x, y \in V(\Gamma') \) and \( |\Gamma'| \leq |\Gamma| \).

Let us go back to the example in Figure 2b: Using Lemma 2.7, we can now remove the six vertices in \( V^- \), add two new ones \( V^+ := \{a, b\} \) as in Figure 2c and assume that the
induction hypothesis gives us an edge guard set $\Gamma'$ with $x, y \in V(\Gamma')$. Then one additional edge is enough to guard the remaining inner faces and $\ell/(|V^-| - |V^+|) = 1/(6 - 2) \leq 2/7$ as desired. This guard set is shown in Figure 2b in red.

In addition to Lemma 2.7, we prove two more of this kind in the master’s thesis [8] and which we list here without a proof. Like the lemma above, they describe how to add new vertices $V^+$ into a stacked triangulation, such that the resulting graph is still a stacked triangulation and that we can assume certain properties of minimal edge guard sets. Combining them, allows to handle all possible ways how the vertices $V^-$ inside $\Delta$ can be connected.

\begin{itemize}
    \item \textbf{Lemma 2.8.} Let $G$ be a stacked triangulation, $v$ be a vertex of degree 3 and $x, y, z$ its neighbors in $G$. Then for any edge guard set $\Gamma$ guarding $G$ we have $|\{v, x, y, z\} \cap V(\Gamma)| \geq 2$.
    \item \textbf{Lemma 2.9.} Let $(x, y, z)$ be a face of a stacked triangulation $G$. By stacking three new vertices into $(x, y, z)$ we can obtain a stacked triangulation $H$ such that for each edge guard set $\Gamma$ of $H$ there is an edge guard set $\Gamma'$ with $x \in V(\Gamma')$ and an edge $vw \in \Gamma'$ with $v \in \{x, y, z\}$ and $w$ inside $(x, y, z)$. Further $|\Gamma'| \leq |\Gamma|$.
\end{itemize}

We conclude this note with the following open problems:

\begin{itemize}
    \item \textbf{Open Problems.} How many edge guards are sometimes necessary and always sufficient for quadrangulations, $(4\text{-}connected)$ triangulations and general plane graphs?
\end{itemize}

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\section*{References}

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